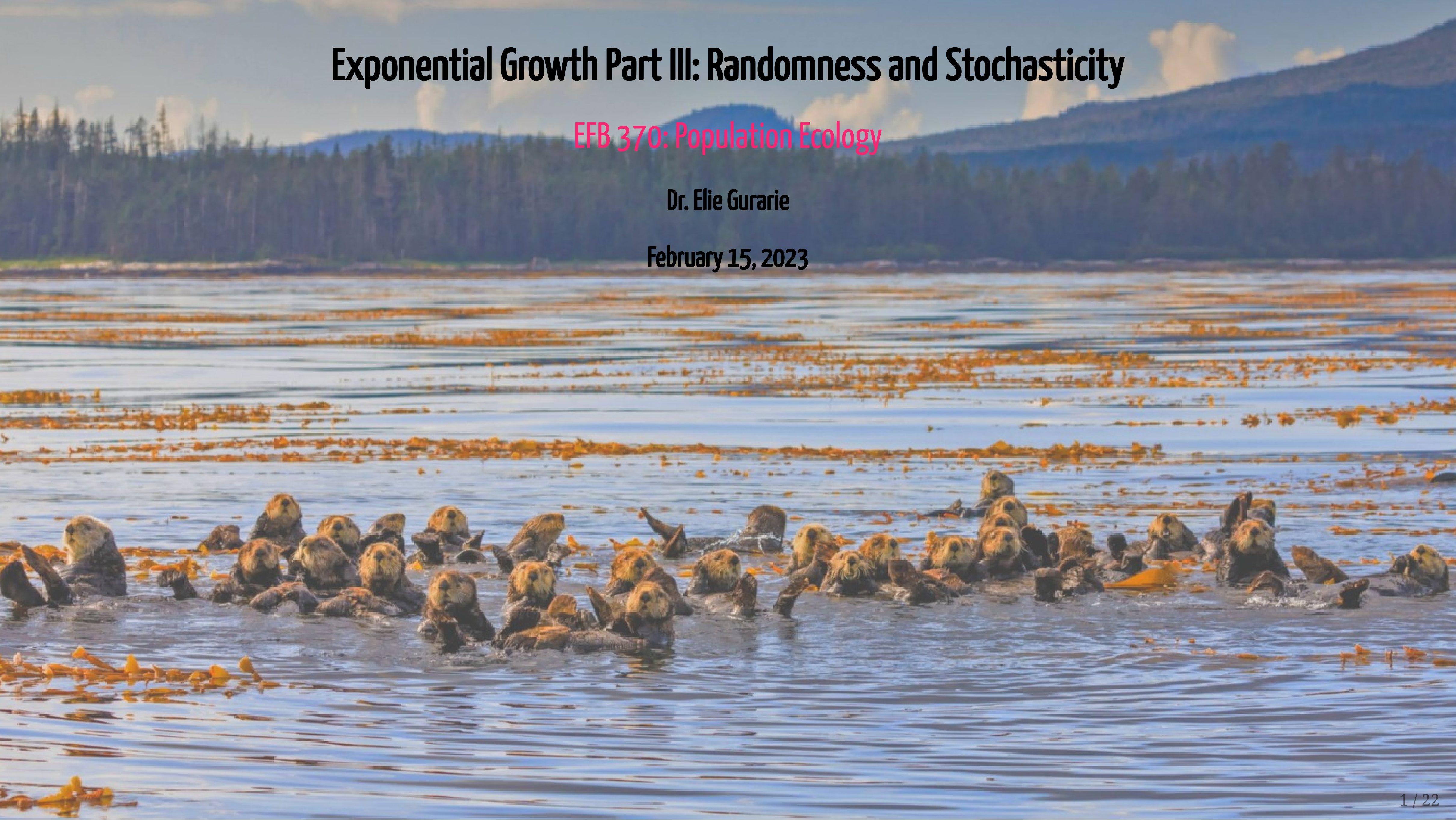
# Exponential Growth Part III: Randomness and Stochasticity

