Range Software Theory

Tomáš Šoltys
Range Software
tomas.soltys@range-software.com

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Introduction

Some introduction ...

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Governing equations

2.1 Heat transfer

2.1.1 Conduction

$$\rho c \frac{\partial T}{\partial t} = -k \nabla T \tag{2.1}$$

$$\rho c \sum_{j=1}^{nf} \frac{\partial T_j}{\partial t} = -k \sum_{j=1}^{nf} \nabla T_j \tag{2.2}$$

2.1.2 Convection

$$q_h = hA\left(T_h - T\right) \tag{2.3}$$

$$q_h^i = h \sum_{j=1}^{nf} A_j \left(T_h - T_j\right) \tag{2.4}$$

2.1.3 Radiation

$$\sum_{j=1}^{np} \left(\frac{\delta_{ij}}{\varepsilon_j} - F_{ij} \frac{1-\varepsilon_j}{\varepsilon_j} \right) q_r^j = \sum_{j=1}^{np} \left(\delta_{ij} - F_{ij}\right) \sigma T_j^4 - \left(1 - \sum_{j=1}^{np} F_{ij}\right) \sigma T_a^4 \tag{2.5}$$

$$A_{ij} = \frac{\delta_{ij}}{\varepsilon_j} - F_{ij} \frac{1-\varepsilon_j}{\varepsilon_j} \tag{2.6}$$

$$B_{ij} = \left(\delta_{ij} - F_{ij}\right) \sigma \tag{2.7}$$

$$q_{ra}^i = \left(1 - \sum_{j=1}^{np} F_{ij}\right) \sigma T_a^4 \tag{2.8}$$

$$\sum_{j=1}^{np} A_{ij} q_r^j = \sum_{j=1}^{np} B_{ij} T_j^4 - q_{ra}^i \tag{2.9}$$

2.1.4 Conservation of energy

$$\rho c \frac{\partial T}{\partial t} + k \nabla T = q_h + q_r + q \tag{2.10}$$

Conduction-Convection equation

$$\rho c \sum_{j=1}^{nf} \frac{\partial T_j}{\partial t} + k \sum_{j=1}^{nf} \nabla T_j = h \sum_{j=1}^{nf} A_j \left(T_h - T_j\right) + q_r^i + q^i \tag{2.11}$$

$$\rho c \sum_{j=1}^{nf} \frac{\partial T_j}{\partial t} + k \sum_{j=1}^{nf} \nabla T_j + h \sum_{j=1}^{nf} A_j T_j = h \sum_{j=1}^{nf} A_j T_h + q_r^i + q^i \ ; \ i \in \langle 1, nf \rangle \tag{2.12}$$

Radiation equation

$$\rho c \sum_{j=1}^{np} B_{ij} T_j^4 = h \sum_{j=1}^{np} A_{ij} \tag{2.13}$$

$$q_r^j = q^j + q_h^j \tag{2.14}$$

$$q_r^j = q^j + h \left(T_h - T_j\right) \tag{2.15}$$

$$\sum_{j=1}^{np} B_{ij} T_j^4 = \sum_{j=1}^{np} A_{ij} \left(q^j + h \left(T_h - T_j\right)\right) + q_{ra}^i \tag{2.16}$$

$$\sum_{j=1}^{np} \left(B_{ij} T_j^4 + A_{ij} h T_j\right) = \sum_{j=1}^{np} A_{ij} \left(q^j + h T_h\right) + q_{ra}^i \ ; \ i \in \langle 1, np \rangle \tag{2.17}$$

$$q_r^i = \sigma T_i^4 - q^i - h \left(T_h - T_i\right) \ ; \ i \in \langle 1, np \rangle \tag{2.18}$$

Coupling algorithm

1. Solve for temperature in equation 2.17

2. Calculate the radiative heat flux 2.18

3. Solve equation 2.12

2.1.5 Notation

Variables description		
T	Temperature	$[K]$
q	Heat flux	$[W \cdot m^{-2}]$
ρ	Material density	$[kg \cdot m^{-3}]$
c	Material heat capacity	$[J \cdot kg^{-3} \cdot K^{-1}]$
k	Material thermal conductivity	$[W \cdot m^{-1} \cdot K^{-1}]$
h	Convection coefficient	$[W \cdot m^{-2} \cdot K^{-1}]$
σ	StefanBoltzmann constant $\sigma = 5.670373\,(21) \times 10^{-8}$	$[W \cdot m^{-2} \cdot K^{-4}]$
ε_j	Emissivity	NA
F_{ij}	View factor $F_{ij} = \langle 0; 1 \rangle$	NA
δ_{ij}	Dirac delta function	NA

