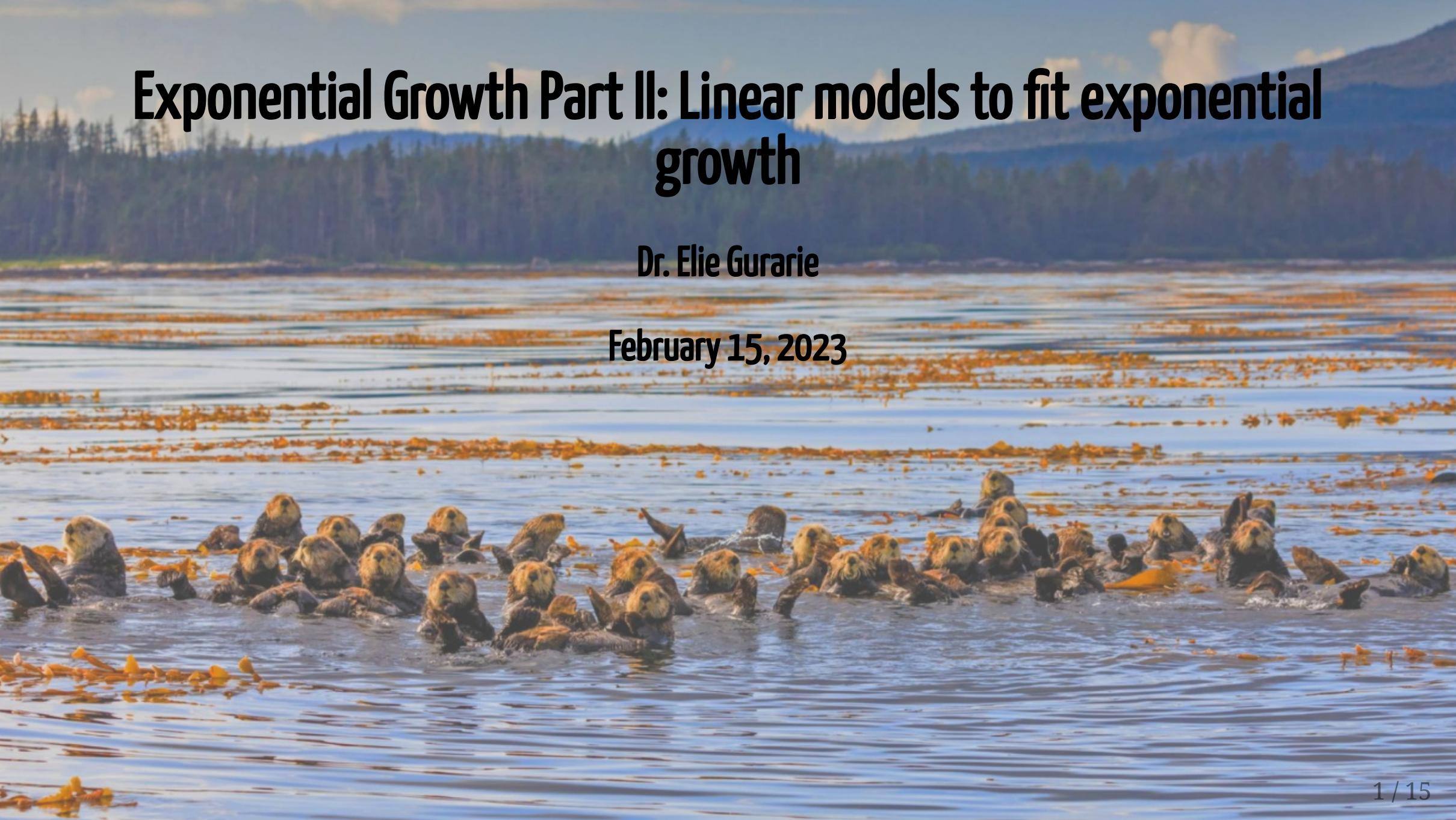


Exponential Growth Part II: Linear models to fit exponential growth

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February 15, 2023



Steller sea lion (*Eumatopias jubatus*) - birth

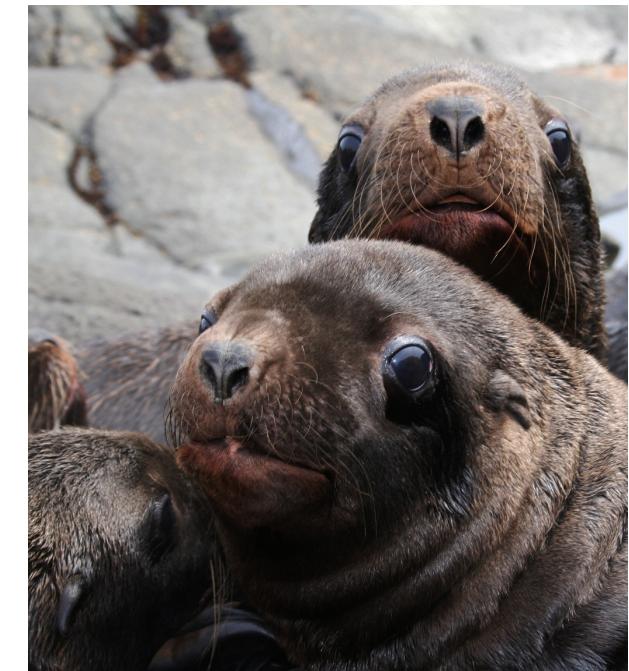