## EFB 370: Population Ecology

Dr. Elie Gurarie

February 15, 2023



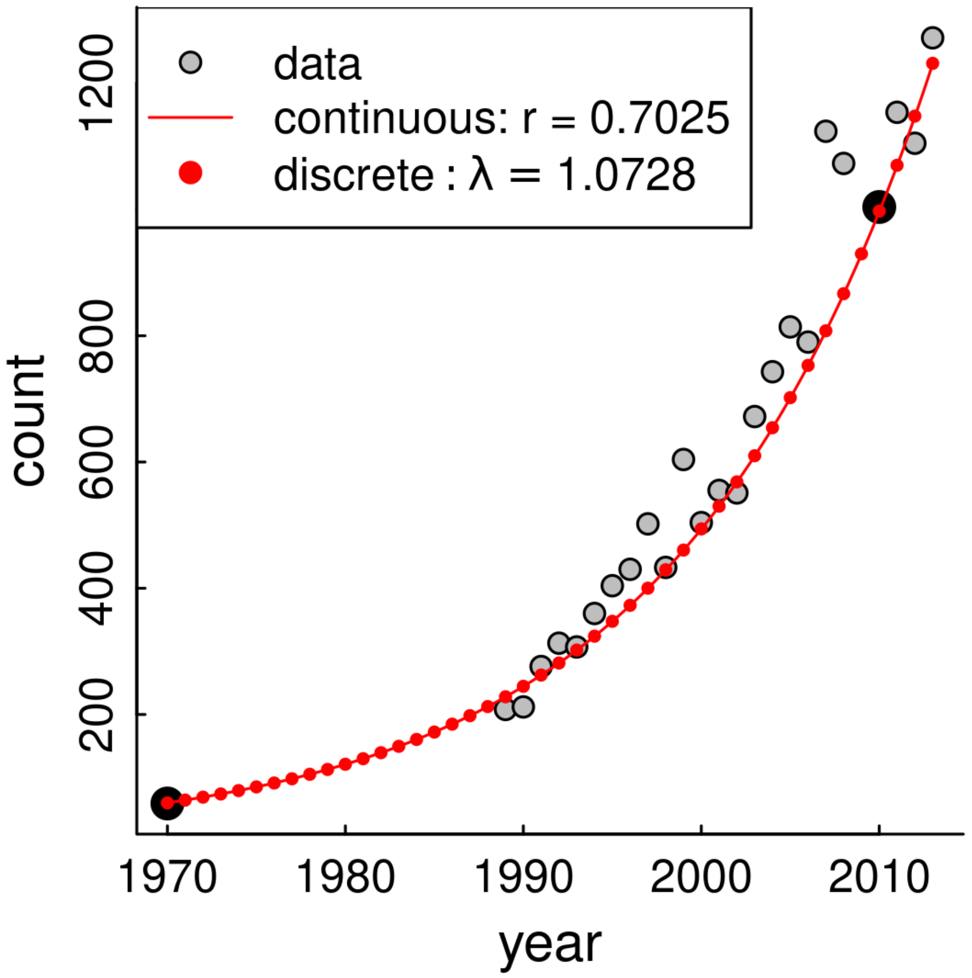
# Goals (concepts / buzzwords)

- I. Random variables:
  - mean / variance / standard deviation / distributions / expected value
  - Binomial and Normal distributions
- II **stochastic** = randomness in time
  - **demographic stochasticity** = randomness in individual processes
  - **environmental stochasticity** = randomness affecting populations

## Suggested reading:

- Gotelli: Chapter 1 (second half)
- Something (anything)

# If the exponential model is so tidy ... why isn't it *perfect*?




## Because ... ***randomness!(?)***

*What are some potential sources of "randomness" in the sea otter population process?*

- Environment good / bad affecting all animals ...
- Randomness in birth-death affecting individual animals ...
- Unexpected immigration / emigration ...
- Observation error ...