2.2 Electromagnetism

2.2.1 Electrostatics

Electrical potential

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_r \varepsilon_0} \tag{2.19}$$

Electric field

$$\mathbf{E} = -\nabla \phi \tag{2.20}$$

Current density

$$\mathbf{J} = \sigma \mathbf{E} \tag{2.21}$$

Joule heating

$$Q = \sigma \int \mathbf{E} \cdot \mathbf{E} \mathrm{d}V \tag{2.22}$$

Energy per unit volume of the electric field

$$u^e = \frac{1}{2} \varepsilon_0 \mathbf{E}^2 \tag{2.23}$$

2.2.2 Magnetostatics

Curl identity operation

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \tag{2.24}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} \tag{2.25}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{2.26}$$

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \tag{2.27}$$

$$\nabla \left(\nabla \cdot \mathbf{F} \right) = \begin{bmatrix} \frac{\partial}{\partial x} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \\ \frac{\partial}{\partial y} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \\ \frac{\partial}{\partial z} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \end{bmatrix} \tag{2.28}$$

$$\nabla^2 \mathbf{F} = \begin{bmatrix} \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2} \\ \frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_y}{\partial z^2} \\ \frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2} \end{bmatrix} \tag{2.29}$$

$$\nabla \times \mathbf{F} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{2.30}$$

Gauss's law for magnetism

$$\nabla \cdot \mathbf{B} = 0 \tag{2.31}$$

Ampere's circuital law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \underbrace{\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}_{if steady state \Rightarrow 0} \tag{2.32}$$

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla \times (\mu_0 \mathbf{J}) \tag{2.33}$$

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla \left(\underbrace{\nabla \cdot \mathbf{B}}_{0} \right) - \nabla^2 \mathbf{B} \tag{2.34}$$

$$-\nabla^2 \mathbf{B} = \nabla \times (\mu_0 \mathbf{J}) \tag{2.35}$$

2.2.3 Notation

Variables description		
ϕ	electric potential	$[J \cdot C^{-1}] \vee [V]$
$\mathbf{E}$	electric field	$[N \cdot C^{-1}] \vee [V \cdot m^{-1}]$
$\mathbf{J}$	current density	$[A \cdot m^{-2}]$
$\mathbf{B}$	magnetic field	$[T]$
Q	Joule heat	$[W]$
u_e	energy per unit volume of the electric field	$[J \cdot m^{-3}]$
ρ	charge density	$[C \cdot m^{-3}]$
ε_r	relative permittivity of the medium (dielectric constant)	$[C^2 \cdot N^{-1} m^{-2})]$
ε_0	vacuum permittivity $\varepsilon_0 = 8.854187817 \times 10^{-12}$	$[C^2 \cdot N^{-1} m^{-2})]$
σ	electrical conductivity	$[S \cdot m^{-1}]$
μ_0	vacuum permeability $\mu_0 = 4\pi \times 10^{-7}$	$[V \cdot s \cdot A^{-1} \cdot m^{-1} \vee N \cdot A^{-2}]$

2.3 Waves

2.3.1 General Wave

$$\frac{\partial^2 u}{\partial t^2} + d\frac{\partial u}{\partial t} = c(u)^2 \left(\nabla^2 u\right) \tag{2.36}$$

For linear case c is a fixed constant equal to the propagation speed of wave. For acoustic $c = 343\,[m{\cdot}s^-1]$ (speed of sound in dry air at $20\,[C]$)