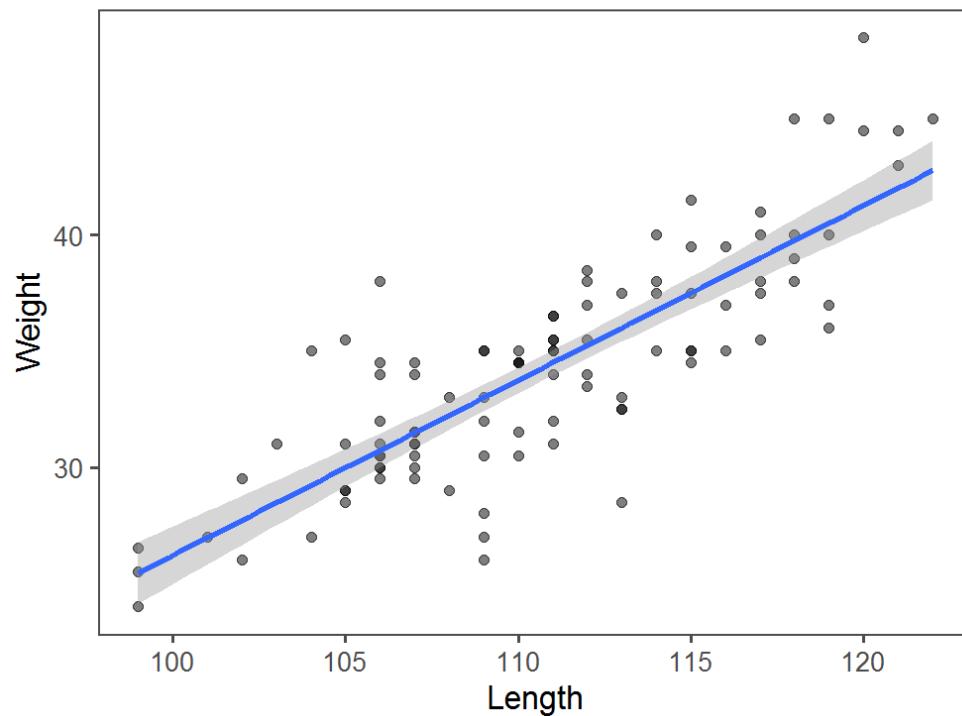


Linear modeling

(aka *REGRESSION*, except I really don't like that term, for a variety of reasons to discuss in class.)

is a very general method to quantifying relationships among variables.

Steller sea lion size



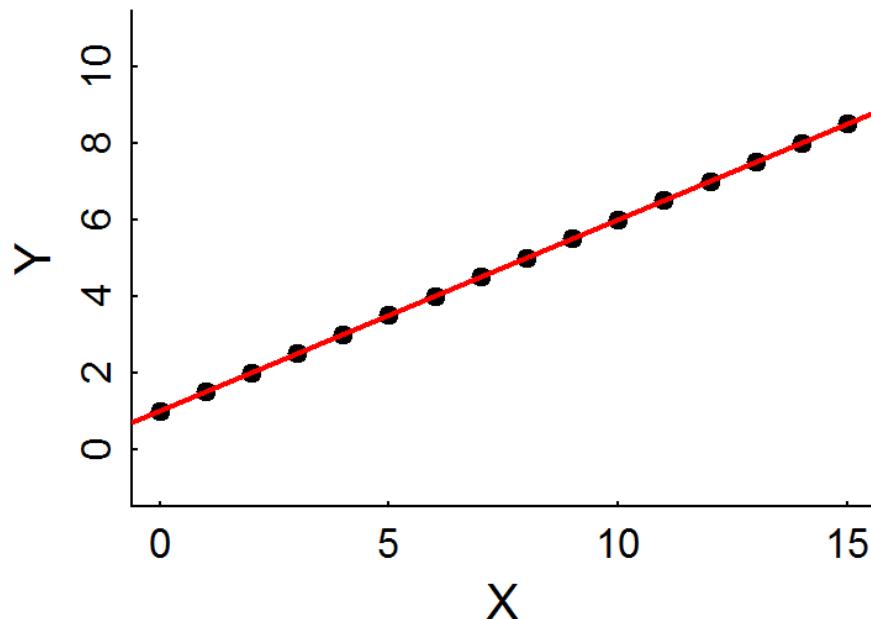
Linear Models

Deterministic:

$$Y_i = a + bX_i$$

a - intercept; b - slope

a = 1; b = 1/2



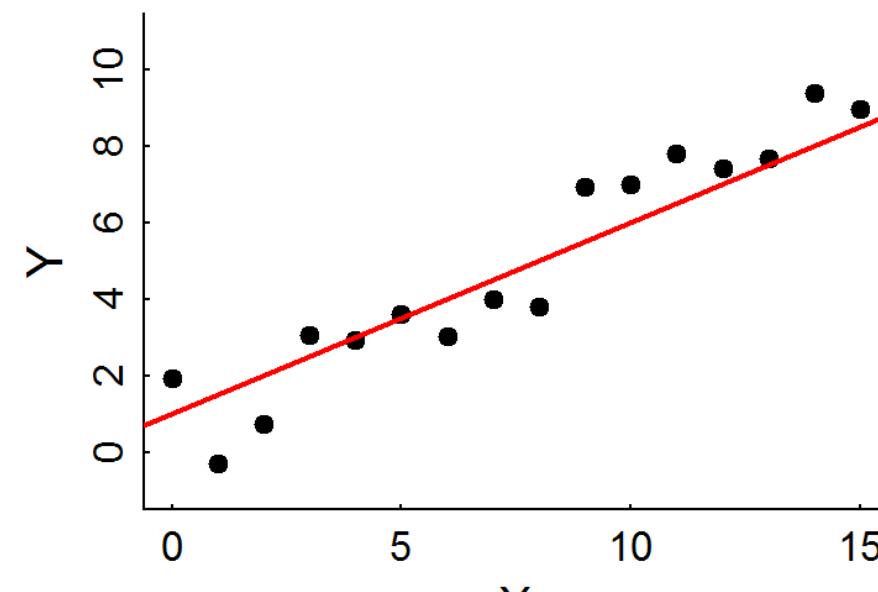
Statistical:

$$Y_i = \alpha + \beta X_i + \epsilon_i$$

α - intercept; β - slope; ϵ - **randomness!**

$$\epsilon_i \sim \mathcal{N}(0, \sigma)$$

$\alpha = 1 ; \beta = 1/2 ; \sigma = 1$



Fitting models is easy in



Point Estimate

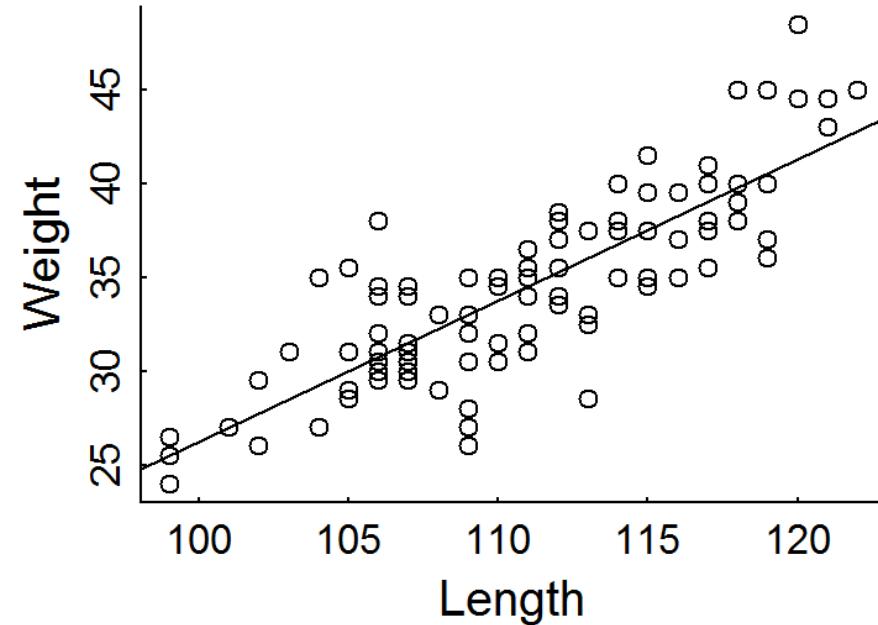
This command fits a model:

```
lm(Weight ~ Length, data = pups)
```

```
##  
## Call:  
## lm(formula = Weight ~ Length, data = pups)  
##  
## Coefficients:  
## (Intercept)      Length  
## -49.1422        0.7535
```

So for **each 1 cm** of length, add another **754 grams**.

```
plot(Weight ~ Length, data = pups)  
abline(my_model)
```



The `abline` puts a line, with intercept a and slope b onto a figure.

Some comments on linear models

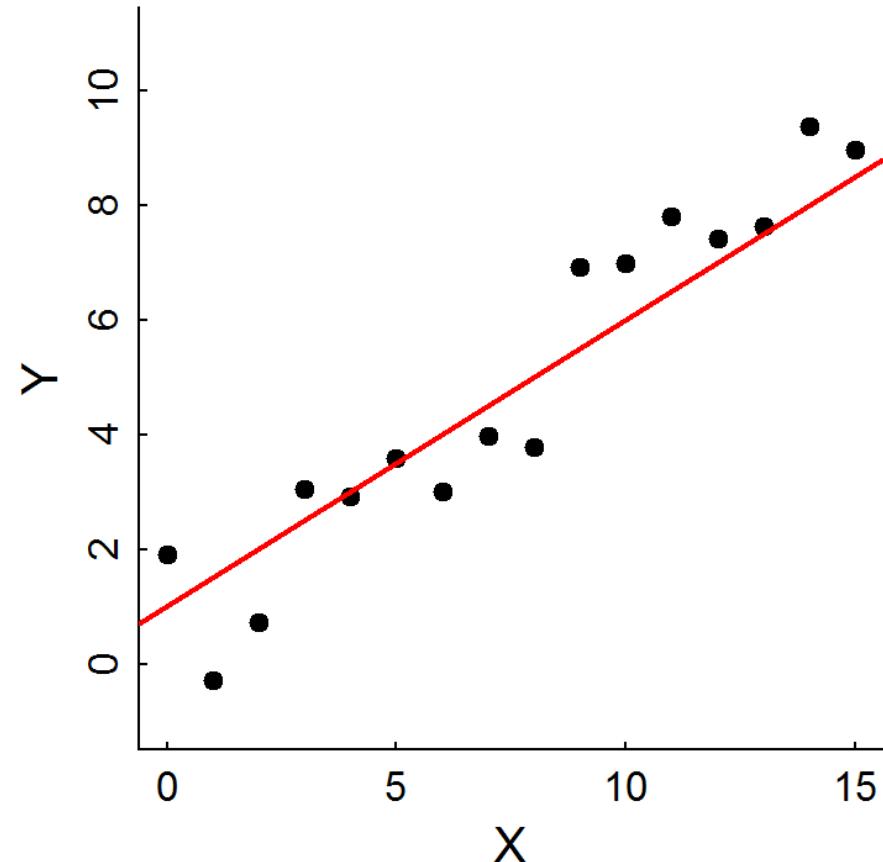
$$Y_i \sim \alpha + \beta X_i + \epsilon_i$$

1. ϵ_i is **unexplained variation** or **residual variance**. It is often (*erroneously*, IMO) referred to as "**error**". It is a **random variable**, NOT a **parameter** or **data**.
2. $\alpha + \beta X_i$ is the **predictor**, or the "**modeled**" portion. There can be any number of variables in the **predictor** and they can have different powers, so:

$$Y_i \sim \mathcal{N}(\alpha + \beta X_i + \gamma Z_i + \delta X_i^2 + \nu X_i Z_i, \sigma)$$

is also a **linear** model.

$$\alpha = 1 ; \beta = 1/2 ; \sigma = 1$$



Statistical inference

Statistical inference is the *science / art* of observing *something* from a **portion of a population** and making statements about the **entire population**.

In practice - this is done by taking **data** and **estimating parameters** of a **model**. (This is also called *fitting* a model).

Two related goals:

1. obtaining a **point estimate** and a **confidence interval** (precision) of the parameter estimate.
2. Assessing whether particular (combinations of) factors, i.e. **models**, provide any **explanatory power**.

This is (almost always) done using **Maximum Likelihood Estimation**, i.e. an algorithm searches through possible values of the parameters that make the model **MOST LIKELY** (have the highest probability) given the data.



Another gratuitous sea lion picture.

Statistical output

```
```

Call:
lm(formula = Weight ~ Length, data = pups %>% subset(Island ==
"Raykoke"))

Residuals:
Min 1Q Median 3Q Max
-7.498 -1.718 0.023 1.764 7.276

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) -49.14222 5.75796 -8.535 1.81e-13 ***
Length 0.75345 0.05193 14.510 < 2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.761 on 98 degrees of freedom
Multiple R-squared: 0.6824, Adjusted R-squared: 0.6791
F-statistic: 210.5 on 1 and 98 DF, p-value: < 2.2e-16
```
```

1. Point estimates and confidence intervals

Intercept (α): -49.14 ± 11.5

Slope (β): 0.75 ± 0.104

2. Is the model a good one?

p-values are very very small, in particular for **slope**

Proportion of variance explained is high:

$$R^2 = 0.68$$

Statistical output

```
...
## Call:
## lm(formula = Weight ~ Length, data = pups %>% subset(Island ==
##      "Raykoke"))
##
## Residuals:
##     Min      1Q Median      3Q     Max
## -7.498 -1.718  0.023  1.764  7.276
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -49.14222   5.75796 -8.535 1.81e-13 ***
## Length       0.75345   0.05193 14.510 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.761 on 98 degrees of freedom
## Multiple R-squared:  0.6824,    Adjusted R-squared:  0.6791
## F-statistic: 210.5 on 1 and 98 DF,  p-value: < 2.2e-16
...
```

Interpreting statistical results

The "standard error" around the **Length** factor is 0.05.

The "true value" lies within **TWO** standard errors of the **point estimate** with 95% probability.

So the estimate of the slope with **confidence interval** is (in g/cm):

$$\hat{\beta} = 754 \text{ g/cm} \pm 104$$

The *p*-value around the **Length** factor is $< 2 \times 10^{-16}$.. i.e. **0** This says that there is NO chance that you would get this steep a slope if there were NO relationship between Length and Weight (the null hypothesis).

So we've performed both **estimation** and **hypothesis testing** with this model.

Models and Hypotheses

Every p -value is a Hypothesis test.

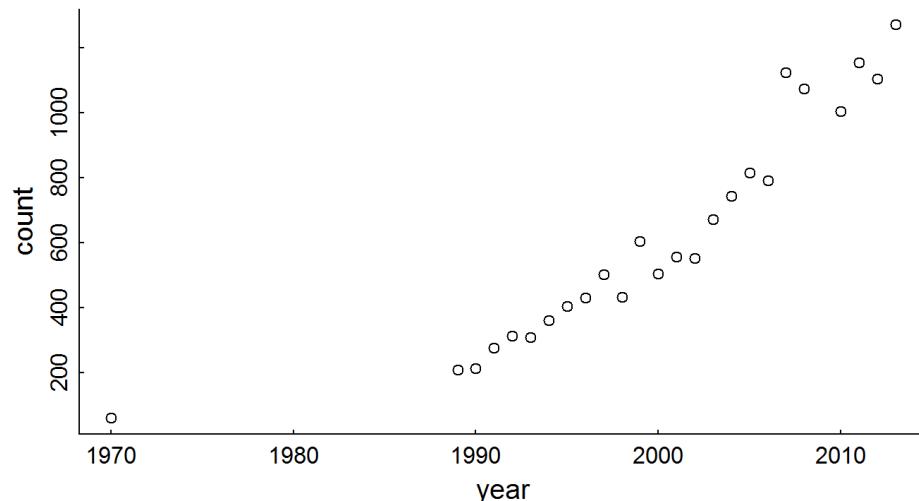
	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	-49.142	5.758	-8.535	0
Length	0.753	0.052	14.510	0

- First hypothesis test: H_0 **intercept = 0**
- Second hypothesis: H_0 **slope = 0**

Both null-hypotheses strongly rejected.

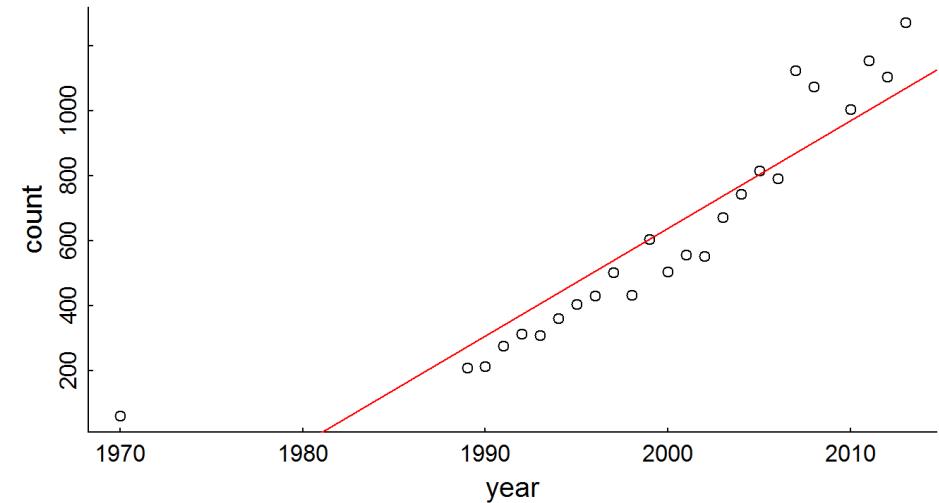
WA sea otter data:

```
WA <- read.csv("data/WA_SeaOtters_PopGrowth.c  
plot(WA)
```



Fit a linear model

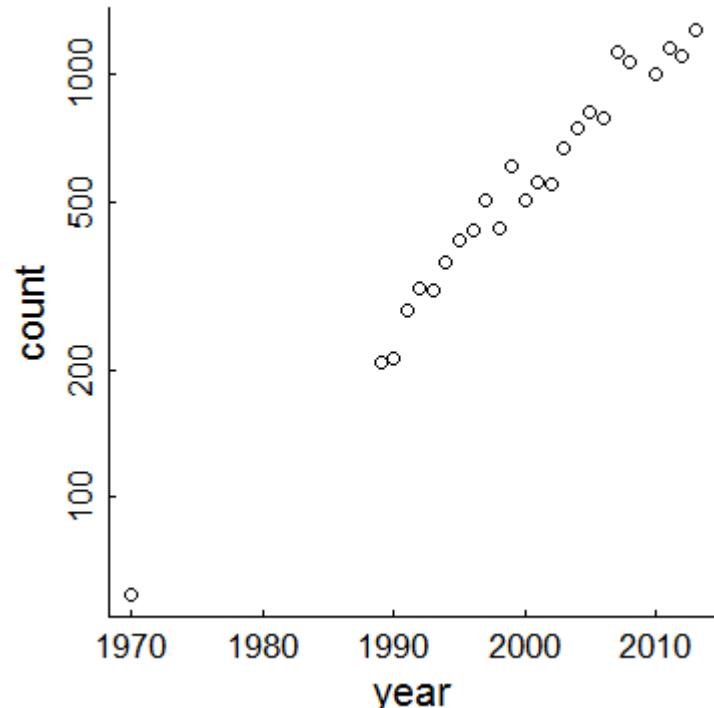
```
WA_lm <- lm(count ~ year, data = WA)  
plot(WA); abline(WA_lm, col = "red")
```



What are some problems with this model?

Plot on Log scale: Much more linear looking!

```
plot(WA, log = "y")
```



Linear model of $\log(count)$

```
logWA_lm <- lm(log(count) ~ year, data = WA)  
logWA_lm
```

```
##  
## Call:  
## lm(formula = log(count) ~ year, data = WA)  
##  
## Coefficients:  
## (Intercept) year  
## -140.22274 0.07325
```

Linear model of $\log(count)$

```
logWA_lm <- lm(log(count) ~ year, data = WA)  
logWA_lm
```

```
##  
## Call:  
## lm(formula = log(count) ~ year, data = WA)  
##  
## Coefficients:  
## (Intercept)      year  
## -140.22274     0.07325
```

A little math:

$$\log(N_i) = \alpha + \beta Y_i$$

$$N_i = \exp(\alpha) \times \exp(\beta Y_i)$$

$$N_i = e^\alpha e^{\beta Y_i}$$

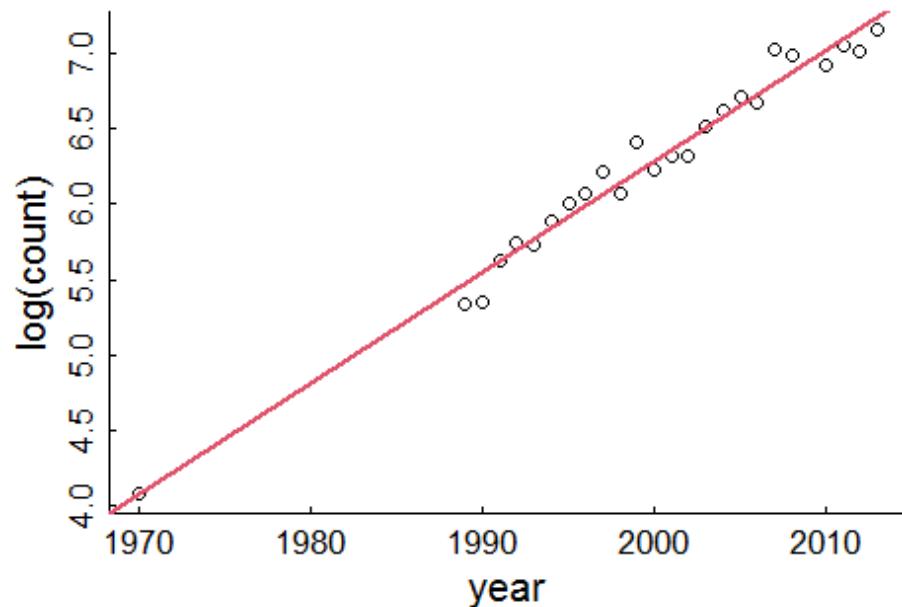
$$N_i = N_0 \lambda^{Y_i}$$

$$\lambda = e^\beta = e^{0.07325} = 1.076$$

SO ... percent rate of growth is about 7.6%.

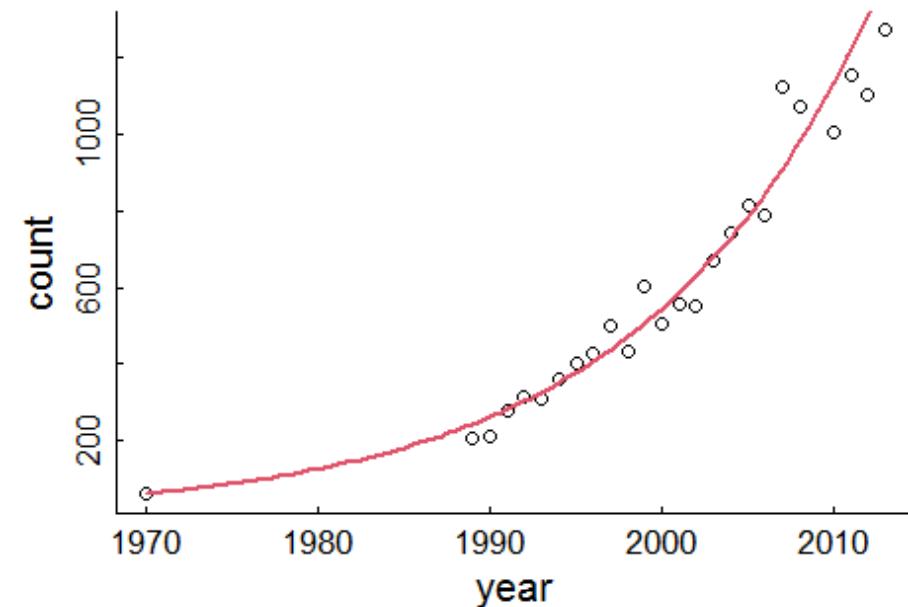
Plot linear model fit

```
plot(log(count)~year, data = WA)
abline(lm(log(count)~year, data = WA), col =
```



Plot exponential growth

```
plot(count~year, data = WA)
curve(exp(-140.2 + 0.07325 * x), add = TRUE,
```



Nice fit!

Summary stats and Confidence intervals

Summary stats

```
summary(logWA_lm)
```

```
##             Estimate Std. Error t value Pr(>|t|)  
## (Intercept) -140.2227    4.7318 -29.6344   0  
## year         0.0733     0.0024  30.9533   0
```

95% confidence intervals

$$\hat{\beta} = 0.073 \pm 2 \times 0.0024 = \{0.068, 0.078\}$$

$$\hat{\lambda} = \exp(0.073 \pm 2 \times 0.0024) = \{1.071, 1.081\}$$

So annual growth rate is $7.6\% \pm 0.5$, with 95% Confidence.

Key takeaway: With linear modeling we can use ALL the data to (a) get a great point estimate and (b) quantify uncertainty on that estimate.