# Brief Intro to Random Variables

Easiest problem in boring arithmetic:

$$1 + 1 = 2$$

Easiest problem in random arithmetic:

 $+$  $= ?$

# One coin flip is Random

Random variables have *possible events* associated with *probabilities*.

| event | numeric | probability |
|---|---|---|
|  | $0$ | $1/2$ |
|  | $1$ | $1/2$ |

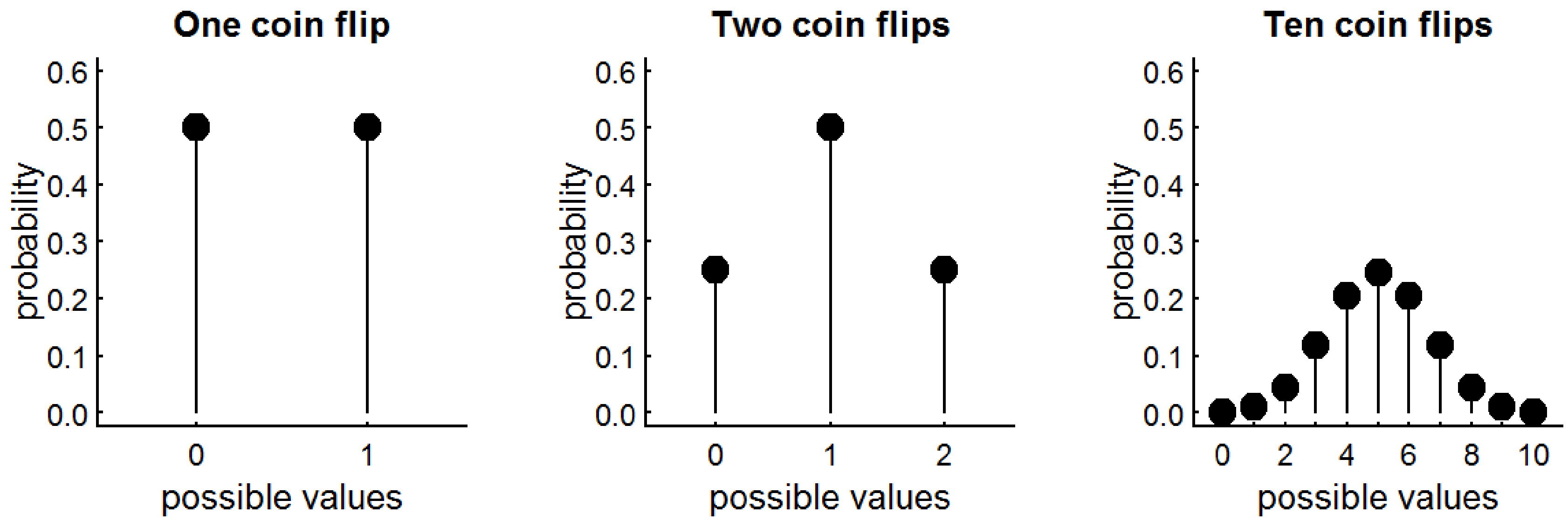
# Two coin flips is also Random

Random variables have *possible events* associated with *probabilities*.

| event | numeric | probability |
|---|---|---|
|  | 0 | 1/4 |
|  OR  | 1 | 1/2 |
|  | 2 | 1/4 |

# Random variables...

... are described by **Probability Distributions**. **Probability Distributions** have **names** and **parameters** that describe the distribution.




| | | |
|---|---|---|
| **One flip** | = | $X \sim Bernoulli(p = 1/2).$ |
| **Two flips** | = | $X_1 + X_2 = Y \sim Binomial(p = 1/2, n = 2).$ |
| $n$ **flips** | = | $\sum_{i=1}^{n} X_i = Y \sim Binomial(p = 1/2, n = n).$ |