Newmark-beta method

$$\ddot{u}_\beta = (1-2\beta)\,\ddot{u}_n + 2\beta\ddot{u}_{n+1} \qquad 0 \leq 2\beta \leq 1 \tag{2.37}$$

$$\dot{u}_{n+1} = \dot{u}_n + \frac{\Delta t}{2}\left(\ddot{u}_n + \ddot{u}_{n+1}\right) \tag{2.38}$$

$$u_{n+1} = u_n + \Delta t\dot{u}_n + \frac{1-2\beta}{2}\Delta t^2\ddot{u}_n + \beta\Delta t^2\ddot{u}_{n+1} \tag{2.39}$$

$$\mathbf{A}\ddot{\mathbf{u}}_{n+1} + \mathbf{B}\dot{\mathbf{u}}_{n+1} + \mathbf{C}\mathbf{u}_{n+1} = \mathbf{F}_{n+1} \tag{2.40}$$

$$\begin{aligned} \mathbf{A}\ddot{\mathbf{u}}_{n+1} + \mathbf{B}\left(\dot{\mathbf{u}}_n + \frac{\Delta t}{2}\left(\ddot{\mathbf{u}}_n + \ddot{\mathbf{u}}_{n+1}\right)\right) \\ +\mathbf{C}\left(\mathbf{u}_n + \Delta t\dot{\mathbf{u}}_n + \frac{1-2\beta}{2}\Delta t^2\ddot{\mathbf{u}}_n + \beta\Delta t^2\ddot{\mathbf{u}}_{n+1}\right) = \mathbf{F}_{n+1} \end{aligned} \tag{2.41}$$

Discretized equation

$$\begin{aligned} \left((1+d\Delta t)\,\mathbf{M} - \Delta t^2\mathbf{K}\right)\mathbf{u}^{(n)} &= \Delta t^2\mathbf{f}^{(n)} \\ &+ \mathbf{M}\left((2+d\Delta t)\,\mathbf{u}^{(n-1)} - \mathbf{u}^{(n-2)}\right) \end{aligned} \tag{2.42}$$

$$\begin{aligned} \left((1+d\Delta t)\,\mathbf{M} - \alpha\Delta t^2\mathbf{K}\right)\mathbf{u}^{(n)} &= \Delta t^2\mathbf{f}^{(n)} \\ &+ \mathbf{M}\left((2+d\Delta t)\,\mathbf{u}^{(n-1)} - \mathbf{u}^{(n-2)}\right) \\ &- \left((1-\alpha)\,\Delta t^2\mathbf{K}\right)\mathbf{u}^{(n-1)} \end{aligned} \tag{2.43}$$

Absorbing boundary

This technique was developed by Higdon (1991).

$$u_{t+1} = -q_x u_{t+1,2} - q_t U_{t,1} - q_{tx} u_{t,2} \tag{2.44}$$

Where:

$$
\begin{aligned}
q_x &= \frac{b\,(B+V)-V}{(B+V)\,(1-b)} \\
q_t &= \frac{b\,(B+V)-B}{(B+V)\,(1-b)} \\
q_{tx} &= \frac{b}{b-1} \\
b &= 0.4 \\
B &= 1 \\
V &= v\frac{\Delta t}{\Delta x} \\
v &- \text{velocity of the wavefront normal to the boundary}
\end{aligned}
$$

2.3.2 Elastic Wave

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{f} + (\lambda + 2\mu)\,\nabla\,(\nabla\cdot\mathbf{u}) - \mu\nabla\times(\nabla\times\mathbf{u}) \tag{2.45}$$

Lamé parameters

$$\lambda = \frac{E\nu}{(1+\nu)\,(1-2\nu)} \tag{2.46}$$

$$\mu = \frac{E}{2\,(1+\nu)} \tag{2.47}$$

2.3.3 Notation

Variables description		
c	Propagation speed of wave	$[m \cdot s^{-1}]$
d	Damping parameter	$[1 \cdot s^{-1}]$
λ	Lamé's first parameter	$[GPa]$
μ	Lamé's second parameter	$[GPa]$
E	Young's modulus	$[GPa]$
ν	Poison's ratio	NA
ρ	Density	$[kg \cdot m^{-3}]$
$\mathbf{f}$	source function (driving force)	NA
$\mathbf{u}$	wave displacement	$NA \vee [m]$
t	time	$[s]$

2.4 Incompressible newtonian fluids

2.4.1 Navier-Stokes equations of incompressible flows

Incompressible fluid flow is described by two equations. These are **momentum equation** 2.48 and **incompressibility equation** 2.49.