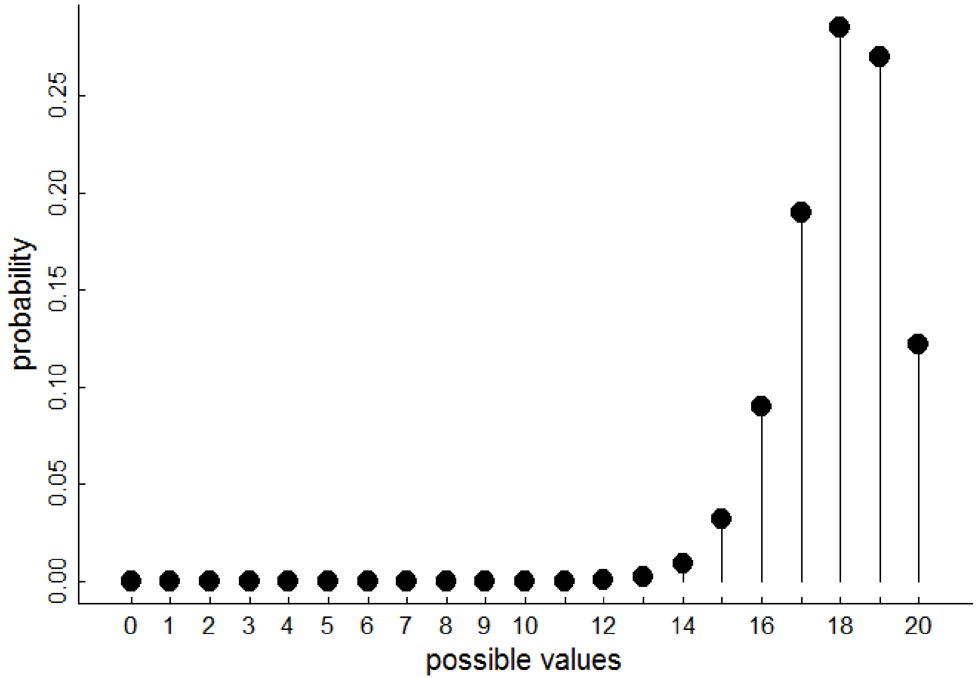
# A bit about the binomial distribution

$X \sim Binomial(n, p)$ describes the sum of $n$ random events, each of which has probability $p$.

## Pop ecology example:

If you have 20 sea otters ( $X_t = 20$ ), each with a 90% probability of survival / year, how many next year?

**Answer:** A random variable with distribution:
$X_{t+1} \sim Binomial(n = 20, p = 0.9)$



# Key questions about a random variable:

$$X_{t+1} \sim Binomial(n = X_t,\ p = p_{surv})$$

## Q1. How many would you *expect* to survive?

$$E(X) = \mu_X = np = 0.9 \times 20 = 18$$

This value is the **Expectation** of the distribution: $E(X)$ or $\mu(X)$. It would be the *mean* value if you could repeat the experiment an infinite amount of times.

## Q2. How much *variability* is there in this process?

$$SD(X) = \sigma_x = \sqrt{n\,p(1-p)} = 1.34$$

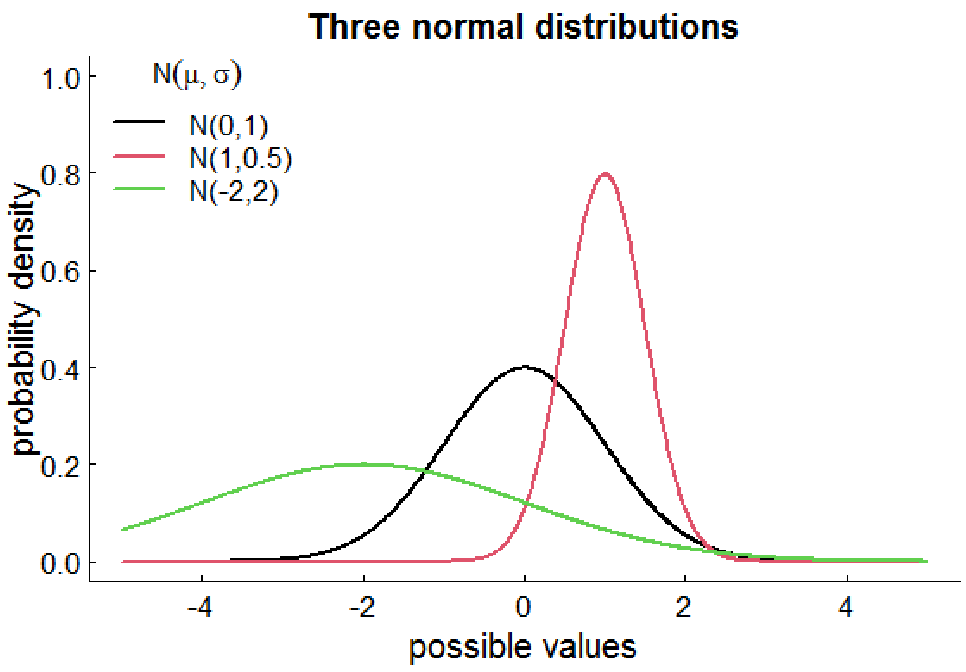
The **standard deviation** of a random variable: $SD(X)$ or $\sigma_x$ quantifies how concentrated the distribution is around the *mean*. *Approximately* 95% of the probability is within 2 standard deviations.

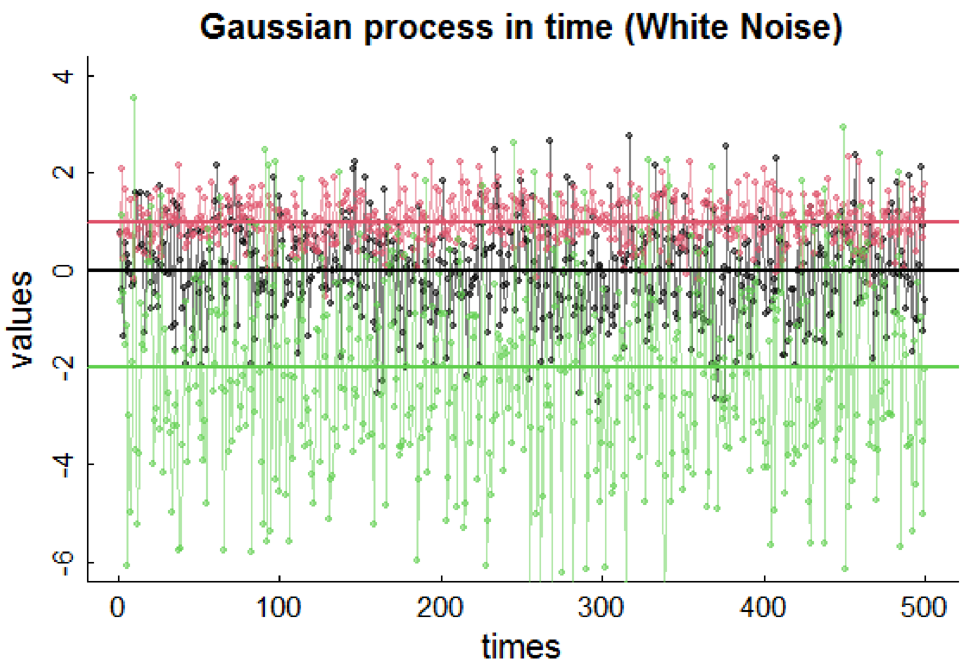
# Continuous random variables

There are a bunch. The most famous is the **Normal** or **Gaussian** distribution:

$$X \sim \mathcal{N}(\mu, \sigma)$$

It has two parameters: $\mu$ and $\sigma$ with are - unsurprisingly - the *mean* (expectation) and *standard deviation* (spread) of the variable.



# Once upon a time in Germany ....

# Some probability distributions {.smaller}

We use these to **model** random processes:

| name | notation | possible values | models | mean | standard deviation |
|---|---|---|---|---|---|
| Normal | $\mathcal{N}(\mu, \sigma)$ | $(-\infty, \infty)$ | bell-shaped | $\mu$ | $\sigma$ |
| Exponential | $Exp(\lambda)$ | $[0, \infty)$ | random events | $\frac{1}{\lambda}$ | $\frac{1}{\lambda}$ |
| Poisson | $Poisson(\lambda)$ | $[0, \infty)$ | positive count data (e.g. births) | $\lambda$ | $\sqrt{\lambda}$ |
| Bernoulli | $Bernoulli(p)$ | $[0, 1)$ | binary outcomes (e.g. deaths) | $p$ | $\sqrt{p(1-p)}$ |
| Binomial | $Binomial(n, p)$ | $[0, n)$ | many binary outcomes | $np$ | $\sqrt{np(1-p)}$ |

# Demographic Stochasticity

- ***Stochasticity*** means: **Randomness** in **time**.
- ***Demography*** is the **Science of Population Dynamics**. Often it refers specifically to **births** and **deaths** (and **movements** ... but we're still looking at closed population).
- Individually, *all* demographic processes are stochastic. An individual has some *probability* of dying at any moment. An individual has some *probability* of reproducting at a given time.

Q: How important is *individual* randomness for a *population* process?