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f}\right) - \nabla\cdot\boldsymbol{\sigma} = 0 \tag{2.48}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.49}$$

For Newtonian fluids stress tenzor $\boldsymbol{\sigma}$ can be expressed with:

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\varepsilon \tag{2.50}$$

Where $\mathbf{I}$ is an identity matrix ε is a strain-rate tensor and μ is dynamic viscosity.

$$\varepsilon = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \tag{2.51}$$

$$\mu = \rho\nu \tag{2.52}$$

$$\nabla \cdot (2\mu\varepsilon) = \mu\nabla \cdot (2\varepsilon) \tag{2.53}$$

$$\begin{aligned}
\nabla \cdot (2\varepsilon) &= \nabla \cdot \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \\
&= \nabla \cdot (\nabla\mathbf{u}) + \nabla \cdot (\nabla\mathbf{u})^T \\
&= \nabla^2\mathbf{u} + \nabla \underbrace{(\nabla \cdot \mathbf{u})}_{0} \\
&= \nabla^2\mathbf{u}
\end{aligned} \tag{2.54}$$

Where ∇^2 is Laplace operator.

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \tag{2.55}$$

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla p + \mu\nabla^2\mathbf{u} \tag{2.56}$$

Applying equation 2.56 to 2.48 results in following set of the **Navier-Stokes equations of incompressible flows**.

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u} - \mathbf{f}\right) + \nabla p - \mu\nabla^2\mathbf{u} = 0 \tag{2.57}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.58}$$

Equations 2.57 and 2.58 written in integral form:

$$\int_\Omega \mathbf{w} \cdot \rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u} - \mathbf{f}\right) d\Omega + \int_\Omega \varepsilon(\mathbf{w}) : \boldsymbol{\sigma} d\Omega + \int_\Omega q\,(\nabla \cdot \mathbf{u})\, d\Omega = \int_\Gamma \mathbf{w} \cdot \mathbf{h} d\Gamma \tag{2.59}$$

2.4.2 Non-dimenzionalization

Non-dimensionalize the equation (steady-state version with $\mathbf{f} = 0$)

$$\mathbf{u} = \mathbf{u}^* U \tag{2.60}$$

$$\nabla = \nabla^* \tfrac{1}{L} \tag{2.61}$$

$$\nabla^2 = (\nabla^*) \tfrac{1}{L} \tag{2.62}$$

2.4.3 Reynolds number

$$R_e = \frac{UL}{\nu} \tag{2.63}$$

If R_e goes to zero, then we are solving stokes flow, if it goes to infinity, then we are ”approaching” inviscid flow.

Solid surface

If viscosity not equals to and it is viscous flow and therefore $\mathbf{u} = \mathbf{0}$.
If unviscous boundary is assumed than only normal component of velocity is equal to zero.

Free surface

$$\mathbf{n} \cdot \boldsymbol{\sigma} = -p_{atm}\mathbf{n} \tag{2.64}$$

If pressure is scaled so that $p_{atm} = 0$, then

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{0} \tag{2.65}$$

If there are two liquids, then surface is not a boundary but interface.
Normal velocity of both liquids on the interface must be equal.

2.4.4 External boundaries

External boundaries are assumed to be sufficiently far from the object, so that we can approximate the boundary conditions with free-stream conditions.

Free-stream conditions

$$\mathbf{u} = \mathbf{u}_\infty \tag{2.66}$$

or

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}_\infty \\ &= -p_\infty \mathbf{I} + 2\mu\boldsymbol{\varepsilon}\left(\mathbf{u}_\infty\right) \\ &= -p_\infty \mathbf{I} + \mu\left(\nabla\mathbf{u}_\infty + \left(\nabla\mathbf{u}_\infty\right)^T\right) \end{aligned} \tag{2.67}$$

More general: $u_i = (u_i nfty)_i$ or $(\mathbf{n} \cdot \boldsymbol{\sigma})_i$ In most cases:

$$\mathbf{u}_\infty = (U, 0, 0) \Rightarrow \nabla\mathbf{u}_\infty = \mathbf{0} \Rightarrow \boldsymbol{\sigma}_\infty = -p_\infty\mathbf{I} \Rightarrow \mathbf{n} \cdot \boldsymbol{\sigma}_\infty = -\mathbf{n}p_\infty$$

2.4.5 Boundary conditions in general

$$\mathbf{u} = \mathbf{q} \tag{2.68}$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{h} \tag{2.69}$$

This means:

$$u_i = g_i \;\text{ or }\; (\mathbf{n} \cdot \boldsymbol{\sigma})_i = h_i$$

If assuming solution in liquid, than hydrostatic pressure should be encounted.
Substitution can be used:

$$p^* = p - p_h(z) \tag{2.70}$$

2.4.6 Spatial discretization

Time-discretization for the case of time-lagging update fot τ, and simultaneous update for $(\mathbf{u}\cdot\nabla)\,\mathbf{w}$:

$$\begin{aligned}\mathbf{M}\frac{\mathbf{U}_{n+1}-\mathbf{U}_n}{\Delta t}+\mathbf{M}_{\widetilde{c}}\left(\mathbf{U}_{n+\alpha}\right)\frac{\mathbf{U}_{n+1}-\mathbf{U}_n}{\Delta t}\\ +\mathbf{N}\left(\mathbf{U}_{n+\alpha}\right)+\mathbf{N}_{\widetilde{k}}\left(\mathbf{U}_{n+\alpha}\right)+\mathbf{K}_e\mathbf{U}_{n+1}+\mathbf{K}\mathbf{U}_{n+\alpha}\\ -\mathbf{G}\mathbf{P}_{n+1}-\mathbf{G}_{\widetilde{\gamma}}\left(\mathbf{U}_{n+\alpha}\right)\mathbf{P}_{n+1}=\left(\mathbf{F}+\mathbf{F}_s\right)_{n+\alpha}\end{aligned} \tag{2.71}$$

$$\mathbf{H}_\beta\frac{\mathbf{U}_{n+1}-\mathbf{U}_n}{\Delta t}+\mathbf{N}_\gamma\left(\mathbf{U}_{n+\alpha}\right)+\mathbf{G}^T\mathbf{U}_{n+1}+\mathbf{L}_\theta\mathbf{P}_{n+1}=\left(\mathbf{E}+\mathbf{E}_s\right)_{n+\alpha} \tag{2.72}$$

Where:

$$\begin{gathered}\mathbf{M}\approx\mathbf{m}\\ \mathbf{M}_{\widetilde{c}}\approx\widetilde{\mathbf{c}}\\ \mathbf{N}\approx\mathbf{w}\cdot\rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\approx\mathbf{c}\\ \mathbf{N}_{\widetilde{k}}\approx\tau_{SUPG}\left(\mathbf{u}\cdot\nabla\right)\mathbf{w}\cdot\rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\approx\widetilde{\mathbf{k}}\\ \mathbf{K}_e\approx\tau_{LSIC}\left(\nabla\cdot\mathbf{w}\right)\rho\left(\nabla\cdot\left(\mathbf{u}\right)\right)\approx\mathbf{e}\\ \mathbf{K}\approx\mathbf{k}\\ \mathbf{G}\approx\mathbf{g}\\ \mathbf{G}_{\widetilde{\gamma}}\approx\tau_{SUPG}\left(\mathbf{u}\cdot\nabla\right)\mathbf{w}\cdot\nabla p\approx\widetilde{\boldsymbol{\gamma}}\\ \mathbf{H}_{\widetilde{\beta}}\approx\beta\\ \mathbf{N}_\gamma\approx\tau_{PSPG}\nabla q\cdot\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\approx\boldsymbol{\gamma}\\ \mathbf{L}_\theta\approx\tau_{PSPG}\nabla q\cdot\nabla p\tfrac{1}{\rho}\approx\boldsymbol{\theta}\end{gathered}$$

Unsteady formulation

Steady-state formulation

2.4.7 Notation

Variables description		
ρ	Density	$[kg\cdot m^{-3}]$
μ	Dynamic viscosity	$[N\cdot s\cdot m^{-2}]\vee[kg\cdot m^{-1}\cdot s^{-1}]$
ν	Kinematic viscosity	$[m^2\cdot s^{-1}]$
t	time	$[s]$

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