More specific Q: **What is the probability of extinction?**

# Demographic Stochasticity: Human Experiment

- 10 students
- Flip a survival coin.
    - If you die (tails) sit down, if you live (heads) stay standing
- Flip a reproduction coin.
    - If you reproduce (heads) call on another student to stand

# What do we predict from this experiment?

Starting with $N_t$: expected number of survivors (S):

$$E(S) = p_s N_t$$

Expected number of new individuals (babies - B):

$$E(B) = p_b E(S)$$

New population equals survivals + new babies:

$$E(N_{t+1}) = E(S) + E(B) = p_s N_t + p_b \, p_s N_t = p_s(1 + p_b) N_t$$

So (in our coin flip example)

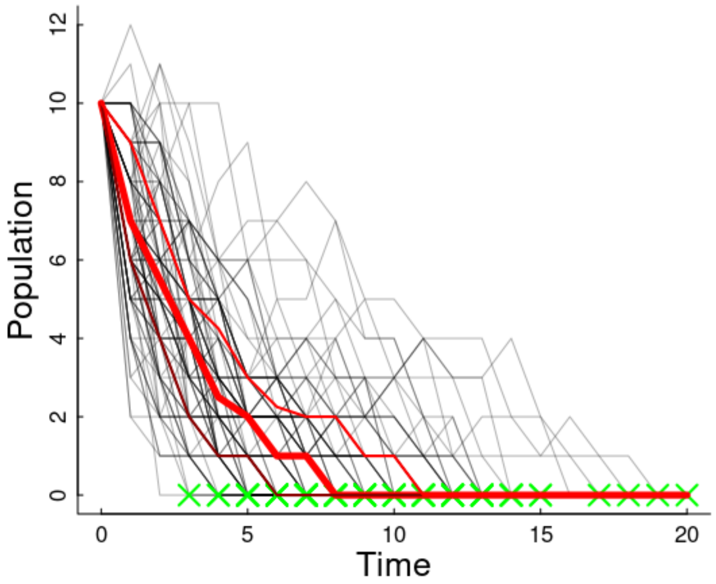
$$\widehat{\lambda} = p_s(1 + p_b) = 0.75$$

.

**What does that mean for population growth!?**

# Cranking this experiment very many times.

https://egurarie.shinyapps.io/StochasticGrowth/



# Some predictions for a similar continuous model

Assume birth rate $b$ and death rate $d$, and growth rate $r = b - d$.

The mean of the process is also exponential growth

$$E(N_t) = N_0 e^{rt}$$

If birth rate = death rate:

$$SD(N_t) \approx \sqrt{2N_0 bt}$$

Note, increases as square root of time. There's a somewhat more complex formula for births $\neq$ deaths.

More importantly:

$$P(extinction) = \begin{cases} \text{if } b > d; & (d/b)^{N_0} \\ \text{if } d > b; & 1 \end{cases}$$

Note that even when birth rate = death rate, $P(extinction) = 1$, i.e. eventual extinction is certain. This is very similar to the eventual probability of fixation for genetic drift.

Also, probability of extinction is lower for smaller $N_0$.

# Environmental Stochasticity...

... refers to some random aspect of the environment affecting $r$ (whether via births or deaths or both) for the **entire population**. :

$$R \sim Dist(\mu_r, \sigma_r)$$

# Environmental Stochasticity: Human Experiment

$N_0 = 16$

If I (the environment) flip **Heads**, the population doubles.

If I flip **Tails**, the population halves.

# Environmental Stochasticity: Analysis

This model can be written:

$$N_{t+1} = N_t \, 2^R$$

Where $R$ is a random variable:

| value | prob. |
|---|---|
| -1 | 1/2 |
| +1 | 1/2 |

$$E(R) = \mu_r = 0$$

So population should not grow, on average.

$$E(N(r)) = N_0 e^{\mu_r t} = N_0$$

And:

$$SD(R) = \sigma_R = 1$$

- Are you doomed to extinction?
- Does this remind you of another process from earlier in class?

Experiment here: https://egurarie.shinyapps.io/StochasticGrowth

# A stochastic population is a random variable!

$$N(t) \sim Dist(\mu_N(t), \sigma_N(t))$$

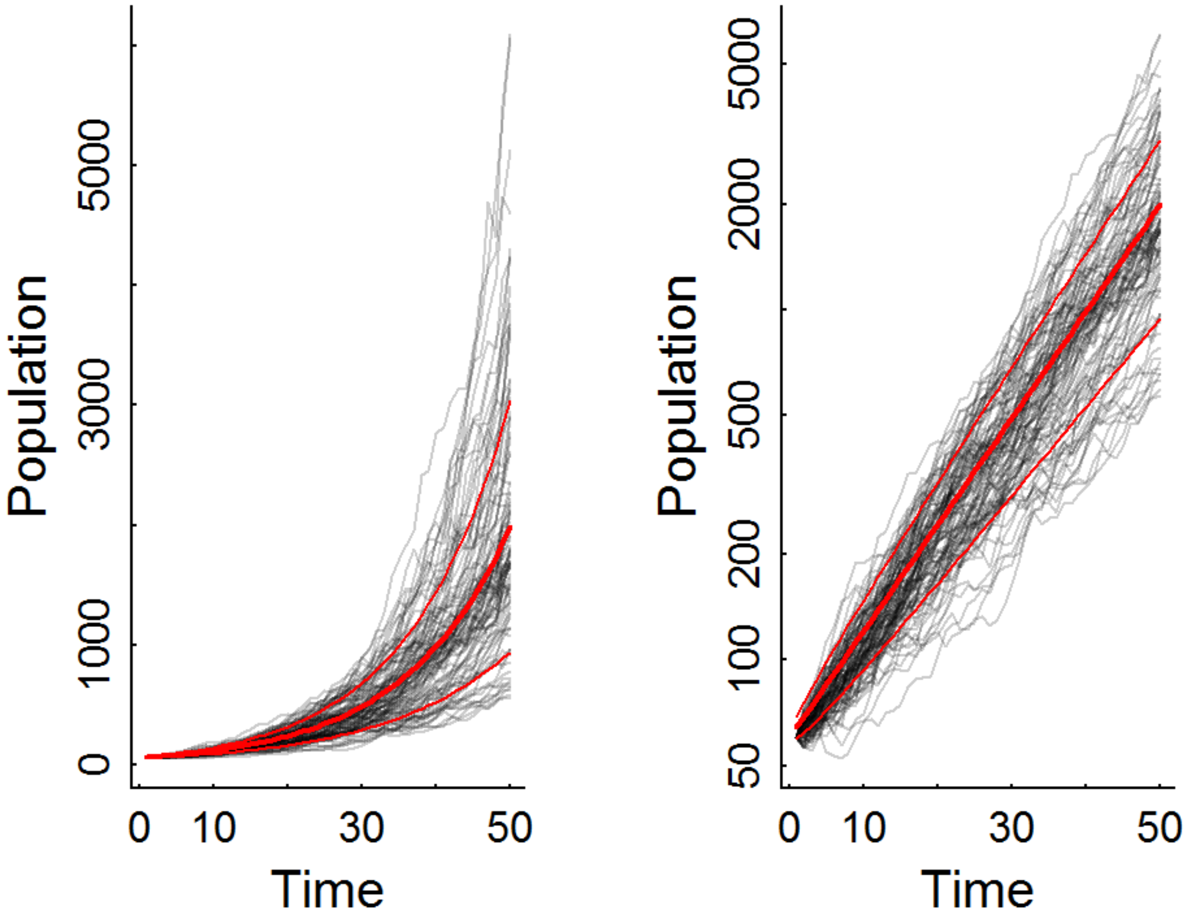
The mean / expectation is the same as for the exponential growth function, but with the mean growth rate substituted:

$$\mu_N(t) = N_0 e^{\mu_r t}$$

But the standard deviation of the function increases with time!

$$\sigma_N^2(t) \approx N_0\, e^{\mu_r t}\sqrt{e^{\sigma_r^2 t} - 1}$$

Sea otter example, with $\mu_r = 0.07$ and $\sigma_r = 0.07$:



# If things are TOO random ...

that spells trouble for a population!

According to theory, if $\sigma_r > 2\mu_r$, extinction is nearly certain - even if $\mu_r$ is positive and on average there is growth.

Opportunities for experimentation: