Estimation of Catch Parameters from a Fishery Observer Program with Multiple Objectives

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Abstract

Estimation of Catch Parameters from a Fishery Observer Program with Multiple Objectives

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Observer programs around the world collect information about fishing activities used to manage species of commercially important fish species as well as untargeted co-occuring species that are impacted by fishing activities. The U. S. government oversees several observer programs including the North Pacific Groundfish Observer Program (NPGOP) that monitors fishing activities in Federal waters off of Alaska. The NPGOP is the largest observer program in the world with respect to yearly numbers of deployed observers and the extent of data collected. Observers working in the NPGOP undergo extensive training and use sampling methods that are rigorous with respect to randomization, however, there are some inadequacies in both the size of samples and the way data are ultimately archived that inhibit unbiased estimation of various catch parameters and associated variances. Others have made important strides in illuminating areas where the data collection procedures can be improved, but there has been no systematic description of appropriate estimators and their corresponding assumptions nor has there been exploration of the levels of sampling effort in the various sectors of the Alaska groundfish fishery that would be optimal with respect to the

many catch parameters estimated from the resulting data. This dissertation provides answers to these important gaps in knowledge and in doing so, also provides a general construct that can be applied in other observer programs with similar deficiencies. Using these derived estimators I provide methods and examples for optimizing observer coverage to different vessel size classes in the Alaskan groundfish fisheries. The optimization can account for any number of catch parameter estimates and provides a transparent and methodical procedure for managers to determine observer coverage allocation. I also recommend changes in data collecting and reporting procedures that will require fewer model assumptions for making inferences from NPGOP data.

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Chapter 1

INTRODUCTION

1.1 Background

1.1.1 The North Pacific Groundfish Observer Program

The marine communities occurring in the Gulf of Alaska (GOA) and Bering Sea (BS) are highly productive and comprised of many desirable species of fish and invertebrates. As such, considerable commercial fishing effort occurs in the GOA and BS and understanding its impact on the marine communities is necessary for the sustainability of both the marine and commercial fishing communities.

The interplay of commercial fishing effort and the emergence of management entities for the waters off Alaska in recent history is a colorful story. A detailed account of fishing activity and regulations in the 19th and early 20th century is given by Fredin (1987). In the late 20th century, but prior to the 1980's, nearly all commercial fishing operations were conducted by foreign vessels from Japan, Russia and Korea. The passage of the Magnuson Fishery Conservation and Management Act (MFCMA) in 1976 caused a rapid transfer of the industry to domestic vessels and brought about significant changes in the process by which Alaska fisheries were managed (Witherell and Pautzke 1997). The state of Alaska has the authority to manage fishing activity in the 3 nautical mile strip along the coastline, but the MSFCMA declared the United States as the sole manager of fishery resources in the 200 nautical miles extending out from the Alaskan management area. The extended range of jurisdiction is known as the Exclusive Economic Zone (EEZ). Furthermore, the MFCMA called for the development of fishery management plans and established a North Pacific Fisheries Management Council (Council) comprised of scientists and industry to make management decisions. Soon after the inception of the MSFCMA and through the 1980's vessels fishing in the EEZ were required by the Council to carry an observer when requested by the National Marine Fisheries Service (NMFS). The program that developed within the NMFS to deploy these observers is the origin of what is known today as the North Pacific Groundfish Observer Program (NPGOP).

The NPGOP collects a wealth of information available to researchers to study the impact of these fishing efforts on the marine communities of the GOA and BS. This information is essential to understanding of (1) population dynamics of harvested groundfish species, (2) effects of commercial fishing on the groundfish populations and (3) effects of commercial fishing efforts on other non-targeted species occurring in the ecosystems in the GOA and BS. In practice, the data are used in activities ranging from day-to-day fishery management to ecological research. Together, these activities attempt to ensure the long-term health of the commercial fishing communities and marine ecosystems in the GOA and BS. Researchers use the collected information to estimate a host of population and fishery parameters and it is important that the data collection procedures (i.e., sampling design) are statistically robust. Furthermore, it is beneficial that data collection procedures yield estimates for the many catch attributes that are as precise as possible.

Catch parameters of particular interest to the NMFS, the groundfish industry and other researchers include total catch in weight of commercially valuable fish species, numbers and/or weight of prohibited species and incidental take of threatened or endangered species including sea birds and marine mammals. Moreover, estimates of total catch of targeted and non-targeted fish species within short time periods and/or small areas of the GOA and BS are needed for management purposes whereas these estimates at larger spatial and/or temporal scales may be sufficient for fishery stock assessments.

Current regulations for the EEZ dictate that large vessels ($\geq$ 125 ft) have observers on board during all fishing activity, medium vessels ($<$ 125 and $\geq$ 60 ft) have observers on board 30% of the days spent fishing in each quarter and small vessels ($<$ 60 ft) are free from observer coverage requirements. Recently, coverage regulations for vessels fishing pot gear were changed so that medium pot vessels must have observers present for 30% of the pots fished each quarter. There are concerns about the statistical robustness of the sampling design with regard to both within-haul estimation and vessel-level estimation (Turnock and Karp 1997; Vølstad et al. 1997; MRAG 2000, 2003). The variance of a within-haul estimator based on the sampling design (design-based) is inestimable because replicate samples are aggregated prior to analysis. As such, design-based inferences or confidence intervals for haul-specific estimates as well as trip-, vessel- or fleet-specific estimates are prohibited. As management of fisheries in the GOA/BS region shifts from a fleet-wide to a co-op or vessel-specific regime, estimation of variance within hauls will become increasingly important because the relative contribution to total uncertainty of an estimate by sampling at the haul level increases as finer scale estimates are needed (Särndal et al. 1992, pgs. 139-140). In addition, there is concern of biased estimation induced by both the lack of coverage for smaller vessels and the lack of randomization of observer coverage in the 30% fleet since vessel operators choose which of their fishing trips are observed.

Some attention has been given to the precision of particular estimators under the current sampling design athough within-haul variability is either unaddressed or addressed with parametric methods. Kappenman (1992) explored a parametric estimator of halibut bycatch. Comparisons of estimates of salmon bycatch based on various mixtures of observer data and industry retention data have also been made (Turnock and Karp 1997). Other investigations have been made into the relationship of precision of total catch estimates for target species to observer coverage levels (Karp 1997) and to proportion of hauls sampled (Vølstad et al. 1997). More recently, an assessment of the bias and precision of estimates of density for hauls made with trawl gear was made by Dorn et al. (1999).

Many estimators are unassessed and those that have been assessed need further investigation. The completed work in this vein has either ignored the complications of the design or made use of a fully parameterized distribution assumed to generate the realized haul data (i.e., Kappenman 1992). On the other hand, the most relevant work by Vølstad et al. (1997) and MRAG (2000, 2003) has noted many of the problems with the current design and suggested some ways to remedy them, but a systematic presentation of appropriate estimators for any of the catch parameters and their respective estimators of uncertainty is lacking. Because of the complex sampling design and the fact that researchers could be interested in estimation at different temporal or spatial scales (e.g, particular vessels, groups of vessels or time periods) that may or may not coincide with the specifications of the design, straighforward application of previously derived estimators (i.e., those in classical sampling theory texts) is prohibited for many catch parameters. Furthermore, there is presently no reference to advise investigators on appropriate methods to make statistically sound estimates from NPGOP data. Because of the complexity of the design and corresponding estimators, the lack of any referable source describing estimation procedures limits ability of investigators to conduct informed analyses.

A unified description of estimation procedures for NPGOP data would be useful, but an idea that logically follows is to somehow assess the efficiency of the design and, if possible, to optimize the efficiency. This is complicated by the fact that the users of NPGOP data have many different objectives (i.e., different users estimate different catch quantities or parameters from the NPGOP data). Optimal efficiency can mean many things, but here I define it as minimized uncertainty in a parameter estimate given the sampling design and estimator. When multiple parameters are estimated from data collected through a single sampling design, maximum precision cannot be achieved for all parameter estimates simultaneously. However, an optimality over all objectives is possible with proper levels of sampling effort at each stage of the design. There is likely to be a lack of consensus on the relative importance of the different objectives, but a presentation of the trade-offs in precision under alternative levels of effort is required. A method to achieve an optimal design with respect to multiple objectives using a prescribed weighting scheme would be useful to those responsible for making decisions on the relative importance of the various catch parameters. Given that the NPGOP operates on a fixed budget, this tool would allow decision makers to objectively allocate resources within the sampling design to maximize the quantitative goals set for the NPGOP. Ultimately, a sampling design that is in some way optimal may be used to collect data more efficiently and potentially provide financial savings to participants in the commercial fishing industry who pay for these services.

1.1.2 NPGOP Sample Design

Suppose the sampling universe is all catches in the EEZ off of Alaska in a given year where observer coverage is in some way required. Strong assumptions of exchangeability and independence for observed and unobserved fishing effort are necessary to make inferences for *all* catches in the EEZ. Differences in regulations for particular gear types, vessel sizes, and fisheries necessitate a multi-stage/phase sampling design implemented by the NPGOP. The elements of the different stages/phases are the following:

1. trips within a fishing activity stratum defined by vessel size and time (vessel-quarters);
2. hauls within trips;
3. clusters within hauls that are defined by gear type;
4. fish within clusters for length measurements; and
5. fish within length measurement samples for otolith retention and ageing.

The primary sampling unit (PSU) are fishing trips (or cruises) made by individual vessels conducting fishing activities that require observer coverage. The level of observer coverage required for a particular stratum is defined by various regulations, but is a function of vessel size, fishery, and time of fishing.

In some circumstances there may be more than one stratum per vessel. This would occur if a vessel performs some fishing activities that require 30% coverage and some that require 100% coverage. For example, a 100ft trawler that fishes 30 days under a Community Development Quota (CDQ) and 45 days under general fishing regulations must have an observer on board all 30 days that it is fishing under a CDQ and at least 14 of its other fishing days. However, whether there is observer coverage on a particular trip is decided by the vessel operator and, therefore unrandomized when coverage is less than 100%. It is sometimes difficult to determine the level of coverage that applies to particular haul records in the NPGOP data archive. Therefore, inferences should technically only be made for fish caught *during coverage* in strata with less than 100% coverage. If we are willing to assume that fishing practices are similar whether an observer is on board a vessel, then we may assume stratified random sampling at the first stage. In strata that require 100% coverage all first-stage elements are sampled. When observer coverage is less than 100% within a vessel-quarter stratum, additional complications include the unknown total number of trips and the occurence of only a single observed (sampled) trip. These problems are unaddressed in previous analyses of the NPGOP sampling design. Instead, it is often assumed that all days spent fishing or all hauls made are randomly sampled by observers which allows variance estimation at this stage. Certain patterns in fishing activity over time such as clustering, may cause estimated variances under these assumptions to be negatively biased.

Within an observed trip, the observer randomly samples hauls, but MRAG (2003) have noted that the present procedure for choosing hauls departs from SRS. Nevertheless, the inclusion probabilities (see Section 1.1.4 below) are likely to render the SRS assumption reasonable. Within-haul sampling procedures vary between the three different gear types of interest and is discussed in Section 1.1.3 below.

1.1.3 Selection of Sample within Haul

Sampling by observers within a haul depends on the type of gear used on board the vessel. We are primarily concerned with the three most predominant gear-types: longline, pot and trawl. The nature of the within-haul sampling universes are similar between trawl and pot gear and between longline and pot gear, but there is less similarity between trawl and longline gear. With all gear-types, model-free estimates of variance are prohibited because of data aggregation.

1.1.3.1 Longlines

Longline sets are composed of groups of hooks known as skates. The number of both skates per set and hooks per skate are variable, but the variability in the number of hooks per skate is generally quite low. Observers periodically count the number of skates per set and the number of hooks per skate for some skates. The number of hooks per skate is assumed to be the average number of hooks per skate on the skates that are counted. The NPGOP instructs observers to count individuals of each species that occur for a minimum of 1/3 of each longline set as it is retrieved (AFSC 2003). A longline set is sampled via (1) simple random sampling of the skates (spatial) or arbitrary time intervals (temporal) during which the longline set is retrieved or (2) systematic sampling of the skates spatially or temporally. If the sample is temporal the observer will either start on the next skate after the random starting time or start at the random starting time and count the number of hooks sampled. The total number of each species (or species group) over the entire sample period is recorded by the observer.

To obtain estimates of total weight for the various species in the set, the observer usually takes a smaller sample of each species that were counted in the tally sample. Observers are instructed to weigh 50 fish for up to 3 predominant species. Observers weigh 15 fish from each non-predominant species. There are special instructions for fish identified as the shortraker/rougheye rockfish (*Sebastes borealis/aleutianus*) complex. If the observer is able to obtain the weight sample from within the tally sample, then 30 fish must be weighed and if the weight sample is taken from the non-tally portion of the set, then 15 fish must be weighed. The total weights for each species are divided by the number of fish in each weight sample to obtain an average weight per species within the haul. The average weight of the sample is multiplied by the total number of the respective species counted in the sampling period. The observer may take the weight sample for any species from either the tally or nontally portions of the set. Whether the weight sample is taken within the tally sample is unrecorded in the NPGOP database nor is the total number of fish in the weight samples.

Observers are instructed to obtain length measurments for species that are visually determined to be predominant in a set. The frequency of sets for which the observer must take these samples and the size of each sample depends on the species. For reasons of practicality, the sample for length measurements is often a subset of the weight sample. For some hauls where length samples are taken, the observer is also instructed to obtain otoliths from a subsample of fish in the length sample.

1.1.3.2 Trawls

When trawl gear is used, the whole haul is brought aboard the vessel at once. The sampling procedures used by an observer depends on what equipment is available. In any case, the observer does not estimate numbers of individuals nor average weights of individuals. Rather, the observer either measures or estimates the total weight of the catch and uses baskets to sample clusters of fish from the haul. On some vessels there are motion-compensated flow (MCF) scales that will weigh the entire catch (less discards) and for small catches the observer can weigh the entire catch even without these scales. In these cases the total weight of the catch may be considered known. When MCF scales are unavailable and the haul is large the observer measures (with error) the volume of the codend holding the catch or the volume of catch in holding bins after the catch is unloaded. The measured volume is multiplied by the density of the sampled fish to obtain a total weight of catch.

Obtaining information on particular species is relatively complicated on trawl vessels. When large fish or mammals are brought aboard the oberver makes a guess for its weight and records it as sample type "X" if not all the remaining individuals of that species are counted and weighed. If all of that species are accounted for, then the sample type for the species is denoted "whole haul." For other species, baskets of fish may be sampled from the catch or groups of fish from unsorted catch in holding bins may be sampled as the holding bins are emptied during processing (catcher/processor vessels only). Whatever sampling method, the observers are instructed to ensure that the the sample weighs a minimum of 300 kg in total when there are several prevalent species. When there are two predominant species in the catch a minimum of 200 kg in total is required to determine the relative abundance and when there is only one predominant species the minimum required is 80 kg. If length measurements are taken for a given species, then the fish in the length sample are generally a subsample of the weight sample. If otoliths are taken, then the otoliths are removed from a subsample of the length sample.

1.1.3.3 Pots

Pots are deployed in groups along a line and a single collection of pots is termed a "string." The number of pots per string is variable, but the total number of pots within a string is known and observers are instructed to sample a minimum of 30% of the pots per string. The pots within a string are sampled via (1) simple random sampling of the pots (spatial) or arbitrary time intervals (temporal) during which the string of pots is retrieved or (2) systematic sampling of the pots spatially or temporally. The total number of each species over all the sampled pots is recorded by the observer.

Among the pots that are chosen for the sample, a random subsample of pots is chosen to take approximately 50 fish for weighing. If less than 50 fish of a particular species are found in the subsample, all of the fish of that species are taken. If for particular species there are far more than 50 fish in the subsample the observer is encouraged to avoid hand selection by randomly picking predefined sections of the subsample. Fish of each species are weighed in groups and the average weights of each of the fish species are multiplied by the total number of the respective species in the sample of pots. Instructions for obtaining length measurements and otoliths are identical to those described above for longline vessels (Section 1.1.3.1). Unlike the longline haul records, the numbers weighed for each species is retained in the NPGOP database.

1.1.3.4 Seabirds and Marine Mammals

The protocols used by observers for recording data on interactions with marine mammals and incidental take of seabirds is similar across all three of the gear types detailed above. When seabirds are encountered during the tally period on longliners and pots or within the species composition sample for trawlers they are treated similar to non-predominant fish species. The numbers and weights of each species of seabird are recorded.

Observers record various types of interactions between marine mammals and fishing vessels of which fishing-induced mortalities are a subset. Observers are instructed to randomly select sets which will be monitored. For longline and pot vessels the observer may sample portions of hauls for marine mammal interactions, but for trawlers the entire haul is observed.

1.1.4 Review of Developments in Finite Sampling Theory

The origin of the field called classical sampling theory is attributed to the influential paper presented by Neyman (1934) to the Royal Statistical Society. For the next few decades there was a great deal of study of various sampling designs such as simple random sampling, stratified simple random sampling and systematic sampling, but it was not until Horvitz and Thompson (1952) presented an approach to forming design-unbiased estimators of the total that a unifying theory of sampling theory was developed. The so-called Horvitz-Thompson estimator is special in that it requires only the first- and second-order inclusion probabilities for elements in the sampling universe to obtain the estimator of the total as well as the unbiased estimator of uncertainty for a design. The first-order inclusion probability for the ith element in the universe is the probability that element is included in the sample and the second-order inclusion probability is the probability that the ith and jth elements are included in the sample ($i = j$ or $i \neq j$). This was a major development because the Horvitz-Thompson form can be applied to any design as long as the first and second-order inclusion probabilities are known.

Although unification was being addressed in sampling theory there were growing concerns about the lack of mathematical foundations for inference in sampling theory.

In the other branches of statistics concepts of sufficiency, completeness and likelihood were being applied, but these ideas had not yet been used to determine any optimality properties of estimators in sampling theory. There arose great debate within the community of statistical scientists of the role of the sampling design in inference. People sorted themselves into two camps: those that favored model-based inference and those that favored design-based inference. Design-based inference relies on the random sampling process used to obtain the sample whereas model-based inference relies on a joint probability distribution function that generates the population. That is, the population is a realization of a vector-valued random variable with length determined by the number of elements in the universe (super-population model).

The real pioneering work that focused on the mathematical properties of design-based inference was accomplished by Godambe (1955). It turns out that the parameter space and sufficient statistic in a finite universe are such that likelihood methods cannot be used to obtain optimal estimators. In particular, the inclusion of the element labels in the sufficient statistic is problematic in the likelihood approach. Under designs where indicators provide no information (e.g., SRS) some in the MBI camp have argued that disregarding them is appropriate and maximum likelihood estimators may be obtained (for example, Hartley and Rao 1968; Royall 1968; Rao 1971). There were also explorations of criteria for estimators such as admissibility and hyperadmissibility (Hanurav 1968) that obtain class-specific optimality.

A strong proponent of model-based inference was Royall (1970). Assuming that $Y_1, \ldots, Y_N$ are iid random variables each with mean, βx_i, variance, $\sigma^2 v(x_i)$, and some plausible condition on $v(x_i)$, Royall (1970) showed that when estimating the total via the ratio-estimator, the model-based variance is minimized by purposely selecting the n population elements with the largest values of x. This result, of course, is strongly dependent on the linearity of Y in x. Royall and Herson (1973a,b) showed that more balanced sampling across the values of x would be robust to model mis-specification.

Efforts to better define asymptotic properties of estimators in the finite population setting were also made. The application of the Central Limit Theorem (CLT) in infinite population sampling requires consistent estimation and finite first and second moments of the population. This theory does not carry over to the finite population without specification of some assumptions on the finite population because different types of consistency are possible. Randomization-consistency was defined by Hansen et al. (1983) which allows application of the CLT. Smith (1994) questioned the necessity of referring to hypothetical populations with randomization consistency and instead proposed the use of Fisher consistency and thought work in the mode of Robinson (1978) would provide conclusive justification for randomization-based inference. However, it appears little advancement in this line has been made. Some have found minimum sample sizes necessary for approximate confidence interval coverage as a function of third and fourth moments of the finite population under SRS (Sugden and Smith 1997; Sugden et al. 2000), but perhaps because of obvious difficulties similar results for more complex designs remain unexplored. In fact, Smith (2001) makes no mention of his earlier thoughts on asymptotic approaches in finite populations. Moreover, in reference to appropriate uses of the various inference approaches for finite population parameters Smith (2001) writes:

> There is no commonly agreed criterion for choosing between these alternative approaches to inference, and so the decision about which approach to use in any given case would appear to be based on metaphysics rather than simply upon statistics.

Although the question of whether design- or model-based inference is more appropriate has not yet been resolved, there have at least been some objective attempts at synthesis of their ideas. Särndal (1978) gives a review and juxtaposition of design-based and model-based inference and Cassel et al. (1977) provide a thorough development of the theoretical underpinnings of both design- and model-based parameter estimation. The papers by Smith (1976, 1994, 2001) give interesting reviews of the state of survey sampling theory over its short history.

1.1.5 Review of Multivariate Optimal Design and its Utility within the NPGOP

Over the latter half of the twentieth century, many have studied optimal sampling with respect to multiple parameters for specific designs. Some of the results of this effort are found in sampling theory references written over the years (see Dalenius 1957; Jessen 1978; Yates 1981; Särndal et al. 1992). Dalenius (1952, 1953) appears to have developed the first sophisticated method using non-linear programming to achieve optimality of a stratified simple random sampling without replacement (SSRS) design where estimation of multiple population parameters are of interest. Many others obtained results for optimal sampling effort for other designs using algorithmic approaches, various loss functions and Bayesian as well as classical methods (e.g., Chakravarti 1955; Kokan 1963; Hartley 1965; Ericson 1965; Draper and Guttman 1968).

Because of the results obtained by those studying inference properties in finite sampling theory, there was also interest in optimal sampling strategy. A sampling strategy consists of both the sampling design and the estimator(s) used with the design. Prior work was focused on optimal levels of effort for given designs, but it became apparent that different combinations of designs and estimators could give equal variance (Solomon and Zacks 1970; Särndal 1978).

In fisheries management there has been considerable interest in optimality of sampling designs when multiple parameters are of interest, especially concerning estimation of proportions-at-age of commercially important species. Ketchen (1949) was interested in finding optimal second-phase sample sizes to estimate proportions-at-age when the second-phase sample size is constant across all size classes encountered at the first phase. Tanaka (1953) appears to be the first to derive optimal sample sizes in estimation of a proportions-at-age under a two-phase design, but there was no optimization over all age proportions. By making some assumptions on the numbers in the population occuring in particular age or length categories and using methods derived by Tanaka (1953), Kutkuhn (1963) found second-phase sample sizes that were optimal for estimation of the age composition of chinook salmon (*Oncorhynchus tshawytscha*). Southward (1976) considered optimal levels of effort using two-phase sampling in estimation of the age composition of Pacific halibut (*Hippoglossus stenolepis*) landings, but like Kutkuhn (1963) there was no attempt to optimize over all ages simultaneously.

Using an objective function that is the sum of the variances of estimated age composition, Kimura (1977) showed that two-phase sampling will do at least as well as SRS when the size of the second-phase sample within the jth length interval is proportional to the number sampled at the first phase that fall into the jth length interval. Kimura (1977) also found two-phase sampling to be less efficient than single-stage SRS when the second-phase sample is constant across all length intervals obtained in the first-phase sample. Lai (1987, 1993) generalized the objective function of Kimura (1977) to a weighted sum of the variances which allows the optimization to favor minization of variances for parameters that are of relatively greater importance. Smith (1989) elaborated on the idea put forth by Kimura (1977).

Sen (1986) showed that under certain conditions two-phase sampling for age composition is virtually no more efficient than SRS, but optimal sampling efforts are obtained for each species rather than over all species. Schweigert and Sibert (1983) also were concerned with estimation of proportions-at-age, but they considered a simplified three-stage sampling design with SRS at each stage. Similar to Sen (1986), they obtained optimal designs for each age class rather than optimizing over all ages simultaneously. Chester and Waters (1985) and Waters and Chester (1987) opted for graphical comparisons of estimated variances rather than analytic methods to find levels of effort that are optimal in a two-stage SRS design for estimation of age composition of Atlantic menhaden (*Brevoortia tyrannus*). Horppila and Peltonen (1992) also used graphical methods to determine optimal levels of effort in a three-stage SRS design for estimation of length or age composition, but, in fact, the optimal levels of effort are obtained for a sum of the variances over all categories and, thus, they use a graphical analog to the method used by Kimura (1977) and the equal weighting case proposed by Lai (1987, 1993).

There has been other results on optimal levels of effort for the age composition problem under the two-phase design that use a more model-based approach. Assuming a multinomial distribution on the numbers in each length class at the first phase and the numbers in each age/length class at the second phase and Dirichlet prior distributions on the corresponding proportions, Smith and Sedransk (1982) found Bayesian optimal sample sizes by minimizing a loss function that places arbitrary weights on the posterior dispersion matrix with sample sizes within length classes at the first phase and a constrained number of samples at the second phase. Jinn et al. (1987) expanded on the earlier work of Smith and Sedransk (1982) to also find the optimal first-phase sample size simultaneously although numerical methods rather than analytic results were necessary.

Andrews and Chen (1997) were interested in optimality of sampling effort in a two-stage SRS design used to estimate both mean size and size composition of blacklip abalone (*Haliotis rubra*). These authors used a Monte Carlo simulation approach to assess changes in error with various levels of effort at the two stages, but there was no cost or sampling effort constraint used to find levels of effort that provided a minimum of uncertainty in any sense.

Manly et al. (2002) produced an iterative method for finding optimal within-strata sample sizes in an unusual two-phase stratified design used to estimate density of several shellfish species. The first phase is an ordinary stratified simple random sample, but estimates of within-stratum variance are used to determine allocation of remaining sampling effort to the various strata. The objective function that Manly et al. (2002) minimized was a weighted sum of the mean coefficient of variation, the maximum coeffecient of variation and the mean of all coefficients of variation over a threshold where the weights and the threshold are arbitrary.

In my opinion, the most innovative use of optimization of effort in sampling theory for fisheries management was illustrated by Smith (1988). He showed how to find levels of sampling effort that minimize the variance component of exploitable biomass estimates obtained through a population dynamics model. This is the only instance where sampling error propogated through a population dynamics model has been the objective to minimize. Moreover, Smith (1988) brought forth the idea of determining whether two-phase sampling or SRS was better as a sampling strategy in this context.

1.2 Objectives

1.2.1 Estimators of Catch Parameters

This dissertation provides a desperately needed compendium of appropriate model-based estimators for catch parameters and respective variance estimators for existing data collected under the NPGOP sampling design. The model-based estimators are couched in the super-population context and measures of uncertainty are based on error in parameter prediction (see Valliant et al. 2000). All model assumptions are made explicit.

Of equal importance, I provide design-based estimators that would be appropriate if various deficiencies in current NPGOP design are removed. Many of these deficiencies are addressed by (MRAG 2003), but others remain unattended. Simple modifications to the way data is archived would alleviate some deficiencies. For example, observers may collect fish randomly for weight within a haul, but weights are recorded in aggregate. If they were, then variation of weights within hauls could be measured. Design-based estimators are derived using finite sampling theory principles and accompanied by a clear presentation of the corresponding assumptions.

The derived estimators will include those appropriate for estimation at the haul, trip, vessel, fleet and arbitrary temporal and spatial domains. The catch parameters I will address are the following:

1. species-specific total catch in numbers;

2. species-specific total catch in weight;

3. species-specific numbers and proportions within arbitrary length classes;

4. species-specific numbers and proportions-at-age;

5. species-specific total mortality for marine mammals;

6. and species-specific total take in numbers for seabirds.

The derived estimators and their corresponding estimators of uncertainty provide a referable source of statistically sound estimation procedures for the catch parameters of interest to researchers using NPGOP data. This in itself should be an asset because uniformity in estimation of catch parameters can be achieved and researchers can easily communicate the method of estimation they use.

1.2.2 Application of Estimators under Current NPGOP Design

To illustrate the utility of the derived estimators I apply them to existing data for some species as examples. In some cases, the results are compared to estimates that are currently used.

1.2.3 Optimization of the NPGOP Design

Finally, I explore optimality of the sampling design that accounts for precision of the various estimators. I do not treat optimality of sampling stategy because the class of practical sampling designs is limited by logistics of training and placing observers. The primary strata in the current design are the vessel-quarters, but there is constant effort assumed in the two (large and medium) size classes of vessels. Therefore, the optimization accounts for the requirement of equal effort (i.e., observer coverage) within these strata. In addition, I determine the optimal size threshold (length criterion) for splitting the two size classes of vessels (currently 125 ft).

I derive optimal effort results with respect to single catch parameters as well as weighted combinations of many catch parameters. The cost constraint I use is the expected number of observed trips. Optimal levels of effort within the vessel, but holding levels of effort within the trip and haul constant, is useful to address the question of optimal observer coverage levels given that decision-makers are satisfied with the levels of effort by observers at the lower stages. These equations for optimality will provide managers tools to allocate resources within the NPGOP sampling design in a transparent and methodical way.

1.2.4 Implications of the Dissertation

This dissertation could advance the study of the Bering Sea and Gulf of Alaska in the following ways:

1. *Defensible methods for estimation of parameters associated with the role of fisheries in the ecosystems.*

   This disseration provides a unique and comprehensive reference for researchers on appropriate estimation procedures for data collected by observers in the Alaskan groundfish fisheries. Anyone working with these data and applying simple estimators found in statistical texts on finite sampling theory is likely to be estimating parameters incorrectly and/or not explicitly stating the simplifying assumptions they are making in doing so. Using inappropriate estimators can result in biased estimates of the parameter as well as the variance of the estimate. In this document, I present a range of defensible estimation procedures along with clear explanations of corresponding assumptions. This should aid in forming a more rigorous basis for scientific analyses.

2. *Explicit and sound methods for estimation of parameters and uncertainty.*

   The estimation procedures should useful to scientists involved with ongoing projects that use NPGOP data. At a minimum, stock assessments scientists and marine ecologists studying populations of animals ranging from fur seals to skates should be able to easily re-evaluate their estimation procedures and outline the assumptions they are making. In other cases, these researchers may find that their estimation procedures could be improved which may alter current approaches and results.

3. *The optimization results illustrate the important role of well-defined, prioritized estimation goals in the construction of an efficient sampling design and allow managers to implement this efficiency once the estimation goals are decided.*

   The work on optimization in the dissertation could have important consequences for the management of the groundfish fisheries in Alaska. Because the industry funds the observer program, they rightfully should expect well-defined goals necessary in dictating observer deployment or coverage. On the other hand, the management agency as a scientific institution with limited funds, should have well-defined goals with respect to the NPGOP. A set of goals and priorities allows methodical decision-making on efficient allocation of resources. The multi-objective optimization results provided by this disseration gives both the industry and the resource managers the tools needed for statistically sound and efficient observer deployment.

4. *Scientists involved with other observer programs or sampling programs with multiple objectives should profit from the ideas on estimation and optimization found in this dissertation.*

   Although this work is specific to the observer sampling in Alaskan groundfish fisheries, it provides an example to other observer programs or any other large-scale, multi-objective sampling project. As seen in the literature review there is little work that has been done in fisheries contexts on optimizing sampling design with respect to multiple objectives and it is hoped that the work here is as an illustration of some possibilities.

1.3 Approach and Methods

1.3.1 Derivation of Estimators

Although there is still considerable division in views among statisticians on the proper method of inference when sampling a finite population there will always be some important advantages for design-based inference. Anyone familiar with statistical modeling is aware of the necessity that various conditions be met so that inferences are valid. However, for design-based estimators the only requirement for unbiased variance estimation is that the sample was collected correctly according to the design and that the first- and second-order inclusion probabilities for the sampled elements are known. There are no assumptions on the mechanism that generated the data.

Assumption-free estimation should be a desirable goal for the users of observer-collected data for at least two reasons. Because of the highly critical atmosphere that scientific results produced by the agency scientists often face, the removal of possible weaknesses of analyses is beneficial. When agency scientists conduct analyses that have an economic impact on the Alaskan groundfish industry, the results are extensively scrutinized and model assumptions can be seen as weaknesses in any analysis. With design-based estimation, there are no model assumptions and these weaknesses are absent.

Although assumption-free, unbiased estimation through design-based estimators is desirable, it is prohibited for data that has been or is currently being collected by observers. Therefore, I use model-based methods in tandem with design-based methods to derive estimators of catch parameters and corresponding estimator of uncertainty for use with current and historic data. I limit the use of model-based methods to situations where design-based methods are prohibited. In contrast, I use completely design-based methods for deriving estimators for use when deficiencies in the NPGOP design alluded to in section 1.2 are removed.

There are a wide array of model-based estimation methods, but the approach I take is as conservative as possible with regard to assumptions. For example, when variances are empirically unestimable within a haul, I assume mean and variance models common to a subset of hauls within a trip rather than a completely defined probability distribution for those hauls. This is a semi-parametric model-based approach rather than a fully parametric approach. In the case of the NPGOP, many estimates are functions of large sample sizes and inferences that depend on assumptions only about the mean and variance tend to be more robust to model mis-specification than the inferences made assuming complete probability distributions.

When I use design-based methods, estimators are formed using the the Horvitz-Thompson theorem (Horvitz and Thompson 1952). This approach is used extensively by Särndal et al. (1992) and provides a nice unified method for deriving unbiased estimators and corresponding estimators of uncertainty. As a simple example, consider the Horvitz-Thompson estimator of a total,

$$\widehat{T} = \sum\nolimits_s \frac{y_i}{\pi_i}$$

and the corresponding unbiased estimator of variance

$$\widehat{V}\left(\widehat{T}\right) = \sum\nolimits_s \frac{\pi_{ij} - \pi_i \pi_j}{\pi_{ij}} \frac{y_i y_j}{\pi_i \pi_j}$$

where $i \in U \equiv \{1, \ldots, N\}$, s is the set of elements that are sampled, π_i is the first-order inclusion probability of the ith element and π_{ij} is the second-order inclusion probability for the ith and jth elements. In the simple case of single-staged SRS, $\pi_i = n/N$, $\forall i \in U$ and

$$\pi_{ij} = \begin{cases} \frac{n(n-1)}{N(N-1)} & \text{if } i \neq j \\ \frac{n}{N} & \text{if } i = j \end{cases}$$

and the estimators of the total and variance can be shown with some algebra to be

$$\widehat{T} = \frac{N}{n} \sum\nolimits_s y_i = N\overline{y}$$

and

$$\widehat{V}\left(\widehat{T}\right) = \frac{N(N-n)}{n} \frac{\sum_s (y_i - \overline{y})^2}{n-1} = N(N-n)\frac{\widehat{S}^2}{n}.$$

Again, this is a simple case, but shows how the Horvitz-Thompson approach is used. In multi-staged or multi-phased sampling designs with methods other than SRS used at times (i.e., the NPGOP sampling design), derivation of the estimators of variance, especially can be more cumbersome and not readily available in existing texts on finite sampling theory. Where appropriate, I also provide approximately unbiased estimators (e.g., forms of ratio estimators) and derive corresponding estimators of uncertainty through Taylor series expansions (i.e., Delta Method).

1.3.2 Derivation of Optimal Design

As with my approach to derivation of estimators, I obtain analytic solutions for optimal sampling whenever possible. These solutions are usually achieved through application of the Cauchy-Schwartz inequality or Lagrange multipliers, however algorithmic methods are necessary in some cases.

The purpose of the optimization is to guide resource managers in effort allocation in the future, but can only be carried out on data already collected. For this reason, the optimization can only ever be an approximation, but this is always the case and does not diminish its utility. Because fishing practices change over time and optimization using earlier data may not be as representative of fishing practices in the immediate future, the optimization I explore will be based on the most current years of data collected by NPGOP.

1.3.3 Organization of this Dissertation

Interest is usually in estimation of catch parameters over large aggregations of fishing effort rather than for specific vessels or trips, but the estimators for larger scales are functions of estimators for finer scales. For example, the estimator of total weight of catch for a commercially important species over an entire season is a function of vessel-specific estimators which are, in turn, functions of trip-specific estimators which are, in turn, functions of haul-specific estimators. Therefore, I begin at the finest scale with haul-specific estimators appropriate to longline, trawl and pot gear in Chapters 2, 3 and 4. In Chapter 5, I develop estimators for larger scales: trips, vessels, fleets, and quarters. I give substantial attention to design- and model-based estimators appropriate when there is interest in totals for areas or time periods for which observer effort is not fixed. I also provide some special estimators such as seabird bycatch rates, proportions-at-age and -length class and length- or age-specific sex ratios that are functions of basic catch parameters. In Chapter 6, I apply appropriate estimators to data collected by observers over the years to illustrate their utility. I explore optimization of observer effort over the various sectors of the Alaskan Groundfish fleet with respect to single and multiple catch parameters in Chapter 7. I conclude with a discussion of the important consequences of the estimators and optimality results I provide (Chapter 8). The Appendix includes some general sampling theory results.

Chapter 2

ESTIMATION WITHIN HAUL FOR LONGLINERS

In this chapter I present estimation approaches for hauls made by longline vessels. For each catch parameter, I develop design- and model-based estimators. I begin with estimation of species-specific total numbers in Section 2.1. Total weight for a given species is the focus of Section 2.2 and I give estimators for total numbers in given length or age classes in Sections 2.3 and 2.4. Finally, I provide estimators for numbers of marine mammal interactions in Section 2.5. Mathematical development of many of the estimators and their properties is found in Section 2.A.

To help anyone reading this document, I provide a list of defintions for many if not all of the notation I use in this chapter (Table 2.1). Many of the terms are also defined as they are needed.

Table 2.1. Definition of terms

Γ_k	number of marine mammal interactions for the kth haul
Γ_{ki}	number of marine mammal interactions for the ith skate in the kth haul
Γ_{ks}	number of marine mammal interactions observed during the tally period in the kth haul
$\overline{\Gamma}_k$	average number of marine mammal interactions per skate in the kth haul
h_k	number of hooks in the tally sample in the kth haul
H_k	number of hooks in the kth haul
$\mathbf{\Lambda}_{ka}$	vector of numbers in each length class in the tally sample of the kth haul
$\mathbf{\Lambda}_{ki}$	vector of numbers in each length class in the ith skate of the kth haul
$\mathbf{\Lambda}_k$	vector of total numbers in each length class in the kth haul
$\boldsymbol{\lambda}_k$	vector of numbers in each length class among fish in the length sample for the kth haul
$\boldsymbol{\lambda}_{ki}$	vector of numbers in each length class among fish in the length sample and obtained from the ith skate of the kth haul
m_k	number of skates in the tally sample of the kth haul

Table 2.1. (Continued)

m_{ka}	number of skates from which the weight sample is taken in the kth haul
M_k	total number of skates in the kth haul
μ_t	mean weight parameter for a model that describes the weights of fish in the tth haul for a given species
N_{ka}	number of fish in the skates from which the weight sample is taken in the kth haul
$N_{ka'}$	number of fish in the skates from which the weight sample is not taken in the kth haul
N_{ki}	number of fish for a given species in the ith skate in the kth haul
N_k	number of fish for a given species in the kth haul
n_k	number of fish for a given species in the tally sample or weigth sample in the kth haul, depending on the context
n_{ki}	number of fish from the ith skate of the kth haul in the weight sample
n_{kL}	number of fish in the length sample of the kth haul
n_{kiL}	number of fish in the length sample obtained from the ith skate of the kth haul
n_{Ok}	number of otoliths sampled for the kth haul
n_{Oki}	number of otoliths sampled in the ith skate for the kth haul
n_{Ak}	number of otoliths sampled in the kth haul that are ultimately aged
n_{Om}	number of all otoliths sampled in the mth management period/region
n_{Am}	number of all otoliths sampled and aged in the mth management period/region
n_t	the number of fish in all weight samples for the tth trip
$n_{t'}$	the number of fish in all weight samples for the tth trip except the kth haul
P_k	ratio of the number of fish for a given species to the number of hooks in the kth haul
p_k	probability of capturing a fish (of a given species) on a given hook from the kth haul
$\mathbf{P}_k$	vector of proportions of fish in length or age classes in the kth haul, depending on the context

Table 2.1. (Continued)

P_{Ok}	the proportion of all otoliths sampled in the mth management period/region that were sampled from the kth haul (n_{Ok}/N_{Om})
$\overline{\mathbf{P}}_k$	average vector of proportions of fish in length or age classes for each skate in the kth haul, depending on the context
$\mathbf{P}_{ki}$	vector of proportions of fish in length or age classes for the ith skate in the kth haul, depending on the context
$\mathbf{P}_{ka}$	vector of proportions of fish in length or age classes for the tally sample in the kth haul, depending on the context
$\mathbf{p}_k$	vector of proportions of sampled fish in either length or age classes for the length or ageing sample for the kth haul, depending on the context
$\mathbf{p}_{\psi k}$	probability vector for a model that describes the age class indication for fish in the kth haul
$\overline{\mathbf{p}}_{ka}$	average vector of proportions of sampled fish in length classes for each skate in the kth haul
$\boldsymbol{\phi}_k$	probability vector for a model that describes the length class indication for fish in the kth haul
ϕ_m	Bernoulli probability that an otolith collected in the mth time period/region will be aged
$\boldsymbol{\Psi}_{ka}$	vector of numbers in each age class in the tally sample for the kth haul
$\boldsymbol{\Psi}_{ki}$	vector of numbers in each age class in the ith skate for the kth haul
$\boldsymbol{\Psi}_k$	vector of total numbers in each age class in the kth haul
$\boldsymbol{\psi}_k$	vector of numbers of aged fish in each age class for the kth haul
$\boldsymbol{\psi}_{ki}$	vector of numbers of aged fish in each age class in the ith skate of the kth haul
$\boldsymbol{\psi}_{Ok}$	vector of numbers of fish in each age class in the otolith sample for the kth haul
$\boldsymbol{\psi}_{Oki}$	vector of numbers of fish in each age class in the otolith sample and ith skate from the kth haul
s_1	set of skates in the tally sample
s_a	set of skates from which the weight sample is taken
$s_{a'}$	set of skates from which the weight sample is not taken

Table 2.1. (Continued)

σ_t^2	variance parameter for a model that describes the weights of fish in the tth haul for a given species
U_1	all skates in a set
W_k	the weight of fish in the kth haul
W_{kt}	the weight of the tth fish in the kth haul
W_{kit}	the weight of the tth fish from the ith skate in the kth haul
W_{ki}	weight of fish in the ith skate from the kth haul
$\overline{W}_{ka}$	average weight of fish in the tally sample kth haul
$\overline{W}_{kb}$	average weight of fish in the weight sample kth haul
$\overline{\overline{W}}_k$	the average of the average weights of fish in each skate for the kth haul

2.1 *Total Number for a Fish or Seabird Species*

2.1.1 Design-based Estimation

The observers are instructed to "randomly" select approximately 30% of the hooks for a given set. Generally, observers take a systematic sample or simple random sample without replacement (SRS) of groups of hooks (skates) which is made systematically or without replacement. If the observer is using the skate as the sample unit, the most obvious estimate of the total number for a given species, N, in the kth set is

$$ {}_1\widehat{N}_k = \frac{M_k}{m_k} \sum_{i=1}^{m_k} N_{ki} \tag{2.1} $$

where m_k is the number of skates sampled, M_k is the total number of skates in the set and N_{ki} is the number in the ith skate. If there is a SRS of skates or hooks, then Eq. 2.1 is the Horvitz-Thompson estimator of the total number of fish (Horvitz and Thompson 1952). The corresponding variance and variance estimator are

$$ V\left({}_1\widehat{N}_k\right) = M_k\left(\frac{M_k}{m_k} - 1\right)\frac{\sum_{i=1}^{M_k}\left(N_{ki} - \overline{N}_k\right)^2}{M_k - 1} = M_k\left(\frac{M_k}{m_k} - 1\right)S_k^2 $$

where $\overline{N}_k = \sum_{i=1}^{M_k} N_{ki}/M_k$ and

$$\widehat{V}\left({}_1\widehat{N}_k\right) = M_k\left(\frac{M_k}{m_k} - 1\right)\frac{\sum_{i=1}^{m_k}\left(N_{ki} - \widehat{\overline{N}}_k\right)^2}{m_k - 1} = M_k\left(\frac{M_k}{m_k} - 1\right)\widehat{S}_k^2, \tag{2.2}$$

where $\widehat{\overline{N}}_k = \sum^{m_k} N_{ki}/m_k$. Derivation of Eq. 2.1 and its properties are obtained by direct application SRS results found in Section A.2. The Horvitz-Thompson estimator of the total under SRS is a well-known result given by various authoritative texts (e.g., Sukhatme and Sukhatme 1970, pg. 59; Cassel et al. 1977, pg. 23; Cochran 1977, pg. 259; Särndal et al. 1992, pg. 43) and will arise repeatedly in design-based estimators I provide in the first four chapters.

As noted in Chapter 1, observers often systematically sample skates (or some other grouping of hooks) within each set and the numbers of each species or group in each skate are summed before reporting which impairs the estimation of variance among the sampled skates. Both of these practices negate the usual design-based estimators for sampling variance of the total under simple random sampling of skates and a less desirable estimate of variance can be obtained by assuming a model for fish capture with a known mean-variance relationship. Data is recorded by observers that can be used to estimate variance under this assumption.

Cochran (1977, pg. 212) and others have advocated the use of the SRS variance estimator for systematic sampling under certain conditions and this may be useful if skate-specific totals were recorded. Under conditions where there may be an increasing or decreasing trend in skate totals the SRS variance estimator would be positively biased (conservative precision). If the skate-specific totals are in random sequence within the haul, then the SRS variance estimator would equal the true sampling variance in expectation over the joint distribution of the sampling design and the process that generate the haul.

2.1.2 Model-based Estimation

Here I use two different approaches to develop model-based estimators of the total number caught for a given species. The first simply applies a design-based estimator when the assumed design is clearly not used to collect data and the second uses the super-population model approach that I described in Section 1.1.4. Interestingly, the resulting estimators (and variance estimators) using either approach are identical functions of the collected data.

If it is safe to assume that there is a SRS of hooks, then the number of fish in the sample, $n_k = \sum^{m_k} N_{ki}$, is a hypergeometric random variable with

$$E_M(n_k) = N_k \frac{h_k}{H_k}$$

and

$$V_M(n_k) = \frac{h_k}{H_k}\left(1 - \frac{h_k}{H_k}\right)\frac{N_k(H_k - N_k)}{H_k(H_k - 1)}$$

where h_k is the number of hooks in the sample and H_k is the total number of hooks in the haul. A model-unbiased estimator of the haul total, N_k, is

$$_2\widehat{N}_k = \frac{H_k}{h_k} n_k. \tag{2.3}$$

The variance of the estimator is

$$V\left({}_2\widehat{N}_k\right) = H_k\left(\frac{H_k}{h_k} - 1\right)\frac{H_k P_k(1 - P_k)}{H_k - 1}$$

where $P_k = N_k/H_k$. An unbiased estimator of the variance of Eq. 2.3 is

$$\widehat{V}\left({}_2\widehat{N}_k\right) = H_k\left(\frac{H_k}{h_k} - 1\right)\frac{h_k\widehat{P}_k(1 - \widehat{P}_k)}{h_k - 1} \approx H_k\left(\frac{H_k}{h_k} - 1\right)\widehat{P}_k(1 - \widehat{P}_k), \tag{2.4}$$

where $\widehat{P}_k = n_k/h_k$. The derivation of Eq. 2.4 is obtained by straightforward application of the results in Section A.2.1.

Using a super-population model approach where the capture of a fish on a hook in a set is assumed independent across all hooks in the set and the probability of capture (p_k) is constant throughout the set, the number of fish counted in the tally sample is a binomial random variable with mean: $E_m(N_k) = h_k p_k$ and variance: $V_m(n_k) = h_k p_k(1 - p_k)$. The best linear unbiased predictor of the total is

$$\widehat{N}_k = n_k + (H_k - h_k)\widehat{p}_k = n_k + (H_k - h_k)\frac{n_k}{h_k} = H_k\widehat{p}_k$$

where $\widehat{p}_k = n_k/h_k$. Valliant et al. (2000) provide a straightforward introduction to prediction with super-population models. The prediction error variance is

$$V\left(\widehat{N}_k\right) = H_k\left(\frac{H_k}{h_k} - 1\right) p_k(1 - p_k).$$

which is unbiasedly estimated by

$$\widehat{V}\left(\widehat{N}_k\right) = H_k\left(\frac{H_k}{h_k} - 1\right) \frac{h_k\widehat{p}_k(1 - \widehat{p}_k)}{h_k - 1} \approx H_k\left(\frac{H_k}{h_k} - 1\right) \widehat{p}_k(1 - \widehat{p}_k).$$

That the variance estimator is unbiased can be shown algebraically with the identity $V(x) = E(x^2) - E(x)^2$.

Notice that this variance estimator is identical in form to Eq. 2.4. Therfore, departures from either model (SRS of hooks or super-population approach) will have the same effect on the properties of the estimators. If there is clustering in the distribution of the fish caught along a given longline set, then the estimators should still be unbiased, but there would be greater variability than expected in the above model which would lead to negatively biased estimation of within-set variance. However, because we cannot obtain an empirical variance estimate and estimation of totals within small spatial/temporal domains (or groups) of hauls is of interest, some model (with estimable parameters) within the haul is better than to ignore the variability within hauls completely. Furthermore, when we are interested in estimating at larger scales the component of variability due to sampling within hauls is likely to be negligible (Särndal et al. 1992, [pg. 139-140).

2.2 Total Weight For a Fish or Seabird Species

2.2.1 Design-based Estimation

With regard to design-based estimation, the sampling for total weight of a species in a haul is fraught with problems. While a longline set is being retrieved, observers often take specimens to weigh from sections of the set that are not part of the tally sample where numbers of each species are recorded. The observer determines the average weight for each species occuring in the weight sample, but the total numbers of each species in the weight sample is unknown. The observers are instructed to weigh a specified number of individuals from each species, however it is doubtful that this number is weighed every haul.

Under an SRS assumption, it is necessary to know the total number of elements being sampled for unbiased estimation of the total weight and corresponding variance. True, when we use single-stage SRS and we are interested in the population mean, the sampling fraction is often ignored if it is near zero. But to estimate the population total, we still need the total number of elements. In two-stage SRS the total number of elements within each of the secondary sampling units is also needed. In addition, the fact that the weight of sampled individuals are recorded in aggregate prohibits estimation of variance among individual fish weights within a set. Lastly, depending on the method used to obtain the weight sample, variance estimation may require the information indicating the skate (or group of hooks) origin for each fish that enters the weight sample.

For developement of design-based estimators of total weight of a given species in the kth haul, we assume a SRS of skates at a first phase. Given the selected skates, a sample of individuals of a particular species is chosen from the individuals within skates sampled at the first phase (tally sample) or the individuals in skates not sampled at the first phase (non-tally sample). Although the total number of individuals to sample for weight is specified in AFSC (2004), exactly how the sample is taken from the chosen skates is not specified. Observers may sample from all fish in the chosen skates (method 1) or observers may randomly sample individual fish within each skate separately to achieve the specified sample number in total (method 2). That there is no indication in the archived data of the sampling method used by observers within each haul further hampers design-based estimation. I further assume that the second-phase sample is collected via SRS. Given the two possible methods described for taking the weight sample and the possibility of the weight sample being taken from the tally sample or the non-tally sample, there are four conceivable within-haul sample designs to obtain weight samples.

Let the set of all skates be U_1, those selected for the tally sample s_1 and those from which the weight sample is ultimately taken, s_a. When the weight sample is taken from the non-tally sample $s_a = U_1 - s_1$. The number of skates in the non-tally section is $m_{ka} = M_k - m_k$ where m_k is the number of skates in the tally sample and M_k is the total number of skates. On the other hand, when the weight sample is taken from the tally sample $s_a = s_1$.

Assuming that method 1 is used and the weight sample is taken from the tally period, the design-based estimator of total weight for a species in the kth haul is

$$ {}_1\widehat{W}_k = \frac{M_k N_{ka}}{m_k n_k} \sum_{t=1}^{n_k} W_{kt} \tag{2.5}$$

where $N_{ka} = \sum_{i=1}^{m_k} N_{ki}$. The observer is instructed to sample a fixed number of a species, so n_k is presumably constant across all designs, but notice that N_{ka} can change across first-phase samples and thus the second-phase inclusion probabilities associated with individual fish are different for given first-phase samples of skates. An unbiased variance estimator is

$$\begin{aligned}\widehat{V}\left({}_1\widehat{W}_k\right) =& M_k\left(\frac{M_k}{m_k}-1\right)\frac{\sum_{i=1}^{m_k}\left(\widehat{W}_{ki}-\widehat{\overline{W}}_k\right)^2}{m_k-1}+\frac{M_k}{m_k}\sum_{i=1}^{m_k}\widehat{V}\left(\widehat{W}_{ki}|s_a\right)\\ &+\frac{M_k(M_k-1)}{m_k(m_k-1)}\sum\sum_{i\neq j}^{m_k}\widehat{Cov}\left(\widehat{W}_{ki},\widehat{W}_{kj}|s_a\right)\end{aligned} \tag{2.6}$$

where $\widehat{W}_{ki} = N_{ka}\sum_{t=1}^{n_{ki}} w_{it}/n_k$, $\widehat{\overline{W}}_k = \sum_{i=1}^{m_k} \widehat{W}_{ki}/m_k$,

$$\widehat{V}\left(\widehat{W}_{ki}|s_a\right) = N_{ka}\left(\frac{N_{ka}}{n_k}-1\right)\left[\frac{\sum_{t=1}^{n_{ki}} W_{it}^2}{n_k-1}-\frac{\left(\sum_{t=1}^{n_{ki}} W_{it}\right)^2}{n_k(n_k-1)}\right], \tag{2.7}$$

$$\widehat{Cov}\left(\widehat{W}_{ki},\widehat{W}_{kj}|s_a\right) = -N_{ka}\left(\frac{N_{ka}}{n_k}-1\right)\frac{\left(\sum_{t=1}^{n_{ki}} W_{it}\right)\left(\sum_{u=1}^{n_{kj}} W_{ju}\right)}{n_k(n_k-1)}, \tag{2.8}$$

and n_{ki} is the number of sample elements in the ith skate. The unbiasedness of Eq. 2.5 and Eq. 2.6 are derived in Section 2.A.1 using results from Sections A.7 and A.6 on two-phase sampling and domain estimation, respectively. Notice in Eq. 2.6 that we need to know weights of individual fish and indication of the skate where they were caught to estimate variance. However, notice that there is no limitation on the number of fish from each skate to be in the sample. The only requirement is that more than one fish is sampled ($n_k \geq 2$).

An alternative estimator of the total weight is obtained by post-stratifying the weight sample at the second phase where the strata are the skates in the tally sample from which the weight sample arises. Post-stratification is a common practice when stratified random sampling cannot be implemented *a priori* and it is believed there is a gain in efficiency by accounting for strata. The post-stratified estimator is

$$ {}_2\widehat{W}_k = \frac{M_k}{m_k} \sum_{i=1}^{m_k} \frac{N_{ki}}{n_{ki}} \sum_{t=1}^{n_{ki}} W_{it}. \tag{2.9} $$

Unbiasedness of Eq. 2.9 requires at least one fish be sampled from each skate in the tally sample, that is, $E({}_2\widehat{W}_k|\Omega_1) = W_k$ where Ω_1 is the event that $n_{ki} \geq 1$ for all $k \in s_a$. The form of the approximate variance of the post-stratified estimator is given in Section A.7.3 which is estimated by substituting sample-based values so that

$$ \begin{aligned} \widehat{V}\left({}_2\widehat{W}_k\right) =& M_k\left(\frac{M_k}{m_k}-1\right)\widehat{S}^2_{k_1} + \frac{m_k}{M_k n_k}\left\{(n_k-1)\left[\left(1-\frac{m_k}{M_k}\right)\frac{\widehat{S}^2_{k_2}}{n_k}\right.\right. \\ &\left.\left. + \widehat{\eta}_k\left(\frac{m_k N_{ka}}{M_k n_k}-1\right)\right] + \left(1-\frac{m_k}{M_k}\right)\frac{\widehat{S}^2_{k_3}}{n_k} + \widehat{S}_k\left(\frac{m_k N_{ka}}{M_k n_k}-1\right)\right\} \end{aligned} \tag{2.10} $$

where

$$ \widehat{S}^2_{k_1} = \frac{\sum_{i=1}^{m_k}\left(\widehat{W}_{ki} - \widehat{\overline{W}}_k\right)^2}{m_k-1} - \frac{1}{m_k}\sum_{i=1}^{m_k}\left[N_{ki}\left(\frac{N_{ki}}{n_{ki}}-1\right)\widehat{S}^2_{ki}\right] $$

$$ \widehat{S}^2_{k_2} = \frac{\sum_{i=1}^{m_k}(N_{ki}-\widehat{\overline{N}}_k)(N_{ki}\widehat{S}^2_{ki} - \widehat{\overline{NS}}_k)}{m_k-1} $$

$$ \widehat{S}^2_{k_3} = \frac{\sum_{i=1}^{m_k}(N_{ki}-\overline{N}_k)(\widehat{S}^2_{ki} - \widehat{\overline{S}}_k)}{m_k-1} $$

$$ \widehat{\eta}_k = \frac{M_k}{m_k}\sum_{i=1}^{m_k} N_{ki}\widehat{S}^2_{ki}, $$

$$ \widehat{S}_k = \frac{M_k}{m_k}\sum_{i=1}^{m_k} \widehat{S}^2_{ki}, $$

$$ \widehat{\overline{N}}_k = \sum_{i=1}^{m_k} N_{ki}/m_k, $$

$$ \widehat{\overline{NS}}_k = \sum_{i=1}^{m_k} N_{ki}\widehat{S}^2_{ki}/m_k $$

and

$$\widehat{\widehat{S}}_k = \sum_{i=1}^{m_k} \widehat{S}^2_{ki}/m_k$$

Notice that the variance estimator requires the condition, Ω_2, that is $n_{ki} \geq 2$ for all sampled skates and thus, it is assumed that $\mathrm{P}(n_{ki} < 2) \approx 0$ which may not be negligible when many skates are sampled at the first phase and/or the species is rare. For derivation of Eq. 2.9 and Eq. 2.10 see Section A.7.3.

Now, consider the case when method 1 is used but the weight sample comes from the non-tally first-phase sample. To use Eq. 2.5, the total number of fish in the selected skates is needed to estimate the total weight as well as the variance, but when the weight sample is taken from the non-tally sample the total number of fish in the selected skates is not known. An estimator that is a function of the total number estimator (Eq. 2.1) is

$$_3\widehat{W}_k = \frac{_1\widehat{N}_k}{n_k} \sum_{t=1}^{n_k} W_t \tag{2.11}$$

and a corresponding approximately unbiased variance estimator is

$$\begin{aligned} \widehat{V}\left(_3\widehat{W}_k\right) =& \widehat{N}_k\left(\frac{\widehat{N}_k}{n_k} - \frac{1}{1-f_k}\right)\widehat{S}^2_{kW_b} + \left(\frac{f_k}{1-f_k}\right)^2 \widehat{V}\left(\widehat{W}_{ka}\right) \\ &+ \left(\frac{_3\widehat{W}_k}{\widehat{N}_k}\right)^2 \left[1 - \frac{f_k}{\widehat{N}_k(1-f_k)}\right]^2 \widehat{V}\left(\widehat{N}_k\right) \\ &+ 2\frac{_3\widehat{W}_k f_k}{\widehat{N}_k(1-f_k)}\left[1 - \frac{f_k}{\widehat{N}_k(1-f_k)}\right] \widehat{Cov}\left(\widehat{N}_k, \widehat{W}_{ka}\right) \end{aligned} \tag{2.12}$$

where $f_k = m_k/M_k$,

$$\widehat{V}\left(\widehat{N}_k\right) = M_k\left(\frac{M_k}{m_k} - 1\right)\frac{\sum_{i=1}^{m_k}(N_{ki} - \widehat{\overline{N}}_{ka'})^2}{m_k - 1}$$

$$\widehat{S}^2_{kWb} = \frac{\sum_{t=1}^{n_k}\left(W_{kt} - \overline{W}_{kb}\right)^2}{n_k - 1},$$

$$\widehat{Cov}\left(\widehat{N}_k, \widehat{W}_{ka}\right) = -M_k\frac{\sum_{i=1}^{m_{ka}}(N_{ki} - \widehat{\overline{N}}_{ka})(\widehat{W}_{ki} - \widehat{\overline{W}}_{ka})}{m_{ka} - 1}, \tag{2.13}$$

$$\hat{V}\left(\widehat{W}_{ka}\right) = M_k\left(\frac{M_k}{m_k}-1\right)\left[\frac{\sum_{i=1}^{m_{ka}}(\widehat{W}_{ki}-\widehat{\overline{W}}_{ka})^2}{m_{ka}-1}\right.$$
$$\left. - \frac{1}{m_{ka}}\sum_{i=1}^{m_{ka}} N_{ki}\left(\frac{N_{ki}}{n_{ki}}-1\right)\frac{\sum_{t=1}^{n_{ki}}(W_{kit}-\widehat{\overline{W}}_{ki})^2}{n_{ki}-1}\right], \tag{2.14}$$

$\overline{W}_{kb} = \sum_{t=1}^{n_k} W_{kit}/n_k$, $\widehat{\overline{W}}_{ka} = \sum_{i=1}^{m_{ka}} \widehat{W}_{ki}/m_{ka}$ and $\hat{\overline{N}}_{ka} = \sum_{i=1}^{m_{ka}} N_{ki}/m_{ka}$. The estimators Eq. 2.13 and Eq. 2.14 would be conditionally unbiased given Ω_2 as defined for Eq. 2.10, but the total number within each of the (non-tally) sampled skates, N_{ki}, is still needed. The approximate unbiasedness of Eq. 2.11 and corresponding asymptotic variance are derived in Section 2.A.2.

If method 2 is used where there is independent sampling within each of the skates chosen in the first-phase, simpler variance estimators are obtained. With this method, observers would spread the number of fish needed for the weight sample over all the skates chosen to sample. If the number taken from within each skate is proportional to the number of fish in the skate and the weight sample is obtained from the skates in the tally sample, then $n_{ki} = n_k N_{ki}/\left(\sum_{i=1}^{m_k} N_{ki}\right)$. If an equal number of fish is chosen from each skate, then $n_{ki} = n_k/m_k$. In any case, the estimator of the total weight and variance estimator result from two-stage SRS which are also used to estimate all catch parameters at the next higher level (trip-specific, see Section 5.1) and are derived in Section A.3.1.

When the fish comprising the weight sample are taken from fish in the tally sample and method 2 is used at the second phase, the unbiased estimator of total weight is

$$_4\widehat{W}_k = \frac{M_k}{m_k}\sum_{i=1}^{m_k}\frac{N_{ki}}{n_{ki}}\sum_{t=1}^{n_{ki}} W_{kit} \tag{2.15}$$

which is identical to the estimator Eq. 2.9 for the post-stratification approach, but here the conditioning on sample sizes within skates is unnecessary because SRS is performed within each skate. The variance estimator is

$$\hat{V}\left({}_4\widehat{W}_k\right) = M_k\left(\frac{M_k}{m_k}-1\right)\frac{\sum_{i=1}^{m_k}(\widehat{W}_{ki}-\widehat{\overline{W}}_k)^2}{m_k-1}$$
$$+\frac{M_k}{m_k}\sum_{i=1}^{m_k} N_{ki}\left(\frac{N_{ki}}{n_{ki}}-1\right)\frac{\sum_{t=1}^{n_{ki}}(W_{kit}-\overline{W}_{ki})^2}{n_{ki}-1} \tag{2.16}$$

where $\widehat{W}_{ki} = N_{ki} \sum_{t=1}^{n_{ki}} W_{kit}/n_{ki}$, $\widehat{\overline{\overline{W}}}_k = \sum_{i=1}^{m_k} \widehat{W}_{ki}/m_k$ and $\overline{W}_{ki} = \sum_{t=1}^{n_{ki}} W_{kit}/n_{ki}$.

Similarly, if method 2 is used, but the weight sample is taken from the non-tally sample, the unbiased estimator of total weight is

$$_5\widehat{W}_k = \frac{M_k}{m_{ka}} \sum_{i=1}^{m_{ka}} \frac{N_{ki}}{n_{ki}} \sum_{t=1}^{n_{ki}} W_{kit}. \tag{2.17}$$

The variance estimator is

$$\widehat{V}\left({}_5\widehat{W}_k\right) = M_k \frac{m_k}{m_{ka}} \frac{\sum_{i=1}^{m_{ka}} (\widehat{W}_{ki} - \widehat{\overline{\overline{W}}}_{ka})^2}{m_{ka} - 1} + \frac{M_k}{m_{ka}} \sum_{i=1}^{m_{ka}} N_{ki} \left(\frac{N_{ki}}{n_{ki}} - 1\right) \widehat{S}^2_{ki}. \tag{2.18}$$

where $\widehat{\overline{\overline{W}}}_k = M_k \sum_{i=1}^{m_{ka}} \widehat{W}_{ki}/m_{ka}$ and $\widehat{W}_{ki}$ and $\widehat{S}^2_{ki}$ are defined as in Eq. 2.16.

However, as mentioned above in the estimation of total weight with method 1 and sampling from the non-tally period, we do not have information on the total number of fish within each sampled skate when we take the weight sample from the non-tally sample. An estimator of the total that is a product of the estimator of the mean of within-skate mean weights and the estimator of the total from Section 2.1,

$$_6\widehat{W}_k = \widehat{N}_k \widehat{\overline{\overline{W}}}_k = \widehat{N}_k \frac{1}{m_{ka}} \sum_{i=1}^{m_{ka}} \frac{1}{n_{ki}} \sum_{t=1}^{n_{ki}} W_{ki}. \tag{2.19}$$

The asymptotically unbiased variance estimator is

$$\widehat{V}\left({}_6\widehat{W}_k\right) = \widehat{\overline{\overline{W}}}^2_k \widehat{V}\left(\widehat{N}_k\right) + \widehat{N}^2_k \widehat{V}\left(\widehat{\overline{\overline{W}}}_k\right) + 2\widehat{\overline{\overline{W}}}_k \widehat{N}_k \widehat{Cov}\left(\widehat{\overline{\overline{W}}}_k, \widehat{N}_k\right)$$

where

$$\widehat{V}\left(\widehat{\overline{\overline{W}}}_k\right) = \frac{m_k}{M_k m_{ka}} \frac{\sum_{i=1}^{m_{ka}} \left(\overline{W}_{ki} - \widehat{\overline{\overline{W}}}_{ka}\right)^2}{m_{ka} - 1} + \frac{1}{m_{ka} M_k} \sum_{i=1}^{m_{ka}} \left(1 - \frac{n_{ki}}{N_{ki}}\right) \frac{\sum_{t=1}^{n_{ki}} \left(W_{kit} - \widehat{\overline{W}}_{ki}\right)^2}{n_{k'i} (n_{ki} - 1)}$$

and

$$\widehat{Cov}\left(\widehat{\overline{\overline{W}}}_k, \widehat{N}_k\right) = -\frac{\sum_{i=1}^{m_{ka}} \left(\widehat{\overline{W}}_{ki} - \widehat{\overline{\overline{W}}}_k\right)\left(N_{ki} - \widehat{\overline{N}}_{ka}\right)}{m_{ka} - 1}.$$

Knowledge of the total number of the species in the skate is still required for the sampling fractions within skates and the covariance term, but for prevalent species the within-skate sampling fraction may be negligible and when size and frequency are uncorrelated the covariance term may be neglible. The properties of Eq. 2.19 are derived in Section 2.A.3.

Regardless of which of the above estimators is deemed appropriate, major changes in the way data are recorded are warranted and perhaps some changes in the sampling instructions given to observers would be beneficial. At the very least, it is critical to know how the observer sampled each haul and that weights of each fish are recorded. Also of importance, is recording the skate origin of each fish to distinguish which fish came from a common skate.

It would behoove both the individual observers and analysts to specify a single method for obtaining weight samples from longline sets. The responsibilities of the observers would be eased by removing the decision of which protocol to use and the analysts would not be required to comb through each haul to apply various estimators for specific methods.

2.2.2 Model-based Estimation

In the absence of design-based methods to estimate variance we have to make stronger assumptions about the distribution of fish in the haul. Let us assume what is termed a "super-population" model for the weight in the tth trip where the expected value and variance of the weight for the ith fish are $E_M(W_i) = \mu_t$ and $V_M(W_i) = \sigma_t^2$. Furthermore, assume that weights of fish in the trip are uncorrelated (for $i \neq j$, $C_M(W_i, W_j) = 0$). In other words, with this model we assume the weight of each fish in the trip is a realization of a random variable with a mean that is specific to the tth trip where the fish is caught and the variance is trip-specific rather than haul-specific. Under this model we also know $E_M\left(\overline{W}_k\right) = \mu_t$ and $V_M\left(\overline{W}_k\right) = \sigma_t^2/n_k$. Remember that the number of fish in the weight sample, n_k, is unknown, but may be assumed to be the number observers are instructed to weigh according to their prevalence in the haul. Therefore, the number weighed for the predominant species in the catch might be assumed to be approximately 50, whereas the number weighed for bycatch species might be about 15. Using the prediction approach (see Royall 1970, 1976; Bolfarine and Zacks 1992; Valliant et al. 2000) with the above information the best linear unbiased predictor (BLUP) of the total weight within the haul is

$$\widehat{W}_k = N_k \widehat{\mu}_t = N_k \frac{\sum_{k=1}^{g_t} n_k \overline{W}_k}{n_t} = N_k \frac{\sum_{k=1}^{g_t} \sum_{i=1}^{n_k} W_{ki}}{n_t}$$

where $n_t = \sum_{k=1}^{g_t} n_k$ and the prediction error variance is

$$V_M\left(\widehat{W}_k\right) = \left(N_k - n_k\right)\left(N_k + n_{t'}\right)\frac{\sigma_t^2}{n_t}$$

where g_t is the number of hauls in the tth trip and

$$n_{t'} = \sum_{\substack{l=1 \\ l \neq k}}^{g_t} n_{tl}$$

is the number of fish in weight samples from hauls other than the one of interest here (i.e., the kth haul). However, we only have an estimate of the total number of fish in the haul so an estimator of the total weight in the haul that is unbiased when the estimators of mean weight and total number of fish are independent is

$$_7\widehat{W}_k = \widehat{N}_k \widehat{\mu}_t \tag{2.20}$$

where the estimator of the total number of fish in the haul may be one presented in Section 2.1. The corresponding variance and unbiased variance estimator are

$$V\left(_7\widehat{W}_k\right) = \left(N_k + n_{t'}\right)\frac{\left(N_k - n_k\right)\sigma_t^2}{n_t} + \left(\mu_t^2 + \frac{\sigma_t^2}{n_t}\right)V\left(\widehat{N}_k\right)$$

and

$$\widehat{V}_1\left(_7\widehat{W}_k\right) = \left[\widehat{N}_k^2 - \widehat{V}\left(\widehat{N}_k\right) + \widehat{N}_k\left(n_{t'} - n_k\right) + n_k n_{t'}\right]\frac{\widehat{\sigma}_t^2}{n_t} + \widehat{\mu}_t^2 \widehat{V}\left(\widehat{N}_k\right) \tag{2.21}$$

where

$$\widehat{\sigma}_t^2 = \frac{g_t}{\sum_{k=1}^{g_t} \frac{1}{n_k}} \frac{\sum_{k=1}^{g_t} \left(\overline{W}_k - \overline{\overline{W}}_t\right)^2}{g_t - 1}$$

and $\overline{\overline{W}}_t = \sum_{k=1}^{g_t} \overline{W}_k / g_t$. A biased but consistent variance estimator is

$$\widehat{V}_2\left({}_7\widehat{W}_k\right) = \left[\widehat{N}_k^2 + \widehat{N}_k\left(n_{t'} - n_k\right) + n_k n_{t'}\right] \frac{\widehat{\sigma}_t^2}{n_t} + \widehat{\mu}_t^2 \widehat{V}\left(\widehat{N}_k\right).$$

The variance estimator Eq. 2.21 is also related to the variance estimator given by Goodman (1960) for the product of independent random variables, but differs slightly in that it treats prediction error variance. See section A.8 in the appendix for derivation of the estimators and variance.

2.3 Total Numbers in Length Classes

2.3.1 Design-based Estimation

As I mentioned in Section 1.1.3.1 the observer is frequently required to record length measurements for some species in a given set and often the observer will use specimens from the weight sample for this purpose. However, lengths are not recorded for all species (see Section 2.2). Whether lengths are measured for a given species is determined by the prevalence of that species in the set and/or the needs of NMFS. If prevalence determines that lengths are recorded for a species, then the universe to which inference is appropriate changes from that considered earlier for total number and total weight. Because length samples are only taken from hauls where the species are prevalent, there is no probability of length sampling in hauls where the species is not prevalent. Therefore, the universe being sampled reduces to catches where the species of interest is prevalent. The subset of hauls where a given species, s, is prevalent will be called s-prevalent. That is, s-prevalent hauls from the tth trip are a subset of all hauls in that trip. This is important because estimation of numbers-at-length for all individuals that are caught based on lengths taken only on individuals from hauls where they are prevalent may be biased with respect to numbers-at-length for *all* hauls. For example, if Pacific cod is found in different species assemblages at different sizes, then those catches where it is predominant may have larger or smaller individuals than those catches where it is not predominant.

Let us assume that the observer takes a SRS of the fish in the weight sample for length measurements and we are interested in estimating the number of individuals in each of L length classes. Because of the multiple ways observers may take weight samples, there were several estimators for total weight developed in Section 2.2. There will be corresponding estimators for total numbers-at-length derived here due to the assumption of the length sample being a subset of the weight sample. As I show in Section A.2.3, when there is a SRS of a SRS the inclusion probabilities are not a function of the first SRS inclusion probabilites. Therefore, we can think of the length sample as a two-phase SRS like the weight sample.

There are two methods described in Section 2.2 to obtain the weight sample (and length sample): method 1 was a SRS of all fish in skates selected in the first phase and method 2 was a SRS of fish within each skate. If method 1 is employed to take the weight sample, the sample comes from the skates in the tally sample and a SRS of skates is assumed for the tally sample, then an unbiased estimator for the vector of total numbers in L length classes, $\mathbf{\Lambda}_k$, is

$$ {}_1\widehat{\mathbf{\Lambda}}_k = \frac{M_k N_{ka}}{m_k n_{kL}} \boldsymbol{\lambda}_k \tag{2.22} $$

and an unbiased variance-covariance matrix (VCM) estimator is

$$ \begin{aligned} \widehat{V}\left({}_1\widehat{\mathbf{\Lambda}}_k\right) =& M_k\left(\frac{M_k}{m_k}-1\right)\frac{\sum_{i=1}^{m_k}\left(\widehat{\mathbf{\Lambda}}_{ki}-\widehat{\overline{\mathbf{\Lambda}}}_{ka}\right)^2}{m_k-1}+\frac{M_k}{m_k}\sum_{i=1}^{m_k}\widehat{V}\left(\widehat{\mathbf{\Lambda}}_{ki}|s_a\right) \\ &+\frac{M_k}{m_k}\left(\frac{M_k}{m_k}-1\right)\frac{\sum\sum_{i\neq i'}^{m_k}\widehat{Cov}\left(\widehat{\mathbf{\Lambda}}_{ki},\widehat{\mathbf{\Lambda}}_{ki'}|s_a\right)}{m_k-1} \end{aligned} \tag{2.23} $$

where $\widehat{\overline{\mathbf{\Lambda}}}_{ka} = \sum_{i=1}^{m_k} \widehat{\mathbf{\Lambda}}_{ki}/m_k$,

$$ \widehat{V}\left(\widehat{\mathbf{\Lambda}}_{ki}|s_a\right) = N_{ka}\left(\frac{N_{ka}}{n_{kL}}-1\right)\frac{\left[\operatorname{diag}\left(\mathbf{p}_{ki}\right)-\mathbf{p}_{ki}\mathbf{p}_{ki}^T\right]}{n_{kL}-1}, $$

$$ \widehat{Cov}\left(\widehat{\mathbf{\Lambda}}_{ki},\widehat{\mathbf{\Lambda}}_{ki'}|s_a\right) = -\left(1-\frac{n_{kL}}{N_{k_a}}\right)\frac{\widehat{\mathbf{\Lambda}}_{ki}\widehat{\mathbf{\Lambda}}_{ki'}^T}{n_{kL}-1}, $$

$\widehat{\mathbf{\Lambda}}_{ki} = N_{k_a}\boldsymbol{\lambda}_{ki}/n_{kL}$ and $\mathbf{p}_{ki} = \boldsymbol{\lambda}_{ki}/n_{kL}$. See Table 2.1 for definition of terms in the variance estimator. Estimators Eq. 2.22 and Eq. 2.23 are analogous to Eq. 2.11 and Eq. 2.12, respectively, and derivations can be found in Section A.7.2. Notice also that $\sum_{l=1}^{L} \Lambda_{kl} = N_k$.

When the length sample comes from the non-tally sample the number of fish from which the length sample is taken is not known. In this case, the asymptotically unbiased estimator analogous to Eq. 2.11 and VCM estimator are

$$ {}_2\widehat{\mathbf{\Lambda}}_k = \widehat{N}_k \frac{\boldsymbol{\lambda}_k}{n_{kL}} = \widehat{N}_k \mathbf{p}_k \tag{2.24} $$

and

$$ \begin{aligned} \widehat{V}\left({}_2\widehat{\mathbf{\Lambda}}_k\right) =& \widehat{N}_k \left(\frac{\widehat{N}_k}{n_{kL}} - \frac{1}{1-f_k}\right) \frac{n_{kL}\left[\operatorname{diag}(\mathbf{p}_k) - \mathbf{p}_k\mathbf{p}_k^T\right]}{n_{kL}-1} + \left(\frac{f_k}{1-f_k}\right)^2 \widehat{V}\left(\widehat{\mathbf{\Lambda}}_{ka}\right) \\ &+ {}_2\widehat{\mathbf{\Lambda}}_k {}_2\widehat{\mathbf{\Lambda}}_k^T (1-\xi_k)^2 \frac{\widehat{V}\left(\widehat{N}_k\right)}{\widehat{N}_k^2} \\ &+ \xi_k (1-\xi_k) \left[{}_2\widehat{\mathbf{\Lambda}}_k \widehat{Cov}\left(\widehat{N}_k, \widehat{\mathbf{\Lambda}}_{ka}^T\right) + \widehat{Cov}\left(\widehat{\mathbf{\Lambda}}_{ka}, \widehat{N}_k\right) {}_2\widehat{\mathbf{\Lambda}}_k^T\right] \end{aligned} \tag{2.25} $$

where $f_k = m_k/M_k$,

$$ \xi_k = \frac{f_k}{\widehat{N}_k(1-f_k)}, $$

$$ \widehat{V}\left(\widehat{N}_k\right) = M_k \left(\frac{M_k}{m_k} - 1\right) \frac{\sum_{i=1}^{m_k} (N_{ki} - \widehat{\overline{N}}_k)^2}{m_k - 1}, $$

$$ \widehat{Cov}\left(\widehat{N}_k, \widehat{\mathbf{\Lambda}}_{ka}\right) = -M_k \frac{\sum_{i=1}^{m_{ka}} \left(N_{ki} - \widehat{\overline{N}}_{ka}\right)\left(\widehat{\mathbf{\Lambda}}_{ki} - \widehat{\overline{\mathbf{\Lambda}}}_{ka}\right)}{m_{ka} - 1}, $$

$$ \begin{aligned} \widehat{V}\left(\widehat{\mathbf{\Lambda}}_{ka}\right) =& M_k \left(\frac{M_k}{m_k} - 1\right) \left\{ \frac{\sum_{i=1}^{m_{ka}} \left(\widehat{\mathbf{\Lambda}}_{ki} - \widehat{\overline{\mathbf{\Lambda}}}_{ka}\right)\left(\widehat{\mathbf{\Lambda}}_{ki} - \widehat{\overline{\mathbf{\Lambda}}}_{ka}\right)^T}{m_{ka} - 1} \right. \\ & \left. - \frac{1}{m_{ka}} \sum_{i=1}^{m_{ka}} N_{ki} \left(\frac{N_{ki}}{n_{ki}} - 1\right) \frac{n_{ki}\left[\operatorname{diag}(\mathbf{p}_{ki}) - \mathbf{p}_{ki}\mathbf{p}_{ki}^T\right]}{n_{ki} - 1} \right\}, \end{aligned} $$

$\widehat{\overline{N}}_{ka} = \sum_{i=1}^{m_{ka}} N_{ki}/m_{ka}$, $\widehat{\overline{\mathbf{\Lambda}}}_{ka} = \sum_{i=1}^{m_{ka}} \widehat{\mathbf{\Lambda}}_{ki}/m_{ka}$ and $m_{ka} = M_k - m_k$.

Now, consider the case where method 2 is used and the length sample comes from the tally period. The estimator of the total numbers in each length class that corresponds to Eq. 2.15 is

$$ {}_3\widehat{\mathbf{\Lambda}}_k = \frac{M_k}{m_k} \sum_{i=1}^{m_k} N_{ki} \mathbf{p}_{ki} \tag{2.26} $$

and the unbiased VCM estimator is

$$ \begin{aligned} \widehat{V}\left({}_3\widehat{\mathbf{\Lambda}}_k\right) =& M_k \left(\frac{M_k}{m_k} - 1\right) \frac{\sum_{i=1}^{m_k} \left(\widehat{\mathbf{\Lambda}}_{ki} - \overline{\mathbf{\Lambda}}_{ka}\right)\left(\widehat{\mathbf{\Lambda}}_{ki} - \overline{\mathbf{\Lambda}}_{ka}\right)^T}{m_k - 1} \\ &+ \frac{M_k}{m_k} \sum_{i=1}^{m_k} \frac{N_{ki}(N_{ki} - n_{kiL})}{n_{kiL}} \frac{n_{kiL}\left[\operatorname{diag}(\mathbf{p}_{ki}) - \mathbf{p}_{ki}\mathbf{p}_{ki}^T\right]}{n_{kiL} - 1}. \end{aligned} \tag{2.27} $$

Similarly, if method 2 is used to collect the length sample, but from the non-tally period, an asymptotically unbiased estimator of the total in each length class analogous to Eq. 2.19 is

$$ {}_4\widehat{\mathbf{\Lambda}}_k = \widehat{N}_k \widehat{\overline{\mathbf{P}}}_k = \frac{\widehat{N}_k}{m_{ka}} \sum_{i=1}^{m_{ka}} \frac{\mathbf{\lambda}_{ki}}{n_{kiL}}. \tag{2.28} $$

The analogous asymptotic VCM estimator is

$$ \begin{aligned} \widehat{V}\left({}_4\widehat{\mathbf{\Lambda}}_k\right) =& \widehat{\overline{\mathbf{P}}}_k \widehat{\overline{\mathbf{P}}}_k^T \widehat{V}\left(\widehat{N}_k\right) + \widehat{N}_k^2 \widehat{V}\left(\widehat{\overline{\mathbf{P}}}_k\right) \\ &+ \widehat{N}_k \left[\widehat{Cov}\left(\widehat{\overline{\mathbf{P}}}_k, \widehat{N}_k\right) \widehat{\overline{\mathbf{P}}}_k^T + \widehat{\overline{\mathbf{P}}}_k \widehat{Cov}\left(\widehat{N}_k, \widehat{\overline{\mathbf{P}}}_k^T\right)\right] \end{aligned} \tag{2.29} $$

where

$$ \begin{aligned} \widehat{V}\left(\widehat{\overline{\mathbf{P}}}_k\right) =& \frac{m_k}{M_k m_{ka}} \frac{\sum_{i=1}^{m_{ka}} \left(\mathbf{p}_{ki} - \overline{\mathbf{p}}_{ka}\right)\left(\mathbf{p}_{ki} - \overline{\mathbf{p}}_{k'}\right)^T}{m_{ka} - 1} \\ &+ \frac{1}{m_{ka} M_k} \sum_{i=1}^{m_{ka}} (1 - f_{ki}) \frac{\operatorname{diag}(\mathbf{p}_{ki}) - \mathbf{p}_{ki}\mathbf{p}_{ki}^T}{n_{ki} - 1}, \end{aligned} $$

$\overline{\mathbf{p}}_{ka} = \sum_{i=1}^{m_{ka}} \mathbf{p}_{ki}/m_{ka}$ and

$$ \widehat{C}\left(\widehat{\overline{\mathbf{P}}}_k, \widehat{N}_k\right) = -\frac{\sum_{i=1}^{m_{ka}} \left(\widehat{\mathbf{P}}_{ki} - \widehat{\overline{\mathbf{P}}}_k\right)\left(N_{ki} - \widehat{\overline{N}}_{ka}\right)}{m_{ka} - 1}. $$

The within-skate sampling fraction, $f_{ki} = n_{ki}/N_{ki}$ and the covariance terms still require knowledge of the total number of the species in the skate, but for prevalent species the sampling fraction may be negligible and the covariance term is neglible when proportions in length classes are uncorrelated with the total number.

The above results would be useful if the required information (such as within-skate numbers in each length class) were known, but as already discussed in Section 2.2 observers are not instructed to record this information. Equally important is the compilation of this information in an accessible database. To make use of data collected and currently being collected by observers to estimate uncertainty model-based methods similar to those in Sections 2.1, 2.2 and 5.1.1.1 are necessary.

2.3.2 Model-based estimation

An alternative to accounting for the complicated sampling methodology used to take the length sample is to assume a simple sampling methodology. Simplification will likely have undesirable consequences for inference (e.g., biased variance estimation) for a particular haul, but over larger domains of interest such as all fishing activities by the vessel or over entire fleets of vessels the biases in variance estimation may be dwarfed by larger scale variability (e.g., between trip variability or between haul variability). Therefore, when inference over larger sets of fishing activity is the focus, simplified assumptions within the haul may be appropriate.

If we assume that the length sample is a SRS of all the fish in the haul, then the estimator for the total numbers in each length class is

$$\widehat{\mathbf{\Lambda}}_k = N_k \frac{\boldsymbol{\lambda}_k}{n_{kL}} = N_k \mathbf{p}_k$$

where $\boldsymbol{\lambda}_k$ is the numbers of sampled fish in each length class. When an independent estimate of the total number in the haul is used the estimator is

$$_5\widehat{\mathbf{\Lambda}}_k = \widehat{N}_k \mathbf{p}_k \tag{2.30}$$

where $\boldsymbol{p}_k = (p_{k1}, \ldots, p_{kl}, \ldots, p_{kL})'$ and $\sum_{l=1}^{L} \lambda_{kl} = n_{kL}$. Under the SRS assumption, the vector of sample counts in the length classes, $\lambda_{\mathbf{k}}$, is a multivariate hypergeometric random variable (see Section A.2.2). The VCM of the unbiased estimator of the total numbers-at-length Eq. 2.30 is

$$V\left({}_5\widehat{\mathbf{\Lambda}}_k\right) = N_k\left(\frac{N_k}{n_{kL}} - 1\right)\frac{N_k\left[\operatorname{diag}\left(\mathbf{P}_k\right) - \mathbf{P}_k\mathbf{P}_k^T\right]}{N_k - 1}$$
$$+ V\left(\widehat{N}_k\right)\left[\left(\frac{1}{n_{kL}} - \frac{1}{N_k}\right)\frac{N_k\left[\operatorname{diag}\left(\mathbf{P}_k\right) - \mathbf{P}_k\mathbf{P}_k^T\right]}{N_k - 1} + \mathbf{P}_k\mathbf{P}_k^T\right]$$

where $\mathbf{P}_k = \mathbf{\Lambda}_k/N_k$. If an unbiased variance estimator for the total number is available, an asymptotically unbiased VCM estimator is

$$\widetilde{V}\left({}_5\widehat{\mathbf{\Lambda}}_k\right) = \widehat{N}_k\left(\frac{\widehat{N}_k}{n_{kL}} - 1\right)\frac{n_{kL}\left[\operatorname{diag}\left(\mathbf{p}_k\right) - \mathbf{p}_k\mathbf{p}_k^T\right]}{n_{kL} - 1} + \widehat{V}\left(\widehat{N}_k\right)\mathbf{p}_k\mathbf{p}_k^T$$

where an unbiased estimator is obtained by replacing $\widehat{N}_k^2$ with $\widehat{N}_k^2 - \widehat{V}\left(\widehat{N}_k\right)$.

Alternatively, suppose we have a super-population model for fish in the haul similar to that in Section 2.2 where, in this case, the expected value for an individual fish is the probability of being in the lth length, ϕ_k. Again, we can consider here all L length classes together so that each fish is an independent multinomial of dimension L. Therefore, the expected value under the model for the ith fish is $(\phi_{k1}, \ldots, \phi_k, \ldots, \phi_{kL})' = \boldsymbol{\phi}_k$ with VCM, $\mathbf{\Sigma}_k = \operatorname{diag}\left(\boldsymbol{\phi}_k\right) - \boldsymbol{\phi}_k\boldsymbol{\phi}_k^T$. Using the independent estimator of the total number of fish in the haul, the model-unbiased estimator for the total numbers-at-length is

$${}_6\widehat{\mathbf{\Lambda}}_k = \widehat{N}_k\frac{\boldsymbol{\lambda}_k}{n_{kL}} = \widehat{N}_k\widehat{\boldsymbol{\phi}}_k \tag{2.31}$$

The prediction error VCM is

$$V\left({}_6\widehat{\mathbf{\Lambda}}_k\right) = V\left(\widehat{N}_k\right)\left[\left(\frac{1}{n_{kL}} - \frac{1}{N_k}\right)\frac{N_k\mathbf{\Sigma}_k}{N_k - 1} + \boldsymbol{\phi}_k\boldsymbol{\phi}_k^T\right] + N_k\left(\frac{N_k}{n_{kL}} - 1\right)\frac{N_k\mathbf{\Sigma}_k}{N_k - 1}$$

and an estimator that is negligibly biased when the number of fish in the haul is large relative to the number of fish in the length sample is

$$\widehat{V}\left({}_6\widehat{\mathbf{\Lambda}}_k\right) = \widehat{V}\left(\widehat{N}_k\right)\left[\widehat{\boldsymbol{\phi}}_k\widehat{\boldsymbol{\phi}}_k^T - \frac{\widehat{\mathbf{\Sigma}}_k}{n_{kL}}\right] + \widehat{N}_k^2\frac{\widehat{\mathbf{\Sigma}}_k}{n_{kL}} \tag{2.32}$$

where

$$\widehat{\Sigma}_k = \frac{n_{kL}\left[\text{diag}\left(\widehat{\phi}_k\right) - \widehat{\phi}_k\widehat{\phi}_k^T\right]}{n_{kL}-1}.$$

Notice that the sample quantities, $\widehat{\phi}_k$ and $\mathbf{p}_k$, are equal which makes Eq. 2.30 and Eq. 2.31 and the corresponding VCM estimators are identical in their use of the data under different assumptions.

2.4 Total Numbers in Age Classes

2.4.1 Design-based Estimation

Observers are instructed to take otoliths from a number of fish in the length sample. For some species subsamples of the otoliths are analyzed to determine the age of the fish and for other species the age is determined for all collected otoliths. Whether and what type of a subsample of otoliths are aged is decided by stock assessment scientists. For example, a SRS of all sablefish otoliths collected in predetermined regions each year are aged. For walleye pollock in previous years both Bernoulli subsampling and simple random subsampling of otolithis collected during particular seasons and/or areas have been performed. For many species all collected otoliths are aged.

As stated before if we assume there is a SRS of fish to obtain otoliths from the length sample, then the otolith sample is effectively a SRS of all the fish from which the length sample is taken. Also, if there is Bernoulli subsampling of particular subsets of otoliths the nature of the subsampling causes inclusion probabilities to be independent across all elements sampled and therefore inference pertaining to each haul is independent. In this case, the overall sampling design by which age data are collected from the kth haul is a Bernoulli sample of a two-phase SRS. When a simple random subsample of all otoliths in a spatial/temporal subset then inference for a particular haul is possible, but the variability in the number of elements in the simple random subsample from the kth haul must also be taken into account.

When the weight sample is a SRS of all fish from the skates (method 1) in the tally sample, the haul is made in the mth time period/region and Bernoulli subsampling is performed, then an unbiased estimator of the total numbers in each age class is

$${}_1\widehat{\mathbf{\Psi}}_k = \frac{M_k N_{ka}}{m_k n_{Ok} \phi_m}\boldsymbol{\psi}_k$$

and the unbiased VCM estimator is

$$\widehat{V}\left({}_1\widehat{\boldsymbol{\Psi}}_k\right) = M_k\left(\frac{M_k}{m_k}-1\right)\widehat{\mathbf{S}}_{ka}^2 + \left(\frac{M_k}{m_k}\right)^2 \frac{N_{ka}}{n_{Ok}}\frac{1-\phi_m}{\phi_m}\boldsymbol{\psi}_k + \left(\frac{M_k}{m_k}\right)^2\left(\frac{N_{ka}}{\phi_m}\right)^2\left(1-\frac{n_{Ok}}{N_{ka}}\right)\frac{\left[\operatorname{diag}\left(\mathbf{p}_k\right)-\mathbf{p}_k\mathbf{p}_k^T\right]}{n_{Ok}-1} \tag{2.33}$$

where $\mathbf{p}_k = \boldsymbol{\psi}_k / n_{Ok}$,

$$\widehat{\mathbf{S}}_{ka}^2 = \frac{\sum_{i=1}^{m_k}\left(\widehat{\boldsymbol{\Psi}}_{ki}-\widehat{\overline{\boldsymbol{\Psi}}}_{ka}\right)\left(\widehat{\boldsymbol{\Psi}}_{ki}-\widehat{\overline{\boldsymbol{\Psi}}}_{ka}\right)^T}{m_k-1} - \frac{1}{m_k}\sum_{i=1}^{m_k}\widehat{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}|s_a\right) + \frac{1}{m_k(m_k-1)}\sum\sum_{i\neq j}^{m_k}\widehat{C}\left(\widehat{\boldsymbol{\Psi}}_{ki},\widehat{\boldsymbol{\Psi}}_{kj}|s_a\right),$$

$$\widehat{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}|s_a\right) = \left(\frac{N_{ka}}{\phi_m}\right)^2\left(1-\frac{n_{Ok}}{N_{ka}}\right)\frac{\operatorname{diag}\left(\mathbf{p}_{ki}\right)-\mathbf{p}_{ki}\mathbf{p}_{ki}^T}{n_{Ok}-1} + \frac{N_{ka}}{n_{Ok}}\left(\frac{1}{\phi_m}-1\right)\boldsymbol{\psi}_{ki},$$

$$\widehat{C}\left(\widehat{\boldsymbol{\Psi}}_{ki},\widehat{\boldsymbol{\Psi}}_{kj}|s_a\right) = -\left(1-\frac{n_{Ok}}{N_{ka}}\right)\frac{\widehat{\boldsymbol{\Psi}}_{ki}\widehat{\boldsymbol{\Psi}}_{kj}^T}{n_{Ok}-1}$$

$\widehat{\boldsymbol{\Psi}}_{ki} = N_{ka}\boldsymbol{\psi}_{ki}/(n_{Ok}\phi_m)$, $\widehat{\overline{\boldsymbol{\Psi}}}_{k_a} = \sum_{i=1}^{m_k}\widehat{\boldsymbol{\Psi}}_{ki}/m_k$ and $\mathbf{p}_{ki} = \boldsymbol{\psi}_{ki}/n_{Ok}$. See Table 2.1 for definitions of terms. Because of the small number of otoliths obtained and even smaller number analyzed for ages there will be many skates for which no otoliths are collected or aged and sample values for skates where no fish are aged will be recorded as zero.

When the weight sample is taken from the non-tally sample, but still is an SRS of all fish in the non-tally skates, the asymptotically unbiased estimator of the total numbers-at-age is

$${}_2\widehat{\boldsymbol{\Psi}}_k = \widehat{N}_k\frac{\boldsymbol{\psi}_k}{\phi_m n_{Ok}} = \widehat{N}_k\frac{\mathbf{p}_{ki}}{\phi_m}$$

and the VCM is

$$\begin{aligned}
V\left({}_2\widehat{\boldsymbol{\Psi}}_k\right) =& V_{p_a}\left\{E_{p_b}\left[E_\phi\left({}_2\widehat{\boldsymbol{\Psi}}_k\right)\right]\right\} + E_{p_a}\left\{V_{p_b}\left[E_\phi\left({}_2\widehat{\boldsymbol{\Psi}}_k\right)\right]\right\} \\
&+ E_{p_a}\left\{E_{p_b}\left[V_\phi\left({}_2\widehat{\boldsymbol{\Psi}}_k\right)\right]\right\} \\
=& V\left({}_1\widehat{N}_k\frac{\boldsymbol{\Psi}_{ka'}}{N_{ka'}}\right) + E\left[{}_1\widehat{N}_k^2\frac{N_{ka'}(N_{ka'}-n_{Ok})}{n_{Ok}}\frac{N_{ka'}\left[\operatorname{diag}\left(\mathbf{P}_{ka'}\right)-\mathbf{P}_{ka'}\mathbf{P}_{ka'}^T\right]}{N_{ka'}-1}\right] \\
&+ \frac{1-\phi_m}{\phi_m n_{Ok}}E\left({}_1\widehat{N}_k^2\frac{\boldsymbol{\Psi}_{ka'}}{N_{ka'}}\right) \\
\approx& \left(\frac{\mathbf{P}_k(1-2f_k)}{1-f_k}\right)^2 V\left({}_1\widehat{N}_k\right) + \left(\frac{f_k}{1-f_k}\right)^2 V\left(\widehat{\boldsymbol{\Psi}}_{k_1}\right) \\
&- \frac{2\mathbf{P}_k f_k(1-2f_k)}{(1-f_k)^2}Cov\left({}_1\widehat{N}_k, \widehat{\boldsymbol{\Psi}}_{k_1}\right) \\
&+ \left[N_k^2(1-f_k)^2\right]\frac{N_k\left[\operatorname{diag}\left(\boldsymbol{\Psi}_k\right)-\boldsymbol{\Psi}_k\boldsymbol{\Psi}_k^T\right]}{n_{Ok}} + \frac{1-\phi_m}{\phi_m n_{Ok}}N_k\boldsymbol{\Psi}_k
\end{aligned}$$

where $f_k = m_k/M_k$, $\widehat{\boldsymbol{\Psi}}_{ka} = M_k\overline{\boldsymbol{\Psi}}_{ka}$, $\overline{\boldsymbol{\Psi}}_{ka} = \sum_{i=1}^{m_k}\boldsymbol{\Psi}_{ki}/m_k$,

$$V\left(\widehat{\boldsymbol{\Psi}}_{ka}\right) = M_k\left(\frac{M_k}{m_k}-1\right)\frac{\sum_{i=1}^{M_k}\left(\boldsymbol{\Psi}_{ki}-\overline{\boldsymbol{\Psi}}_{ka}\right)\left(\boldsymbol{\Psi}_{ki}-\overline{\boldsymbol{\Psi}}_{ka}\right)^T}{M_k-1},$$

$$Cov\left(\widehat{\boldsymbol{\Psi}}_{ka}, {}_1\widehat{N}_k\right) = M_k\left(\frac{M_k}{m_k}-1\right)\frac{\sum_{i=1}^{M_k}\left(\boldsymbol{\Psi}_{ki}-\overline{\boldsymbol{\Psi}}_{ka}\right)\left(N_{ki}-\overline{N}_{ka}\right)}{M_k-1}$$

$\overline{N}_{ka} = \sum_{i=1}^{M_k}N_{ki}/M_k$ and $\overline{\boldsymbol{\Psi}}_{ka} = \sum_{i=1}^{M_k}\boldsymbol{\Psi}_{ki}/M_k$. However, this asymptotic result may not be useful because the number of otoliths sampled from a given haul is not large.

When the weight sample is obtained via a SRS of all fish in the tally period and a SRS of all otoliths collected in a time period/region are analyzed for ages, then an unbiased estimator of the total numbers-at-age in the haul is

$$_3\widehat{\boldsymbol{\Psi}}_k = \frac{M_k}{m_k}\frac{N_{ka}}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\psi_{ki} \tag{2.34}$$

where N_{Om} and n_{Am} are the total number of sampled otoliths and number of aged otoliths in the spatial/temporal domain and n_{Ok} is the number of otoliths collected from the kth haul. The VCM of the estimator is

$$\begin{aligned}
V\left({}_3\widehat{\mathbf{\Psi}}_{ki}\right) =& M_k\left(\frac{M_k}{m_k}-1\right)\frac{\sum_{i=1}^{M_k}\left(\mathbf{\Psi}_{ki}-\overline{\mathbf{\Psi}}_{ka}\right)\left(\mathbf{\Psi}_{ki}-\overline{\mathbf{\Psi}}_{ka}\right)^T}{M_k-1} \\
&+\left(\frac{M_k}{m_k}\right)^2 E\left[N_{ka}^2\left(\frac{N_{ka}}{n_{Ok}}-1\right)\frac{\operatorname{diag}\left(\mathbf{P}_{ka}\right)-\mathbf{P}_{ka}\mathbf{P}_{ka}^T}{N_{ka}-1}\right] \\
&+\left(\frac{M_k}{m_k}\right)^2 E\left[\left(\frac{N_{ka}}{n_{Ok}}\right)^2 N_{Om}^2\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{\operatorname{diag}\left(\mathbf{p}_{Ok}\right)-\mathbf{p}_{Ok}\mathbf{p}_{Ok}^T}{N_{Om}-1}\right]
\end{aligned}$$

where $\mathbf{P}_{ka}=\sum_{i=1}^{m_k}\psi_{ki}/N_{ka}$ and $\mathbf{p}_{Ok}=\psi_{Ok}/N_{Om}$. Estimation of the VCM will require skate identification for each of the aged otoliths which is not available from archived observer data.

When the weight sample is obtained by randomly sampling fish within each skate in either the tally or non-tally group, it is doubtful that there is a subsequent otolith subsample within each of the skates selected for weight sampling. The number of total ototiths to take from a given haul is usually ≤ 5 and it would be imposible to properly split an already small sample across the selected skates.

2.4.2 Model-based Estimation

If we assume a super-population model similar to that described in Section 2.3, then we can form a model-based estimator of numbers-at-age with the cells of the multinomial corresponding to ages of interest. Because, in general, a subsample of collected otoliths are aged, the number of aged otoliths in each longline haul is random and I account for this variability when deriving an estimator of numbers-at-age in the haul and the corresponding variance estimator.

Since it appears that the subsample of collected otoliths that are aged is often obtained by SRS in the mth time period/region are aged. For a given species, the number of otoliths in all hauls sampled in the mth time period/region is N_{Om}, the number in the kth haul is n_{Ok}, the number aged in the mth period/region is n_{Am}, and the number aged in the kth haul is n_{Ak}. Assuming independence of fish in the haul and the estimator of total number of fish, $\widehat{N}_k$, is independent of the estimator for the probability vector for the A age classes, the unbiased estimator of the total numbers-at-age is

$$ {}_4\widehat{\boldsymbol{\Psi}}_k = \frac{N_{Om}}{n_{\psi m}} \frac{\widehat{N}_k}{n_{Ok}} \boldsymbol{\psi}_k = \frac{N_{Om}}{n_{Am}} \widehat{N}_k \widehat{\mathbf{p}}_{\psi k} \tag{2.35} $$

where $\boldsymbol{\psi}_k$ is the vector of counts in the age classes in the read sample and $\mathbf{p}_{\psi k}$ is the vector of probabilities for the age classes in the super-population model. Assuming that the ratio of aged fish to total number of fish in the haul is negligle, the prediction error VCM of Eq. 2.35 is

$$ \begin{aligned} V\left({}_4\widehat{\boldsymbol{\Psi}}_k\right) =& \boldsymbol{\Psi}_k \boldsymbol{\Psi}_k^T \frac{V\left(\widehat{n}_{Ok}\right)}{n_{Ok}^2} + \mathbf{p}_{\psi k} \mathbf{p}_{\psi k}^T V\left(\widehat{N}_k\right) \left[\frac{V\left(\widehat{n}_{Ok}\right)}{n_{Ok}^2} + 1\right] \\ &+ \left[\operatorname{diag}(\mathbf{p}_{\psi k}) - \mathbf{p}_{\psi k} \mathbf{p}_{\psi k}^T\right] \frac{N_{Om}\left[N_k^2 + V\left(\widehat{N}_k\right)\right]}{n_{Ok} n_{Am}} \end{aligned} \tag{2.36} $$

where $\boldsymbol{\Psi}_k = N_k \mathbf{p}_{\psi k}$,

$$ V\left(\widehat{n}_{Ok}\right) = N_{Om}^2 \left(\frac{N_{Om}}{n_{Am}} - 1\right) \frac{P_{Ok}(1 - P_{Ok})}{N_{Om} - 1} $$

and $P_{Ok} = n_{Ok}/N_{Om}$. The VCM estimator is

$$ \begin{aligned} \widehat{V}\left({}_4\widehat{\boldsymbol{\Psi}}_k\right) =& \left[\widehat{V}\left(\widehat{N}_k\right) + \widehat{N}_k^2\right] \left\{\frac{V\left(\widehat{n}_{Ok}\right)}{n_{Ok}^2} \widehat{\mathbf{p}}_{\psi k} \widehat{\mathbf{p}}_{\psi k}^T + \frac{N_{Om}}{n_{Ok} n_{\psi m}} \left[\operatorname{diag}(\widehat{\mathbf{p}}_{\psi k}) - \widehat{\mathbf{p}}_{\psi k} \widehat{\mathbf{p}}_{\psi k}^T\right]\right\} \\ &+ \widehat{V}\left(\widehat{N}_k\right) \widehat{\mathbf{p}}_{\psi k} \widehat{\mathbf{p}}_{\psi k}^T \end{aligned} \tag{2.37} $$

where

$$ \widehat{\mathbf{p}}_{\psi k} = \frac{N_{Om}}{n_{Am}} \frac{\boldsymbol{\psi}_k}{n_{Ok}}. $$

See Section 2.A.4 for derivation of Eq. 2.37 and the unbiasedness of Eq. 2.35.

2.5 *Total Number of Marine Mammal Interactions*

2.5.1 Design-based Estimation

Observers monitor marine mammal interactions with the vessel while they are tallying species of seabirds and fish as the longline gear is retrieved (tally period). Sometimes observers also note interactions outside the tally period, but this is at the discretion of the observer unless a marine mammal is actually captured. The interactions are further classified by the observer into various types ranging from presence to mortality and any remarks are also recorded. The estimation techniques presented here only make use of the interactions that are recorded during the tally period.

If the observer randomly samples skates for the tally period, then the estimator for the total number of marine mammal interactions in the ith haul, Γ_k, is identical to that derived for the total number of a given species (see Section 2.1),

$$ {}_1\widehat{\Gamma}_k = \frac{M_k}{m_k} \sum_{i=1}^{m_k} \Gamma_{ki} \tag{2.38} $$

where M_k is the total number of skates, m_k is the number of skates sampled and Γ_{ki} is the number of interactions in the ith skate. The variance estimator is

$$ \widehat{V}\left({}_1\widehat{\Gamma}_k\right) = M_k \left(\frac{M_k}{m_k} - 1\right) \frac{\sum_{i=1}^{m_k} \left(\Gamma_{ki} - \widehat{\overline{\Gamma}}_k\right)^2}{m_k - 1} \tag{2.39} $$

where $\overline{\Gamma}_k = \sum^{M_k} \Gamma_{ki}/M_k$ and $\widehat{\overline{\Gamma}}_k = \sum_{i=1}^{m_k} \Gamma_{ki}/m_k$.

2.5.2 Model-Based Estimation

The same lack of recorded information mentioned previously in Section 2.1 that prevents design-based inference within the haul for estimation of variance of total numbers of species of fish and seabirds also prevents estimation of variance of the total number of marine mammal interactions within the haul. To reiterate, the lack of identification of which groups of hooks in which the interactions were observed and how many groups of hooks sampled disallows design-based estimation of variance when the interactions occur. However, to make some use of the data that has been and is currently being collected by observers, models with minimal assumptions can be assumed to generate the data that are observed and also allow model-based inferences to be made. It is important to realize, that without studies where the mentioned information is actually recorded when marine mammal intractions occur the validity of model-based inference cannot be tested.

If we assume that the rate of marine mammal interactions, ν_k, is constant during the gear retrieval and we consider the the hook as a measure of time, then the number of marine mammal interactions that occur during which h_{ki} hooks from the ith skate are retrieved is a Poisson random variable with mean, $h_{ki}\nu_k$. Furthermore, because of the memoryless property of the Poisson process, the expected number (and variance) of interactions during the entire tally period is $\nu_k \sum_{i=1}^{m_k} h_{ki} \equiv \nu_k h_k$. Using the prediction approach (Royall 1976; Bolfarine and Zacks 1992; Valliant et al. 2000) described in Section 2.2 and developed in Section A.8, the best linear unbiased predictor of the total number of interactions for the entire haul is

$$
{}_2\widehat{\Gamma}_k = \frac{H_k}{h_k}\Gamma_{ks} \tag{2.40}
$$

where H_k is the total number of hooks in the haul, h_k is the total number of hooks sampled during the tally period and Γ_{ks} is the number of marine mammal interactions observed during the tally period. The prediction error variance and variance estimator are

$$
V\left({}_2\widehat{\Gamma}_k\right) = H_k\left(\frac{H_k}{h_k} - 1\right)\nu_k
$$

and

$$
\widehat{V}\left({}_2\widehat{\Gamma}_k\right) = \frac{H_k}{h_k}\left(\frac{H_k}{h_k} - 1\right)\Gamma_{ks}, \tag{2.41}
$$

respectively.

2.6 Choosing an Estimator

The numbers of available estimators for various catch parameters can be overwhelming and in this section I summarize more succinctly the sampling scenarios under which each estimator is appropriate and recommend particular estimators when more than one is available for a given scenario. I will discuss each catch parameter that I treated in the previous sections of this chapter.

For longline gear, the simplest estimators turned out to be those for total numbers in the catch for a given fish or seabird species caught and total number of interactions for a given marine mammal species. Furthermore, there were few necessary sampling scenarios for these estimators. Figures 2.1 and 2.2 show that the only real criterion for determining which of these estimators to use is whether the data are aggregated when archived by the NPGOP. That is, whether or not separate counts for each randomly sampled skate or skate group are recorded determines which estimator and variance estimator to use.

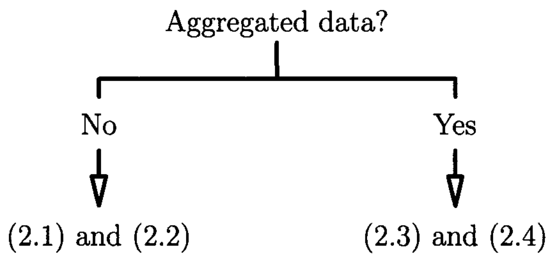

Figure 2.1. Decision tree for determining which estimators to use for total numbers of a particular fish or seabird species in a longline haul.

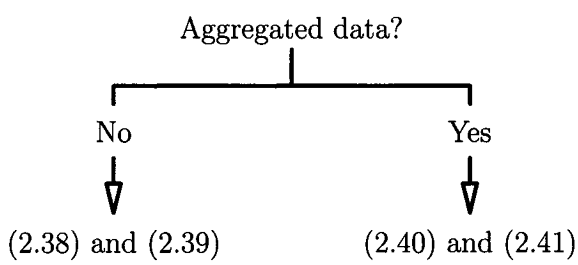

Figure 2.2. Decision tree for determining which estimators to use for interactions of a marine mammal species in a longline haul.

For total catch weight of a given species, I presented many estimators under various scenarios. As with the total number in the catch of a given species, the appropriate estimator depends on whether data are aggregated. However, whether the fish entering the weight sample are from the tally sample is important. Whether the fish entering the weight sample are randomly sampled from all fish in the selected skates (tally or non-tally skates) or from fish within each skate is also a criterion (Figure 2.3). I suggested that a post-stratified estimator where skates are the strata (Eq. 2.9) is a possiblity when fish in the weight sample come from those in the tally sample and random sampling of all fish in the tally sample, but using this estimator requires at least some fish from every skate in the tally sample enter the weight sample. This is not guaranteed to occur especially with species that are not abundant in the catch.

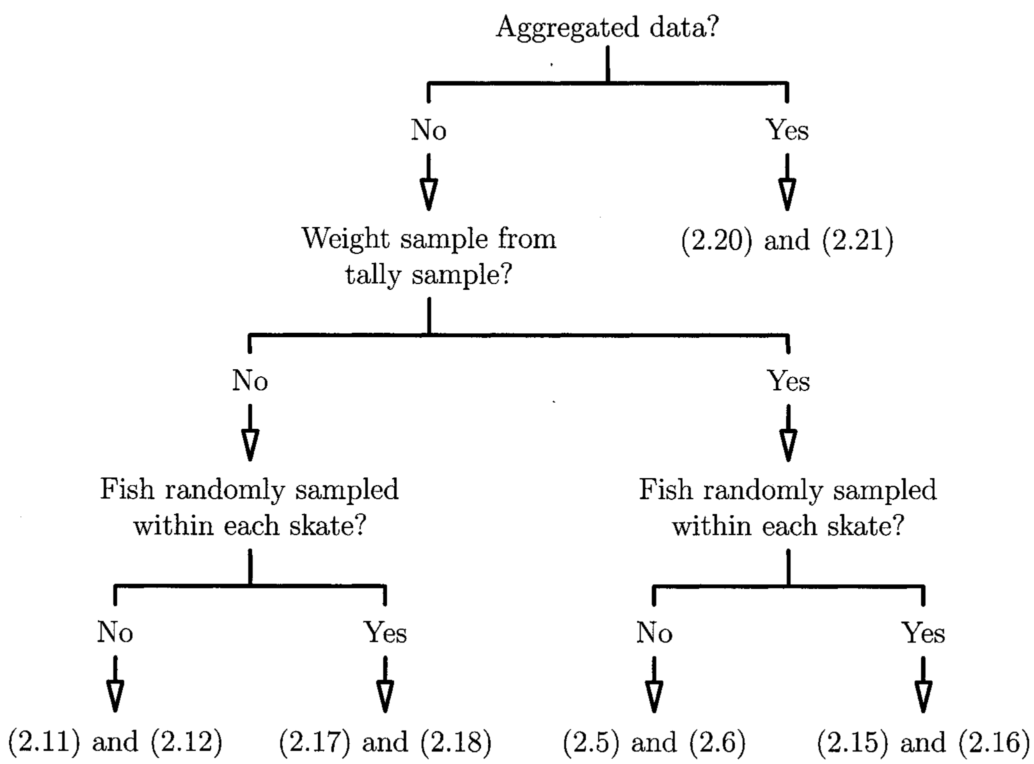

Figure 2.3. Decision tree for determining which estimators to use for total weight of a particular fish species in a longline haul.

Because the length sample is generally obtained by randomly subsampling the weight sample, the determination of the appropriate estimators for numbers in length classes is similar to that of total catch weight for a species. Figure 2.4 shows how to decide which of the pairs of estimator and variance estimator to use for numbers in length classes. As with the other catch parameters the model-based estimators will be used for data that has been collected by observers because data is aggregated within hauls, but the other estimators may be appropriate if changes are made in the way data are archived.

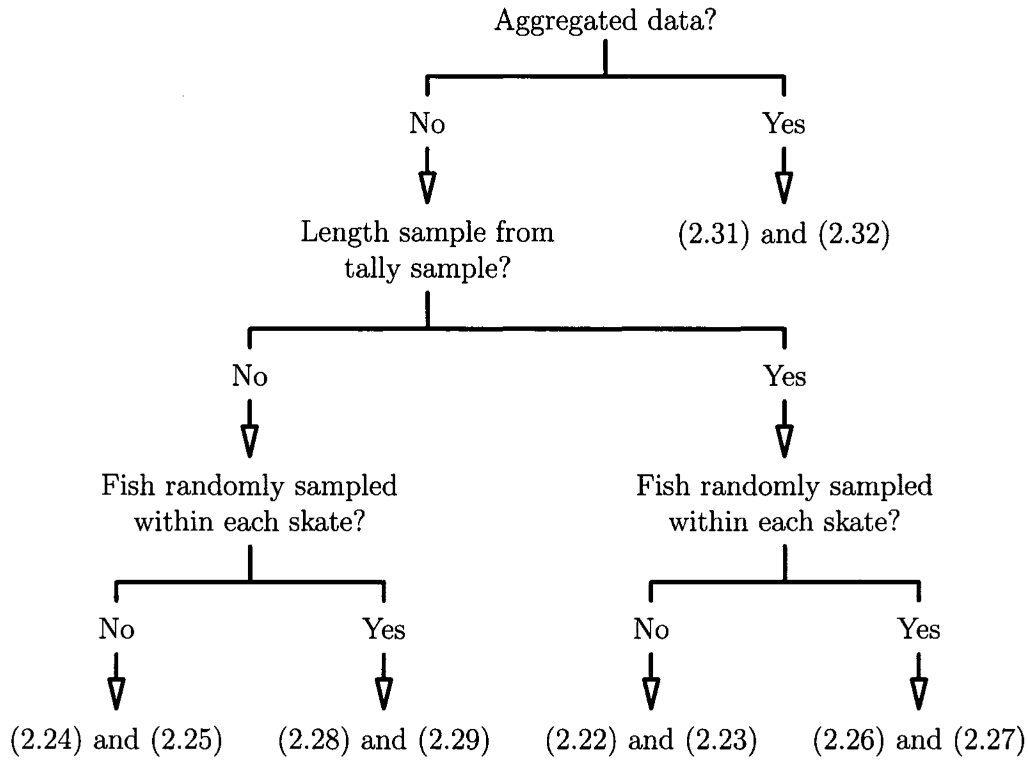

Figure 2.4. Decision tree for determining which estimators to use for total length class numbers of a particular fish species in a longline haul.

I do not give a decision tree for estimators of numbers in age classes because even if data were not aggregated, there are so few otoliths that are collected from any given haul that variance estimation would be difficult if it all possible. Furthermore, estimation of numbers-at-age is generally not of interest at very small scales (i.e., hauls). Therefore, I recommend using the model-based estimators (Eqs. 2.35 and 2.37) in all cases. So long as the otoliths are collected in a random fashion from fish within the haul, the haul-specific estimates will be unbiased. The within-haul variance may be underestimated, but this will be a negligble contributor to the overall variance compared to variability between sampling elements at higher levels (e.g., variation between hauls and between trips).

2.A Derivation of Estimators

2.A.1 Derivation of ${}_1\widehat{W}_k$

The first total weight estimator (Eq. 2.5) results from two-phase sampling. First there is a SRS of m_k clusters (skates) and subsequently there is a SRS of n_k elements (fish) from all N_{ka} elements in m_k clusters obtained in the first phase. Let a denote the sampling distribution at the first phase and b denote the conditional sampling distribution at the second phase. The inclusion probabilities are provided for a general case of this two-phase sampling design in Section A.7.2 and many of the results there are analagous to those presented here.

The estimator is unbiased because

$$\begin{aligned} E\left({}_1\widehat{W}_k\right) =& E_a\left[\frac{M_k}{m_k}E_b\left(\frac{N_{ka}}{n_k}\sum_{t=1}^{n_k}W_t|s_a\right)\right] \\ =& E_a\left(\frac{M_k}{m_k}\sum_{t=1}^{N_{ka}}W_t\right) = E_a\left(\frac{M_k}{m_k}\sum_{i=1}^{m_k}W_i\right) = \sum_{i=1}^{M_k}W_i = W_k. \end{aligned}$$

The variance of Eq. 2.5 is

$$\begin{aligned} V\left({}_1\widehat{W}_k\right) =& V_a\left[\frac{M_k}{m_k}E_b\left(\frac{N_{ka}}{n_k}\sum_{t=1}^{n_k}W_t|s_a\right)\right] + E_a\left[\left(\frac{M_k}{m_k}\right)^2 V_b\left(\frac{N_{ka}}{n_k}\sum_{t=1}^{n_k}W_t|s_a\right)\right] \\ =& \underbrace{V_a\left[\frac{M_k}{m_k}\sum_{i=1}^{m_k}W_i\right]}_{V_1} + \underbrace{E_a\left[\left(\frac{M_k}{m_k}\right)^2 N_{ka}\left(\frac{N_{ka}}{n_k}-1\right)\frac{\sum_{t=1}^{N_{ka}}\left(W_t-\overline{W}_{ka}\right)^2}{N_{ka}-1}\right]}_{V_2} \end{aligned} \tag{2.42}$$

where

$$V_1 = M_k \left(\frac{M_k}{m_k} - 1 \right) \frac{\sum_{i=1}^{M_k} \left(W_i - \overline{W}_k \right)^2}{M_k - 1} \tag{2.43}$$

and the variance result makes use of the identity,

$$\sum_{t=1}^{N_{ka}} W_t = \sum_{i=1}^{m_k} \sum_{t=1}^{N_{ki}} W_{it} = \sum_{i=1}^{m_k} W_i$$

where N_{ki} is the the number of elements in the ith cluster as well as SRS results derived in Section A.2. The expectation in the second term is over all first-phase samples and cannot be written explicitly because of the variation in N_{ka} across first-phase samples. This variability in the conditional sampling design distinguishes the overall design as multi-phased (in this case just two-phase) with respect to estimation of total weight rather than the simpler multi-*staged* design to estimate total numbers.

Below, I give estimators, $\widehat{V}_1$ and $\widehat{V}_2$, for the two components of Eq. 2.42 and show that they are unbiased. The sum of the component estimators yields Eq. 2.6. An estimator for Eq. 2.43 is

$$\begin{aligned} \widehat{V}_1 =& M_k \left(\frac{M_k}{m_k} - 1 \right) \left\{ \frac{\sum_{i=1}^{m_k} \left[\widehat{W}_{ki}^2 - \widehat{V}_b \left(\widehat{W}_{ki} | s_a \right) \right]}{m_k} \right. \\ & \left. - \frac{\sum\sum_{i \neq j}^{m_k} \left[\widehat{W}_{ki} \widehat{W}_{kj} - \widehat{Cov}_b \left(\widehat{W}_{ki}, \widehat{W}_{kj} | s_a \right) \right]}{m_k(m_k - 1)} \right\} \end{aligned}$$

where $\widehat{W}_{ki} = N_{ka} \sum_{t=1}^{n_{ki}} W_{it}/n_k$ and $\widehat{V}_b \left(\widehat{W}_{ki} | s_a \right)$ and $\widehat{Cov}_b \left(\widehat{W}_{ki}, \widehat{W}_{kj} | s_a \right)$ are defined in Eq. 2.7 and Eq. 2.8, respectively. The estimator for the first component, $\widehat{V}_1$, is unbiased because

$$E\left(\widehat{V}_1\right) = M_k\left(\frac{M_k}{m_k}-1\right)E_a\left[\frac{\sum_{i=1}^{m_k} W_{ki}^2}{m_k} - \frac{\sum\sum_{i\neq j}^{m_k} W_{ki}W_{kj}}{m_k(m_k-1)}\right]$$
$$= M_k\left(\frac{M_k}{m_k}-1\right)\left[\frac{\sum_{i=1}^{M_k} W_{ki}^2}{M_k} - \frac{\sum\sum_{i\neq j}^{M_k} W_{ki}W_{kj}}{M_k(M_k-1)}\right]$$

which can be rewritten in the form appropriate for V_1. Notice that the second component can be written as

$$V_2 = E_a\left[\left(\frac{M_k}{m_k}\right)^2 V_b\left(\sum_{i=1}^{m_k}\frac{N_{ka}}{n_k}\sum_{t=1}^{n_{ki}} w_{it}|s_a\right)\right] = E_a\left[\left(\frac{M_k}{m_k}\right)^2 V_b\left(\sum_{i=1}^{m_k}\widehat{W}_i|s_a\right)\right]$$
$$= E_a\left\{\left(\frac{M_k}{m_k}\right)^2\left[\sum_{i=1}^{m_k} V_b\left(\widehat{W}_i|s_a\right) + \sum\sum_{i\neq j}^{m_k} Cov_b\left(\widehat{W}_i,\widehat{W}_j|s_a\right)\right]\right\}$$

which is unbiasedly estimated by

$$\widehat{V}_2 = \left(\frac{M_k}{m_k}\right)^2\left[\sum_{i=1}^{m_k}\widehat{V}_b\left(\widehat{W}_i|s_a\right) + \sum\sum_{i\neq j}^{m_k}\widehat{Cov}_b\left(\widehat{W}_i,\widehat{W}_j|s_a\right)\right].$$

Unbiasedness of $\widehat{V}_2$ follows directly given unbiasedness of the within-cluster variance and covariance estimators, $\widehat{V}_b\left(\widehat{W}_{ki}|s_a\right)$ and $\widehat{Cov}_b\left(\widehat{W}_{ki},\widehat{W}_{kj}|s_a\right)$. The clusters can be thought of as domains with regard to the second-phase sampling distribution because sample sizes are not controlled within each cluster. The variance and covariance estimators for within-cluster totals are shown unbiased in the Section A.6 (See especially results A.20 and A.21).

2.A.2 Derivation of $_3\widehat{W}_k$

The expected value of Eq. 2.11 is

$$\begin{aligned} E_{p_a}\left({}_3\widehat{W}_k\right) =& E_{p_a}\left[\widehat{N}_k E\left(\frac{\sum_{i=1}^{n_k} w_{ki}}{n_k}\middle| s_a\right)\right] \\ =& E_{p_a}\left(\widehat{N}_k \frac{\sum_{i=1}^{m_{ka}} W_{ki}}{N_{ka}}\right) = E_{p_a}\left(\widehat{N}_k \frac{\sum_{i=1}^{m_{ka}} W_{ki}}{\sum_{i=1}^{m_{k'}} N_{ki}}\right) \\ =& E_{p_a}\left(\widehat{N}_k \frac{W_k - \frac{m_k}{M_k}\widehat{W}_{k_a}}{N_k - \frac{m_k}{M_k}\widehat{N}_k}\right). \end{aligned}$$

where $\widehat{W}_{ka} = \frac{M_k}{m_k}\sum_{i=1}^{m_k} W_{ki}$, N_{ka} is the number of fish in the non-tally portion of the haul and p_a is the probability distribution of first-phase samples. Note that the sample comes from the non-tally period, $s_a \neq s_1$ and $m_{ka} = M_k - m_k$ here. This is an expectation of a nonlinear function of two random variables and, using a first-order Taylor series expansion about (N_k, W_k),

$$E_{p_a}\left(\widehat{N}_k \frac{W_k - \frac{m_k}{M_k}\widehat{W}_{k_a}}{N_k - \frac{m_k}{M_k}\widehat{N}_k}\right) \equiv E\left({}_3\widehat{W}_k\right) \approx W_k.$$

The variance of Eq. 2.11 is

$$V\left({}_3\widehat{W}_k\right) = E_{p_a}\left[V\left({}_3\widehat{W}_k | s_a\right)\right] + V_{p_a}\left[E\left({}_3\widehat{W}_k | s_a\right)\right]$$

The first term on the right side is

$$
\begin{aligned}
E_{p_a}\left[V\left({}_3\widehat{W}_k|s_a\right)\right] =& E_{p_a}\left[\widehat{N}_k^2 V\left(\frac{\sum_{i=1}^{n_k} W_{ki}}{n_k}\right)\right] \\
=& E_{p_a}\left[\widehat{N}_k^2(1-f_{k_b})\frac{S_{kWb}^2}{n_k}\right] = E_{p_a}\left[\widehat{N}_k^2\left(1-\frac{n_k}{N_{ka}}\right)\frac{S_{kWb}^2}{n_k}\right] \\
=& E_{p_a}\left[\frac{\widehat{N}_k^2\left(N_k-\frac{m_k}{M_k}\widehat{N}_k-n_k\right)}{N_k-\frac{m_k}{M_k}\widehat{N}_k}\frac{S_{kWb}^2}{n_k}\right] \\
=& E_{p_a}\left[\widehat{N}_k^2\frac{S_{kWb}^2}{n_k}\left(1-\frac{n_k}{N_k-\frac{m_k}{M_k}\widehat{N}_k}\right)\right] \\
\approx& N_k\left(\frac{N_k}{n_k}-\frac{1}{1-f_k}\right)E_{p_a}\left(S_{kWb}^2\right)
\end{aligned}
$$

where

$$
S_{kWb}^2 = \frac{\sum_{i=1}^{N_{ka}}\left(W_{ki}-\overline{W}_{kb}\right)^2}{N_{ka}}
$$

and $\overline{W}_{kb} = \sum_{i=1}^{N_{ka}} W_{ki}/N_{ka}$. The second term on the right side is

$$
\begin{aligned}
V_{p_a}\left[E\left({}_3\widehat{W}_k|s_a\right)\right] \approx& \left(\frac{f_k}{1-f_k}\right)^2 V\left(\widehat{W}_{ka}\right)+\left(\frac{W_k}{N_k}\right)^2\left[1-\frac{f_k}{N_k(1-f_k)}\right]^2 V\left(\widehat{N}_k\right) \\
&+2W_k\frac{f_k}{N_k(1-f_k)}\left[1-\frac{f_k}{N_k(1-f_k)}\right]Cov\left(\widehat{N}_k,\widehat{W}_{ka}\right)
\end{aligned}
$$

where

$$
V(\widehat{W}_{k_a}) = M_k\left(\frac{M_k}{m_k}-1\right)\frac{\sum_{i=1}^{M_k}\left(W_{ki}-\overline{W}_k\right)^2}{M_k-1},
$$

$$
V(\widehat{N}_k) = M_k\left(\frac{M_k}{m_k}-1\right)\frac{\sum_{i=1}^{M_k}\left(N_{ki}-\overline{N}_k\right)^2}{M_k-1}
$$

and

$$
Cov\left(\widehat{N}_k,\widehat{W}_{k_a}\right) = -M_k\frac{\sum_{i=1}^{M_k}\left(N_{ki}-\overline{N}_k\right)\left(W_{ki}-\overline{W}_k\right)}{M_k-1}.
$$

The asymptotic variance then would be

$$
\begin{aligned}
V\left({}_3\widehat{W}_k\right) \approx & N_k\left(\frac{N_k}{n_k}-\frac{1}{1-f_k}\right)E_{p_a}\left(S^2_{kW_b}\right)+\left(\frac{f_k}{1-f_k}\right)^2 V\left(\widehat{W}_{k_a}\right)\\
&+\left(\frac{W_k}{N_k}\right)^2\left[1-\frac{f_k}{N_k(1-f_k)}\right]^2 V\left(\widehat{N}_k\right)\\
&+\frac{2W_kf_k}{N_k(1-f_k)}\left[1-\frac{f_k}{N_k(1-f_k)}\right]Cov\left(\widehat{N}_k,\widehat{W}_{k_a}\right)
\end{aligned}
\tag{2.44}
$$

2.A.3 Derivation of ${}_6\widehat{W}_k$

The expected value of $\widehat{\overline{\overline{W}}}_k$ is

$$E\left(\widehat{\overline{\overline{W}}}_k\right)=\frac{1}{M_k}\sum_{i=1}^{M_k}\overline{W}_{ki}=\overline{\overline{W}}_k$$

where $\overline{W}_{ki} = \sum_{t=1}^{N_{ki}} W_{kit}/N_{ki}$. It might be presumed that the mean weight of individuals within each skate varies little or not at all from skate to skate and if so, $\sum_{i=1}^{M_k}\overline{W}_{ki}/M_k \approx \sum_{i=1}^{N_k} W_{ki}/N_k = W_k$. Assuming $\widehat{N}_k$ is given in Eq. 2.1,

$$E\left({}_6\widehat{W}_k\right)=E\left(\widehat{N}_k\widehat{\overline{\overline{W}}}_k\right)\approx N_k\overline{W}_k=W_k,$$

based on a first-order Taylor Series (Delta Method), and the approximate variance of the estimator is

$$V\left({}_6\widehat{W}_k\right)\approx\overline{\overline{W}}{}^2_kV\left(\widehat{N}_k\right)+N_k^2V\left(\widehat{\overline{\overline{W}}}_k\right)+2\overline{\overline{W}}_kN_kCov\left(\widehat{\overline{\overline{W}}}_k,\widehat{N}_k\right)\tag{2.45}$$

where

$$V\left(\widehat{\overline{\overline{W}}}_k\right)=\frac{1-f_{ka}}{m_{ka}}\frac{\sum_{i=1}^{M_k}\left(\overline{W}_{ki}-\overline{\overline{W}}_k\right)^2}{M_k-1}+\frac{1}{m_{ka}M_k}\sum_{i=1}^{M_k}\frac{1-f_{ki}}{n_{ki}}\frac{\sum_{t=1}^{N_{ki}}(W_{kit}-\overline{W}_{ki})^2}{N_{ki}-1}$$

and

$$Cov\left(\widehat{\overline{\overline{W}}}_k,\widehat{N}_k\right)=-\frac{\sum_{i=1}^{M_k}\left(\overline{W}_{ki}-\overline{\overline{W}}_k\right)\left(N_{ki}-\overline{N}_k\right)}{M_k-1}.$$

The variance estimator is obtained by substituting $\widehat{\overline{\overline{W}}}_k$, $\widehat{N}_k$, $\widehat{V}\left(\widehat{N}_k\right)$, $\widehat{V}\left(\widehat{\overline{\overline{W}}}_k\right)$ and $\widehat{Cov}\left(\widehat{N}_k, \widehat{\overline{\overline{W}}}_k\right)$ for the respective values they estimate.

2.A.4 Derivation of ${}_4\widehat{\mathbf{\Psi}}_k$

Let M_N denote probability distribution assumed for total numbers and M_ψ be that for the independent vector-valued identifier of the age of a given fish. Also let S_A denote the sample distribution for the simple random sampling of collected otoliths for ageing in the mth region. The estimator can be written as

$$ {}_4\widehat{\mathbf{\Psi}}_k = \frac{N_{Om}\widehat{N}_k}{n_{Am}n_{Ok}} \sum_{i=1}^{n_{Ak}} \boldsymbol{\psi}_i $$

where $\boldsymbol{\psi}_i$ is the vector-valued age identifier for the ith aged fish. The element of the vector that corresponds to the age of the ith fish takes on the value 1 and all other elements are zero. The expected value of Eq. 2.35 is

$$ \begin{aligned} E\left({}_4\widehat{\mathbf{\Psi}}_k\right) =& E_{M_N}\left(\widehat{N}_k\right) E_{S_A}\left[\frac{N_{Om}}{n_{Am}n_{Ok}} E_{M_\psi}\left(\sum_{i=1}^{n_{Ak}} \boldsymbol{\psi}_i\right)\right] = N_k E_{S_A}\left(\frac{N_{Om}}{n_{Am}} n_{Ak}\right)\frac{\mathbf{p}_{\psi k}}{n_{Ok}} \\ =& N_k \mathbf{p}_{\psi k} = \mathbf{\Psi}_k. \end{aligned} $$

The variance is

$$ \begin{aligned} V\left({}_4\widehat{\mathbf{\Psi}}_k\right) =& \underbrace{V_{S_A}\left\{E_{M_N}\left[E_{M_\psi}\left({}_4\widehat{\mathbf{\Psi}}_k\right)\right]\right\}}_{V_1} + \underbrace{E_{S_A}\left\{V_{M_N}\left[E_{M_\psi}\left({}_4\widehat{\mathbf{\Psi}}_k\right)\right]\right\}}_{V_2} \\ &+ \underbrace{E_{S_A}\left\{E_{M_N}\left[V_{M_\psi}\left({}_4\widehat{\mathbf{\Psi}}_k\right)\right]\right\}}_{V_3} \end{aligned} $$

Now, component-wise,

$$
\begin{aligned}
V_1 =& V_{S_A}\left\{\frac{N_{Om}}{n_{Am}n_{Ok}}E_{M_N}\left[\widehat{N}_k E_{M_\psi}\left(\sum_{i=1}^{n_{Ak}}\psi_i\right)\right]\right\}\\
=& V_{S_A}\left(\frac{N_{Om}}{n_{Am}n_{Ok}}n_{\psi k}N_k\mathbf{p}_k\right)\\
=& \left(\frac{N_k}{n_{Ok}}\right)^2\mathbf{p}_{\psi k}\mathbf{p}_{\psi k}^T\underbrace{\left[N_{Om}\left(\frac{N_{Om}}{n_{\psi m}}-1\right)\frac{N_{Om}P_{Ok}\left(1-P_{Ok}\right)}{N_{Om}-1}\right]}_{V(\widehat{n}_{Ok})}
\end{aligned}
$$

where $P_{Ok}=\frac{n_{Ok}}{N_{Om}}$,

$$
\begin{aligned}
V_2 =& E_{S_A}\left\{\left(\frac{N_{Om}}{n_{Am}n_{Ok}}\right)^2 V_{M_N}\left[\widehat{N}_k E_{M_\psi}\left(\sum_{i=1}^{n_{Ak}}\psi_i\right)\right]\right\}\\
=& \frac{\mathbf{p}_k\mathbf{p}_k^T}{n_{Ok}^2}V\left(\widehat{N}_k\right)E_{S_A}\left[\left(\frac{N_{Om}}{n_{Am}}n_{Ak}\right)^2\right]\\
=& V\left(\widehat{N}_k\right)\left[\frac{V\left(\widehat{n}_{Ok}\right)}{n_{Ok}^2}+1\right]\mathbf{p}_{\psi k}\mathbf{p}_{\psi k}^T
\end{aligned}
$$

and

$$
\begin{aligned}
V_3 =& E_{S_A}\left\{\left(\frac{N_{Om}}{n_{Am}n_{Ok}}\right)^2 E_{M_N}\left[\widehat{N}_k^2 V_{M_\psi}\left(\sum_{i=1}^{n_{Ak}}\psi_i\right)\right]\right\}\\
=& \frac{V\left(\widehat{N}_k\right)+N_k^2}{n_{Ok}^2}E_{S_A}\left[\left(\frac{N_{Om}}{n_{Am}}\right)^2 n_{Ak}\left(1-\frac{n_{Ak}}{N_k}\right)\right]\left[\operatorname{diag}\left(\mathbf{p}_{\psi k}\right)-\mathbf{p}_{\psi k}\mathbf{p}_{\psi k}^T\right]\\
\approx& \frac{N_{Om}}{n_{Am}}\left[V\left(\widehat{N}_k\right)+N_k^2\right]\frac{\left[\operatorname{diag}\left(\mathbf{p}_{\psi k}\right)-\mathbf{p}_{\psi k}\mathbf{p}_{\psi k}^T\right]}{n_{Ok}}
\end{aligned}
$$

where the approximation holds when the number of aged fish in the haul is small relative to the total number of fish in the haul. Summing the component-wise results will give Eq. 2.36. The number of aged fish in the kth haul is hypergeometrically distributed when a SRS of otoliths from the mth region are aged, thus, the expected number of aged fish in the kth haul is $E_{S_A}(n_{Ak})=n_{Am}n_{Ok}/N_{Om}$. See Section A.2.1 in the appendix for further details. The prediction error VCM estimator (Eq. 2.37) is obtained by substituting sample-based estimates of all parameters so that

$$
\begin{aligned}
\widehat{V}\left({}_4\widehat{\boldsymbol{\Psi}}_k\right) =& \widehat{V}_1 + \widehat{V}_2 + \widehat{V}_3 \\
=& \widehat{N}_k^2 \frac{V\left(\widehat{n}_{Ok}\right)}{n_{Ok}^2} \widehat{\mathbf{p}}_{\psi k} \widehat{\mathbf{p}}_{\psi k}^T + \widehat{V}\left(\widehat{N}_k\right) \left[\frac{V\left(\widehat{n}_{Ok}\right)}{n_{Ok}^2} + 1\right] \widehat{\mathbf{p}}_{\psi k} \widehat{\mathbf{p}}_{\psi k}^T \\
&+ \frac{N_{Om}}{n_{\psi m}} \left[\widehat{V}\left(\widehat{N}_k\right) + \widehat{N}_k^2\right] \left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{\psi k}\right) - \widehat{\mathbf{p}}_{\psi k} \widehat{\mathbf{p}}_{\psi k}^T\right] \\
=& \left[\widehat{V}\left(\widehat{N}_k\right) + \widehat{N}_k^2\right] \left[\frac{V\left(\widehat{n}_{Ok}\right)}{n_{Ok}^2} \widehat{\mathbf{p}}_{\psi k} \widehat{\mathbf{p}}_{\psi k}^T + \frac{N_{Om}}{n_{Ok} n_{\psi m}} \left(\operatorname{diag}\{\widehat{\mathbf{p}}_{\psi k}\} - \widehat{\mathbf{p}}_{\psi k} \widehat{\mathbf{p}}_{\psi k}^T\right)\right] \\
&+ \widehat{V}\left(\widehat{N}_k\right) \widehat{\mathbf{p}}_{\psi k} \widehat{\mathbf{p}}_{\psi k}^T
\end{aligned}
$$

where

$$\widehat{\mathbf{p}}_{\psi k} = \frac{N_{Om}}{n_{Am}} \frac{\boldsymbol{\psi}_k}{n_{Ok}}.$$

Notice that when no otoliths are aged for a particular haul ($n_{\psi k} = 0$), $\widehat{\mathbf{p}}_{\psi k} = 0$.

2.A.5 Performance of Model-based Estimators

The models I have proposed for use in estimating catch parameters are as limited in assumptions as possible. Only the mean and variance of the super-population generating processes are specified rather than complete probability distributions. Furthermore, the mean and variance parameters are defined at the finest scales possible for variance estimation. However, the models do not truly represent the sampling process and some inconsistencies should be expected.

The most important qualities of an estimator are its bias and variance. When there is no relationship of the parameter of interest with the sampling process, the model-based estimators will be consistent for the parameters that they estimate. On the other hand, there are potential biases of the variance estimators that may be of concern. In particular, for estimators based on Bernoulli, Poisson or multinomial models, the variance of the estimators will be underestimated when the assumption of independence for sampled elements is incorrect. That is, variance estimates will be negatively biased. When there are products of these types of estimators (e.g., numbers in length or age classes) the variance estimator is a function of multiple negatively biased variance estimators and the problem is compounded.

There is silver lining to the unfortunate behavior of the variance estimators. Most of the catch parameters are really only of interest at larger scales such as seasonal or yearly estimates rather than haul-specific estimates. The variance of these larger scale estimates are functions of variability between larger scale elements as well as the within-haul estimates. Moreover, the variability between the larger scale elements dominates the variability between smaller scale elements and disparities between the expected values of within-haul variance estimates and the true variances will be a neglible component of the overall variance. In fact, even if unbiased variance estimation were possible within hauls, the small sample sizes and multiple sources of uncertainty for many catch parameters may yield estimates that are so variable as to make inference at such fine scales useless.

Chapter 3

ESTIMATION WITHIN HAUL FOR TRAWLERS

In this chapter I first develop estimators for total weight, total numbers and numbers-at-length and -age using a (approximate) design-based approach (Sections 3.1 to 3.4). As I will discuss in Section 3.5, when observers concern themselves with marine mammals they search the entire haul and so there is no sampling variability within hauls for marine mammal mortalities. In Section 3.6, I present two general model-based ratio estimators and then give results for the specific catch paramters for which design-based results are presented in preceding sections. Finally, in a short chapter appendix, Section 3.A, I give some derivation of properties of the more complicated estimators.

To help anyone reading this document, I provide a list of defintions for many if not all of the notation I use in this chapter (Table 3.1). Many of the terms are also defined as they are needed.

Table 3.1. Definition of terms

$\boldsymbol{\Lambda}_k$	vector of total numbers in each length class in the kth haul
$\boldsymbol{\lambda}_k$	vector of numbers in each length class among fish in the length sample for the kth haul
$\boldsymbol{\lambda}_{ki}$	vector of numbers in each length class among fish in the ith basket or cluster of fish for the kth haul, depending on the context
$\overline{\boldsymbol{\lambda}}_k$	average vector of numbers in each length class per basket or cluster of fish for the kth haul, depending on the context
m_k	number of baskets or clusters of fish sampled in the kth haul, depending on the context
$m_{\lambda k}$	number of baskets or clusters of fish used in the length sample for the kth haul, depending on the context
M_k	total number of baskets or clusters of fish in the kth haul, depending on the context

Table 3.1. (Continued)

N_k	number of fish for a given species in the kth haul
N_{ki}	number of fish in the ith basket or cluster from the kth haul, depending on the context
N_{sk}	the number of fish (for a given species) in the sampled portion of the kth haul
$n_{\lambda,k}$	number of fish in the length sample of the kth haul
n_{Ok}	number of otoliths sampled for the kth haul
n_{Ak}	number of otoliths sampled in the kth haul that are ultimately aged
n_{Ai}	number of otoliths sampled in the ith basket or cluster of fish that are ultimately aged, depending on the context
n_{Om}	number of all otoliths sampled in the mth management period/region
n_{Am}	number of all otoliths sampled and aged in the mth management period/region
$\mathbf{p}_k$	vector of proportions of fish in age classes for the sampled portion of the kth haul
$\mathbf{p}_{\lambda,k}$	vector of proportions of sampled fish in each length class for the kth haul
$\mathbf{p}_{\psi,k}$	vector of proportions of aged otoliths in each age class and the kth haul among all aged otoliths in the mth region
$\mathbf{p}_{Ok}$	vector of proportions of sampled otoliths in each age class and the kth haul among all otoliths collected in the mth region
P_{Ok}	the proportion of all otoliths sampled in the mth management period/region that were sampled from the kth haul (n_{Ok}/N_{Om})
$\phi_{\theta,t}$	a mean parameter that describes the rate of increase in the catch parameter (Θ) for the tth trip with respect to a given measured covariate (usually catch weight v or volume v)
$\mathbf{\Psi}_k$	vector of total numbers in each age class in the kth haul
$\boldsymbol{\psi}_k$	vector of numbers of aged fish in each age class for the kth haul
$\boldsymbol{\psi}_i$	vector of numbers of aged fish in each age class for the ith basket or cluster of fish, depending on the context
$\overline{\boldsymbol{\psi}}_k$	average vector of numbers of fish in each age class per basket or cluster of fish in the kth haul, depending on the context

Table 3.1. (Continued)

ψ_t	indicatior vector for age class of the tth fish
ψ_{Ok}	vector of numbers of fish in each age class in the otolith sample for the kth haul
ρ_k	density of catch for the kth haul
$\widehat{\rho}_k^*$	prescribed density of catch for the kth haul
$\sigma^2_{\mathrm{V}'}$	variance parameter for a model describing the haul volume measurement errors
σ^2_ρ	variance parameter for a model describing the haul density measurement errors
$\Sigma^2_{\boldsymbol{\theta},t}$	a variance-covariance matrix parameter for a super-population model that describes the values the parameter vector $\boldsymbol{\theta}$ takes on in the tth trip
$\boldsymbol{\theta}_k$	a generic vector-valued catch parameter for the sampled portion of the kth haul
Υ_k	total weight of the kth haul (for all species)
υ_k	weight of the sampled portion of the kth haul (for all species)
V_k	total volume of the kth haul
V'_k	measurement of the total volume of the kth haul
v_k	volume of a basket of fish for the kth haul
W_k	the weight of fish in the kth haul for a given species
W_{ki}	weight of fish in the ith basket or cluster from the kth haul, depending on the context
w_{ki}	weight of ith fish in the kth haul
$\overline{w}_k$	average weight fish (for a given species) in the kth haul
W_{sk}	the weight of fish (for a given species) in the sampled portion of the kth haul
$\overline{W}_k$	average weight of fish per basket or cluster (for a given species) in the kth haul, depending on the context
x_k	a covariate total for the sampled portion of the catch made in the kth haul, usually total weight (υ_k) or total volume (v_k) of the catch made in the kth haul

Table 3.1. (Continued)

$x_{k'}$	a covariate total for the unsampled portion of the catch made in the kth haul, usually total weight ($\upsilon_{k'}$) or total volume ($\mathrm{v}_{k'}$) of the catch made in the kth haul
x_t	a covariate total for the sampled portions of catches made on the tth fishing trip, usually total weight (υ_t) or total volume (v_t) of the catch made on the tth trip

3.1 Total Weight

Observers use various methods to obtain information that is subsequently used in estimation of total weight of a species in the catch. When large fish or mammals are brought aboard, the oberver makes a guess for its weight and records it as sample type "X" if not all the remaining individuals of that species are weighed. If all other individuals of the species are counted and weighed the sample type is denoted "whole haul" and the guessed weight of the large individual is added to the measured weight of smaller individuals. The whole haul method can also be used for non-targeted species in hauls that have little species heterogeneity and for any species in hauls that are small in total weight. In many circumstances, a partial haul sample is taken where baskets of fish are sampled from the catch and the total numbers and weight of each species in the basket samples is used to determine the total weights of those species in the haul. The baskets used for sampling by observers have known volumes. Other often-used methods for obtaining a partial haul sample, are a random section of fish from holding bins or a random section of a conveyor belt that moves fish to where they are processed. For species that are abundant in the catch, observers will sometimes count all the fish of that species in the catch and estimate the average weight from a simple random sample without replacement (SRS).

For species that are "whole haul" sampled, the total weight in the catch may be assumed known, but there will be sampling uncertainty for species that are "partial haul" sampled. We can consider the haul as a set of clusters of fish that could potentially be sampled. The clusters take various forms that depend on the sampling scenario. When baskets are used, the baskets are the clusters. When portions of a holding bin are sampled, each portion is a cluster. Finally, when sections of a conveyor belt are sampled, the sections are the clusters. Under any of the possible sampling scenarios, the observer uses SRS or systematic sampling. For reasons similar to those discussed elsewhere (e.g. Section 2.1) I will use SRS variance estimators for both SRS and systematic sampling scenarios. Assuming that the total volume of the haul is known, then an unbiased esitmator of the total weight is

$$ {}_1\widehat{W}_k = \frac{\mathrm{V}_k}{\mathrm{v}_k m_k} \sum_{i=1}^{m_k} W_{ki} \tag{3.1} $$

where V_k is the total volume of the haul, v_k is the volume of a basket of fish, m_k is the total number of sampled baskets and W_{ki} is the weight of the species of interest in the ith basket of fish. The ratio $\mathrm{V}_k/\mathrm{v}_k = M_k$ is the (approximate) total number of basket-sized groups of fish in the haul and thus Eq. 3.1 is a Horvitz-Thompson estimator like many of the estimators discussed for other gear types. Under the SRS assumption, the variance and variance estimator are

$$ V\left({}_1\widehat{W}_k\right) = M_k \left(\frac{M_k}{m_k} - 1\right) \frac{\sum_{i=1}^{M_k} \left(W_{ki} - \overline{W}_k\right)^2}{M_k - 1} \tag{3.2} $$

and

$$ \widehat{V}\left({}_1\widehat{W}_k\right) = M_k \left(\frac{M_k}{m_k} - 1\right) \frac{\sum_{i=1}^{m_k} \left(W_{ki} - \widehat{\overline{W}}_k\right)^2}{m_k - 1} \tag{3.3} $$

where $\overline{W}_k = \sum_{i=1}^{M_k} W_{ki}/M_k$ and $\widehat{\overline{W}}_k = \sum_{i=1}^{m_k} W_{ki}/m_k$. Of course, it is unlikely that the ratio, $\mathrm{V}_k/\mathrm{v}_k = M_k$, would be an integer, but when basket sampling is used, the total volume of the haul is large enough that there is no important consequences of ignoring the remainder (i.e., one group of fish that is less than one basket in volume). As such, the estimate and variance are negligibly affected whether $\mathrm{V}_k/\mathrm{v}_k$ is rounded to the nearest integer or left as a rational number.

A potential source of bias in Eq. 3.1 is differences in density between fish in the basket and those in the codend of the trawl. If density is higher in the codend, then the estimates will be negatively biased. A further reality is that the volume of the total catch is measured with error (and possibly bias). Biases in volume measurements could propogate in estimation of total catches at higher levels (e.g., quarters, years, etc.) and Dorn et al. (1999) found that small biases may occur for the codend volume measurements of the largest walleye pollock catches.

If the measured volume, V'_k, is unbiased, then an unbiased estimator of the total weight is identical, but with the error-ridden measured volume substituted for the true volume so that

$$_2\widehat{W}_k = \frac{\mathrm{V}'_k}{\mathrm{v}_k m_k} \sum_{i=1}^{m_k} W_{ki}, \tag{3.4}$$

but the variance is

$$V\left(_2\widehat{W}_k\right) = \frac{\sigma^2_{\mathrm{V}'}}{\mathrm{V}^2_k}\left[V\left(_1\widehat{W}_k\right) + W^2_k\right] + V\left(_1\widehat{W}_k\right) \tag{3.5}$$

where $\sigma^2_{\mathrm{V}'}$ is the variance of the volume measurement errors. Dorn et al. (1999) also find that volume estimates made in holding bins are more precise than those made in codends, but their measures of precision of the volume estimates include confounded variance in density of catches of a given total weight and variance of measurement errors. There are no other previous or proposed experiments to estimate bias and precision of volume estimates. Even if the estimates were found to be unbiased and measurement error variance estimates were available, an approximately model-unbiased variance prediction could not be made because there is no way of unbiasedly estimating the variance of weights in each basket,

$$S^2_{w,k} = \frac{\sum_{i=1}^{M_k}\left(W_{ki} - \overline{W}_k\right)^2}{M_k - 1}.$$

The variance of weights is not estimable because either the basket samples are mixed before data collection or only one bin or conveyor belt section is sampled. However, if the data were not recorded in sum and an independent experiment-based volume measurement error variance estimate, $\widehat{\sigma}^2_{\mathrm{V}'}$ were available, then the within-haul variance could be unbiasedly estimated by

$$\widehat{V}\left(_2\widehat{W}_k\right) = \frac{\widehat{\sigma}^2_{\mathrm{V}'}}{\mathrm{v}^2_k}\left[\widehat{\overline{W}}^2_k - \frac{\widehat{S}^2_{W,k}}{m_k}\right] + \frac{\mathrm{V}'_k}{\mathrm{v}_k}\left(\frac{\mathrm{V}'_k}{\mathrm{v}_k m_k} - 1\right)\widehat{S}^2_{W,k} \tag{3.6}$$

where

$$\widehat{S}^2_{W,k} = \frac{\sum_{i=1}^{m_k} \left(W_{ki} - \widehat{\overline{W}}_k\right)^2}{m_k - 1}.$$

Details regarding the properties of Eq. 3.1 and Eq. 3.6 can be found in Section 3.A.1.

A small complication arises when large "X" category individuals are caught. When codend volume is measured, these large individuals make up part of that volume, but in the estimation methodology the entire volume defines the number of clusters of fish that might be sampled for total weight of particular species. Therefore, when "X" category individuals occur and codend volume is used, estimation of total weight for any species using Eq. 3.1 or Eq. 3.4 would be positively biased if the volume of the "X" category individuals are not removed from the total catch volume.

When a motion-compensated flow scale (MCFS) weighs all unsorted fish before processing, observers will often take a random sample of known weight. Because the MCFS will weigh all fish in the catch (except "X" category fish), the total weight of the entire catch will be known and another approach for species total weight estimation uses the known entire catch weight instead of total volume. It may be typical that just one section of fish are randomly sampled which would lead to the species total weight estimator,

$${}_3\widehat{W}_k = \frac{\Upsilon_k}{\upsilon_k} W_{sk}, \tag{3.7}$$

where Υ_k is the total weight measured from the MCFS, υ_k is the weight of fish in the sample and W_{sk} is the weight of the species of interest in the sample. There would be no unbiased variance estimator in this case because only one sample is taken. If more than one sample were taken, then it would be important that the total weights of each sample were the same so that an estimator identical in form to Eq. 3.1 could be used where total catch weights (Υ_k) and sample unit weights (υ_k) are used instead of total catch volumes (V_k) and sample unit volumes (v_k). Otherwise, the total number of sampling units in the catch would be unknown. Corresponding variance and variance estimators would follow Eq. 3.2 and Eq. 3.3, respectively.

When partial haul sampling is performed so that the total number of the species of interest in the catch is known and a SRS of those fish is weighed, an estimator of total weight is

$${}_4\widehat{W}_k = \frac{N_k}{n_k} \sum_{i=1}^{n_k} W_{ki} \tag{3.8}$$

where N_k is the total number of fish and n_k is the number of fish sampled. The corresponding variance and variance estimator are

$$V\left({}_4\widehat{W}_k\right) = N_k\left(\frac{N_k}{n_k} - 1\right)\frac{\sum_{i=1}^{N_k}\left(w_{ki} - \overline{w}_k\right)^2}{N_k - 1}$$

where $\overline{W}_k = \sum_{i=1}^{N_k} w_{ki}/N_k$ and

$$\widehat{V}\left({}_4\widehat{W}_k\right) = N_k\left(\frac{N_k}{n_k} - 1\right)\frac{\sum_{i=1}^{n_k}\left(w_{ki} - \widehat{\overline{w}}_k\right)^2}{n_k - 1} \tag{3.9}$$

where $\widehat{\overline{w}}_k = \sum_{i=1}^{n_k} w_{ki}/n_k$. The unbiasedness of the estimator and variance estimator follow directly from SRS results in Section A.2.

It so happens that observers are instructed to make estimates of the weight of the entire catch more frequently than total weight of particular species. The observer may measure the total weight of the catch with motion-compensated scales or, for small catches, the observer can weigh the entire catch with scales provided by the NPGOP. Otherwise, the observer estimates the total weight of the catch. Estimates are made by multiplying a measurement of the volume of catch by a measurement of the density of the catch. The density measurement is generally obtained by weighing an unsorted sample of the total catch and dividing by the known volume of the sample. However, when walleye pollock comprises a majority of the catch, prescribed densities are used. The total catch of a particular species might be more precisely estimated at higher levels of the NPGOP design by making use of the more frequent entire catch estimates in a ratio estimator fashion. Therefore, I will also discuss estimation of total catch weight. The total catch weight is a function of the volume and density and has the form

$$\Upsilon_k = \mathrm{V}_k \rho_k$$

where ρ_k is the density. As mentioned above, there may be volume measurement error, but there is also sampling error when the density is estimated using a sample of the catch. Suppose there is a SRS of m_k clusters of fish from the haul. The unbiased estimator of the total weight of the entire catch is

$$_1\widehat{\Upsilon}_k = \frac{\mathrm{V}'_k}{\mathrm{v}_k m_k} \sum_{i=1}^{m_k} \Upsilon_{ki} \equiv \mathrm{V}'_k \widehat{\rho}_k$$

where V'_k is the measured volume of the catch, v_k is the constant volume of each cluster of fish sampled from the catch and Υ_{ki} is the total weight of the ith basket. The variance of this estimator,

$$V\left({}_1\widehat{\Upsilon}_k\right) = M_k \left(\frac{M_k}{m_k} - 1\right) S^2_{\Upsilon,k} \left[\frac{\sigma^2_{\mathrm{V}'}}{\mathrm{V}^2_k} + 1\right] + \frac{\sigma^2_{\mathrm{V}'}}{\mathrm{V}^2_k} \Upsilon^2_k$$

where

$$S^2_{\Upsilon,k} = \frac{\sum_{i=1}^{M_k} \left(\Upsilon_{ki} - \overline{\Upsilon}_k\right)^2}{M_k - 1}$$

and $\overline{\Upsilon}_k = \sum_{i=1}^{M_k} \Upsilon_{ki}/M_k$, is analogous in form to Eq. 3.5. If independent experiment-based volume measurement error variances are available and at least two density samples, then the variance of the total weight estimate could be predicted by

$$\widehat{V}\left({}_1\widehat{\Upsilon}_k\right) = \frac{\widehat{\sigma}^2_{\mathrm{V}'}}{\mathrm{v}^2_k} \left[\widehat{\overline{\Upsilon}}{}^2_k - \frac{\widehat{S}^2_{\Upsilon,k}}{m_k}\right] + \frac{\mathrm{V}'_k}{\mathrm{v}_k} \left(\frac{\mathrm{V}'_k}{\mathrm{v}_k m_k} - 1\right) \widehat{S}^2_{\Upsilon,k}$$

where

$$\widehat{S}^2_{\Upsilon,k} = \frac{\sum_{i=1}^{m_k} \left(\Upsilon_{ki} - \widehat{\overline{\Upsilon}}_k\right)^2}{m_k - 1}$$

and $\widehat{\overline{\Upsilon}}_k = \sum_{i=1}^{m_k} \Upsilon_{ki}/m_k$. The derivation of this variance estimator is identical to that provided for Eq. 3.4 in Section 3.A.1.

In their investigation, Dorn et al. (1999) find different densities appropriate for volumes measured in holding bins or codends when walleye pollock is the major species in the catch. Thus, in these hauls an alternative predictor of total weight using the prescribed densities is

$$_2\widehat{\Upsilon}_k = \mathrm{V}'_k \widehat{\rho}^*_k$$

where $\widehat{\rho}^*_k$ is the prescribed density. If an independent estimate of the volume measurement variance is available, then the prediction error variance estimator is

$$\widehat{V}\left({}_2\widehat{\Upsilon}_k\right) = 2\widehat{\sigma}^2_{V'}\left[(\widehat{\rho}^*_k)^2 - V\left(\widehat{\rho}^*_k\right)\right] + (V'_k)^2\left[\widehat{\sigma}^2_\rho + V\left(\widehat{\rho}^*_k\right)\right]$$

where $\widehat{\sigma}^2_\rho$ is the estimated variance of the experimental density measurements and $V\left(\widehat{\rho}^*_k\right)$ is the estimated variance of the density estimate made by Dorn et al. (1999). Density is typically estimated from one sample of the catch analogous to volume measurements, there is no ability to estimate variance of the density estimates. Therefore, variance estimation for the total weight estimate would require further assumptions.

3.2 Total Number of a Fish or Seabird Species

Unlike longline hauls, the nature of trawl catch sampling actually makes estimators of total numbers of a particular species virtually identical in form to those for total weight. See Section 3.1 for more detailed explanation of the various sampling methods used by observers for trawl species compositon sampling.

When baskets are sampled or when large clusters of fish are sampled from bins or the conveyor belt, the total number in the baskets or the bin or converor belt sample is recorded by the observer. So, just as for total weight, the estimator for the total number when the total volume is known is

$${}_1\widehat{N}_k = \frac{\mathrm{V}_k}{\mathrm{v}_k m_k}\sum_{i=1}^{m_k} N_{ki} = \frac{M_k}{m_k}\sum_{i=1}^{m_k} N_{ki} \tag{3.10}$$

where N_{ki} is the number of fish of the appropriate species in the ith cluster of fish that is sampled and $M_k = \mathrm{V}_k/\mathrm{v}_k$ is an approximation of the total number of sampling units in the haul. Similarly, to estimate the variance of Eq. 3.10 with

$$\widehat{V}\left({}_1\widehat{N}_k\right) = M_k\left(\frac{M_k}{m_k} - 1\right)\frac{\sum_{i=1}^{m_k}\left(N_{ki} - \widehat{\overline{N}}_k\right)^2}{m_k - 1}, \tag{3.11}$$

the numbers in each basket must be recorded separately when basket sampling is performed. When observers sample holding bins or conveyor belts, more than one section must be randomly sampled and the numbers in each section must be recorded separately.

When the codend or holding bin volume is measured with error, but the measurements are unbiased, the total number estimator is

$$ {}_2\widehat{N}_k = \frac{\mathrm{V}'_k}{\mathrm{v}_k m_k} \sum_{i=1}^{m_k} N_{ki} \tag{3.12} $$

and the unbiased variance estimator,

$$ \widehat{V}\left({}_2\widehat{N}_k\right) = \frac{\widehat{\sigma}^2_{\mathrm{V}'}}{\mathrm{v}_k^2}\left[\widehat{\overline{N}}_k^2 - \frac{\widehat{S}^2_{N,k}}{\mathrm{v}_k m_k}\right] + \frac{\mathrm{V}'_k}{\mathrm{v}_k}\left(\frac{\mathrm{V}'_k}{\mathrm{v}_k m_k} - 1\right)\widehat{S}^2_{N,k} \tag{3.13} $$

where

$$ \widehat{S}^2_{N,k} = \frac{\sum_{i=1}^{m_k}\left(N_{ki} - \widehat{\overline{N}}_k\right)^2}{m_k - 1}. $$

and $\widehat{\overline{N}}_k = \sum_{i=1}^{m_k} N_{ki}/m_k$, requires some unbiased predictor of the volume measurement error variance.

When a section of fish are sampled from the motion-compensated flow scale where the total weight of the sampled section is known an unbiased estimator of the total number of a particular species is

$$ {}_3\widehat{N}_k = \frac{\Upsilon_k}{\upsilon_k} N_{sk} \tag{3.14} $$

where Υ_k is the total weight of fish known from the flow scale, υ_k is the weight of fish in the sampled section and N_{sk} is the number of the species of interest in the sample. As mentioned for the analogous total weight estimator, there is no unbiased estimator available because of the single sampled section.

The one instance where estimating total number and total weight is different for trawlers is when a "partial haul" sample is completed by counting all the individuals of a species in the catch, but a SRS of those individuals is weighed. In this instance, the total number is known, but the total weight is not. Therefore, an estimator of total number is unnecessary.

3.3 Numbers in Length Classes

The numbers of fish to sample and which species to measure for lengths is the same across gear types. Therefore, all the nuances described for determining which species to measure for lengths in Section 2.3 for numbers in length classes in longline hauls apply to numbers in length classes in trawler hauls. For trawl hauls, observers take length measurements from prevalent species whenever they take samples to determine the total weight of particular species. The observer may subsample the fish that comprise the weight/numbers sample or they may randomly sample individuals from outside that sample. Whether observers use fish inside or outside the total number/weight sample is not indicated in the NPGOP data archive.

When an observer uses basket sampling for the total number and weight AFSC (2003) suggests observers randomly choose one or more of the baskets to use for the length measurement sample. When the total volume of the catch is known, the unbiased estimator of the $L \times 1$ vector of numbers in L length classes in the catch is

$$ {}_1\widehat{\mathbf{\Lambda}}_k = \frac{\mathrm{V}_k}{\mathrm{v}_k m_{\lambda k}} \sum_{i=1}^{m_{\lambda k}} \boldsymbol{\lambda}_{ki} = \frac{M_k}{m_{\lambda k}} \sum_{i=1}^{m_{\lambda k}} \boldsymbol{\lambda}_{ki} \tag{3.15} $$

where $m_{\lambda k}$ is the number of baskets of fish used for the length sample and $\boldsymbol{\lambda}_{ki}$ is the vector of numbers-at-length in the ith basket. Because the baskets used for the length sample are a simple random subsample of the baskets used for the total number/weight sample, $m_{\lambda k} \leq m_k$. The $L \times L$ variance-covariance matrix (VCM) and VCM estimator are

$$ V\left({}_1\widehat{\mathbf{\Lambda}}_k\right) = M_k \left(\frac{M_k}{m_k} - 1\right) \frac{\sum_{i=1}^{M_k} \left(\boldsymbol{\lambda}_{ki} - \overline{\boldsymbol{\lambda}}_k\right)\left(\boldsymbol{\lambda}_{ki} - \overline{\boldsymbol{\lambda}}_k\right)^T}{M_k - 1} $$

and

$$ \widehat{V}\left({}_1\widehat{\mathbf{\Lambda}}_k\right) = M_k \left(\frac{M_k}{m_k} - 1\right) \frac{\sum_{i=1}^{m_{\lambda k}} \left(\boldsymbol{\lambda}_{ki} - \widehat{\overline{\boldsymbol{\lambda}}}_k\right)\left(\boldsymbol{\lambda}_{ki} - \widehat{\overline{\boldsymbol{\lambda}}}_k\right)^T}{m_{\lambda k} - 1} \tag{3.16} $$

where $\overline{\boldsymbol{\lambda}}_k = \sum_{i=1}^{M_k} \boldsymbol{\lambda}_{ki}/M_k$ and $\widehat{\overline{\boldsymbol{\lambda}}}_k = \sum_{i=1}^{m_{\lambda k}} \boldsymbol{\lambda}_{ki}/m_{\lambda k}$. As mentioned for total weight and numbers, unbiased variance estimation requires that the numbers in each length class be recorded separately for each basket sample. When observers randomly sample sections of holding bins the same results apply and more than one section must be randomly sampled. When unbiased measurement error of the catch volume is considered and some independent estimate of measurement error variance is available, the estimator and variance estimator are

$$ {}_2\widehat{\mathbf{\Lambda}}_k = \frac{\mathrm{V}'_k}{\mathrm{v}_k m_k} \sum_{i=1}^{m_k} \mathbf{\Lambda}_{ki} \tag{3.17} $$

and

$$ \widehat{V}\left({}_2\widehat{\mathbf{\Lambda}}_k\right) = \frac{\widehat{\sigma}^2_{\mathrm{V}'}}{\mathrm{v}_k^2}\left[\widehat{\mathbf{\Lambda}}_k \widehat{\mathbf{\Lambda}}_k^T - \frac{\widehat{\mathbf{S}}^2_{\lambda,k}}{\mathrm{v}_k m_k}\right] + \frac{\mathrm{V}'_k}{\mathrm{v}_k}\left(\frac{\mathrm{V}'_k}{\mathrm{v}_k m_k} - 1\right)\widehat{\mathbf{S}}^2_{\lambda,k} \tag{3.18} $$

where

$$ \widehat{\mathbf{S}}^2_{\lambda,k} = \frac{\sum_{i=1}^{m_k}\left(\boldsymbol{\lambda}_{ki} - \widehat{\overline{\boldsymbol{\lambda}}}_k\right)\left(\boldsymbol{\lambda}_{ki} - \widehat{\overline{\boldsymbol{\lambda}}}_k\right)^T}{m_k - 1}. $$

When a total weight is known such as when sampling from a motion-compensated flow scale, the estimator of numbers-at-length is

$$ {}_3\widehat{\mathbf{\Lambda}}_k = \frac{\Upsilon_k}{\upsilon_k}\boldsymbol{\lambda}_k \tag{3.19} $$

where Υ_k is the known total weight of the haul, υ_k is the weight of the sample and $\boldsymbol{\lambda}_k$ is the numbers-at-length in the sample. Estimation of the VCM is not possible when only one large cluster of fish is sampled and if more than one cluster is sampled, then it is important that the clusters are identical in total weight. That is, the total weight of each sampled cluster of fish must be constant so that the total number of clusters in the haul, $M_k = \Upsilon_k/\upsilon_k$, is known.

When partial haul sampling is completed by counting all the fish of the species of interest, a SRS of those fish is taken for the weight sample and simple random subsample of the fish in the weight sample are measured for length, an unbiased estimator of the numbers-at-length is

$$ {}_4\widehat{\mathbf{\Lambda}}_k = \frac{N_k}{n_{\lambda,k}}\boldsymbol{\lambda}_k \tag{3.20} $$

where N_k is the number of fish in the haul, $n_{\lambda,k}$ is the number of fish measured for lengths and $\boldsymbol{\lambda}_k$ is the vector of numbers-at-length in the length sample. The unbiased

VCM estimator is

$$\widehat{V}\left({}_4\widehat{\boldsymbol{\Lambda}}_k\right) = N_k\left(\frac{N_k}{n_{\lambda,k}} - 1\right)\frac{n_{\lambda,k}}{n_{\lambda,k}-1}\left[\operatorname{diag}\left(\mathbf{p}_{\lambda,k}\right) - \mathbf{p}_{\lambda,k}\mathbf{p}_{\lambda,k}^T\right] \tag{3.21}$$

where $\mathbf{p}_{\lambda,k} = \boldsymbol{\lambda}_k / n_{\lambda,k}$.

In situations where observers take the length sample outside of the weight sample or where the number in the weight sample is insufficient and some fish outside the weight sample are used to supplement the numbers of fish in the weight sample, estimation becomes complicated. Whether observers sample fish inside or outside the weight sample for the length sample is not denoted in the NPGOP database. Furthermore, if observers only take a fraction of the length sample outside of the weight sample, then distinction of the individuals that come from the respective portions of the haul is lacking.

3.4 Total Numbers in Age Classes

As for longlines and pots, otoliths are generally removed from a simple random subsampled from the sample of fish measured for lengths. Which species and the number of fish of those species from which otoliths are removed is indicated in AFSC (2003), but can change from time to time depending on the needs of stock assessment scientists. How the otoliths are subsampled for ageing is the same across gear types and is described in Section 2.4.

I assume here that a simple random subsample (n_{Am}) of all otoliths collected in the mth management region (N_{Om}) are aged and that the haul of interest (i.e., the kth haul) is made in the mth region. The first numbers-at-age estimator assumes the length sample is a simple random subsample of the baskets used for the weight sample, a simple random subsample of $m_{Ok} \leq m_{\lambda k}$ of the length sample baskets are chosen to sample otoliths and n_i otoliths are sampled from the ith otolith basket. Under these assumptions, the unbiased estimator of the $A \times 1$ vector of numbers-at-age in the catch is

$${}_1\widehat{\boldsymbol{\Psi}}_k = \frac{N_{Om}}{n_{Am}}\frac{\mathrm{V}_k}{\mathrm{v}_k m_{Ok}}\sum_{i=1}^{m_{Ok}}\frac{N_i}{n_i}\boldsymbol{\psi}_i = \frac{N_{Om}}{n_{Am}}\frac{M_k}{m_{Ok}}\sum_{i=1}^{m_{Ok}}\frac{N_i}{n_i}\sum_{t=1}^{n_i}\boldsymbol{\psi}_t \tag{3.22}$$

where N_i is the number of fish in the ith basket and $\boldsymbol{\psi}_t$ is the $A \times 1$ vector that identifies the age of the tth fish so that the element that corresponds to the age of the fish takes on the value of one and all other elements are zero. Thus, summing the vectors for all fish in the haul, $\sum_{i=1}^{M_k}\sum_{t=1}^{N_i}\boldsymbol{\psi}_t = \boldsymbol{\Psi}_k$, gives the numbers-at-age. When $n_{Am} \ll N_{Om}$, few sampled otoliths are aged in each basket on average and many numbers-at-age vectors for particular baskets will be $\boldsymbol{\psi}_i = \mathbf{0}$. The $A \times A$ VCM is a function of the three random sampling events: baskets from the haul (s_1), fish in each basket for otoliths (s_2) and otoliths to age (s_3). Thus,

$$
\begin{aligned}
V\left({}_1\widehat{\boldsymbol{\Psi}}_k\right) =& E_1\left\{E_2\left[V_3\left({}_1\widehat{\boldsymbol{\Psi}}_k\right)\right]\right\} + E_1\left\{V_2\left[E_3\left({}_1\widehat{\boldsymbol{\Psi}}_k\right)\right]\right\} + V_1\left\{E_2\left[E_3\left({}_1\widehat{\boldsymbol{\Psi}}_k\right)\right]\right\} \\
=& E_1\left\{\left(\frac{M_k}{m_{Ok}}\right)^2 E_2\left[\sum_{i=1}^{m_{Ok}}\left(\frac{N_i}{n_i}\right)^2 V_3\left(\widehat{\boldsymbol{\Psi}}_{Oi}|s_1,s_2\right)\right.\right. \\
&\left.\left.+\sum_{i\neq j}^{m_{Ok}}\sum\left(\frac{N_iN_j}{n_in_j}\right) Cov_3\left(\widehat{\boldsymbol{\Psi}}_{Oi},\widehat{\boldsymbol{\Psi}}_{Oj}|s_1,s_2\right)\right]\right\} \\
&+E_1\left\{\left(\frac{M_k}{m_{Ok}}\right)^2 \sum_{i=1}^{m_{Ok}} V_2\left(\widehat{\boldsymbol{\Psi}}_i|s_1\right)\right\} \\
&+M_k\left(\frac{M_k}{m_{Ok}}-1\right)\frac{\sum_{i=1}^{M_k}\left(\boldsymbol{\psi}_{ki}-\overline{\boldsymbol{\psi}}_k\right)\left(\boldsymbol{\psi}_{ki}-\overline{\boldsymbol{\psi}}_k\right)^T}{M_k-1}
\end{aligned}
$$

where $\overline{\boldsymbol{\psi}}_k = \sum_{i=1}^{M_k}\psi_i/M_k$,

$$
V_2\left(\widehat{\boldsymbol{\Psi}}_i|s_1\right) = N_i^2\left(\frac{N_i}{n_i}-1\right)\frac{\left[\mathrm{diag}(\mathbf{p}_i)-\mathbf{p}_i\mathbf{p}_i^T\right]}{N_i-1},
$$

$$
V_3\left(\widehat{\boldsymbol{\Psi}}_{Oi}|s_1,s_2\right) = N_{Om}^2\left(\frac{N_{Om}}{n_{\psi m}}-1\right)\frac{\left[\mathrm{diag}(\mathbf{p}_{Oi})-\mathbf{p}_{Oi}\mathbf{p}_{Oi}^T\right]}{N_{Om}-1},
$$

$$
Cov_3\left(\widehat{\boldsymbol{\Psi}}_{Oi},\widehat{\boldsymbol{\Psi}}_{Oj}|s_1,s_2\right) = -N_{Om}^2\left(\frac{N_{Om}}{n_{\psi m}}-1\right)\frac{\mathbf{p}_{Oi}\mathbf{p}_{Oj}^T}{N_{Om}-1},
$$

$\mathbf{p}_i = \boldsymbol{\psi}_i / N_i$ and $\mathbf{p}_{Oi} = \boldsymbol{\psi}_{Oi} / N_{Om}$. The unbiased VCM estimator is

$$
\begin{aligned}
\widehat{V}\left({}_1\widehat{\boldsymbol{\Psi}}_k\right) =& \frac{M_k}{m_{Ok}} \left[\sum_{i=1}^{m_{Ok}} \left(\frac{N_i}{n_i}\right)^2 \widehat{V}_3\left(\widehat{\boldsymbol{\Psi}}_{Oi} | s_1, s_2\right) \right. \\
&\left. + \frac{M_k - 1}{m_{Ok} - 1} \sum\sum_{i \neq j}^{m_{Ok}} \frac{N_i}{n_i} \frac{N_j}{n_j} \widehat{Cov}_3\left(\widehat{\boldsymbol{\Psi}}_{Oi}, \widehat{\boldsymbol{\Psi}}_{Oj} | s_1, s_2\right) \right] \\
&+ \frac{M_k}{m_{Ok}} \sum_{i=1}^{m_{Ok}} \widehat{V}_2\left(\widehat{\boldsymbol{\Psi}}_i | s_1\right) \\
&+ M_k \left(\frac{M_k}{m_{Ok}} - 1\right) \frac{\sum_{i=1}^{m_{Ok}} \left(\widehat{\boldsymbol{\psi}}_i - \widehat{\overline{\boldsymbol{\psi}}}_k\right)\left(\widehat{\boldsymbol{\psi}}_i - \widehat{\overline{\boldsymbol{\psi}}}_k\right)^T}{m_{Ok} - 1}
\end{aligned}
\tag{3.23}
$$

where

$$
\widehat{V}_2\left(\widehat{\boldsymbol{\Psi}}_i | s_1\right) = N_i \left(\frac{N_i}{n_i} - 1\right) \frac{N_{Om}}{n_{Am}} \left[\frac{\sum_{t=1}^{n_{Ai}} \boldsymbol{\psi}_t \boldsymbol{\psi}_t^T}{n_i} - \frac{N_{Om} - 1}{n_{Am} - 1} \frac{\sum\sum_{t \neq u}^{n_{Ai}} \boldsymbol{\psi}_t \boldsymbol{\psi}_u^T}{n_i(n_i - 1)} \right],
$$

$$
\widehat{V}_3\left(\widehat{\boldsymbol{\Psi}}_{Oi} | s_1, s_2\right) = N_{Om} \left(\frac{N_{Om}}{n_{Am}} - 1\right) \left[\frac{\sum_{t=1}^{n_{Ai}} \boldsymbol{\psi}_t \boldsymbol{\psi}_t^T}{n_{Am} - 1} - \frac{\left(\sum_{t=1}^{n_{Ai}} \boldsymbol{\psi}_t\right)\left(\sum_{t=1}^{n_{Ai}} \boldsymbol{\psi}_t\right)^T}{n_{Am}\left(n_{Am} - 1\right)} \right],
$$

$$
\widehat{Cov}_3\left(\widehat{\boldsymbol{\Psi}}_{Oi}, \widehat{\boldsymbol{\Psi}}_{Oj} | s_1, s_2\right) = -N_{Om} \left(\frac{N_{Om}}{n_{Am}} - 1\right) \frac{\left(\sum_{t=1}^{n_{Ai}} \boldsymbol{\psi}_t\right)\left(\sum_{t=1}^{n_{Ai}} \boldsymbol{\psi}_t\right)^T}{n_{Am}\left(n_{Am} - 1\right)},
$$

$$
\widehat{\boldsymbol{\psi}}_i = \frac{N_i}{n_i} \frac{N_{Om}}{n_{Am}} \sum_{t=1}^{n_{Ai}} \boldsymbol{\psi}_t,
$$

$\widehat{\overline{\boldsymbol{\psi}}}_k = \sum_{i=1}^{m_{Ok}} \widehat{\boldsymbol{\psi}}_i / m_{Ok}$ and n_{Ai} is the number of fish from the ith basket that are aged. Section 3.A.2 in the appendix shows that Eq. 3.22 and Eq. 3.23 are unbiased.

When there is measurement error of the catch volume and an independent estimate of measurement error variance is available, the estimator and VCM estimator are

$$
{}_2\widehat{\boldsymbol{\Psi}}_k = \frac{N_{Om}}{n_{Am}} \frac{\mathrm{V}_k'}{\mathrm{v}_k m_k} \sum_{i=1}^{m_k} \frac{N_i}{n_i} \sum_{t=1}^{n_{Ai}} \boldsymbol{\psi}_t \tag{3.24}
$$

and

$$\widehat{V}\left({}_2\widehat{\boldsymbol{\Psi}}_k\right)=\widehat{\sigma}^2_{\mathrm{V}'}\left[\frac{\left({}_1\widehat{\boldsymbol{\Psi}}_k\right)\left({}_1\widehat{\boldsymbol{\Psi}}_k\right)^T}{\left(\mathrm{v}_k M_k\right)^2}-\frac{\widetilde{\mathbf{S}}_1^2}{\mathrm{v}_k m_{Ok}}\right]+\frac{\mathrm{V}'_k}{\mathrm{v}_k}\left[\widehat{V}_R+\left(\frac{\mathrm{V}'_k}{\mathrm{v}_k m_{Ok}}-1\right)\widetilde{\mathbf{S}}_1^2\right], \quad (3.25)$$

where

$$\widetilde{\mathbf{S}}_1^2=\frac{\sum_{i=1}^{m_{Ok}}\left(\widehat{\psi}_i-\widehat{\overline{\psi}}_k\right)\left(\widehat{\psi}_i-\widehat{\overline{\psi}}_k\right)^T}{m_{Ok}-1}$$

and

$$\widehat{V}_R=\frac{\widehat{V}\left({}_1\widehat{\boldsymbol{\Psi}}_k\right)-M_k\left(\frac{M_k}{m_k}-1\right)\widetilde{\mathbf{S}}_1^2}{M_k}.$$

These results are analogous to Eq. 3.4 and Eq. 3.6 but there are extra terms here due to the multiple sampling events for numbers-at-age in the haul.

When a total weight is known such as when sampling from a motion-compensated flow scale and a cluster or section of the stream of fish passing over the scale is sampled, the total number of clusters of fish available to the sampler is $\Upsilon_k/\upsilon_k = M_k$ where Υ_k is the known total weight of the haul, υ_k is the weight of the sample. The estimator of numbers-at-age is

$${}_3\widehat{\boldsymbol{\Psi}}_k=M_k\frac{N_{Om}}{n_{Am}}\frac{N_k}{n_{Ok}}\sum_{t=1}^{n_{Ak}}\psi_t \quad (3.26)$$

where N_k is the number of fish (of the species) in the sampled section/cluster, n_{Ok} is the number of fish in the otolith sample and n_{Ak} is the number of aged fish. The VCM is

$$\begin{aligned}V\left({}_3\widehat{\boldsymbol{\Psi}}_k\right)=&E_1\left\{\left(\frac{M_kN_k}{n_{Ok}}\right)^2N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{N_{Om}\left[\operatorname{diag}\left(\mathbf{p}_{Ok}\right)-\mathbf{p}_{Ok}\mathbf{p}_{Ok}^T\right]}{N_{Om}-1}\right\}\\&+E_2\left\{M_k^2N_k\left(\frac{N_k}{n_{Ok}}-1\right)\frac{N_k\left[\operatorname{diag}\left(\mathbf{p}_k\right)-\mathbf{p}_k\mathbf{p}_k^T\right]}{N_k-1}\right\}\\&+M_k\sum_{i=1}^{M_k}\left(\psi_{ki}-\overline{\psi}_k\right)\left(\psi_{ki}-\overline{\psi}_k\right)^T\end{aligned}$$

where $\mathbf{p}_{Ok} = \sum_{t=1}^{n_{Ok}}\psi_t/N_{Om}$, $\mathbf{p}_k = \sum_{t=1}^{N_k}\psi_t/N_k$ and $\overline{\psi}_k = \boldsymbol{\Psi}_k/M_k$. The second expectation (E_2) is with respect to the sampling distribution of otoliths conditional on the randomly selected section/cluster from the haul whereas the first expectation (E_1) is with respect to the sampling distribution associated with selection of the section/cluster from the haul. The variance cannot be written explicitly because the total number of collected otoliths (N_{Om}) depends on sampling in other hauls and the total number of fish in the sampled cluster (N_k) depends on which cluster is sampled.

Estimation of the VCM is not possible when only one large cluster of fish is sampled and if more than one cluster is sampled, then it is important that the clusters are identical in total weight. That is, the total weight of each sampled cluster of fish must be constant so that the total number of clusters in the haul, $M_k = \Upsilon_k/\upsilon_k$, is known. If multiple clusters of fish of the same weight are possible, then the clusters are analogous to baskets and the estimator of the total and the corresponding VCM estimator are identical in form to Eq. 3.22 and Eq. 3.23.

When partial haul sampling is completed by counting all the fish of the species of interest and a simple random subsample of fish in the length sample are determined for otolith removals,

$$ {}_4\widehat{\boldsymbol{\Psi}}_k = \frac{N_{Om}}{n_{Am}} \frac{N_k}{n_{Ok}} \sum_{t=1}^{n_{Ak}} \boldsymbol{\psi}_t \tag{3.27} $$

is an unbiased estimator of the total numbers-at-age in the haul, where N_k is the number of fish in the haul, n_{Ok} is the number of fish in the otolith sample. The VCM estimator is

$$ \begin{aligned} \widehat{V}\left({}_4\widehat{\boldsymbol{\Psi}}_k\right) =& N_k\left(\frac{N_k}{n_{Ok}} - 1\right)\frac{N_{Om}}{n_{Am}}\left[\frac{\sum_{t=1}^{n_{Ak}} \boldsymbol{\psi}_t\boldsymbol{\psi}_t^T}{n_{Ok}} - \left(\frac{N_{Om}-1}{n_{Am}-1}\right)\frac{\sum\sum_{t\neq u}^{n_{Ak}} \boldsymbol{\psi}_t\boldsymbol{\psi}_u^T}{n_{Ok}(n_{Ok}-1)}\right] \\ &+ \left(\frac{N_k}{n_{Ok}}\right)^2 N_{Om}\left(\frac{N_{Om}}{n_{Am}} - 1\right)\frac{n_{Am}\left[\operatorname{diag}\left(\mathbf{p}_{\psi,k}\right) - \mathbf{p}_{\psi,k}\mathbf{p}_{\psi,k}^T\right]}{n_{Am}-1} \end{aligned} \tag{3.28} $$

where $\mathbf{p}_{\psi,k} = \boldsymbol{\psi}_k/n_{Am}$. For a proof of unbiasedness of Eq. 3.27 and Eq. 3.28 see Section 3.A.3 in the appendix.

As mentioned at the end of Section 3.3, when observers take some portion of the length sample outside of the weight sample, design-based estimation becomes problematic. This is also the case for otolith samples since otoliths are obtained from the length sample.

3.5 Total Number of Marine Mammal Interactions

For trawl gear, marine mammals may feed on fish from the codend as it is being retrieved. Sometimes individuals may be swept into the codend and brought aboard the vessel. Observers may record other types of interactions if they witness the deployment and/or retrieval of gear, but for many hauls the observer only sees the contents of the codend. The type of marine mammal interactions that I will focus on here are those where individuals are brought on board in the trawl and which may or may not lead to mortality.

Unlike longline and pot gear, observers will not sample portions of a haul for marine mammals aboard trawlers or motherships. Observers randomly select hauls to observe for marine mammals and search the entire codend for any animals. Therefore, there is no uncertainty in the number of those types of marine mammal interactions considered here.

3.6 Model-based Estimation

Model-based estimation for longline hauls is facilitated by the ability to model each hook. Unfortunately, analogous observations on segments within trawl hauls are unavailable, hence forming simple model-based estimators requires stronger assumptions. I develop here a ratio estimator for total numbers and weight of fish and seabirds which uses a model-based approach with the information obtained by sampling hauls during a trip. However, an important simplification is the assumption of known total weight (all species) or volume of each haul made during a trip. The weight measurement error may be negligible for some hauls, such as when a motion-compensated flow scale is used, but volume measurements may have substantial errors (Dorn et al. 1999). Measurement error in either weight or volume may be an important consideration when volume and density methods are used to measure total weight.

3.6.1 The Generic Model

Consider the observed value for a generic vector-valued catch parameter measured on the sample within the kth haul, $\boldsymbol{\theta}_k$, where the total weight of the sample, υ_k, or volume or the sample, v_k, is known. I propose two similar models, the first of which is

$$\begin{aligned} E_M(\boldsymbol{\theta}_k) =& \boldsymbol{\mu}_k = x_k \boldsymbol{\phi}_{\boldsymbol{\theta},t} \\ V_M(\boldsymbol{\theta}_k) =& \boldsymbol{\Sigma}_{\boldsymbol{\theta},t} \\ Cov_M(\boldsymbol{\theta}_k, \boldsymbol{\theta}_l) =& \mathbf{0} \quad \text{for} \quad k \neq l. \end{aligned} \tag{3.29}$$

The expected value is a function of an unknown trip-specific parameter, $\boldsymbol{\phi}_{\boldsymbol{\theta},t}$ and x_k, the known value of either weight or volume, the VCM is trip-specific and all samples are uncorrelated. The slope parameter, $\boldsymbol{\phi}_{\boldsymbol{\theta},t}$, can be understood as the rate of increase in $\boldsymbol{\theta}$ with x_k. Notice that when total weight of a particular species is the catch parameter of interest and the weight of the sample is the covariate, the slope parameter can be viewed as the average proportion of the sample weight attributable to the species and thus $0 \leq \phi_{w,t} \leq 1$. For other catch parameters, however, there is not necessarily an obvious constraint on the upper bound of $\boldsymbol{\phi}_{\boldsymbol{\theta},t}$. Under this semiparametric model, an estimating function approach is useful for determining the optimal estimator for $\boldsymbol{\phi}_{\boldsymbol{\theta},t}$. The optimal estimating function (Godambe 1960) for the parameter, $\boldsymbol{\phi}_{\boldsymbol{\theta},t}$, is

$$\sum_{k=1}^{g_t} \left(\frac{\partial \boldsymbol{\mu}_k}{\partial \boldsymbol{\phi}_{\boldsymbol{\theta},t}} \right)^T \boldsymbol{\Sigma}_{\boldsymbol{\theta},t}^{-1} (\boldsymbol{\theta}_k - x_k \boldsymbol{\phi}_{\boldsymbol{\theta},t}) = \mathbf{0}. \tag{3.30}$$

This leads to the familiar slope estimator in a classical zero-intercept regression,

$$_1\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t} = \frac{\sum_{k=1}^{g_t} x_k \boldsymbol{\theta}_k}{\sum_{k=1}^{g_t} x_k^2} \tag{3.31}$$

and the best linear unbiased predictor of the parameter total for the kth haul is

$$_1\widehat{\boldsymbol{\Theta}}_k = \boldsymbol{\theta}_k + {}_1\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t} x_{k'}$$

where $\boldsymbol{\theta}_k$ is the value of the catch parameter for the sample and $x_{k'}$ is the covariate total over the unsampled portion of the haul (See Table 5.1 for definition of $x_{k'}$) (Valliant et al. 2000).

The estimating function (Eq. 3.30) is optimal in that it is has minimum variance in the class of unbiased linear estimating functions for the given model (see McCullagh and Nelder 1989, Chapter 9). The prediction error VCM is

$$V_M\left({}_1\widehat{\Theta}_k\right) = \left[\frac{x_{k'}^2}{\sum_{k=1}^{g_t} x_k^2} + (M_k - 1)\right]\boldsymbol{\Sigma}_{\boldsymbol{\theta},t}$$

which can be estimated by subtituting the VCM of the observations with an unbiased estimator,

$$\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta},t} = \frac{\sum_{k=1}^{g_t}\left(\boldsymbol{\theta}_k - x_{k1}\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}\right)\left(\boldsymbol{\theta}_k - x_{k1}\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}\right)^T}{g_t - 1}. \tag{3.32}$$

An alternative model supposes that variance increases proportionally with the covariate (weight υ or volume v of the sample). Thus,

$$\begin{aligned} E_M(\boldsymbol{\theta}_k) &= x_k\boldsymbol{\phi}_{\boldsymbol{\theta},t}, \\ V_M(\boldsymbol{\theta}_k) &= x_k\boldsymbol{\Sigma}_{\boldsymbol{\theta},t} \\ Cov_M(\boldsymbol{\theta}_k, \boldsymbol{\theta}_l) &= \mathbf{0} \quad \text{for} \quad k \neq l. \end{aligned} \tag{3.33}$$

Using Eq. 3.30, the optimal estimator of $\boldsymbol{\phi}_{\boldsymbol{\theta},t}$ is now

$${}_2\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t} = \frac{\boldsymbol{\theta}_t}{x_t} \tag{3.34}$$

where $\boldsymbol{\theta}_t = \sum_{k=1}^{g_t}\boldsymbol{\theta}_k$ and $x_t = \sum_{k=1}^{g_t} x_k$. The estimator of the total is still

$${}_2\widehat{\boldsymbol{\Theta}}_k = \boldsymbol{\theta}_k + {}_2\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}x_{k'}, \tag{3.35}$$

but the variance of the estimator under this model is

$$V_M\left({}_2\widehat{\boldsymbol{\Theta}}_k\right) = \left(\frac{x_{k'}}{x_t} + 1\right)x_{k'}\boldsymbol{\Sigma}_{\boldsymbol{\theta},t}.$$

Estimation of the variance is achieved using an unbiased estimator of the model variance parameter, $\boldsymbol{\Sigma}_{\boldsymbol{\theta},t}$,

$$\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta},t} = \frac{x_t}{x_t^2 - \sum_{k=1}^{g_t} x_k^2}\sum_{k=1}^{g_t}\left(\boldsymbol{\theta}_k - {}_2\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}x_k\right)\left(\boldsymbol{\theta}_k - {}_2\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}x_k\right)^T. \tag{3.36}$$

This second estimator (Eq. 3.35) is the analog to the more familiar design-based ratio estimator, but notice that both estimators use $\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}$, only to predict the mean of unsampled elements. The value of the sampled element(s) is taken as known. The variance estimator (Eq. 3.36) is derived independently by Valliant et al. (2000, Chapter 3) as a modified sandwich estimator.

The vector form of both model-based estimators, draws attention to the fact that any of the catch parameters that I discuss individually below can be modeled jointly. The joint model allows a researcher to estimate the totals of the parameters of interest and corresponding variances, just as when they are modelled individually, but covariance estimates can also be obtained. Covariance can be of interest on it own, but may also be useful for obtaining approximate variance estimates for functions of the individual parameters.

I provide derivations of both model-based estimators and their properties in Section 3.A.4 at the end of this chapter. In the following sections, I will present applications of the general estimators and variance estimators to specific catch parameters using weight as a covariate. Using a different covariate such as volume is possible by simple substitution. Numbers-at-length and -age will require further model developments because of subsampling of the fish that comprise the weight and numbers sample, but the ratio estimator results will also be used.

An important yet subtle requirement for the use of the above estimators is that the sample weight or volume x_k is known *even when none of the species of interest is present.* NPFMC (2004) notes that this is not necessarily true specifically regarding seabird bycatch estimation aboard trawlers. Observers may have more than one sample size for different species in a given haul, but they are instructed to count and weigh seabirds for the largest sample size they use. Therefore, I will assume that the maximum sample size of a given haul is used for seabirds when none are present. As for fish species, there is no specific instructions for particular species and I, therefore, assume that the smallest sample size is used for species when they are not present. However, in the case of fish species, if observers consistently search larger samples of fish than the smallest sample size for a given species that is not present then this would make the estimates positively biased.

3.6.2 Total Number in Catch for a Species of Fish or Seabird

When the first model (Eq. 3.29) with a constant variance parameter is assumed, for the total number of a fish or seabird species we replace Θ and θ by N and n and use Eq. 3.31 so that the optimal estimator of the total number is

$$ {}_1\widehat{N}_k = n_k + {}_1\widehat{\phi}_{n,t}\upsilon_{k'} = n_k + \frac{\sum_{k=1}^{g_t} n_k \upsilon_k}{\sum_{k=1}^{g_t} \upsilon_k^2}\upsilon_{k'} $$

where n_k is the number of animals for the species of interest in the sample collected from the kth haul. The total weight of the unsampled portion of the haul is $\upsilon_{k'} = \Upsilon_k - \upsilon_k$ and the prediction error variance estimator is

$$ \widehat{V}_M\left({}_1\widehat{N}_k\right) = \left[\frac{\upsilon_{k'}^2}{\sum_{k=1}^{g_t} \upsilon_k^2} + (M_k - 1)\right]\widehat{\sigma}_{n,t}^2 $$

where

$$ \widehat{\sigma}_{n,t}^2 = \frac{\sum_{k=1}^{g_t}\left(n_k - {}_1\widehat{\phi}_{n,t}\upsilon_k\right)^2}{g_t - 1}. \tag{3.37} $$

When the second model (Eq. 3.33) with variance proportional to the size of the covariate is assumed, we use Eq. 3.34 and the estimator of the total number is

$$ {}_2\widehat{N}_k = n_k + {}_2\widehat{\phi}_{n,t}\upsilon_{k'} = n_k + \frac{\sum_{k=1}^{g_t} n_k}{\sum_{k=1}^{g_t} \upsilon_k}\upsilon_{k'} \tag{3.38} $$

and the prediction error variance estimator is

$$ \widehat{V}_M\left({}_2\widehat{N}_k\right) = \left(\frac{\upsilon_{k'}}{\upsilon_t} + 1\right)\upsilon_{k'}\widehat{\sigma}_{n,t}^2 \tag{3.39} $$

where

$$ \widehat{\sigma}_{n,t}^2 = \frac{\upsilon_t}{\upsilon_t^2 - \sum_{k=1}^{g_t} \upsilon_k^2}\sum_{k=1}^{g_t}\left(n_k - {}_2\widehat{\phi}_{n,t}\upsilon_k\right)^2. \tag{3.40} $$

3.6.3 Total Weight for a Species of Fish or Seabird

When Eq. 3.29 is assumed, for the total weight of a species we replace Θ and θ by W and w and use Eq. 3.31 so that the optimal estimator of the total weight is

$$_1\widehat{W}_k = w_k + {}_1\widehat{\phi}_{w,t} v_{k'} = w_k + \frac{\sum_{k=1}^{g_t} w_k v_k}{\sum_{k=1}^{g_t} v_k^2} v_{k'}$$

where w_k is the number of the species in the sample collected from the kth haul. The prediction error variance estimator is

$$\widehat{V}\left({}_1\widehat{W}_k\right) = \left[\frac{v_{k'}^2}{\sum_{k=1}^{g_t} v_k^2} + (M_k - 1)\right] \widehat{\sigma}_{w,t}^2$$

where

$$\widehat{\sigma}_{w,t}^2 = \frac{\sum_{k=1}^{g_t}\left(w_k - {}_1\widehat{\phi}_{w,t} v_k\right)^2}{g_t - 1}.$$

When the second model (Eq. 3.33) is assumed, we use (Eq. 3.34) and the estimator of the total weight is

$$_2\widehat{W}_k = w_k + {}_2\widehat{\phi}_{w,t} v_{k'} = w_k + \frac{\sum_{k=1}^{g_t} w_k}{\sum_{k=1}^{g_t} v_k} v_{k'} \tag{3.41}$$

and the prediction error variance estimator is

$$\widehat{V}_M\left({}_2\widehat{W}_k\right) = \left(\frac{v_{k'}}{v_t} + 1\right) v_{k'} \widehat{\sigma}_{w,t}^2 \tag{3.42}$$

where

$$\widehat{\sigma}_{w,t}^2 = \frac{v_t}{v_t^2 - \sum_{k=1}^{g_t} v_k^2} \sum_{k=1}^{g_t}\left(w_k - {}_2\widehat{\phi}_{w,t} v_k\right)^2.$$

3.6.4 Total Numbers in Length Classes

In general, lengths are not measured for all fish that enter the weight sample and using weight or volume of the weight sample as a covariate leads to negatively biased estimation. If the total weight or volume of the fish in the length sample is known, then the ratio estimator I used for the total weight and number is appropriate. When the weight or volume of the length sample is unknown, an option is to use the product of the model-based estimator for total number and the proportions-at-length in the length sample,

$$ {}_1\widehat{\mathbf{\Lambda}}_k = \widehat{N}_k \frac{\boldsymbol{\lambda}_k}{n_{\lambda,k}} = \widehat{N}_k \widehat{\mathbf{p}}_{\lambda,k} \tag{3.43} $$

where $n_{\lambda,k}$ is the number of fish in the length sample. The form of $\widehat{N}_k$ depends on the variance model assumption (see Section 3.6.2). For estimation of numbers in length classes, I have modified the model for total numbers so that the parameters are specific to hauls where the species of interest is prevalent. See Section 5.1.2.5 for more details on the estimator of total number and corresponding variance estimator. When the models for total numbers and numbers-at-length are independent, Eq. 3.43 is unbiased and it is asymptotically so, when they are dependent. As described in Section 2.3.2, the numbers-at-length are multinomially distributed when fish in the haul are assumed independent. When the estimators for total numbers and proportions-at-length are independent a variance estimator that has negligible bias when the number of fish in the haul is large relative to the number in the length sample is

$$ \widehat{V}\left({}_1\widehat{\mathbf{\Lambda}}_k\right) = \widehat{V}\left(\widehat{N}_k\right)\left[\widehat{\mathbf{p}}_{\lambda,k}\widehat{\mathbf{p}}_{\lambda,k}^T - \frac{\widehat{\mathbf{\Sigma}}_{\lambda,k}}{n_{\lambda,k}}\right] + \widehat{N}_k^2 \frac{\widehat{\mathbf{\Sigma}}_{\lambda,k}}{n_{\lambda,k}} \tag{3.44} $$

where

$$ \widehat{\mathbf{\Sigma}}_{\lambda,k} = \frac{n_{\lambda,k}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{\lambda,k}\right) - \widehat{\mathbf{p}}_{\lambda,k}\widehat{\mathbf{p}}_{\lambda,k}^T\right]}{n_{\lambda,k} - 1} $$

The form of Eq. 3.44 is identical to the model-based estimator for longline hauls, Eq. 2.32. because of similar independence assumptions. However, the form of the component variance estimator, $\widehat{V}\left(\widehat{N}_k\right)$, differs. The variance estimator (Eq. 3.44) is also related to the variance estimator given by Goodman (1960) for the product of independent random variables, but differs slightly in that it treats prediction error variance.

3.6.5 Total Numbers in Age Classes

The total weight of fish in the otolith sample is known as is the weight of each fish and the general ratio estimator could be applied, but any additional variance due to otolith subsampling for ageing must be taken into account. An approach that uses the model-based estimator of the total number like Eq. 3.43, but still accounts for variablity due to subsampling, is more straightforward than one that uses the ratio estimator. As I mentioned for numbers in length classes, I use a modified model for total numbers so that the parameters are specific to hauls where the species of interest is prevalent. See Section 5.1.2.5 for more details on the estimator of total number and corresponding variance estimator. The model-unbiased estimator of the total numbers-at-age is

$$ {}_1\widehat{\mathbf{\Psi}}_k = \widehat{N}_k \frac{N_{Om}}{n_{Om}} \frac{\boldsymbol{\psi}_k}{n_{Ok}} = \widehat{N}_k \widehat{\mathbf{p}}_{\psi,k} \tag{3.45} $$

where $\boldsymbol{\psi}_k$ is the numbers-at-age for the species in the aged sample collected from the kth haul. The approximately unbiased variance estimator is

$$ \widehat{V}\left(\widehat{\mathbf{\Psi}}_k\right) = \left[\widehat{V}\left(\widehat{N}_k\right) + \widehat{N}_k^2\right] \left\{ V\left(\widehat{n}_{Ok}\right) \widehat{\mathbf{p}}_{\psi,k} \widehat{\mathbf{p}}_{\psi,k}^T + \frac{N_{Om}}{n_{Ok} n_{Om}} \left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{\psi,k}\right) - \widehat{\mathbf{p}}_{\psi,k} \widehat{\mathbf{p}}_{\psi,k}^T\right] \right\} $$
$$ + \widehat{V}\left(\widehat{N}_k\right) \widehat{\mathbf{p}}_{\psi,k} \widehat{\mathbf{p}}_{\psi,k}^T \tag{3.46} $$

where

$$ V\left(\widehat{n}_{Ok}\right) = N_{Om} \left(\frac{N_{Om}}{n_{Om}} - 1\right) \frac{N_{Om} P_{Ok} (1 - P_{Ok})}{N_{Om} - 1} $$

and $P_{Ok} = n_{Ok}/N_{Om}$.

I noted the similarities between the model-based numbers-at-length estimators and variance estimators for trawl and longline hauls in Section 3.6.4 and the same holds for Eq. 3.45 and Eq. 3.46 here with Eq. 2.35 and Eq. 2.36, respectively. The difference between the gear-types is the form of $\widehat{N}_k$ and $\widehat{V}\left(\widehat{N}_k\right)$. See the derivation of the properties of the longline-specific results in Section 2.A.4 for motivation of the trawl-specific results presented here.

3.7 Choosing an estimator

The numbers of available estimators for various catch parameters can be overwhelming and in this section I summarize more succinctly the sampling scenarios under which each estimator is appropriate and recommend particular estimators when more than one is available for a given scenario.

Unlike longline gear, the estimators I presented for each catch parameter when trawl gear is used have a good deal of commonality, in that, each estimator can be a function of the sampled and total volumes or the sampled and total weight of the catch. All model-based estimators rely on a similar zero-intercept mean model as well.

Similarities between the estimators also carries over to the criteria for determining which estimators to use in a given sampling scenario. Figure 3.1 shows the decision tree for choosing the catch weight estimator and variance estimator and, by inspection of the decision trees for the other catch parameters, we see that the decisions are identical (Figures 3.2 through 3.4).

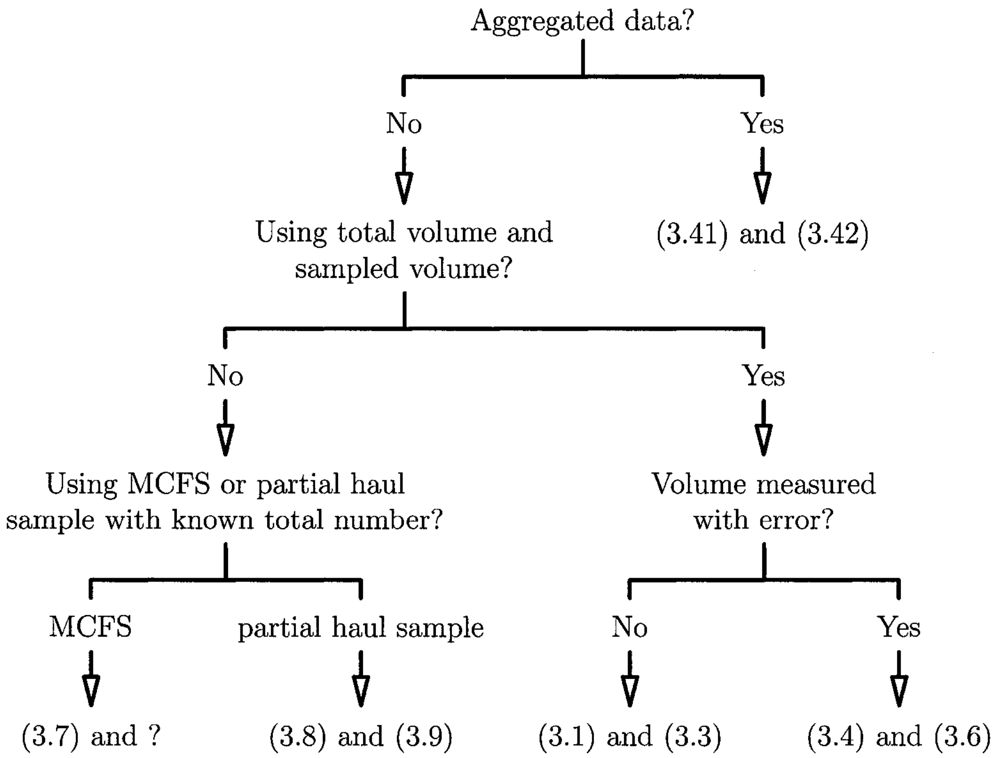

Figure 3.1. Decision tree for determining which estimators to use for total weight of a particular fish species in a trawl haul.

As with longline gear, the appropriate choice of estimators depends on whether data are aggregated. However, for trawl gear we now have different estimation depending on whether we measure total volume of the catch and whether we are using a MCFS or partial haul sampling. Notice also that, in general, I have supplied no vari-

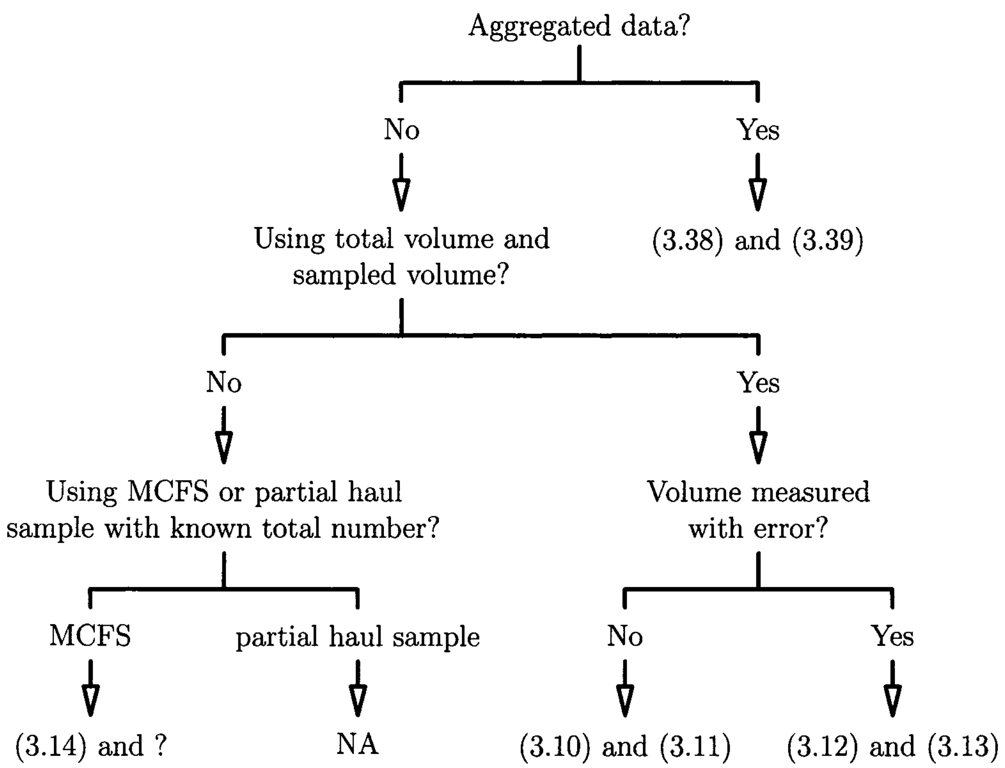

Figure 3.2. Decision tree for determining which estimators to use for total number of a particular fish species in a trawl haul.

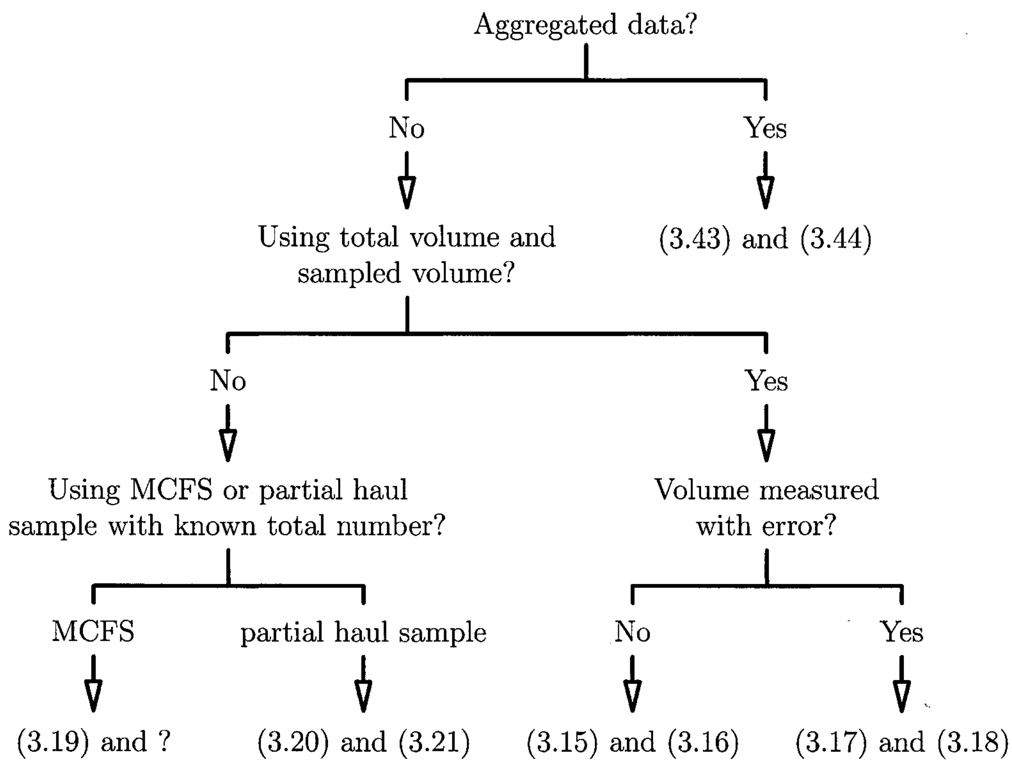

Figure 3.3. Decision tree for determining which estimators to use for total numbers in length classes in a trawl haul.

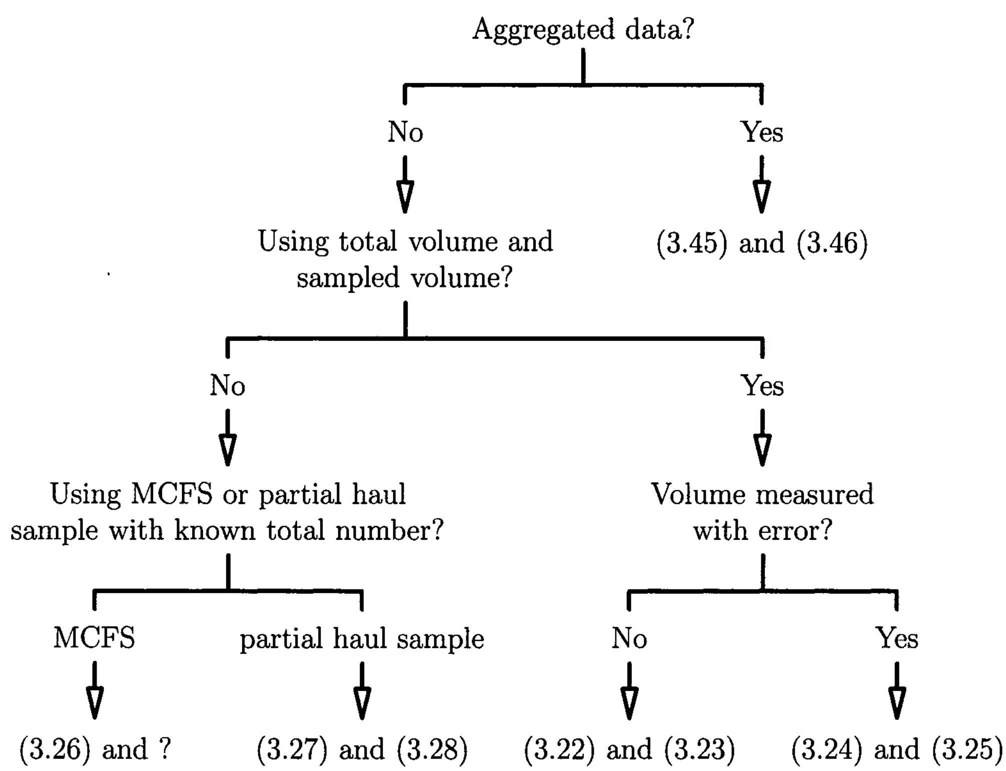

Figure 3.4. Decision tree for determining which estimators to use for total numbers in age classes in a trawl haul.

ance estimator when a MCFS is used. Furthermore, the total number for a particular species is known without error when partial haul sampling and all fish are counted.

There were two models I proposed for model-based estimation, but in the decision trees I recommend using the model that assumes a variance that is proportional to the weight or volume of the sample elements (Eq. 3.33). I recommend estimators based on this model because they can actually be used and the variance estimates will on average be larger than those obtained under the model in Eq. 3.29. Because the variance estimates will be larger on average the variance estimator is the most conservative of the two available.

3.A Derivation of Estimators

3.A.1 Derivation of ${}_2\widehat{W}_k$

The volume estimates are assumed to have mean V_k and variance $\sigma^2_{\mathrm{V}'}$. Denoting expectation with respect to the volume measurement model and sampling distribution within the haul as $E_M(\cdot)$ and $E_S(\cdot)$, the expected value of Eq. 3.4 is

$$E\left({}_2\widehat{W}_k\right) = E_S\left[\left(\frac{\sum_{i=1}^{m_k} w_{ki}}{m_k \mathrm{v}_k}\right) E_M\left(\mathrm{V}'_k\right)\right] = E_S\left(\frac{M_k}{m_k}\sum_{i=1}^{m_k} w_{ki}\right) = \sum_{i=1}^{M_k} w_{ki} = W_k$$

and the variance is

$$\begin{aligned} V\left({}_2\widehat{W}_k\right) =& E_S\left[V_M\left({}_2\widehat{W}_k|S\right)\right] + V_S\left[E_M\left({}_2\widehat{W}_k|S\right)\right] \\ =& E_S\left[\left(\frac{\sum_{i=1}^{m_k} w_{ki}}{m_k \mathrm{v}_k}\right)^2 \sigma^2_{\mathrm{V}'}\right] + V_S\left(\frac{M_k}{m_k}\sum_{i=1}^{m_k} w_{ki}\right) \\ =& \sigma^2_{\mathrm{V}'}\left[\frac{V\left({}_1\widehat{W}_k\right) + W_k^2}{\left(M_k \mathrm{v}_k\right)^2}\right] + V\left({}_1\widehat{W}_k\right). \end{aligned}$$

The variance estimator is unbiased because

$$
\begin{aligned}
E\left[\widehat{V}\left({}_2\widehat{W}_k\right)\right] =& \frac{E_M\left(\widehat{\sigma}^2_{\mathrm{V}'}\right)}{\mathrm{v}_k^2} E_S\left[\left(\frac{\sum^{m_k} w_{ki}}{m_k}\right)^2 - \frac{\widehat{S}^2_{w,k}}{m_k}\right] \\
&+ \left[\frac{E_M\left(\mathrm{V}'^2_k\right)}{\mathrm{v}_k^2 m_k} - \frac{E_M\left(\mathrm{V}'_k\right)}{\mathrm{v}_k}\right] E_S\left(\widehat{S}^2_{w,k}\right) \\
=& \frac{\sigma^2_{\mathrm{V}'}}{\mathrm{v}_k^2}\left[\frac{V\left({}_1\widehat{W}_k\right) + W_k^2}{M_k^2} - \frac{S^2_{w,k}}{m_k}\right] + \left[\frac{\mathrm{V}_k^2 + \sigma^2_{\mathrm{V}'}}{\mathrm{v}_k^2 m_k} - \frac{\mathrm{V}_k}{\mathrm{v}_k}\right] S^2_{w,k} \\
=& \sigma^2_{\mathrm{V}'}\left[\frac{V\left({}_1\widehat{W}_k\right) + W_k^2}{\left(M_k \mathrm{v}_k\right)^2}\right] + \frac{\mathrm{V}_k}{\mathrm{v}_k}\left(\frac{\mathrm{V}_k}{\mathrm{v}_k m_k} - 1\right) S^2_{w,k}.
\end{aligned}
$$

3.A.2 Derivation of ${}_1\widehat{\boldsymbol{\Psi}}_k$

Let the sampling distributions for baskets within hauls, fish for otoliths within a basket and otoliths for ageing be denoted 1,2, and 3, respectively. Also, let s_1, s_2 and s_3 represent given samples from the respective distributions. The within-haul total numbers-at-age estimator (Eq. 3.22) is unbiased because

$$
\begin{aligned}
E\left({}_1\widehat{\boldsymbol{\Psi}}_k\right) =& E\left(\frac{M_k}{m_{Ok}} \sum_{i=1}^{m_{Ok}} \frac{N_i}{n_i} \frac{N_{Om}}{n_{Am}} \sum_{t=1}^{n_{Ai}} \boldsymbol{\psi}_{it}\right) \\
=& E_1\left\{\frac{M_k}{m_{Ok}} \sum_{i=1}^{m_{Ok}} E_2\left[\frac{N_i}{n_i} E_3\left(\frac{N_{Om}}{n_{Am}} \sum_{t=1}^{n_{Ai}} \boldsymbol{\psi}_{it} | s_1, s_2\right)\right]\right\} \\
=& E_1\left[\frac{M_k}{m_{Ok}} \sum_{i=1}^{m_{Ok}} E_2\left(\frac{N_i}{n_i} \sum_{t=1}^{n_i} \boldsymbol{\psi}_{it} | s_1\right)\right] \\
=& E_1\left(\frac{M_k}{m_{Ok}} \sum_{i=1}^{m_{Ok}} \sum_{t=1}^{N_i} \boldsymbol{\psi}_{it}\right) = \sum_{i=1}^{M_k} \sum_{t=1}^{N_i} \boldsymbol{\psi}_{it} = \boldsymbol{\Psi}_k
\end{aligned}
$$

where $\boldsymbol{\psi}_{it}$ is the $A \times 1$ vector identifying the age of the tth fish by taking the value, 1, where the element in the vector corresponds to the age of the fish and all other values in the vector are zero. Thus, the sum of the vectors for each fish in the haul will yield the vector of numbers-at-age in the haul. The variance of the estimator is

$$
\begin{aligned}
V\left({}_1\widehat{\boldsymbol{\Psi}}_k\right) =& E_1\left\{E_2\left[V_3\left({}_1\widehat{\boldsymbol{\Psi}}_k|s_1,s_2\right)\right]\right\} + E_1\left\{V_2\left[E_3\left({}_1\widehat{\boldsymbol{\Psi}}_k|s_1,s_2\right)\right]\right\} \\
&+ V_1\left\{E_2\left[E_3\left({}_1\widehat{\boldsymbol{\Psi}}_k|s_1,s_2\right)\right]\right\} \\
=& E_1\left\{\left(\frac{M_k}{m_{Ok}}\right)^2 E_2\left[V_3\left(\sum_{i=1}^{m_{Ok}}\frac{N_i}{n_i}\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ai}}\boldsymbol{\psi}_{it}|s_1,s_2\right)\right]\right\} \\
&+ E_1\left[\left(\frac{M_k}{m_{Ok}}\right)^2\sum_{i=1}^{m_{Ok}} V_2\left(\frac{N_i}{n_i}\sum_{t=1}^{n_i}\boldsymbol{\psi}_{it}|s_1\right)\right] \\
&+ V_1\left[\frac{M_k}{m_{Ok}}\sum_{i=1}^{m_{Ok}} E_2\left(\frac{N_i}{n_i}\sum_{t=1}^{n_i}\boldsymbol{\psi}_{it}|s_1\right)\right] \\
=& E_1\left\{\left(\frac{M_k}{m_{Ok}}\right)^2 E_2\left[\sum_{i=1}^{m_{Ok}}\left(\frac{N_i}{n_i}\right)^2 V_3\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ai}}\boldsymbol{\psi}_{it}|s_1,s_2\right)\right.\right. \\
&\left.\left.+\sum_{i\neq j}^{m_{Ok}}\sum\frac{N_i}{n_i}\frac{N_j}{n_j}Cov_3\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ai}}\boldsymbol{\psi}_{it},\frac{N_{Om}}{n_{Am}}\sum_{u=1}^{n_{Aj}}\boldsymbol{\psi}_{ju}|s_1,s_2\right)\right]\right\} \\
&+ E_1\left[\left(\frac{M_k}{m_{Ok}}\right)^2\sum_{i=1}^{m_{Ok}} N_i\left(\frac{N_i}{n_i}-1\right)\frac{\sum_{t=1}^{N_i}\left(\boldsymbol{\psi}_{it}-\overline{\boldsymbol{\psi}}_i\right)\left(\boldsymbol{\psi}_{it}-\overline{\boldsymbol{\psi}}_i\right)^T}{N_i-1}\right] \\
&+ M_k\left(\frac{M_k}{m_{Ok}}-1\right)\frac{\sum_{i=1}^{M_k}\left(\boldsymbol{\psi}_i-\overline{\boldsymbol{\psi}}_k\right)\left(\boldsymbol{\psi}_i-\overline{\boldsymbol{\psi}}_k\right)^T}{M_k-1}
\end{aligned}
$$

where

$$
\begin{aligned}
&V_3\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ai}}\boldsymbol{\psi}_{it}|s_1,s_2\right) \\
&= N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\left[\frac{\sum_{t=1}^{n_i}\boldsymbol{\psi}_t\boldsymbol{\psi}_t^T}{N_{Om}-1}-\frac{\left(\sum_{t=1}^{n_i}\boldsymbol{\psi}_t\right)\left(\sum_{t=1}^{n_i}\boldsymbol{\psi}_t\right)^T}{N_{Om}\left(N_{Om}-1\right)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&Cov_3\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ai}}\boldsymbol{\psi}_{it},\frac{N_{Om}}{n_{Am}}\sum_{u=1}^{n_{Aj}}\boldsymbol{\psi}_{ju}|s_1,s_2\right) \\
&= -N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{\left(\sum_{t=1}^{n_i}\boldsymbol{\psi}_t\right)\left(\sum_{t=1}^{n_i}\boldsymbol{\psi}_t\right)^T}{N_{Om}\left(N_{Om}-1\right)}.
\end{aligned}
$$

There is covariance of samples within separate baskets because of the otolith subsampling across all otoliths collected in the region. If otoliths were subsampled within each basket independently, then there would be no covariance terms. The form of the variance and covariance terms follows directly form results on domain estimation because each basket can be viewed as a domains in the universe of all otoliths sampled in the region (see Section A.6). The results for the other components of the variance follow directly from simple random sampling (see Section A.2). It is also straighforward to show

$$\frac{\sum_{t=1}^{N_i}\left(\boldsymbol{\psi}_{it}-\overline{\boldsymbol{\psi}}_i\right)\left(\boldsymbol{\psi}_{it}-\overline{\boldsymbol{\psi}}_i\right)^T}{N_i-1}=\frac{N_i\left[\operatorname{diag}\left(\mathbf{p}_{\psi i}\right)-\mathbf{p}_{\psi i}\mathbf{p}_{\psi i}^T\right]}{N_i-1}$$

where $\mathbf{p}_{\psi i}=\sum_{t=1}^{N_i}\boldsymbol{\psi}_t/N_i$ and

$$\frac{\sum_{t=1}^{n_i}\boldsymbol{\psi}_t\boldsymbol{\psi}_t^T}{N_{Om}-1}-\frac{\left(\sum_{t=1}^{n_i}\boldsymbol{\psi}_t\right)\left(\sum_{t=1}^{n_i}\boldsymbol{\psi}_t\right)^T}{N_{Om}\left(N_{Om}-1\right)}=\frac{N_{Om}\left[\operatorname{diag}\left(\mathbf{p}_{\psi Oi}\right)-\mathbf{p}_{\psi Oi}\mathbf{p}_{\psi Oi}^T\right]}{N_{Om}-1}$$

where $\mathbf{p}_{\psi Oi} = \sum_{t=1}^{n_i} \psi_t / N_{Om}$ (see Section A.2.2). The variance estimator (Eq. 3.23) is unbiased because

$$
\begin{aligned}
&E\left[\frac{\sum_{i=1}^{m_{Ok}}\left(\widehat{\psi}_i-\widehat{\overline{\psi}}_k\right)\left(\widehat{\psi}_i-\widehat{\overline{\psi}}_k\right)^T}{m_{Ok}-1}\right] \\
&= E_1\left\{\frac{1}{m_{Ok}}\sum_{i=1}^{m_{Ok}} E_2\left[\left(\frac{N_i}{n_i}\right)^2\left[\left(\sum_{t=1}^{n_i}\psi_t\right)\left(\sum_{t=1}^{n_i}\psi_t\right)^T + V_3\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ai}}\psi_t|s_1,s_2\right)\right]\right]\right\} \\
&- E_1\left\{\frac{\sum\sum_{i\neq j}^{m_{Ok}}\psi_i\psi_j^T}{m_{Ok}(m_{Ok}-1)}\right. \\
&\left.+\frac{E_2\left[\sum\sum_{i\neq j}^{m_{Ok}}\frac{N_i}{n_i}\frac{N_j}{n_j}Cov_3\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ai}}\psi_t,\frac{N_{Om}}{n_{Am}}\sum_{u=1}^{n_{Aj}}\psi_u|s_1,s_2\right)\right]}{m_{Ok}(m_{Ok}-1)}\right\} \\
&= \frac{\sum_{i=1}^{M_k}\left(\psi_i-\overline{\psi}_k\right)\left(\psi_i-\overline{\psi}_k\right)^T}{M_k-1} \\
&+ E_1\left[\frac{1}{m_{Ok}}\sum_{i=1}^{m_{Ok}} N_i\left(\frac{N_i}{n_i}-1\right)\frac{\sum_{t=1}^{N_i}\left(\psi_{it}-\overline{\psi}_i\right)\left(\psi_{it}-\overline{\psi}_i\right)^T}{N_i-1}\right] \\
&+ E_1\left\{\frac{1}{m_{Ok}}\sum_{i=1}^{m_{Ok}} E_2\left[\left(\frac{N_i}{n_i}\right)^2 V_3\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ai}}\psi_t|s_1,s_2\right)\right]\right\} \\
&- E_1\left\{\frac{E_2\left[\sum\sum_{i\neq j}^{m_{Ok}}\frac{N_i}{n_i}\frac{N_j}{n_j}Cov_3\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ai}}\psi_t,\frac{N_{Om}}{n_{Am}}\sum_{u=1}^{n_{Au}}\psi_u|s_1,s_2\right)\right]}{m_{Ok}(m_{Ok}-1)}\right\},
\end{aligned}
$$

$$E_2\left[\widehat{V}_2\left(\psi_i|s_1\right)\right] = N_i\left(\frac{N_i}{n_i}-1\right)E_2\left[\frac{E_3\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ai}}\psi_{it}\psi_{it}^T\right)}{n_i} - \frac{E_3\left(\frac{N_{Om}(N_{Om}-1)}{n_{Am}(n_{Am}-1)}\sum\sum_{t\neq u}^{n_{Ai}}\psi_{it}\psi_{iu}^T\right)}{n_i(n_i-1)}\right]$$

$$= N_i\left(\frac{N_i}{n_i}-1\right)E_2\left[\frac{\sum_{t=1}^{n_i}\psi_{it}\psi_{it}^T}{n_i} - \frac{\sum\sum_{t\neq u}^{n_i}\psi_{it}\psi_{iu}^T}{n_i(n_i-1)}\right] = V_2\left(\psi_i|s_1\right)$$

and the unbiasedness of the conditional variance and covariance estimators follows from the results on domain estimation (see Section A.6). Substituting the expectations for the components in the variance estimator will show that the variance estimator is unbiased.

3.A.3 Derivation of ${}_4\widehat{\boldsymbol{\Psi}}_k$

The otoliths are obtained by a SRS of all of the counted fish in the haul and a simple random subsample of all otoliths in the region (m) are aged. Thus, conditional on the sampling of fish in hauls to obtain otoliths (s_1), the otoliths from the kth haul is a domain of all of the otoliths collected in the region and results of Section A.6 apply directly. Therefore,

$$E\left({}_4\widehat{\boldsymbol{\Psi}}_k\right) = E_1\left[E_2\left(\frac{N_{Om}}{n_{Am}}\frac{N_k}{n_{Ok}}\sum_{t=1}^{n_{Ak}}\psi_t|s_1\right)\right]$$

$$= E_1\left(\frac{N_k}{n_{Ok}}\sum_{t=1}^{n_{Ok}}\psi_t\right) = \sum_{t=1}^{N_k}\psi_t = \boldsymbol{\Psi}_k$$

The variance of the estimator is

$$
\begin{aligned}
V\left({}_4\widehat{\boldsymbol{\Psi}}_k\right) =& E_1\left[V_2\left({}_4\widehat{\boldsymbol{\Psi}}_k|s_1\right)\right] + V_1\left[E_2\left({}_4\widehat{\boldsymbol{\Psi}}_k|s_1\right)\right] \\
=& E_1\left[\left(\frac{N_k}{n_{Ok}}\right)^2 V_2\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ak}}\boldsymbol{\psi}_t|s_1\right)\right] + V_1\left[\frac{N_k}{n_{Ok}}E_2\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ak}}\boldsymbol{\psi}_t|s_1\right)\right] \\
=& E_1\left\{\left(\frac{N_k}{n_{Ok}}\right)^2 N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{N_{Om}\left[\operatorname{diag}(\mathbf{p}_{Ok})-\mathbf{p}_{Ok}\mathbf{p}_{Ok}^T\right]}{N_{Om}-1}\right\} \\
&+ N_k\left(\frac{N_k}{n_{Ok}}-1\right)\frac{N_k\left[\operatorname{diag}(\mathbf{p}_k)-\mathbf{p}_k\mathbf{p}_k^T\right]}{N_k-1}
\end{aligned}
\tag{3.47}
$$

Now, the expectation of Eq. 3.28 is

$$
\begin{aligned}
&E\left[\widehat{V}\left({}_4\widehat{\boldsymbol{\Psi}}_k\right)\right] = \\
&N_k\left(\frac{N_k}{n_{Ok}}-1\right)E_1\left\{\frac{E_2\left(\frac{N_{Om}}{n_{Am}}\sum_{t=1}^{n_{Ak}}\boldsymbol{\psi}_t\boldsymbol{\psi}_t^T|s_1\right)}{n_{Ok}} - \frac{E_2\left(\frac{N_{Om}(N_{Om}-1)}{n_{Am}(n_{Am}-1)}\sum\sum_{t\neq u}^{n_{Ak}}\boldsymbol{\psi}_t\boldsymbol{\psi}_u^T|s_1\right)}{n_{Ok}(n_{Ok}-1)}\right\} \\
&+ E_1\left\{\left(\frac{N_k}{n_{Ok}}\right)^2 N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)E_2\left[\frac{n_{Am}\left[\operatorname{diag}(\mathbf{p}_{\psi k})-\mathbf{p}_{\psi k}\mathbf{p}_{\psi k}^T\right]}{n_{Am}-1}|s_1\right]\right\} \\
&= N_k\left(\frac{N_k}{n_{Ok}}-1\right)E_1\left[\frac{\sum_{t=1}^{n_{Ok}}\boldsymbol{\psi}_t\boldsymbol{\psi}_t^T}{n_{Ok}} - \frac{\sum\sum_{t\neq u}^{n_{Ok}}\boldsymbol{\psi}_t\boldsymbol{\psi}_u^T}{n_{Ok}(n_{Ok}-1)}\right] \\
&+ E_1\left[\left(\frac{N_k}{n_{Ok}}\right)^2 N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{N_{Om}\left[\operatorname{diag}(\mathbf{p}_{Ok})-\mathbf{p}_{Ok}\mathbf{p}_{Ok}^T\right]}{N_{Om}-1}\right]
\end{aligned}
$$

and result Eq. 3.47 is obtained if we realize that

$$
\frac{\sum_{t=1}^{n_{Ok}}\boldsymbol{\psi}_t\boldsymbol{\psi}_t^T}{n_{Ok}} - \frac{\sum\sum_{t\neq u}^{n_{Ok}}\boldsymbol{\psi}_t\boldsymbol{\psi}_u^T}{n_{Ok}(n_{Ok}-1)} = \frac{\sum_{t=1}^{n_{Ok}}\left(\boldsymbol{\psi}_t-\overline{\boldsymbol{\psi}}_k\right)\left(\boldsymbol{\psi}_t-\overline{\boldsymbol{\psi}}_k\right)^T}{n_{Ok}-1}
$$

and the expectation of that term follows from simple random sampling result Eq. A.10 in Section A.2 and the fact that

$$\frac{\sum_{t=1}^{N_k}\left(\boldsymbol{\psi}_t-\overline{\boldsymbol{\psi}}_k\right)\left(\boldsymbol{\psi}_t-\overline{\boldsymbol{\psi}}_k\right)^T}{N_k-1}=\frac{N_k\left[\operatorname{diag}\left(\mathbf{p}_k\right)-\mathbf{p}_k\mathbf{p}_k^T\right]}{N_k-1}.$$

3.A.4 Derivation of model-based $\widehat{\boldsymbol{\Theta}}_k$

This presentation will be in vector/matrix form. Scalar catch parameters are special cases. The population is defined as all clusters of captured animals in all hauls made during the tth trip where the number of clusters in each haul is M_k and the number for the entire trip is $\sum_{k=1}^{G_t} M_k = M_t$. Also let the number of clusters sampled in the kth haul be m_k, the number of unsampled hauls be $m_{k'} = M_k - m_k$ and the total number of sampled clusters be $\sum_{i=1}^{g_t} m_k = m_t$. Under the first model, the best linear unbiased predictor of the total for the kth haul is

$$\widehat{\boldsymbol{\Theta}}_k=\sum_{i=1}^{m_k}\boldsymbol{\theta}_{ki}+\widehat{\boldsymbol{\phi}}_{\theta,t}\sum_{i=1}^{m_{k'}}x_{k'i}$$

where $\widehat{\boldsymbol{\phi}}_{\theta,t}$ is defined in Eq. 3.31 (see Valliant et al. 2000, Section 2.2). Let, $x_{k'} = \sum_{i=1}^{m_{k'}} x_{k'i}$ and $x_k = \sum_{i=1}^{m_k} x_{ki}$, then the prediction error variance is

$$\begin{aligned}V_M\left[x_{k'}\widehat{\boldsymbol{\phi}}_{\theta,t}-\sum_{i=1}^{m_k'}\boldsymbol{\theta}_{k'i}\right] &= x_{k'}^2\frac{\sum_{i=1}^{m_t}x_i^2V_M\left(\boldsymbol{\theta}_i\right)}{\left(\sum_{i=1}^{m_t}x_i^2\right)^2}+\sum_{i=1}^{m_k'}V_M\left(\boldsymbol{\theta}_{k'i}\right)\\ &=\left[\frac{x_{k'}^2}{\sum_{i=1}^{m_t}x_i^2}+m_{k'}\right]\boldsymbol{\Sigma}_{\theta,t}=\left[\frac{x_{k'}^2}{\sum_{i=1}^{m_t}x_i^2}+(M_k-m_k)\right]\boldsymbol{\Sigma}_{\theta,t}\end{aligned}$$

When only one cluster is sampled in each haul, then the parameter estimator and prediction error variance can be written as

$$\widehat{\boldsymbol{\Theta}}_k=\boldsymbol{\theta}_k+\widehat{\boldsymbol{\phi}}_{\theta,t}\sum_{i=1}^{m_{k'}}x_{k'i}=\boldsymbol{\theta}_k+\widehat{\boldsymbol{\phi}}_{\theta,t}x_{k'}$$

and

$$V_M\left[x_{k'}\widehat{\boldsymbol{\phi}}_{\theta,t}-\sum_{i=1}^{m_k'}\boldsymbol{\theta}_{k'i}\right]=\left[\frac{x_{k'}^2}{\sum_{k=1}^{g_t}x_k^2}+(M_k-1)\right]\boldsymbol{\Sigma}_{\theta,t}.$$

The familiar unbiased estimator of the variance parameter for a scalar catch paramter is

$$\widehat{\sigma}^2_{\theta,t} = \frac{\sum_{i=1}^{m_t}(\theta_i - \widehat{\mu}_k)^2}{m_t - 1} = \frac{\sum_{i=1}^{m_t}\left(\theta_i - x_k\widehat{\phi}_{\theta,t}\right)^2}{m_t - 1}$$

and the analogous estimator for a covariance matrix corresponding to a vector-valued catch parameter is

$$\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta},t} = \frac{\sum_{k=1}^{m_t}\left(\boldsymbol{\theta}_i - x_k\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}\right)\left(\boldsymbol{\theta}_i - x_k\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}\right)^T}{m_t - 1}$$

When the second model is assumed where variance is proportional to the size of the covariate (See Table 5.1 for definition), the variance is

$$\begin{aligned} V_M\left[x_{k'}\widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t} - \sum_{i=1}^{m'_k}\boldsymbol{\theta}_{k'i}\right] =& x^2_{k'}\frac{\sum_{i=1}^{m_t}V_M(\boldsymbol{\theta}_i)}{\left(\sum_{i=1}^{m_t}x_i\right)^2} + \sum_{i=1}^{m'_k}V_M(\boldsymbol{\theta}_{k'i}) \\ =& \left[\frac{x^2_{k'}}{\sum_{i=1}^{m_t}x_i} + x_{k'}\right]\boldsymbol{\Sigma}_{\boldsymbol{\theta},t} = x_{k'}\left(\frac{x_{k'}}{x_t} + 1\right)\boldsymbol{\Sigma}_{\boldsymbol{\theta},t}. \end{aligned}$$

To show that the estimator (Eq. 3.36) is unbiased first notice that

$$\begin{aligned} E\left[\left(\boldsymbol{\theta}_i - \widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}x_i\right)\left(\boldsymbol{\theta}_i - \widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}x_i\right)^T\right] =& E\left[\boldsymbol{\theta}_i\boldsymbol{\theta}_i^T - \frac{x_i}{x_t^2}\left(\boldsymbol{\theta}_i\boldsymbol{\theta}_t^T + \boldsymbol{\theta}_t\boldsymbol{\theta}_i^T\right) + x_i^2\frac{\boldsymbol{\theta}_t\boldsymbol{\theta}_t^T}{x_t^2}\right] \\ =& \left(x_i - \frac{x_i^2}{x_t}\right)\boldsymbol{\Sigma}_{\boldsymbol{\theta},t}. \end{aligned}$$

Therefore,

$$\begin{aligned} E\left[\sum_{i=1}^{m_t}\left(\boldsymbol{\theta}_i - \widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}x_i\right)\left(\boldsymbol{\theta}_i - \widehat{\boldsymbol{\phi}}_{\boldsymbol{\theta},t}x_i\right)^T\right] =& \left(\sum_{i=1}^{m_t}x_i - \frac{\sum_{i=1}^{m_t}x_i^2}{x_t}\right)\boldsymbol{\Sigma}_{\boldsymbol{\theta},t} \\ =& \frac{x_t^2 - \sum_{i=1}^{m_t}x_i^2}{x_t}\boldsymbol{\Sigma}_{\boldsymbol{\theta},t} \end{aligned}$$

and simple algebra yields the unbiased estimator (Eq. 3.36) when it is understood that there is only one sample per haul.

3.A.5 Performance of Model-based Estimators

As with the models used for estimation when longline gear is used, the models I have proposed for use in estimating catch parameters for trawl gear are as limited in assumptions as possible. Only the mean and variance of the super-population generating processes are specified rather than complete probability distributions. Furthermore, the mean and variance parameters are defined at the finest scales possible for variance estimation. However, the models do not truly represent the sampling process and some inconsistencies should be expected. See Section 2.A.5 for more details on the circumstances under which bias of catch estimators and corresponding variance estimators is neglible.

Chapter 4

ESTIMATION WITHIN HAUL FOR POT VESSELS

This chapter is organized similarly to the preceding chapter regarding trawler vessels. From a sampling perspective, pot hauls are a sort of intermediate of longline and trawl hauls. Like longline hauls there is a discreteness of the components of the haul (pots), but like trawls, the total number of animals that can be caught per discrete component (pot) is not binary. First I develop estimators for total number, total weight, numbers in length classes and age classes and numbers of marine mammal interactions using a (approximate) design-based approach. In Section 4.6, I present two general model-based ratio-type estimators and then give results for the specific catch paramters for which design-based results are presented in preceding sections. Finally, in a short chapter appendix, Section 4.A, I give some derivation of properties of the more complicated estimators provided in earlier sections.

There are several nuances unique to data collection on pot vessels. Depending on the number of pots per string, observers may or may not sample every string of pots deployed by a vessel. When vessels fish several short string, observers are instructed to randomly sample the strings and and sample at least 30% of each string for data on specific species. When vessels fish few long strings, observers are instructed to sample at least 33% of every string. I find these instructions confusing because the observers are ultimately instructed to sample more of the total fishing effort when few long string are fished than when many short strings are fished.

When observers cannot count and weigh all animals in a string of pots they are instructed to group pots into sampling units (pot-groups) so that effort required to collect information from each pot-group is manageable. Observers may collect what amounts to either a SRS or systematic sample of the pot-groups. It is well-noted that design-based inference from systematic samples is not possible when only one systematic sample is collected. However, when systematic sampling is performed people often assume there is no pattern in the characteristic of interest for the pot-groups that coincides with the period used in the systematic sampling process and estimators resulting under SRS are used as approximations (e.g., Cochran 1977, pg. 212).

To help anyone reading this document, I provide a list of defintions for many if not all of the notation I use in this chapter (Table 4.1). Many of the terms are also defined as they are needed.

Table 4.1. Definition of terms

Γ_k	number of marine mammal interactions for the kth haul
Γ_{ki}	number of marine mammal interactions in the ith pot-group of the kth haul
κ_k	number of pots in the sampled portion of the kth haul
$\kappa_{k'}$	number of pots in the unsampled portion of the kth haul
κ_t	number of sampled pots in the tth trip
$\mathbf{\Lambda}_k$	vector of total numbers in each length class in the kth haul
$\boldsymbol{\lambda}_k$	vector of numbers in each length class among fish in the length sample for the kth haul
$\boldsymbol{\lambda}_{ki}$	vector of numbers in each length class among fish in the ith pot-group for the kth haul
$\overline{\boldsymbol{\lambda}}_k$	average vector of numbers in each length class per pot-group for the kth haul
m_k	number of pot-groups sampled in the kth haul
m_{2k}	number of pot-groups used in the length sample for the kth haul
M_k	total number of pot-groups in the kth haul
N_k	number of fish for a given species in the kth haul
N_{2k}	number of fish in the weight sample of the kth haul
N_{ki}	number of fish in the ith pot-group from the kth haul
n_k	number of fish in the length sample of the kth haul
n_{ki}	number of fish in the length sample from the ith pot-group of the kth haul
n_{Ok}	number of otoliths sampled for the kth haul
n_{Ak}	number of otoliths sampled in the kth haul that are ultimately aged
n_{Ai}	number of otoliths sampled in the ith pot-group that are ultimately aged
n_{Om}	number of all otoliths sampled in the mth management period/region

Table 4.1. (Continued)

n_{Am}	number of all otoliths sampled and aged in the mth management period/region
$\mathbf{p}_{\boldsymbol{\lambda},k}$	probability vector for a model describing the length class indication of fish (of a given species) in the kth haul
$\mathbf{p}_{\boldsymbol{\psi},k}$	probability vector for a model describing the age class indication of fish (of a given species) in the kth haul
$\mathbf{p}_{Lki}$	vector of proportions of length sample fish in each age class and the ith pot-group of the kth haul
$\mathbf{p}_{2ki}$	vector of proportions of length sample fish in each age class and the ith pot-group of the kth haul
$\mathbf{p}_{Oki}$	vector of proportions of sampled otoliths in each age class and the ith pot-group of the kth haul among all otoliths collected in the mth region
$\mathbf{P}_{ki}$	vector of proportions of fish in each age class for the ith pot-group of the kth haul
P_{Ok}	the proportion of all otoliths sampled in the mth management period/region that were sampled from the kth haul (n_{Ok}/N_{Om})
$\phi_{\theta,t}$	a mean parameter that describes the rate of increase in the catch parameter (Θ) for the tth trip with respect to a given measured covariate (usually pots, κ)
$\boldsymbol{\Psi}_k$	vector of total numbers in each age class in the kth haul
$\boldsymbol{\psi}_k$	vector of numbers of aged fish in each age class for the kth haul
$\boldsymbol{\psi}_i$	vector of numbers of aged fish in each age class for the ith pot-group
$\overline{\boldsymbol{\psi}}_k$	average vector of numbers of fish in each age class per pot-group in the kth haul
$\Sigma^2_{\boldsymbol{\theta},t}$	a variance-covariance matrix parameter for a super-population model that describes the values the parameter vector $\boldsymbol{\theta}$ takes on in the tth trip
$\boldsymbol{\theta}_k$	a generic vector-valued catch parameter for the sampled portion of the kth haul
W_k	the weight of fish in the kth haul for a given species
W_{ki}	weight of fish in the ith pot-group from the kth haul
$\overline{W}_k$	average weight of fish per pot-group (for a given species) in the kth haul

4.1 Total Number in Catch for a Fish or Seabird Species

Observers count the total number of each species in each pot-group and when the SRS assumption is made, the estimator for the total of that catch parameter over the entire string and corresponding variance estimator are the same whether the observer performs systematic sampling or SRS.

Let the total number of pot-groups be M_k and the number sampled be m_k. If all pot-groups are sampled, then there is no sampling variability and the total number in the catch for the species of interest in the haul is known. When a subset of pot-groups are sampled, the estimator for the total number caught in the kth haul is

$$\widehat{N}_k = \frac{M_k}{m_k} \sum_{i=1}^{m_k} N_{ki} \tag{4.1}$$

where N_{ki} is the number caught in the ith pot-group. The corresponding variance estimator is

$$\widehat{V}\left(\widehat{N}_k\right) = M_k \left(\frac{M_k}{m_k} - 1\right) \frac{\sum_{i=1}^{m_k} \left(N_{ki} - \widehat{\overline{N}}_k\right)^2}{m_k - 1} \tag{4.2}$$

where $\widehat{\overline{N}}_k = \widehat{N}_k / M_k$. The estimator of the total under SRS that is used here is common in many contexts throughout this dissertation. See Section A.2 for details.

4.2 Total Weight of a Fish or Seabird Species

Although observers count all animals of each species in each sampled pot-group, they may or may not weigh all animals in each sampled pot-group. When catches are large, observers are instructed to weigh all animals for species that are not prevalent in the catch, but must weigh a minimum of 50 individuals of prevalent species. Observers are encouraged to collect the weight sample for prevalent species in a random manner, but exact methodology is left up to the observer. The most practical method is weighing all animals in a subsample of pot-groups for which animals are counted and the subsample may be obtained by SRS.

Suppose that the simple random subsample contains m_{k2} pot-groups and we know that $m_{k2} \leq m_k$ where m_k is the number of pot-groups originally chosen and for which all animals are counted. An unbiased estimator of the total weight under the SRS assumption is

$$_1\widehat{W}_k = \frac{M_k}{m_{k2}} \sum_{i=1}^{m_{k2}} W_{ki} \tag{4.3}$$

where M_k is the total number of pot-groups in the string of pots. The corresponding variance estimator is

$$\widehat{V}\left(_1\widehat{W}_k\right) = M_k\left(\frac{M_k}{m_{k2}} - 1\right)\frac{\sum_{i=1}^{m_{k2}}\left(W_{ki} - \widehat{\overline{W}}_k\right)^2}{m_{k2} - 1} \tag{4.4}$$

where $\widehat{\overline{W}}_k = {}_1\widehat{W}_k / M_k$.

Although Eq. 4.3 is unbiased it does not use any of the information about total numbers of animals counted in the other pot-groups. An alternative asymptotically unbiased estimator is

$$_2\widehat{W}_k = \widehat{N}_k \frac{\sum_{i=1}^{m_{k2}} W_{ki}}{\sum_{i=1}^{m_{k2}} N_{ki}} \tag{4.5}$$

with approximate unbiased variance estimator,

$$\widehat{V}\left(_2\widehat{W}_k\right) = {}_2\widehat{W}_k^2\left[\frac{\widehat{V}\left(_1\widehat{W}_k\right)}{_2\widehat{W}_k^2} + M_k^2\left(\frac{1}{m_{k2}} - \frac{1}{m_k}\right)\left(\frac{\widehat{S}_{N,k}^2}{\widehat{N}_k^2} - 2\frac{\widehat{S}_{NW,k}}{\widehat{N}_{k2}\widehat{W}_k}\right)\right] \tag{4.6}$$

where

$$\widehat{S}_{N,k}^2 = \frac{\sum_{i=1}^{m_k}\left(N_{ki} - \widehat{\overline{N}}_k\right)^2}{m_k - 1}$$

and

$$\widehat{S}_{NW,k} = \frac{\sum_{i=1}^{m_{k2}}\left(n_{ki} - \widehat{\overline{N}}_k\right)\left(W_{ki} - \widehat{\overline{W}}_k\right)}{m_{k2} - 1}.$$

Note that this variance estimator is analogous to the approximate variance given by Cochran (1977, pg. 344) for an estimator of the mean (weight) rather than total for ratio estimation in double (two-phase) sampling. See Section 4.A.1 for derivation of the unbiasedness and variance.

4.3 Total Numbers in Length Classes

When lengths are measured for a species, the fish comprising the length sample are presumably a subset of the fish that comprise the weight sample. The likely sampling scenario to obtain the weight sample is discussed in Section 4.2 and two likely sampling scenarios for obtaining the length sample are a simple random subsample of all of the fish comprising the weight sample and separate simple random subsamples of all fish in each pot-group that enters the weight sample.

Let N_{ki} be the number of fish in the ith pot-group and m_{2k} be the number of pot-groups used for the weight sample. When there are separate simple random subsamples of size n_{ki} within the ith pot-group, then there is a two-stage SRS design within the haul (i.e., string of pots) and the estimator of the total numbers-at-length is

$$ {}_1\widehat{\mathbf{\Lambda}}_k = \frac{M_k}{m_{2k}} \sum_{i=1}^{m_{2k}} \frac{N_{ki}}{n_{ki}} \boldsymbol{\lambda}_{ki}. \tag{4.7} $$

The variance estimator is

$$ \begin{aligned} \widehat{V}\left({}_1\widehat{\mathbf{\Lambda}}_k\right) =& M_k \left(\frac{M_k}{m_{2k}} - 1\right) \frac{\sum_{i=1}^{m_{2k}} \left(\widehat{\mathbf{\Lambda}}_{ki} - \widehat{\overline{\mathbf{\Lambda}}}_k\right)\left(\widehat{\mathbf{\Lambda}}_{ki} - \widehat{\overline{\mathbf{\Lambda}}}_k\right)^T}{m_{2k} - 1} \\ &+ \frac{M_k}{m_{2k}} \sum_{i=1}^{m_{2k}} N_{ki} \left(\frac{N_{ki}}{n_{ki}} - 1\right) \frac{n_{ki}\left[\operatorname{diag}\left(\mathbf{p}_{\boldsymbol{\lambda},i}\right) - \mathbf{p}_{\boldsymbol{\lambda},i}\mathbf{p}_{\boldsymbol{\lambda},i}^T\right]}{n_{ki} - 1} \end{aligned} \tag{4.8} $$

where $\mathbf{p}_{\boldsymbol{\lambda},i} = \boldsymbol{\lambda}_{ki}/n_{ki}$ is the numbers-at-length in the length sample from the ith pot-group and

$$ \widehat{\overline{\mathbf{\Lambda}}}_k = \frac{1}{m_{2k}} \sum_{i=1}^{m_{2k}} \frac{N_{ki}}{n_{ki}} \boldsymbol{\lambda}_{ki}. $$

Note that Eq. 4.7 is identical in form to the third numbers-at-length estimator for longline hauls (Eq. 2.26).

An estimator of the numbers-at-length that uses the counts of fish from all sampled pot-groups is

$$ {}_2\widehat{\mathbf{\Lambda}}_k = \widehat{N}_k \frac{\sum_{i=1}^{m_{2k}} N_{ki}\mathbf{p}_{\boldsymbol{\lambda},i}}{\sum_{i=1}^{m_{2k}} N_{ki}} = \widehat{N}_k \frac{\sum_{i=1}^{m_{2k}} \widehat{\mathbf{\Lambda}}_{ki}}{\sum_{i=1}^{m_{2k}} N_{ki}} \tag{4.9} $$

and a corresponding approximately unbiased variance estimator is

$$\widehat{V}\left({}_2\widehat{\mathbf{\Lambda}}_k\right) = V\left({}_1\widehat{\mathbf{\Lambda}}_k\right) + M_k^2\left(\frac{1}{m_{2k}} - \frac{1}{m_k}\right)\left[\frac{{}_2\widehat{\mathbf{\Lambda}}_{k2}\widehat{\mathbf{\Lambda}}_k^T}{\widehat{N}_k^2}\widehat{S}_{N,k}^2 - \frac{{}_2\widehat{\mathbf{\Lambda}}_k\widehat{\mathbf{S}}_{N\mathbf{\Lambda},k}^T + \widehat{\mathbf{S}}_{N\mathbf{\Lambda},k2}\widehat{\mathbf{\Lambda}}_k^T}{\widehat{N}_k}\right] \tag{4.10}$$

where

$$\widehat{S}_{N,k}^2 = \frac{\sum_{i=1}^{m_k}\left(N_{ki} - \widehat{\overline{N}}_k\right)^2}{m_k - 1}$$

and

$$\widehat{\mathbf{S}}_{N\mathbf{\Lambda},k} = \frac{\sum_{i=1}^{m_{2k}}\left(N_{ki} - \widehat{\overline{N}}_k\right)\left(\widehat{\mathbf{\Lambda}}_{ki} - \widehat{\overline{\mathbf{\Lambda}}}_k\right)}{m_{2k} - 1}.$$

The form of the estimator (Eq. 4.9) and its approximate variance are similar to that of the second total weight estimator (Eq. 4.5) except that here lengths are known only for a subsample of the fish within each of the pot-groups. See Section 4.A.2 for derivation of the approximate variance.

When the length sample is obtained by a simple random subsample of all fish in the weight sample, the unbiased estimator of the total numbers-at-length is

$${}_3\widehat{\mathbf{\Lambda}}_k = \frac{M_k}{m_{2k}}\frac{N_{2k}}{n_k}\boldsymbol{\lambda}_k \tag{4.11}$$

where N_{2k} and n_k are the numbers of fish in the weight sample and length sample, respectively, and $\boldsymbol{\lambda}_k$ is the numbers-at-length in the length sample. The corresponding unbiased variance estimator is

$$\begin{aligned}\widehat{V}\left({}_3\widehat{\mathbf{\Lambda}}_k\right) =& M_k\left(\frac{M_k}{m_{2k}} - 1\right)\frac{\sum_{i=1}^{m_{2k}}\left(\widehat{\mathbf{\Lambda}}_{ki} - \widehat{\overline{\mathbf{\Lambda}}}_k\right)\left(\widehat{\mathbf{\Lambda}}_{ki} - \widehat{\overline{\mathbf{\Lambda}}}_k\right)^T}{m_{2k} - 1} \\ &+ \frac{M_k}{m_{2k}}\left[\sum_{i=1}^{m_{2k}}\widehat{V}\left(\widehat{\mathbf{\Lambda}}_{ki}\right) + \frac{M_k - 1}{m_{2k} - 1}\sum_{i\neq j}^{m_{2k}}\sum\widehat{Cov}\left(\widehat{\mathbf{\Lambda}}_{ki}, \widehat{\mathbf{\Lambda}}_{kj}\right)\right]\end{aligned} \tag{4.12}$$

where

$$\widehat{V}\left(\widehat{\mathbf{\Lambda}}_{ki}\right) = N_{2k}\left(\frac{N_{2k}}{n_k} - 1\right)\frac{n_k\left[\text{diag}\left(\mathbf{p}_{ki}\right) - \mathbf{p}_{ki}\mathbf{p}_{ki}^T\right]}{n_k - 1},$$

$$\widehat{Cov}\left(\widehat{\mathbf{\Lambda}}_{ki}, \widehat{\mathbf{\Lambda}}_{kj}\right) = -N_{2k}\left(\frac{N_{2k}}{n_k} - 1\right)\frac{n_k \mathbf{p}_{ki}\mathbf{p}_{kj}^T}{n_k - 1}$$

and $\mathbf{p}_{ki} = \boldsymbol{\lambda}_{ki}/n_k$. The estimator (Eq. 4.11) is identical in form to the first numbers-at-length estimator for longline hauls, Eq. 2.22. The longline skates parallel the pot-groups here and the length sample for the longline haul was taken from all fish in the skates that were randomly sampled for total numbers whereas the length sample here is taken from all fish in the pot-groups selected for the weight sample.

An estimator that makes use of the counts of fish in all sampled pot-groups is

$$_4\widehat{\mathbf{\Lambda}}_k = \widehat{N}_k \frac{\boldsymbol{\lambda}_k}{n_k} \tag{4.13}$$

and a corresponding approximately unbiased variance estimator is

$$\widehat{V}\left(_4\widehat{\mathbf{\Lambda}}_k\right) = V\left(_3\widehat{\mathbf{\Lambda}}_k\right) + M_k^2\left(\frac{1}{m_{2k}} - \frac{1}{m_k}\right)\left[\frac{_4\widehat{\mathbf{\Lambda}}_{k4}\widehat{\mathbf{\Lambda}}_k^T}{\widehat{N}_k^2}\widehat{S}_{N,k}^2 - \frac{_4\widehat{\mathbf{\Lambda}}_k\widehat{\mathbf{S}}_{N\mathbf{\Lambda},k}^T + \widehat{\mathbf{S}}_{N\mathbf{\Lambda},k4}\widehat{\mathbf{\Lambda}}_k^T}{\widehat{N}_k}\right] \tag{4.14}$$

where

$$\widehat{S}_{N,k}^2 = \frac{\sum_{i=1}^{m_k}\left(N_{ki} - \widehat{\overline{N}}_k\right)^2}{m_k - 1}, \tag{4.15}$$

and

$$\widehat{\mathbf{S}}_{N\mathbf{\Lambda},k} = \frac{\sum_{i=1}^{m_{2k}}\left(N_{ki} - \widehat{\overline{N}}_k\right)\left(\widehat{\mathbf{\Lambda}}_{ki} - \widehat{\overline{\mathbf{\Lambda}}}_k\right)}{m_{2k} - 1}. \tag{4.16}$$

Also, $\widehat{\overline{N}}_k = \sum_{i=1}^{m_k} N_{ki}/m_k$ for Eq. 4.15 and $\widehat{\overline{N}}_k = \sum_{i=1}^{m_{2k}} N_{ki}/m_{2k}$ for Eq. 4.16,

$$\widehat{\mathbf{\Lambda}}_{ki} = \frac{N_{2k}}{n_k}\boldsymbol{\lambda}_{ki}$$

and

$$\widehat{\overline{\mathbf{\Lambda}}}_k = \frac{1}{m_{2k}}\sum_{i=1}^{m_{2k}}\widehat{\mathbf{\Lambda}}_{ki}.$$

4.4 *Total Numbers in Age Classes*

Generally, the fish selected for otolith removals are taken from the length sample and two general sampling methods were described in Section 4.3 for collecting the length sample. A relatively large number (n_k) fish make up the length sample and subampling within each of the m_{2k} pot-groups for the length sample is feasible. But because generally very few fish from a given haul are selected for otolith removal regardless of species, it is doubtful that there will be simple random subsampling within each pot-group. Thus, the only likely sampling methods are selection of all fish for the otolith sample from all fish in the length sample. However, as discussed in sections on within-haul estimation of numbers-at-age for the other gear types, in general, not all collected otoliths are aged. Rather, a sample of all otoliths collected in the mth management region/period is aged.

Assuming there is a simple random sample within each pot-group for the length sample, a simple random subsample of all fish in the length sample for the otolith sample and a simple random subsample of all otoliths collected in the mth management region/period is aged, the unbiased estimator for the total numbers-at-age is

$$ {}_1\widehat{\boldsymbol{\Psi}}_k = \frac{N_{Om}}{n_{Am}} \frac{M_k}{m_{2k}} \frac{n_k}{n_{Ok}} \sum_{i=1}^{m_{2k}} \frac{N_{ki}}{n_{ki}} \boldsymbol{\psi}_{ki} \tag{4.17} $$

where N_{Om} is the number of otoliths obtained by observers in the mth management region/period, n_{Am} is the number of aged otoliths in the mth region/period, n_k is the number of fish in the length sample, N_{ki} is the number of fish in the ith pot-group, n_{ki} is the number of fish in the length sample from the ith pot-group, n_{Ok} is the number of otoliths obtained from the kth haul and $\boldsymbol{\psi}_{ki}$ is the numbers-at-age for the aged otoliths from the ith pot-group of the kth haul. The variance estimator is

$$ \begin{aligned} \widehat{V}\left({}_1\widehat{\boldsymbol{\Psi}}_k\right) =& M_k \left(\frac{M_k}{m_{2k}} - 1\right) \frac{\sum_{i=1}^{m_{2k}} \left(\widehat{\boldsymbol{\Psi}}_{ki} - \widehat{\overline{\boldsymbol{\Psi}}}_k\right)\left(\widehat{\boldsymbol{\Psi}}_{ki} - \widehat{\overline{\boldsymbol{\Psi}}}_k\right)^T}{m_{2k}-1} \\ &+ \frac{M_k}{m_{2k}} \left\{ \sum_{i=1}^{m_{2k}} \widehat{\mathbf{V}}_{1i} + \sum_{i=1}^{m_{2k}} \frac{N_{ki}(N_{ki}-1)}{n_{ki}(n_{ki}-1)} \left[\widehat{\mathbf{V}}_{2i} + \frac{n_k(n_k-1)}{n_{Ok}(n_{Ok}-1)} \widehat{\mathbf{V}}_{3i} \right] \right\} \\ &+ \frac{M_k(M_k-1)}{m_{2k}(m_{2k}-1)} \left\{ \sum\sum_{i \neq j}^{m_{2k}} \frac{N_{ki}N_{kj}}{n_{ki}n_{kj}} \left[\widehat{\mathbf{C}}_{2ij} + \frac{n_k(n_k-1)}{n_{Ok}(n_{Ok}-1)} \widehat{\mathbf{C}}_{3ij} \right] \right\} \end{aligned} \tag{4.18} $$

where

$$ \widehat{\boldsymbol{\Psi}}_{ki} = \frac{N_{ki}}{n_{ki}} \frac{n_k}{n_{Ok}} \frac{N_{Om}}{n_{Am}} \boldsymbol{\psi}_{ki}, $$

$\widehat{\overline{\Psi}}_k = \sum_{i=1}^{m_{2k}} \widehat{\Psi}_{ki}/m_{2k}$,

$$\widehat{\mathbf{V}}_{1i} = N_{ki}\left(\frac{N_{ki}}{n_{ki}} - 1\right)\frac{n_{ki}\left[\mathrm{diag}\left(\widehat{\mathbf{P}}_{ki}\right) - \widehat{\mathbf{P}}_{ki}\widehat{\mathbf{P}}_{ki}^T\right]}{n_{ki} - 1},$$

$$\widehat{\mathbf{P}}_{ki} = \frac{1}{n_{ki}}\frac{n_k N_{Om}}{n_{Ok} n_{Am}}\psi_{ki}$$

$$\widehat{\mathbf{V}}_{2i} = n_k\left(\frac{n_k}{n_{Ok}} - 1\right)\frac{n_{Ok}\left[\mathrm{diag}\left(\widehat{\mathbf{p}}_{Lki}\right) - \widehat{\mathbf{p}}_{Lki}\widehat{\mathbf{p}}_{Lki}^T\right]}{n_{Ok} - 1},$$

$$\widehat{\mathbf{C}}_{2ij} = -n_k\left(\frac{n_k}{n_{Ok}} - 1\right)\frac{n_{Ok}\widehat{\mathbf{p}}_{Lki}\widehat{\mathbf{p}}_{Lkj}^T}{n_{Ok} - 1},$$

$$\widehat{\mathbf{p}}_{Lki} = \frac{1}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\psi_{ki}$$

$$\widehat{\mathbf{V}}_{3i} = N_{Om}\left(\frac{N_{Om}}{n_{Am}} - 1\right)\frac{n_{Am}\left[\mathrm{diag}\left(\widehat{\mathbf{p}}_{Oki}\right) - \widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Oki}^T\right]}{n_{Am} - 1},$$

$$\widehat{\mathbf{C}}_{3ij} = -N_{Om}\left(\frac{N_{Om}}{n_{Am}} - 1\right)\frac{n_{Am}\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Okj}^T}{n_{Am} - 1}$$

and

$$\widehat{\mathbf{p}}_{Oki} = \frac{1}{n_{Am}}\psi_{ki}.$$

The covariance terms exist because of the sampling across all fish in the length sample for the otolith sample ($\widehat{\mathbf{C}}_{2ij}$) and sampling across all otoliths collected in the mth region for the ageing sample ($\widehat{\mathbf{C}}_{3ij}$). For details on the derivation of the estimator (Eq. 4.17) and its variance, see Section 4.A.4.

Under an alternative scenario, the length sample is collected by SRS of all fish in all pot-groups comprising the weight sample. In this case, the unbiased estimator of the numbers-at-age in the haul is

$$_2\widehat{\Psi}_k = \frac{M_k}{m_{2k}}\frac{N_{2k}}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\psi_k \tag{4.19}$$

where N_{2k} is the total number of fish in the pot-groups comprising the weight sample (i.e. $N_{2k} = \sum_{i=1}^{m_{2k}} N_{ki}$) and ψ_k is the numbers-at-age for the aged otoliths from the kth haul. The form of Eq. 4.19 is identical to that for the third numbers-at-age estimator in longline hauls (Eq. 2.34). As such, the variance estimator also has the corresponding form,

$$
\begin{aligned}
\widehat{V}\left({}_{2}\widehat{\boldsymbol{\Psi}}_{k}\right) =& \widehat{\mathbf{V}}_1 + \frac{M_k}{m_{2k}}\left[\sum_{i=1}^{m_{2k}}\widehat{\mathbf{V}}_i + \frac{M_k-1}{m_{2k}-1}\sum\sum_{i\neq j}^{m_{2k}}\widehat{\mathbf{C}}_{ij}\right] \\
&+ \frac{M_k N_{2k}(N_{2k}-1)}{m_{2k}n_{Ok}(n_{Ok}-1)}\left[\sum_{i=1}^{m_{2k}}\widehat{\mathbf{V}}_{Oi} + \frac{M_k-1}{m_{2k}-1}\sum\sum_{i\neq j}^{m_{2k}}\widehat{\mathbf{C}}_{Oij}\right] \qquad (4.20)
\end{aligned}
$$

where

$$
\widehat{\mathbf{V}}_1 = M_k\left(\frac{M_k}{m_{2k}}-1\right)\frac{\sum_{i=1}^{m_{2k}}\left(\widehat{\boldsymbol{\Psi}}_{ki}-\widehat{\overline{\boldsymbol{\Psi}}}_k\right)\left(\widehat{\boldsymbol{\Psi}}_{ki}-\widehat{\overline{\boldsymbol{\Psi}}}_k\right)^T}{m_{2k}-1},
$$

$$
\widehat{\mathbf{V}}_i = N_{2k}\left(\frac{N_{2k}}{n_{Ok}}-1\right)\frac{n_{Ok}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{2ki}\right)-\widehat{\mathbf{p}}_{2ki}\widehat{\mathbf{p}}_{2ki}^T\right]}{n_{Ok}-1},
$$

$$
\widehat{\mathbf{C}}_{ij} = -N_{2k}\left(\frac{N_{2k}}{n_{Ok}}-1\right)\frac{n_{Ok}\left[\widehat{\mathbf{p}}_{2ki}\widehat{\mathbf{p}}_{2kj}^T\right]}{n_{Ok}-1},
$$

$$
\widehat{\mathbf{V}}_{Oi} = N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{Oki}\right)-\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Oki}^T\right]}{n_{Am}-1},
$$

$$
\widehat{\mathbf{C}}_{Oij} = -N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\left[\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Okj}^T\right]}{n_{Am}-1},
$$

$$
\widehat{\boldsymbol{\Psi}}_{ki} = \frac{N_{2k}}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\psi_{ki},
$$

$$
\widehat{\mathbf{p}}_{2ki} = \frac{1}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\psi_{ki}
$$

and

$$
\widehat{\mathbf{p}}_{Oki} = \frac{1}{n_{Am}}\psi_{ki}.
$$

See Section 4.A.5 for a derivation of the unbiasedness of the numbers-at-age estimator (Eq. 4.19) and its variance.

4.5 Total Number of Marine Mammal Interactions

For marine mammal interactions the observers are instructed to randomly select the pot-groups for observation of marine mammals. Observers will record any marine mammals caught in pots within the selected pot-groups and if the pot-groups for marine mammal interactions are the same as those selected for tallying numbers of fish and seabird species caught, then the sampling design is identical to that for total numbers caught of fish and seabird species. Observers may choose to sample for marine mammals more frequently than species of fish and seabirds, but assuming a SRS for the pot-groups, the estimation approach is still identical to that for total numbers of fish or seabirds. The estimator then of the total number of marine mammal interactions for a haul is

$$\widehat{\Gamma}_k = \frac{M_k}{m_k} \sum_{i=1}^{m_k} \Gamma_{ki}$$

where M_k is the total number of pot-groups, m_k is the number sampled and Γ_{ki} is the number of marine mammals in the ith pot-group. The variance estimator is

$$V\left(\widehat{\Gamma}_k\right) = M_k \left(\frac{M_k}{m_k} - 1\right) \frac{\sum_{i=1}^{m_k} \left(\Gamma_{ki} - \widehat{\overline{\Gamma}}_k\right)^2}{m_k - 1}$$

where $\widehat{\overline{\Gamma}}_k = \widehat{\Gamma}_k / M_k$.

4.6 Model-based Estimation

Like longline hauls, pot hauls are comprised of a known number of discrete units where the units are hooks for the former and pots for the latter. From a modelling perspective, the important distinction is that each hook was assumed to be a Bernoulli trial for capturing a fish of a given species whereas each pot can capture many fish. Thus, pots are more similar to the arbitrary, but necessary, sampling units I used for model-based estimation within trawl hauls.

In my development of model-based estimators of total number and weight for trawlers in Section 3.6, recall that the weight or volume of the entire haul as well as the sampled portion was assumed known. I remarked that weight and volume of trawl hauls are, in many circumstances, measured with error and may induce a significant increase in variance of the estimates, however, for pot vessels the total number of units (pots) is actually known for each haul and the assumption is not so artificial.

The model-based estimators for total numbers and weight I present in the following sections are based on the ratio estimators of Section 3.6 where they are fully developed. Because within-haul estimation of marine mammal interactions is necessary for pot hauls, model-based estimators are provided as well. As is the case for design-based estimation, the model-based estimators of marine mammal interactions will be identical to those for total numbers of fish and seabirds. The only difference in the ratio estimator is that the covariate is the number of pots rather than weight or volume. The estimators of total numbers in length and age classes are also identical in form to those presented for trawl hauls in Sections 3.6.4 and 3.6.5, respectively.

4.6.1 Total Number for a Species of Fish or Seabird

When the constant variance model (Eq. 3.29) is assumed for the total number of a fish or seabird species (see Section 3.6.1), the optimal estimator of the total number is

$$ {}_1\widehat{N}_k = n_k + {}_1\widehat{\phi}_{n,t}\kappa_{k'} $$

where

$$ {}_1\widehat{\phi}_{n,t}\kappa_{k'} = \frac{\sum_{k=1}^{g_t} n_k \kappa_k}{\sum_{k=1}^{g_t} \kappa_k^2}, $$

n_k is the number of the species in the sample collected from the kth haul, κ_k is the number of sampled pots and $\kappa_{k'}$ is the number of unsampled pots in the haul. The variance estimator is

$$ \widehat{V}_M\left({}_1\widehat{N}_k\right) = \left[\frac{\kappa_{k'}^2}{\sum_{k=1}^{g_t} \kappa_k^2} + (M_k - 1)\right]\widehat{\sigma}_{n,t}^2 = \left[\frac{\kappa_{k'}}{\sum_{k=1}^{g_t} \kappa_k^2} + \frac{1}{\kappa_k}\right]\kappa_{k'}\widehat{\sigma}_{n,t}^2 $$

where

$$ \widehat{\sigma}_{n,t}^2 = \frac{\sum_{k=1}^{g_t}\left(n_k - {}_1\widehat{\phi}_{n,t}\kappa_k\right)^2}{g_t - 1}. $$

and $M_k = 1 + \kappa_{k'}/\kappa_k$.

When the variance is assumed proportional to the number of pots (Eq. 3.33), the estimator of the total number is

$$ {}_2\widehat{N}_k = n_k + {}_2\widehat{\phi}_{n,t}\kappa_{k'} = n_k + \frac{\sum_{k=1}^{g_t} n_k}{\sum_{k=1}^{g_t} \kappa_k}\kappa_{k'} \tag{4.21} $$

where

$$ {}_2\widehat{\phi}_{n,t}\kappa_{k'} = \frac{\sum_{k=1}^{g_t} n_k}{\sum_{k=1}^{g_t} \kappa_k} = \frac{n_t}{\kappa_t} $$

and the variance estimator is

$$\widehat{V}_M\left({}_2\widehat{N}_k\right) = \left(\frac{\kappa_{k'}}{\kappa_t} + 1\right)\kappa_{k'}\widehat{\sigma}^2_{n,t} \tag{4.22}$$

where

$$\widehat{\sigma}^2_{n,t} = \frac{\kappa_t}{\kappa_t^2 - \sum_{k=1}^{g_t}\kappa_k^2}\sum_{k=1}^{g_t}\left(n_k - {}_2\widehat{\phi}_{n,t}\kappa_k\right)^2.$$

4.6.2 Total Weight for a Species of Fish or Seabird

The model-based estimators of total weight are identical in form to those for total number. When constant variance is assumed for the total weight of a species (Eq. 3.29), the optimal estimator of the total weight is

$${}_1\widehat{W}_k = w_k + {}_1\widehat{\phi}_{w,t}\kappa_{k'} = w_k + \frac{\sum_{k=1}^{g_t} w_k\kappa_k}{\sum_{k=1}^{g_t}\kappa_k^2}\kappa_{k'}$$

where w_k is the number of the species in the sample collected from the kth haul. The variance estimator is

$$\widehat{V}\left({}_1\widehat{W}_k\right) = \left[\frac{\kappa_{k'}}{\sum_{k=1}^{g_t}\kappa_k^2} + \frac{1}{\kappa_k}\right]\kappa_{k'}\widehat{\sigma}^2_{w,t}.$$

where

$$\widehat{\sigma}^2_{w,t} = \frac{\sum_{k=1}^{g_t}\left(w_k - {}_1\widehat{\phi}_{w,t}\kappa_k\right)^2}{g_t - 1}.$$

When the variance is proportional to the number of pots (Eq. 3.33), the estimator of the total weight is

$${}_2\widehat{W}_k = w_k + {}_2\widehat{\phi}_{w,t}\kappa_{k'} = w_k + \frac{\sum_{k=1}^{g_t} w_k}{\sum_{k=1}^{g_t}\kappa_k}\kappa_{k'} \tag{4.23}$$

and the variance estimator is

$$\widehat{V}_M\left({}_2\widehat{W}_k\right) = \left(\frac{\kappa_{k'}}{\kappa_t} + 1\right)\kappa_{k'}\widehat{\sigma}^2_{w,t} \tag{4.24}$$

where

$$\widehat{\sigma}_{w,t}^2 = \frac{\kappa_t}{\kappa_t^2 - \sum_{k=1}^{g_t} \kappa_k^2} \sum_{k=1}^{g_t} \left(w_k - {}_2\widehat{\phi}_{w,t}\kappa_k \right)^2 .$$

4.6.3 Total Numbers in Length Classes

As mentioned for other gear types, lengths are not measured for all fish that enter the samples used to estimate total weight or numbers and thus using the number of sampled pots is inappropriate for directly estimating the numbers in each length class in the haul. The option considered for other gear types is to use the product of the model-based estimator for total number and the proportions of the length sample in each length class,

$$ {}_1\widehat{\mathbf{\Lambda}}_k = \widehat{N}_k \widehat{\mathbf{p}}_{\lambda,k} \tag{4.25}$$

where $\widehat{\mathbf{p}}_{\lambda,k} = \mathbf{\lambda}_k / n_k$. The form of $\widehat{N}_k$ depends on the variance model assumption (see Section 4.6.1). For estimation of numbers in length classes, I have modified the model for total numbers so that the parameters are specific to hauls where the species of interest is prevalent. See Section 5.1.2.5 for more details on the estimator of total number and corresponding variance estimator. The numbers-at-length estimator (Eq. 4.25) is unbiased when the estimators for total numbers and length class proportions are independent and it is asymptotically unbiased when they are dependent. Under the assumption that all fish in the haul are independent, the numbers in each length class in the haul are assumed multinomially distributed (see Section 2.3.2) and the resulting variance estimator is

$$\widehat{V}\left({}_1\widehat{\mathbf{\Lambda}}_k\right) = \widehat{V}\left(\widehat{N}_k\right)\left[\widehat{\mathbf{p}}_{\lambda,k}\widehat{\mathbf{p}}_{\lambda,k}^T - \frac{\widehat{\mathbf{\Sigma}}_{\lambda,k}}{n_{\lambda,k}}\right] + \widehat{N}_k^2 \frac{\widehat{\mathbf{\Sigma}}_{\lambda,k}}{n_{\lambda,k}} \tag{4.26}$$

where

$$\widehat{\mathbf{\Sigma}}_{\lambda,k} = \frac{n_{\lambda,k}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{\lambda,k}\right) - \widehat{\mathbf{p}}_{\lambda,k}\widehat{\mathbf{p}}_{\lambda,k}^T\right]}{n_{\lambda,k} - 1}$$

The form of Eq. 4.26 is identical to the model-based estimator for longline hauls (Eq. 2.32) and trawl hauls (Eq. 3.44) but the form of the component variance estimator, $\widehat{V}\left(\widehat{N}_k\right)$ depends on the gear type.

4.6.4 Total Numbers in Age Classes

Using the same approach as longline and trawl hauls where fish are assumed independent within the haul and the otoliths that are aged in the mth management region/period are chosen by SRS from all the otoliths obtained in the region/period, the model-based estimator of the total numbers-at-age is

$$_1\widehat{\boldsymbol{\Psi}}_k = \widehat{N}_k \frac{N_{Om}}{n_{Am}} \frac{\boldsymbol{\psi}_k}{n_{Ok}} \tag{4.27}$$

where $\boldsymbol{\psi}_k$ is the numbers-at-age for the species in the aged sample collected from the kth haul. See Sections 2.4.2 and 3.6.5 for more details. As I mentioned for numbers in length classes, I use a modified model for total numbers so that the parameters are specific to hauls where the species of interest is prevalent. See Section 5.1.2.5 for more details on the estimator of total number and corresponding variance estimator. The approximately unbiased variance estimator is

$$\widehat{V}\left(\widehat{\boldsymbol{\Psi}}_k\right) = \left[\widehat{V}\left(\widehat{N}_k\right) + \widehat{N}_k^2\right]\left\{V\left(\widehat{n}_{Ok}\right)\widehat{\mathbf{p}}_{\psi,k}\widehat{\mathbf{p}}_{\psi,k}^T + \frac{N_{Om}}{n_{Ok}n_{Am}}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{\psi,k}\right) - \widehat{\mathbf{p}}_{\psi,k}\widehat{\mathbf{p}}_{\psi,k}^T\right]\right\}$$
$$+ \widehat{V}\left(\widehat{N}_k\right)\widehat{\mathbf{p}}_{\psi,k}\widehat{\mathbf{p}}_{\psi,k}^T \tag{4.28}$$

where

$$\widehat{\mathbf{p}}_{\psi,k} = \frac{N_{Om}}{n_{Am}} \frac{\boldsymbol{\psi}_k}{n_{Ok}},$$

$$V\left(\widehat{n}_{Ok}\right) = N_{Om}\left(\frac{N_{Om}}{n_{Am}} - 1\right)\frac{N_{Om}P_{Ok}(1-P_{Ok})}{N_{Om}-1}$$

and $P_{Ok} = n_{Ok}/N_{Om}$. Section 2.A.4 provides development of the results for longline hauls which is analogous to those presented here.

4.7 Choosing an Estimator

The numbers of available estimators for various catch parameters can be overwhelming and in this section I summarize more succinctly the sampling scenarios under which each estimator is appropriate and recommend particular estimators when more than one is available for a given scenario.

As with trawl gear, the estimators I presented for each catch parameter when pot gear is used have a good deal of commonality, in that, each estimator can be a function of the sampled and total numbers of pot-groups that comprise a given haul (string of pots). All model-based estimators rely on a similar zero-intercept mean model as well.

Similarities between the estimators also carries over to the criteria for determining which estimators to use in a given sampling scenario. Figure 4.1 shows the decision tree for choosing the estimator and variance estimator for numbers in the catch of a species of interest. The only criterion for determining the appropriate set of estimators is whether the data are aggregated in the NPGOP archive. However, for other catch parameters there are more criteria given that the data are not aggregated which would hypothetically be the case in future archiving of observer data (Figures 4.2 to 4.4).

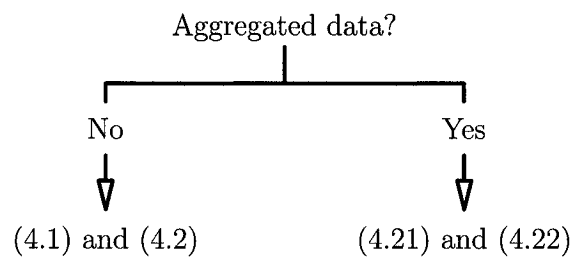

Figure 4.1. Decision tree for determining which estimators to use for total number of a particular fish species in a haul where pot gear is used.

For estimation of total weight in the catch a particular species, I presented an unbiased estimator and an asymptotically unbiased estimator. The asymptotically unbiased estimator is more efficient, but requires that the number of sampled elements is large and the species is not a small component of the total catch (Figure 4.2).

When data are not aggregated, the appropriate estimators for total numbers in length or age classes depends on how the length sample is obtained from the fish that comprise the weight sample. For numbers in length classes, I recommend using the asymptotically unbiased results Eq. 4.13 and Eq. 4.14 rather than the unbiased results Eq. 4.11 and Eq. 4.12 because lengths are only measured when the species is abundant in the haul.

As with trawl hauls, there were two models I proposed for model-based estimation,

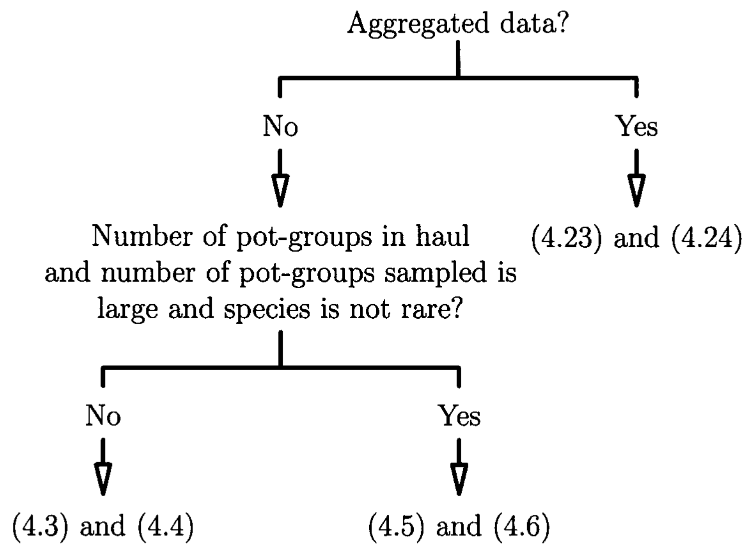

Figure 4.2. Decision tree for determining which estimators to use for total weight of a particular fish species in a haul where pot gear is used.

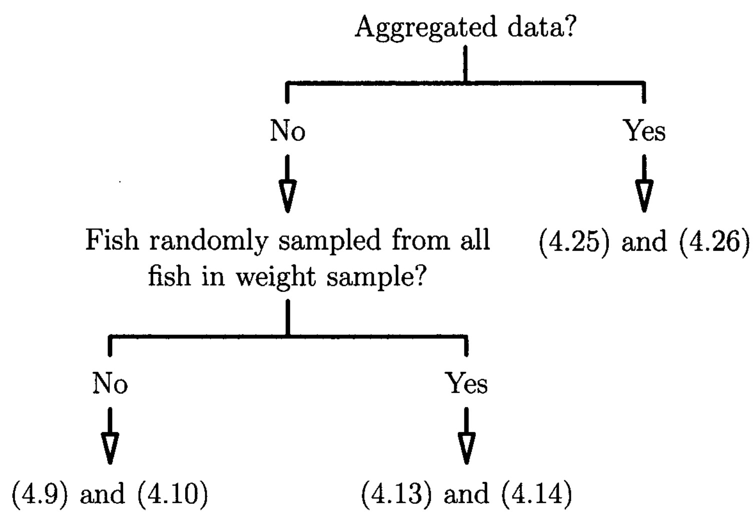

Figure 4.3. Decision tree for determining which estimators to use for total numbers in length classes in a haul made using pot gear.

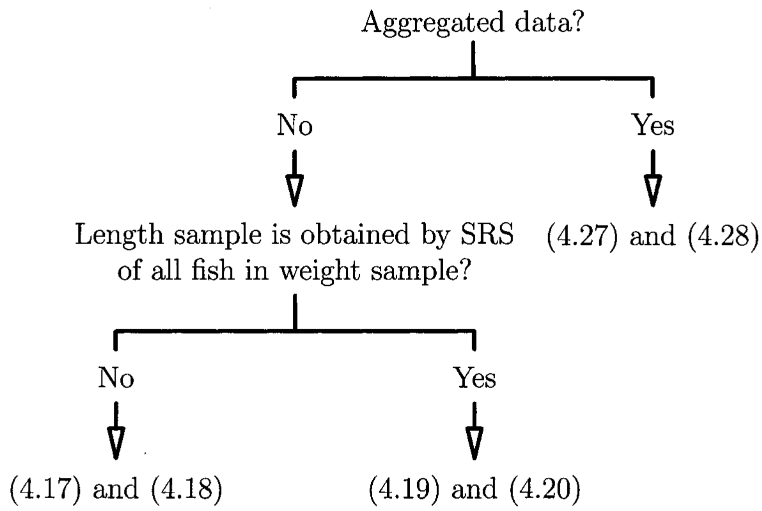

Figure 4.4. Decision tree for determining which estimators to use for total numbers in age classes in a haul made using pot gear.

and in the decision trees I recommend using the model that assumes a variance that is proportional to the weight or volume of the sample elements (Eq. 3.33). I recommend estimators based on this model because they can actually be used and the variance estimates will on average be larger than those obtained under the model in Eq. 3.29. Because the variance estimates will be larger on average the variance estimator is the most conservative of the two available.

4.A Derivation of Estimators

4.A.1 Derivation of ${}_2\widehat{W}_k$

The estimator is asymptotically unbiased as sample and subsample sizes (m_k and m_{2k}) approach M_k and by Taylor series approximation (Delta Method),

$$\begin{aligned}
V\left({}_2\widehat{W}_k\right) =& V\left(\widehat{N}_k \frac{\sum_{i=1}^{m_{2k}} w_{ki}}{\sum_{i=1}^{m_{2k}} n_{ki}}\right) = V\left(\widehat{N}_k \frac{\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}} w_{ki}}{\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}} n_{ki}}\right) = V\left(\widehat{N}_k \frac{{}_1\widehat{W}_k}{{}_1\widehat{N}_k}\right) \\
\approx& \left(\frac{W_k}{N_k}\right)^2 \left[V\left(\widehat{N}_k\right) + V\left({}_1\widehat{N}_k\right) - 2Cov\left(\widehat{N}_k, {}_1\widehat{N}_k\right)\right] + V\left({}_1\widehat{W}_k\right) \\
& - 2\frac{W_k}{N_k}\left[Cov\left({}_1\widehat{N}_k, {}_1\widehat{W}_k\right) - Cov\left(\widehat{N}_k, {}_1\widehat{W}_k\right)\right] \\
=& \left(\frac{W_k}{N_k}\right)^2 \left[V\left({}_1\widehat{N}_k\right) - V\left(\widehat{N}_k\right)\right] + V\left({}_1\widehat{W}_k\right) \\
& - 2\frac{W_k}{N_k}\left[Cov\left({}_1\widehat{N}_k, {}_1\widehat{W}_k\right) - Cov\left(\widehat{N}_k, \widehat{W}_k\right)\right]
\end{aligned}$$

where $\widehat{W}_k = M_k \sum_{i=1}^{m_k} W_{ki}/m_k$,

$$V\left(\widehat{W}_k\right) = M_k \left(\frac{M_k}{m_k} - 1\right) \frac{\sum_{i=1}^{M_k}\left(W_{ki} - \overline{W}_k\right)^2}{M_k - 1} = M_k \left(\frac{M_k}{m_k} - 1\right) S^2_{W,k}$$

$V\left(\widehat{N}_k\right)$ is identical in form to $V\left(\widehat{W}_k\right)$, ${}_1\widehat{W}_k$ is given in Eq. 4.3, ${}_1\widehat{N}_k$ is identical to ${}_1\widehat{W}_k$ in form,

$$V\left({}_1\widehat{W}_k\right) = M_k \left(\frac{M_k}{m_{2k}} - 1\right) S^2_{W,k}$$

and $\widehat{V}\left({}_1\widehat{N}_k\right)$ is identical in form to $V\left({}_1\widehat{W}_k\right)$.

The covariance terms are

$$\begin{aligned}
Cov\left({}_1\widehat{W}_k, {}_1\widehat{N}_k\right) =& M_k \left(\frac{M_k}{m_{2k}} - 1\right) \frac{\sum_{i=1}^{M_k}\left(N_{ki} - \overline{N}_k\right)\left(W_{ki} - \overline{W}_k\right)}{M_k - 1} \\
=& M_k \left(\frac{M_k}{m_{2k}} - 1\right) S_{NW,k}
\end{aligned}$$

and

$$Cov\left(\widehat{W}_k, \widehat{N}_k\right) = M_k\left(\frac{M_k}{m_k} - 1\right) S_{NW,k}.$$

After combining terms the variance approximation is

$$V\left({}_2\widehat{W}_k\right) \approx W_k^2\left[\frac{V\left({}_1\widehat{W}_k\right)}{W_k^2} + M_k^2\left(\frac{1}{m_{2k}} - \frac{1}{m_k}\right)\left(\frac{S_{N,k}^2}{N_k^2} - 2\frac{S_{NW,k}}{N_k W_k}\right)\right]$$

and a variance estimator is obtained by substituting sample-based estimates for corresponding quantities in the approximate variance. All of the component variance and covariance terms derive from single-stage SRS (see Section A.2) as do unbiased estimators thereof.

4.A.2 Derivation of ${}_2\widehat{\mathbf{\Lambda}}_k$

The derivation of the asymptotic variance of this estimator parallels that of ${}_2\widehat{W}_k$, but the difference is that there is subsampling within each of the m_{2k} pot-groups. Thus, we have to estimate within pot-group totals, $\mathbf{\Lambda}_{ki}$. The approximate variance is

$$V\left({}_2\widehat{\mathbf{\Lambda}}_k\right) = V\left(\widehat{N}_k \frac{\sum_{i=1}^{m_{2k}} \widehat{\mathbf{\Lambda}}_{ki}}{\sum_{i=1}^{m_{2k}} N_{ki}}\right) = V\left(\widehat{N}_k \frac{\frac{M_k}{m_{2k}} \sum_{i=1}^{m_{2k}} \widehat{\mathbf{\Lambda}}_{ki}}{\frac{M_k}{m_{2k}} \sum_{i=1}^{m_{2k}} N_{ki}}\right) = V\left(\widehat{N}_k \frac{\widehat{\mathbf{\Lambda}}_{2k}}{\widehat{N}_{2k}}\right)$$

where $\widehat{\mathbf{\Lambda}}_{2k}$ is identical to Eq. 4.7. Now, by first-order Taylor series approximation,

$$\begin{aligned}
V\left({}_2\widehat{\mathbf{\Lambda}}_k\right) \approx & \mathbf{\Lambda}_k\mathbf{\Lambda}_k^T\left[\frac{V\left(\widehat{N}_k\right)+V\left(\widehat{N}_{2k}\right)-2Cov\left(\widehat{N}_k,\widehat{N}_{2k}\right)}{N_k^2}\right]+V\left(\widehat{\mathbf{\Lambda}}_{2k}\right)\\
&+\frac{\mathbf{\Lambda}_k}{N_k}\left[Cov\left(\widehat{N}_k,\widehat{\mathbf{\Lambda}}_{2k}^T\right)-Cov\left(\widehat{N}_{2k},\widehat{\mathbf{\Lambda}}_{2k}^T\right)\right]\\
&+\left[Cov\left(\widehat{\mathbf{\Lambda}}_{2k},\widehat{N}_k\right)-Cov\left(\widehat{\mathbf{\Lambda}}_{2k},\widehat{N}_{2k}\right)\right]\frac{\mathbf{\Lambda}_k^T}{N_k}\\
\approx & \mathbf{\Lambda}_k\mathbf{\Lambda}_k^T\left[\frac{V\left(\widehat{N}_{2k}\right)-V\left(\widehat{N}_k\right)}{N_k^2}\right]+V\left(\widehat{\mathbf{\Lambda}}_{2k}\right)\\
&+\frac{\mathbf{\Lambda}_k}{N_k}\left[Cov\left(\widehat{N}_k,\widehat{\mathbf{\Lambda}}_k^T\right)-Cov\left(\widehat{N}_{2k},\widehat{\mathbf{\Lambda}}_{2k}^T\right)\right]\\
&+\left[Cov\left(\widehat{\mathbf{\Lambda}}_k,\widehat{N}_k\right)-Cov\left(\widehat{\mathbf{\Lambda}}_{2k},\widehat{N}_{2k}\right)\right]\frac{\mathbf{\Lambda}_k^T}{N_k}
\end{aligned}\tag{4.29}$$

where

$$V\left(\widehat{N}_{2k}\right)=M_k\left(\frac{M_k}{m_{2k}}-1\right)\frac{\sum_{i=1}^{M_k}\left(N_{ki}-\overline{N}_k\right)^2}{M_k-1}=M_k\left(\frac{M_k}{m_{2k}}-1\right)S_{N,k}^2,$$

$$\begin{aligned}
Cov\left(\widehat{N}_k,\widehat{\mathbf{\Lambda}}_k\right)=&M_k\left(\frac{M_k}{m_k}-1\right)\frac{\sum_{i=1}^{M_k}\left(N_{ki}-\overline{N}_k\right)\left(\mathbf{\Lambda}_{ki}-\overline{\mathbf{\Lambda}}_k\right)}{M_k-1}\\
=&M_k\left(\frac{M_k}{m_k}-1\right)\mathbf{S}_{N\mathbf{\Lambda},k},
\end{aligned}\tag{4.30}$$

$$Cov\left(\widehat{N}_{2k},\widehat{\mathbf{\Lambda}}_{2k}\right)=M_k\left(\frac{M_k}{m_{2k}}-1\right)\mathbf{S}_{N\mathbf{\Lambda},k}$$

and $V\left(\widehat{\mathbf{\Lambda}}_{2k}\right)$ is given in Eq. 4.8. The variance (Eq. 4.29) can be simplified by noting that,

$$V\left(\widehat{N}_{2k}\right)-V\left(\widehat{N}_k\right)=M_k^2\left(\frac{1}{m_{2k}}-\frac{1}{m_k}\right)S_{N,k}^2,$$

and

$$Cov\left(\widehat{N}_k,\widehat{\mathbf{\Lambda}}_k\right)-Cov\left(\widehat{N}_{2k},\widehat{\mathbf{\Lambda}}_{2k}\right)=-M_k^2\left(\frac{1}{m_{2k}}-\frac{1}{m_k}\right)\mathbf{S}_{N\mathbf{\Lambda},k},$$

giving

$$V\left({}_2\widehat{\mathbf{\Lambda}}_k\right) \approx V\left(\widehat{\mathbf{\Lambda}}_{2k}\right) + M_k^2\left(\frac{1}{m_{2k}} - \frac{1}{m_k}\right)\left[S_{N,k}^2\frac{\mathbf{\Lambda}_k\mathbf{\Lambda}_k^T}{N_k^2} - \frac{\mathbf{\Lambda}_k}{N_k}\mathbf{S}_{N\mathbf{\Lambda},k}^T - \mathbf{S}_{N\mathbf{\Lambda},k}\frac{\mathbf{\Lambda}_k^T}{N_k}\right]$$

The variance estimator is obtained using unbiased estimators of the components in Eq. 4.29. In particular,

$$\widehat{V}\left(\widehat{N}_{2k}\right) = M_k\left(\frac{M_k}{m_{2k}} - 1\right)\frac{\sum_{i=1}^{m_k}\left(N_{ki} - \widehat{\overline{N}}_k\right)^2}{m_k - 1}$$

is a more efficient variance estimator than that using the sum of squared difference only for the m_{2k} pot-groups chosen for the weight/length sample.

4.A.3 Derivation of ${}_4\widehat{\mathbf{\Lambda}}_k$

The derivation of the asymptotic variance for ${}_4\widehat{\mathbf{\Lambda}}_k$ follows that for ${}_2\widehat{\mathbf{\Lambda}}_k$. In fact, the form is identical to Eq. 4.29. Specifically, here we have

$$V\left({}_4\widehat{\mathbf{\Lambda}}_k\right) = V\left(\widehat{N}_k\frac{\widehat{\mathbf{\Lambda}}_k^*}{\widehat{N}_k^*}\right)$$

where

$$\widehat{\mathbf{\Lambda}}_k^* = \frac{M_k}{m_{2k}}\frac{N_{2k}}{n_k}\boldsymbol{\lambda}_k = \frac{M_k}{m_{2k}}\frac{N_{2k}}{n_k}\sum_{i=1}^{m_k}\boldsymbol{\lambda}_{ki},$$

which is identical to Eq. 4.11, and

$$\widehat{N}_k^* = \frac{M_k}{m_{2k}}\frac{N_{2k}}{n_k}\sum_{i=1}^{m_{2k}} n_{ki} = \frac{M_k}{m_{2k}}N_{2k} = \frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}} N_{ki}$$

which is identical to Eq. 4.1.

As mentioned, the first-order Taylor series approximatation,

$$V\left({}_4\widehat{\mathbf{\Lambda}}_k\right) \approx \mathbf{\Lambda}_k\mathbf{\Lambda}_k^T \left[\frac{V\left(\widehat{N}_k^*\right) - V\left(\widehat{N}_k\right)}{N_k^2}\right] + V\left(\widehat{\mathbf{\Lambda}}_k^*\right)$$
$$+ \frac{\mathbf{\Lambda}_k}{N_k}\left[Cov\left(\widehat{N}_k, \widehat{\mathbf{\Lambda}}_k^T\right) - Cov\left(\widehat{N}_k^*, \widehat{\mathbf{\Lambda}}_k^{*T}\right)\right]$$
$$+ \left[Cov\left(\widehat{\mathbf{\Lambda}}_k, \widehat{N}_k\right) - Cov\left(\widehat{\mathbf{\Lambda}}_k^*, \widehat{N}_k^*\right)\right]\frac{\mathbf{\Lambda}_k^T}{N_k} \tag{4.31}$$

is identical in form to the approximation for the variance of Eq. 4.29, except that $V\left(\widehat{\mathbf{\Lambda}}_k^*\right)$ is given by

$$V\left(\widehat{\mathbf{\Lambda}}_k^*\right) = M_k\left(\frac{M_k}{m_{2k}} - 1\right)\frac{\sum_{i=1}^{M_k}\left(\mathbf{\Lambda}_{ki} - \overline{\mathbf{\Lambda}}_k\right)\left(\mathbf{\Lambda}_{ki} - \overline{\mathbf{\Lambda}}_k\right)^T}{M_k - 1}$$
$$+ \left(\frac{M_k}{m_{2k}}\right)^2 E\left[\sum_{i=1}^{M_k} V\left(\widehat{\mathbf{\Lambda}}_{ki}|s_1\right) + \sum\sum_{i\neq j}^{M_k} Cov\left(\widehat{\mathbf{\Lambda}}_{ki}, \widehat{\mathbf{\Lambda}}_{kj}|s_1\right)\right].$$

where

$$V\left(\widehat{\mathbf{\Lambda}}_{ki}|s_1\right) = N_{2k}\left(\frac{N_{2k}}{n_k} - 1\right)\frac{N_{2k}\left[\text{diag}\left(\mathbf{p}_{ki}\right) - \mathbf{p}_{ki}\mathbf{p}_{ki}^T\right]}{N_{2k} - 1},$$
$$Cov\left(\widehat{\mathbf{\Lambda}}_{ki}, \widehat{\mathbf{\Lambda}}_{kj}|s_1\right) = -N_{2k}\left(\frac{N_{2k}}{n_k} - 1\right)\frac{N_{2k}\mathbf{p}_{ki}\mathbf{p}_{kj}^T}{N_{2k} - 1}$$

and $\mathbf{p}_{ki} = \mathbf{\Lambda}_{ki}/N_{2k}$. The variance estimator (Eq. 4.14) is obtained by substituting sample-based estimators.

4.A.4 Derivation of ${}_1\widehat{\mathbf{\Psi}}_k$

There are four phases of sampling for estimation of total numbers-at-age in a haul. Let s_A, s_O, s_{Li} and s_w denote samples of otoliths for ageing, otoliths from fish in the length sample, fish in each sampling unit for the length sample and sampling units for the weight sample, respectively. Also, let ψ_{Aki}, ψ_{Oki} and ψ_{Lki} be the numbers-at-age among aged otoliths in the ith sampling unit, among all collected otoliths in the ith sampling unit and among all fish in the length sample and ith sampling unit, respectively. The expectation of Eq. 4.17 is

$$
\begin{aligned}
E\left({}_1\widehat{\boldsymbol{\Psi}}_k\right) =& E_{s_w}\left\{\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}} E_{s_{L_i}}\left[\frac{N_{ki}}{n_{ki}}E_{s_O}\left(\frac{n_k}{n_{Ok}}E_{s_A}\left(\frac{N_{Om}}{n_{Am}}\boldsymbol{\psi}_{Aki}\right)\right)\right]\right\} \\
=& E_{s_w}\left\{\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}} E_{s_{L_i}}\left[\frac{N_{ki}}{n_{ki}}E_{s_O}\left(\frac{n_k}{n_{Ok}}\boldsymbol{\psi}_{Oki}\right)\right]\right\} \\
=& E_{s_w}\left[\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}} E_{s_{L_i}}\left(\frac{N_{ki}}{n_{ki}}\boldsymbol{\psi}_{Lki}\right)\right] = E_{s_w}\left(\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}}\boldsymbol{\Psi}_{ki}\right) = \boldsymbol{\Psi}_k.
\end{aligned}
$$

The variance of the estimator can be written as

$$
\begin{aligned}
V\left({}_1\widehat{\boldsymbol{\Psi}}_k\right) =& \underbrace{V_{s_w}\left\{E_{s_{L_i}}\left[E_{s_O}\left(E_{s_A}\left({}_1\widehat{\boldsymbol{\Psi}}_k\right)\right)\right]\right\}}_{V_1} + \underbrace{E_{s_w}\left\{V_{s_{L_i}}\left[E_{s_O}\left(E_{s_A}\left({}_1\widehat{\boldsymbol{\Psi}}_k\right)\right)\right]\right\}}_{V_2} \\
&+ \underbrace{E_{s_w}\left\{E_{s_{L_i}}\left[V_{s_O}\left(E_{s_A}\left({}_1\widehat{\boldsymbol{\Psi}}_k\right)\right)\right]\right\}}_{V_3} + \underbrace{E_{s_w}\left\{E_{s_{L_i}}\left[E_{s_O}\left(V_{s_A}\left({}_1\widehat{\boldsymbol{\Psi}}_k\right)\right)\right]\right\}}_{V_4}.
\end{aligned}
$$

Component-wise,

$$
V_1 = V_{s_w}\left(\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}}\boldsymbol{\Psi}_{ki}\right) = M_k\left(\frac{M_k}{m_{2k}}-1\right)\frac{\sum_{i=1}^{M_k}\left(\boldsymbol{\Psi}_{ki}-\overline{\boldsymbol{\Psi}}_k\right)\left(\boldsymbol{\Psi}_{ki}-\overline{\boldsymbol{\Psi}}_k\right)^T}{M_k-1},
$$

$$
\begin{aligned}
V_2 =& E_{s_w}\left[\left(\frac{M_k}{m_{2k}}\right)^2\sum_{i=1}^{m_{2k}} V_{s_{L_i}}\left(\frac{N_{ki}}{n_{ki}}\boldsymbol{\psi}_{Lki}\right)\right] \\
=& \frac{M_k}{m_{2k}}\sum_{i=1}^{M_k} N_{ki}\left(\frac{N_{ki}}{n_{ki}}-1\right)\frac{N_{ki}\left[\operatorname{diag}\left(\mathbf{P}_{ki}\right)-\mathbf{P}_{ki}\mathbf{P}_{ki}^T\right]}{N_{ki}-1},
\end{aligned}
$$

$$
\begin{aligned}
V_3 =& E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\right)^2 E_{s_{L_i}}\left[V_{s_O}\left(\sum_{i=1}^{m_{2k}}\frac{N_{ki}}{n_{ki}}\frac{n_k}{n_{Ok}}\psi_{Oki}\right)\right]\right\} \\
=& E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\right)^2 \sum_{i=1}^{m_{2k}} E_{s_{L_i}}\left[\left(\frac{N_{ki}}{n_{ki}}\right)^2 V_{s_O}\left(\frac{n_k}{n_{Ok}}\psi_{Oki}\right)\right]\right\} \\
&+ E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\right)^2 \sum_{i\neq j}^{m_{2k}}\sum E_{s_{L_i}}\left[\frac{N_{ki}}{n_{ki}}\frac{N_{kj}}{n_{kj}} Cov_{s_O}\left(\frac{n_k}{n_{Ok}}\psi_{Oki}, \frac{n_k}{n_{Ok}}\psi_{Okj}\right)\right]\right\} \\
=& E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\right)^2 \sum_{i=1}^{m_{2k}} E_{s_{L_i}}\left[\left(\frac{N_{ki}}{n_{ki}}\right)^2 n_k\left(\frac{n_k}{n_{Ok}}-1\right)\frac{n_k\left[\mathrm{diag}\left(\mathbf{p}_{Lki}\right)-\mathbf{p}_{Lki}\mathbf{p}_{Lki}^T\right]}{n_k-1}\right]\right\} \\
&- E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\right)^2 \sum_{i\neq j}^{m_{2k}}\sum E_{s_{L_i}}\left[\frac{N_{ki}}{n_{ki}}\frac{N_{kj}}{n_{kj}} n_k\left(\frac{n_k}{n_{Ok}}-1\right)\frac{n_k\mathbf{p}_{Lki}\mathbf{p}_{Lkj}^T}{n_k-1}\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
V_4 =& E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\right)^2 E_{s_{L_i}}\left[E_{s_O}\left(V_{s_A}\left(\sum_{i=1}^{m_{2k}}\frac{N_{ki}}{n_{ki}}\frac{n_k}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\psi_{Aki}\right)\right)\right]\right\} \\
=& E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\right)^2 E_{s_{L_i}}\left[\sum_{i=1}^{m_{2k}} E_{s_O}\left[\left(\frac{N_{ki}}{n_{ki}}\frac{n_k}{n_{Ok}}\right)^2 V_{s_A}\left(\frac{N_{Om}}{n_{Am}}\psi_{Aki}\right)\right]\right.\right. \\
&\left.\left.+ \sum_{i\neq j}^{m_{2k}}\sum E_{s_O}\left[\frac{N_{ki}}{n_{ki}}\frac{N_{kj}}{n_{kj}}\left(\frac{n_k}{n_{Ok}}\right)^2 Cov_{s_A}\left(\frac{N_{Om}}{n_{Am}}\psi_{Aki}, \frac{N_{Om}}{n_{Am}}\psi_{Akj}\right)\right]\right]\right\} \\
=& E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\right)^2 \sum_{i=1}^{m_{2k}} E_{s_{L_i}}\left[\left(\frac{N_{ki}}{n_{ki}}\frac{n_k}{n_{Ok}}\right)^2 N_{Om}^2\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{\left[\mathrm{diag}\left(\mathbf{p}_{Oki}\right)-\mathbf{p}_{Oki}\mathbf{p}_{Oki}^T\right]}{N_{Om}-1}\right]\right\} \\
&- E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\right)^2 \sum_{i\neq j}^{m_{2k}}\sum E_{s_{L_i}}\left[\frac{N_{ki}}{n_{ki}}\frac{N_{kj}}{n_{kj}}\left(\frac{n_k}{n_{Ok}}\right)^2 N_{Om}^2\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{\mathbf{p}_{Oki}\mathbf{p}_{Okj}^T}{N_{Om}-1}\right]\right\}
\end{aligned}
$$

where $\mathbf{P}_{ki} = \mathbf{\Psi}_{ki}/N_{ki}$, $\mathbf{p}_{Lki} = \psi_{Lki}/n_k$ and $\mathbf{p}_{Oki} = \psi_{Oki}/N_{Om}$.

Now, working backward, unbiased estimators of the components are

$$
\begin{aligned}
\widehat{V}_4 = & \left(\frac{M_k}{m_{2k}}\frac{n_k}{n_{Ok}}\right)^2 \left[\sum_{i=1}^{m_{2k}} \left(\frac{N_{ki}}{n_{ki}}\right)^2 N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{Oki}\right)-\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Oki}^T\right]}{n_{Am}-1}\right. \\
& \left. -\sum\sum_{i\neq j}^{m_{2k}}\frac{N_{ki}}{n_{ki}}\frac{N_{kj}}{n_{kj}}N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Okj}^T}{n_{Am}-1}\right] \\
= & \left(\frac{M_k}{m_{2k}}\frac{n_k}{n_{Ok}}\right)^2 \left[\sum_{i=1}^{m_{2k}} \left(\frac{N_{ki}}{n_{ki}}\right)^2 \widehat{V}\left(\widehat{\boldsymbol{\psi}}_{Oki}\right) + \sum\sum_{i\neq j}^{m_{2k}}\frac{N_{ki}}{n_{ki}}\frac{N_{kj}}{n_{kj}}\widehat{Cov}\left(\widehat{\boldsymbol{\psi}}_{Oki},\widehat{\boldsymbol{\psi}}_{Okj}\right)\right]
\end{aligned}
$$

where $\widehat{\mathbf{p}}_{Oki} = \boldsymbol{\psi}_{Aki}/n_{Am}$,

$$
\begin{aligned}
\widehat{V}_3 = & \left(\frac{M_k}{m_{2k}}\right)^2 \sum_{i=1}^{m_{2k}} \left(\frac{N_{ki}}{n_{ki}}\right)^2 n_k\left(\frac{n_k}{n_{Ok}}-1\right)\frac{n_{Ok}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{Lki}\right)-\widehat{\mathbf{p}}_{Lki}\widehat{\mathbf{p}}_{Lki}^T+\widehat{V}\left(\widehat{\mathbf{p}}_{Lki}\right)\right]}{n_{Ok}-1} \\
& -\left(\frac{M_k}{m_{2k}}\right)^2 \sum\sum_{i\neq j}^{m_{2k}}\frac{N_{ki}}{n_{ki}}\frac{N_{kj}}{n_{kj}}n_k\left(\frac{n_k}{n_{Ok}}-1\right)\frac{n_{Ok}\left[\widehat{\mathbf{p}}_{Lki}\widehat{\mathbf{p}}_{Lkj}^T-\widehat{Cov}\left(\widehat{\mathbf{p}}_{Lki},\widehat{\mathbf{p}}_{Lkj}\right)\right]}{n_{Ok}-1} \\
= & \left(\frac{M_k}{m_{2k}}\right)^2 \left\{\sum_{i=1}^{m_{2k}} \left(\frac{N_{ki}}{n_{ki}}\right)^2 \widehat{V}\left(\widehat{\boldsymbol{\psi}}_{Lki}\right) + \sum\sum_{i\neq j}^{m_{2k}}\frac{N_{ki}}{n_{ki}}\frac{N_{kj}}{n_{kj}}\widehat{Cov}\left(\widehat{\boldsymbol{\psi}}_{Lki},\widehat{\boldsymbol{\psi}}_{Lkj}\right)\right\}
\end{aligned}
$$

where

$$
\widehat{\mathbf{p}}_{Lki} = \frac{N_{Om}}{n_{Ok}n_{Am}}\boldsymbol{\psi}_{Aki},
$$

$$
\begin{aligned}
\widehat{V}\left(\widehat{\mathbf{p}}_{Lki}\right) = & \frac{1}{n_{Ok}^2}N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{Oki}\right)-\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Oki}^T\right]}{n_{Am}-1} \\
= & \frac{\widehat{V}\left(\widehat{\boldsymbol{\psi}}_{Oki}\right)}{n_{Ok}^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{Cov}\left(\widehat{\mathbf{p}}_{Lki},\widehat{\mathbf{p}}_{Lkj}\right) = & -\frac{1}{n_{Ok}^2}N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Okj}^T}{n_{Am}-1} \\
= & \frac{\widehat{Cov}\left(\widehat{\boldsymbol{\psi}}_{Oki},\widehat{\boldsymbol{\psi}}_{Okj}\right)}{n_{Ok}^2}
\end{aligned}
$$

$$\widehat{V}_2 = \left(\frac{M_k}{m_{2k}}\right)^2 \sum_{i=1}^{m_{2k}} N_{ki} \left(\frac{N_{ki}}{n_{ki}} - 1\right) \frac{n_{ki}\left[\operatorname{diag}\left(\widehat{\mathbf{P}}_{ki}\right) - \widehat{\mathbf{P}}_{ki}\widehat{\mathbf{P}}_{ki}^T + \widehat{V}\left(\widehat{\mathbf{P}}_{ki}\right)\right]}{n_{ki}-1}$$
$$= \left(\frac{M_k}{m_{2k}}\right)^2 \sum_{i=1}^{m_{2k}} \widehat{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}\right)$$

where

$$\widehat{\mathbf{P}}_{ki} = \frac{n_k N_{Om}}{n_{ki} n_{Ok} n_{Am}} \psi_{Aki}$$

and

$$\widehat{V}\left(\widehat{\mathbf{P}}_{ki}\right) = \frac{1}{n_{ki}^2}\left[n_k\left(\frac{n_k}{n_{Ok}} - 1\right)\frac{n_{Ok}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{Lki}\right) - \widehat{\mathbf{p}}_{Lki}\widehat{\mathbf{p}}_{Lki}^T + \widehat{V}\left(\widehat{\mathbf{p}}_{Lki}\right)\right]}{n_{Ok}-1}\right.$$
$$\left. + \left(\frac{n_k}{n_{Ok}}\right)^2 N_{Om}\left(\frac{N_{Om}}{n_{Am}} - 1\right)\frac{n_{Am}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{Oki}\right) - \widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Oki}^T\right]}{n_{Am}-1}\right]$$
$$= \frac{1}{n_{ki}^2}\left[\widehat{V}\left(\widehat{\psi}_{Lki}\right) + \left(\frac{n_k}{n_{Ok}}\right)^2 \widehat{V}\left(\widehat{\psi}_{Oki}\right)\right]$$

and finally,

$$\widehat{V}_1 = M_k\left(\frac{M_k}{m_{2k}} - 1\right)\left[\frac{\sum_{i=1}^{m_{2k}} \widehat{\boldsymbol{\Psi}}_{ki}\widehat{\boldsymbol{\Psi}}_{ki}^T - \widehat{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}|s_w\right)}{m_{2k}}\right.$$
$$\left. - \frac{\sum\sum_{i\neq j}^{m_{2k}} \widehat{\boldsymbol{\Psi}}_{ki}\widehat{\boldsymbol{\Psi}}_{kj}^T - \widehat{Cov}\left(\widehat{\boldsymbol{\Psi}}_{ki}, \widehat{\boldsymbol{\Psi}}_{kj}|s_w\right)}{m_{2k}(m_{2k}-1)}\right]$$
$$= M_k\left(\frac{M_k}{m_{2k}} - 1\right)\frac{\sum_{i=1}^{m_{2k}}\left(\widehat{\boldsymbol{\Psi}}_{ki} - \widehat{\widehat{\boldsymbol{\Psi}}}_k\right)\left(\widehat{\boldsymbol{\Psi}}_{ki} - \widehat{\widehat{\boldsymbol{\Psi}}}_k\right)^T}{m_{2k}-1}$$
$$- \frac{M_k}{m_{2k}}\left(\frac{M_k}{m_{2k}} - 1\right)\left[\sum_{i=1}^{m_{2k}} \widehat{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}|s_w\right) - \frac{1}{m_{2k}-1}\sum\sum_{i\neq j}^{m_{2k}} \widehat{Cov}\left(\widehat{\boldsymbol{\Psi}}_{ki}, \widehat{\boldsymbol{\Psi}}_{kj}|s_w\right)\right]$$

where

$$\widehat{\boldsymbol{\Psi}}_{ki} = \frac{N_{ki} n_k N_{Om}}{n_{ki} n_{Ok} n_{Am}} \psi_{Aki},$$

$$\begin{aligned}
\widehat{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}|s_w\right) &= \left(\frac{N_{ki}n_k}{n_{ki}n_{Ok}}\right)^2 N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{Oki}\right)-\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Oki}^T\right]}{n_{Am}-1} \\
&+\left(\frac{N_{ki}}{n_{ki}}\right)^2 n_k\left(\frac{n_k}{n_{Ok}}-1\right)\frac{n_{Ok}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{Lki}\right)-\widehat{\mathbf{p}}_{Lki}\widehat{\mathbf{p}}_{Lki}^T+\widehat{V}\left(\widehat{\mathbf{p}}_{Lki}\right)\right]}{n_{Ok}-1} \\
&+N_{ki}\left(\frac{N_{ki}}{n_{ki}}-1\right)\frac{n_{ki}\left[\operatorname{diag}\left(\widehat{\mathbf{P}}_{ki}\right)-\widehat{\mathbf{P}}_{ki}\widehat{\mathbf{P}}_{ki}^T+\widehat{V}\left(\widehat{\mathbf{P}}_{ki}\right)\right]}{n_{ki}-1} \\
&=\left(\frac{N_{ki}n_k}{n_{ki}n_{Ok}}\right)^2\widehat{V}\left(\widehat{\psi}_{Oki}\right)+\left(\frac{N_{ki}}{n_{ki}}\right)^2\widehat{V}\left(\widehat{\psi}_{Lki}\right)+\widehat{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}\right)
\end{aligned}$$

and

$$\begin{aligned}
\widehat{Cov}\left(\widehat{\boldsymbol{\Psi}}_{ki},\widehat{\boldsymbol{\Psi}}_{kj}|s_w\right) &= -\frac{N_{ki}N_{kj}}{n_{ki}n_{kj}}\left[\left(\frac{n_k}{n_{Ok}}\right)^2 N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Okj}^T}{n_{Am}-1}\right. \\
&\left.+n_k\left(\frac{n_k}{n_{Ok}}-1\right)\frac{n_{Ok}\left[\widehat{\mathbf{p}}_{Lki}\widehat{\mathbf{p}}_{Lkj}^T-\widehat{Cov}\left(\widehat{\mathbf{p}}_{Lki},\widehat{\mathbf{p}}_{Lkj}\right)\right]}{n_{Ok}-1}\right] \\
&=\frac{N_{ki}N_{kj}}{n_{ki}n_{kj}}\left[\left(\frac{n_k}{n_{Ok}}\right)^2\widehat{Cov}\left(\widehat{\psi}_{Oki},\widehat{\psi}_{Okj}\right)\right. \\
&\left.+\widehat{Cov}\left(\widehat{\psi}_{Lki},\widehat{\psi}_{Lkj}\right)\right]
\end{aligned}$$

Combining all of the component estimators yields Eq. 4.18.

4.A.5 Derivation of ${}_2\widehat{\boldsymbol{\Psi}}_k$

There are three phases of sampling for estimation of total numbers-at-age in a haul assuming the given sampling design and using Eq. 4.19. Because the length sample is obtained by a SRS of all fish in the weight sample and the otolith sample is obtained by a SRS of all fish in the length sample the sampling distribution of otoliths conditional on the weight sample is marginally a SRS from all fish in the weight sample. Let s_A, s_O, and s_w denote samples of otoliths for ageing, otoliths from fish in the weight sample and sampling units for the weight sample, respectively. Also, let ψ_{Aki}, ψ_{Oki} and ψ_{Lki} be the numbers-at-age among aged otoliths in the ith sampling unit, among all collected otoliths in the ith sampling unit and among all fish in the ith sampling unit of the weight sample, respectively. To show the unbiasedness of Eq. 4.19 first note that

$$_2\widehat{\Psi}_k = \frac{M_k}{m_{2k}}\frac{N_{2k}}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\psi_{Ak} = \frac{M_k}{m_{2k}}\frac{N_{2k}}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\sum_{i=1}^{m_{2k}}\psi_{Aki}.$$

The expectation of Eq. 4.19 is

$$\begin{aligned} E\left({}_2\widehat{\Psi}_k\right) =& E_{s_w}\left\{\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}} E_{s_O}\left[\frac{N_{2k}}{n_{Ok}}E_{s_A}\left(\frac{N_{Om}}{n_{Am}}\psi_{Aki}\right)\right]\right\} \\ =& E_{s_w}\left[\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}} E_{s_O}\left(\frac{N_{2k}}{n_{Ok}}\psi_{Oki}\right)\right] \\ =& E_{s_w}\left(\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}}\Psi_k\right) = \Psi_k. \end{aligned}$$

The variance of the estimator is

$$\begin{aligned} V\left({}_2\widehat{\Psi}_k\right) =& \underbrace{V_{s_w}\left\{E_{s_O}\left[E_{s_A}\left({}_2\widehat{\Psi}_k\right)\right]\right\}}_{V_1} + \underbrace{E_{s_w}\left\{V_{s_O}\left[E_{s_A}\left({}_2\widehat{\Psi}_k\right)\right]\right\}}_{V_2} \\ &+ \underbrace{E_{s_w}\left\{E_{s_O}\left[V_{s_A}\left({}_2\widehat{\Psi}_k\right)\right]\right\}}_{V_3} \end{aligned}$$

Component-wise,

$$V_1 = V_{s_w}\left(\frac{M_k}{m_{2k}}\sum_{i=1}^{m_{2k}}\Psi_{ki}\right) = M_k\left(\frac{M_k}{m_{2k}}-1\right)\frac{\sum_{i=1}^{M_k}\left(\Psi_{ki}-\overline{\Psi}_k\right)\left(\Psi_{ki}-\overline{\Psi}_k\right)^T}{M_k-1},$$

$$\begin{aligned} V_2 =& E_{s_w}\left[\left(\frac{M_k}{m_{2k}}\right)^2\sum_{i=1}^{m_{2k}} V_{s_O}\left(\frac{N_{2k}}{n_{Ok}}\psi_{Oki}\right) + \sum\sum_{i\neq j}^{m_{2k}} Cov_{s_O}\left(\frac{N_{2k}}{n_{Ok}}\psi_{Oki}, \frac{N_{2k}}{n_{Ok}}\psi_{Okj}\right)\right] \\ =& E_{s_w}\left[\left(\frac{M_k}{m_{2k}}\right)^2 N_{2k}\left(\frac{N_{2k}}{n_{Ok}}-1\right)\frac{N_{2k}\left[\operatorname{diag}\left(\mathbf{p}_{2k}\right)-\mathbf{p}_{2k}\mathbf{p}_{2k}^T\right]}{N_{2k}-1}\right], \end{aligned}$$

and

$$
\begin{aligned}
V_3 =& E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\right)^2 E_{s_O}\left[V_{s_A}\left(\sum_{i=1}^{m_{2k}}\frac{N_{2k}}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\boldsymbol{\psi}_{Aki}\right)\right]\right\} \\
=& E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\frac{N_{2k}}{n_{Ok}}\right)^2 E_{s_O}\left[\sum_{i=1}^{m_{2k}} V_{s_A}\left(\frac{N_{Om}}{n_{Am}}\boldsymbol{\psi}_{Aki}\right)\right.\right. \\
&\left.\left.+\sum\sum_{i\neq j}^{m_{2k}} Cov_{s_A}\left(\frac{N_{Om}}{n_{Am}}\boldsymbol{\psi}_{Aki}, \frac{N_{Om}}{n_{Am}}\boldsymbol{\psi}_{Akj}\right)\right]\right\} \\
=& E_{s_w}\left\{\left(\frac{M_k}{m_{2k}}\frac{N_{2k}}{n_{Ok}}\right)^2 N_{Om}^2\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{\left[\mathrm{diag}\left(\mathbf{p}_{Ok}\right)-\mathbf{p}_{Ok}\mathbf{p}_{Ok}^T\right]}{N_{Om}-1}\right\}
\end{aligned}
$$

where $\mathbf{p}_{Ok} = \boldsymbol{\psi}_{Ok}/N_{Om}$ and $\boldsymbol{\psi}_{Ok}$ is the numbers-at-age among all otoliths sampled from the kth haul.

The estimator of the third component is

$$
\begin{aligned}
\widehat{V}_3 =& \left(\frac{M_k}{m_{2k}}\frac{N_{2k}}{n_{Ok}}\right)^2 N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\left[\mathrm{diag}\left(\widehat{\mathbf{p}}_{Ok}\right)-\widehat{\mathbf{p}}_{Ok}\widehat{\mathbf{p}}_{Ok}^T\right]}{n_{Am}-1} \\
=& \left(\frac{M_k}{m_{2k}}\frac{N_{2k}}{n_{Ok}}\right)^2 \widehat{V}\left(\widehat{\boldsymbol{\psi}}_{Ok}\right) \\
=& \left(\frac{M_k}{m_{2k}}\frac{N_{2k}}{n_{Ok}}\right)^2 N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}}{n_{Am}-1}\left[\sum_{i=1}^{m_{2k}}\left[\mathrm{diag}\left(\widehat{\mathbf{p}}_{Oki}\right)-\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Oki}^T\right]\right. \\
&\left.-\sum\sum_{i\neq j}^{m_{2k}}\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Okj}^T\right] \\
=& \left(\frac{M_k}{m_{2k}}\frac{N_{2k}}{n_{Ok}}\right)^2\left[\sum_{i=1}^{m_{2k}}\widehat{V}\left(\widehat{\boldsymbol{\psi}}_{Oki}\right)+\sum\sum_{i\neq j}^{m_{2k}}\widehat{Cov}\left(\widehat{\boldsymbol{\psi}}_{Oki},\widehat{\boldsymbol{\psi}}_{Okj}\right)\right]
\end{aligned}
$$

where $\widehat{\mathbf{p}}_{Ok} = \boldsymbol{\psi}_{Ak}/n_{Am}$ and $\widehat{\mathbf{p}}_{Oki} = \boldsymbol{\psi}_{Aki}/n_{Am}$. The estimator of the second component is

$$\widehat{V}_2 = \left(\frac{M_k}{m_{2k}}\right)^2 N_{2k}\left(\frac{N_{2k}}{n_{Ok}}-1\right)\frac{n_{Ok}\left[\text{diag}\left(\widehat{\mathbf{p}}_{2k}\right)-\widehat{\mathbf{p}}_{2k}\widehat{\mathbf{p}}_{2k}^T+\widehat{V}\left(\widehat{\mathbf{p}}_{2k}\right)\right]}{n_{Ok}-1}$$

$$= \left(\frac{M_k}{m_{2k}}\right)^2 N_{2k}\left(\frac{N_{2k}}{n_{Ok}}-1\right)\left[\sum_{i=1}^{m_{2k}}\frac{n_{Ok}\left[\text{diag}\left(\widehat{\mathbf{p}}_{2ki}\right)-\widehat{\mathbf{p}}_{2ki}\widehat{\mathbf{p}}_{2ki}^T+\widehat{V}\left(\widehat{\mathbf{p}}_{2ki}\right)\right]}{n_{Ok}-1}\right.$$

$$\left.-\sum\sum_{i\neq j}^{m_{2k}}\frac{n_{Ok}\left[\widehat{\mathbf{p}}_{2ki}\widehat{\mathbf{p}}_{2kj}^T-\widehat{Cov}\left(\widehat{\mathbf{p}}_{2ki},\widehat{\mathbf{p}}_{2kj}\right)\right]}{n_{Ok}-1}\right]$$

$$= \left(\frac{M_k}{m_{2k}}\right)^2\left\{\sum_{i=1}^{m_{2k}}\left[\widetilde{V}\left(\widehat{\mathbf{\Psi}}_{ki}\right)+N_{2k}\left(\frac{N_{2k}}{n_{Ok}}-1\right)\frac{n_{Ok}\widehat{V}\left(\widehat{\mathbf{p}}_{2ki}\right)}{n_{Ok}-1}\right]\right.$$

$$\left.+\sum\sum_{i\neq j}^{m_{2k}}\left[\widetilde{Cov}\left(\widehat{\mathbf{\Psi}}_{ki},\widehat{\mathbf{\Psi}}_{kj}\right)+N_{2k}\left(\frac{N_{2k}}{n_{Ok}}-1\right)\frac{n_{Ok}\widehat{Cov}\left(\widehat{\mathbf{p}}_{2ki},\widehat{\mathbf{p}}_{2kj}\right)}{n_{Ok}-1}\right]\right\}$$

where

$$\widehat{\mathbf{p}}_{2k} = \frac{1}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\boldsymbol{\psi}_{Ak},$$

$$\widehat{V}\left(\widehat{\mathbf{p}}_{ka}\right) = \frac{1}{n_{Ok}^2}N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\left[\text{diag}\left(\widehat{\mathbf{p}}_{Ok}\right)-\widehat{\mathbf{p}}_{Ok}\widehat{\mathbf{p}}_{Ok}^T\right]}{n_{Am}-1} = \frac{\widehat{V}\left(\widehat{\boldsymbol{\psi}}_{Ok}\right)}{n_{Ok}^2},$$

$$\widehat{\mathbf{p}}_{2ki} = \frac{1}{n_{Ok}}\frac{N_{Om}}{n_{Am}}\boldsymbol{\psi}_{Aki},$$

$$\widehat{V}\left(\widehat{\mathbf{p}}_{2ki}\right) = \frac{1}{n_{Ok}^2}N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\left[\text{diag}\left(\widehat{\mathbf{p}}_{Oki}\right)-\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Oki}^T\right]}{n_{Am}-1} = \frac{\widehat{V}\left(\widehat{\boldsymbol{\psi}}_{Oki}\right)}{n_{Ok}^2}$$

and

$$\widehat{Cov}\left(\widehat{\mathbf{p}}_{2ki},\widehat{\mathbf{p}}_{2kj}\right) = -\frac{1}{n_{Ok}^2}N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Oki}^T}{n_{Am}-1} = \frac{\widehat{Cov}\left(\widehat{\boldsymbol{\psi}}_{Oki},\widehat{\boldsymbol{\psi}}_{Okj}\right)}{n_{Ok}^2}.$$

Finally, the first component is estimated by

$$
\begin{aligned}
\widehat{V}_1 =& M_k\left(\frac{M_k}{m_{2k}}-1\right)\left[\frac{\sum_{i=1}^{m_{2k}}\widehat{\boldsymbol{\Psi}}_{ki}\widehat{\boldsymbol{\Psi}}_{ki}^T-\widehat{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}|s_w\right)}{m_{2k}}\right.\\
&\left.-\frac{\sum\sum_{i\neq j}^{m_{2k}}\widehat{\boldsymbol{\Psi}}_{ki}\widehat{\boldsymbol{\Psi}}_{kj}^T-\widehat{Cov}\left(\widehat{\boldsymbol{\Psi}}_{ki},\widehat{\boldsymbol{\Psi}}_{kj}|s_w\right)}{m_{2k}(m_{2k}-1)}\right]\\
=& M_k\left(\frac{M_k}{m_{2k}}-1\right)\frac{\sum_{i=1}^{m_{2k}}\left(\widehat{\boldsymbol{\Psi}}_{ki}-\widehat{\overline{\boldsymbol{\Psi}}}_k\right)\left(\widehat{\boldsymbol{\Psi}}_{ki}-\widehat{\overline{\boldsymbol{\Psi}}}_k\right)^T}{m_{2k}-1}\\
&-\frac{M_k}{m_{2k}}\left(\frac{M_k}{m_{2k}}-1\right)\left[\sum_{i=1}^{m_{2k}}\widehat{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}|s_w\right)-\frac{1}{m_{2k}-1}\sum\sum_{i\neq j}^{m_{2k}}\widehat{Cov}\left(\widehat{\boldsymbol{\Psi}}_{ki},\widehat{\boldsymbol{\Psi}}_{kj}|s_w\right)\right]
\end{aligned}
$$

where $\widehat{\boldsymbol{\Psi}}_{ki}=N_{2k}\mathbf{p}_{2ki}$,

$$
\begin{aligned}
\widehat{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}|s_w\right)=&\left(\frac{N_{2k}}{n_{Ok}}\right)^2N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{Oki}\right)-\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Oki}^T\right]}{n_{Am}-1}\\
&+N_{2k}\left(\frac{N_{2k}}{n_{Ok}}-1\right)\frac{n_{Ok}\left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{2ki}\right)-\widehat{\mathbf{p}}_{2ki}\widehat{\mathbf{p}}_{2ki}^T+\widehat{V}\left(\widehat{\mathbf{p}}_{2ki}\right)\right]}{n_{Ok}-1}\\
=&\frac{N_{2k}(N_{2k}-1)}{n_{Ok}(n_{Ok}-1)}\widehat{V}\left(\widehat{\boldsymbol{\psi}}_{Oki}\right)+\widetilde{V}\left(\widehat{\boldsymbol{\Psi}}_{ki}\right),
\end{aligned}
$$

$$
\begin{aligned}
\widehat{Cov}\left(\widehat{\boldsymbol{\Psi}}_{ki},\widehat{\boldsymbol{\Psi}}_{kj}|s_w\right)=&-\left(\frac{N_{2k}}{n_{Ok}}\right)^2N_{Om}\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{n_{Am}\widehat{\mathbf{p}}_{Oki}\widehat{\mathbf{p}}_{Okj}^T}{n_{Am}-1}\\
&-N_{2k}\left(\frac{N_{2k}}{n_{Ok}}-1\right)\frac{n_{Ok}\left[\widehat{\mathbf{p}}_{2ki}\widehat{\mathbf{p}}_{2kj}^T-\widehat{Cov}\left(\widehat{\mathbf{p}}_{2ki},\widehat{\mathbf{p}}_{2kj}\right)\right]}{n_{Ok}-1}\\
=&\widetilde{Cov}\left(\widehat{\boldsymbol{\Psi}}_{ki},\widehat{\boldsymbol{\Psi}}_{kj}\right)+\frac{N_{2k}(N_{2k}-1)}{n_{Ok}(n_{Ok}-1)}\widehat{Cov}\left(\widehat{\boldsymbol{\psi}}_{Oki},\widehat{\boldsymbol{\psi}}_{Okj}\right)
\end{aligned}
$$

Combining all of the component estimators yields Eq. 4.20.

4.A.6 Performance of Model-based Estimators

As with the models used for estimation when longline gear or trawl gear is used, the models I have proposed for use in estimating catch parameters for pot gear are as limited in assumptions as possible. Only the mean and variance of the super-population generating processes are specified rather than complete probability distributions. Furthermore, the mean and variance parameters are defined at the finest scales possible for variance estimation. However, the models do not truly represent the sampling process and some inconsistencies should be expected. See Section 2.A.5 for more details on the circumstances under which bias of catch estimators and corresponding variance estimators is neglible.

Chapter 5

ESTIMATION AT LARGER SCALES

Here I build on the previous three chapters that develop estimators for hauls made using specific gear-types. Estimation at trip, vessel and fleet levels is, in general, not gear-specific, however, many of the models used for within-haul estimation necessitate some terms that are unique to each gear type. In Section 5.1, I present trip-specific estimators of the various catch parameters for cases when design-based estimation is possible within hauls (Section 5.1.1) and cases when model-based estimation within-hauls is necessary (Section 5.1.2). Section 5.1.3 deals with the very important topic of estimation for specific time periods or regions which may pertain to only a subset of hauls made within trips. There are always instances when insufficient numbers of hauls are sampled during a trip and appropriate estimators for those cases are presented in Section 5.1.4. In Section 5.2, I move on to vessel-specific estimation which also includes an approach to dealing with the unknown number of trips made by vessels with less than 100% observer coverage. I present estimators for parameters over all effort made by several vessels within a quarter and over multiple quarters in Sections 5.3 and 5.4. Some special estimators that are functions of those presented earier in the chapter given in Sections 5.5 and 5.6. I present Estimators at different scales (trips, vessels, etc.) primarily because estimators over long time periods and/or large regions (which are usually of interest) are functions of the more basic estimators. However, infrequently estimation at the trip- or vessel-level is also of interest. There is also an appendix that derives many estimators given in the chapter. As the chapter is laden with special notation, a list of terms and definitions is provided in Table 5.1.

Table 5.1. Definition of terms

a	the subset of hauls where a given species is prevalent (s-prevalence), that is, length measurements of that species are obtained
α_m	probability a hook in the mth management period/region is from a s-prevalent haul
c_v	number of observed/sampled fishing trips made by the vth vessel
c_v^*	number of observed/sampled fishing trips made by the vth vessel that are undersampled
c_v^{**}	number of observed/sampled fishing trips made by the vth vessel that are sufficiently sampled ($c_v^{**} = c_v - c_v^*$)
C_v	number of fishing trips made by the vth vessel
$\overline{c}_M$	average number of trips observed/sampled for all vessels in the medium size class in a given quarter
G_t	number of hauls made in the tth trip
g_t	number of sampled hauls in the tth trip
g_t'	number of unsampled hauls in the tth trip
G_{at}	number of hauls made in the tth trip where a given species is s-prevalent
g_{at}	number of sampled hauls in the tth trip where a given species is s-prevalent
G_{dt}	number of hauls made in the tth trip in the dth time period or region
g_{dt}	number of hauls sampled in the tth trip in the dth time period or region
$g_{dt'}$	number of hauls not sampled in the tth trip in the dth time period or region
G_{mt}	number of hauls made in the tth trip in the mth management region/period
g_{mt}	number of hauls sampled in the tth trip in the mth management region/period
g_m	total number of hauls sampled in the mth management region/period
g_{am}	total number of hauls sampled in the mth management region/period that are prevalent in a given species (s-prevalent)
g_{Ot}	number of sampled hauls made in the tth trip where a given species is s-prevalent and otolith samples are obtained

Table 5.1. (Continued)

G_{amt}	number of hauls in the tth trip in the mth management region/period that are prevalent in a given species (s-prevalent)
g_{Omt}	number of sampled hauls made in the tth trip that occur in the mth management period/region where a given species is s-prevalent and otolith samples are obtained
g_{amt}	number of sampled hauls in the tth trip in the mth management region/period that are prevalent in a given species (s-prevalent)
γ_k	number of marine mammal interactions (for a given species) during sampling in the kth haul
Γ_k	number of marine mammal interactions in the kth haul for a given species
Γ_t	number of marine mammal interactions in the tth trip for a given species
$\overline{\Gamma}_t$	average number of marine mammal interactions per haul in the tth trip for a given species
H_{mt}	number of hooks fished in the tth trip in the mth management region/period
h_{mt}	number of hooks sampled in the tth trip in the mth management region/period
h_m	total number of hooks sampled in the mth management region/period
h_{am}	total number of hooks sampled from hauls in the mth management region/period where a given species is prevalent (s-prevalent)
κ_k	total number of sampled pots in the kth haul when pot gear is used
K_k	total number of pots in the kth haul when pot gear is used
κ'_k	total number of unsampled pots in the kth haul when trawl or pot gear is used
$\boldsymbol{\lambda}_k$	is the vector of numbers in L length classes among fish of a given species in the length sample from the kth haul
$\boldsymbol{\Lambda}_k$	vector of numbers in L length classes caught in the kth haul for a given species
$\boldsymbol{\Lambda}_{at}$	vector of numbers in L length classes caught in the tth trip for a given species where the species is s-prevalent

Table 5.1. (Continued)

$\overline{\mathbf{\Lambda}}_{at}$	vector of average numbers in L length classes caught per haul in the tth trip for a given species where the species is s-prevalent
μ_{Wt}	mean parameter for a super-population model that describes the weights that fish (of a given species) in the tth trip take on
μ_{Wm}	mean parameter for a super-population model that describes the weights that fish (of a given species) in the mth management period/region take on
n_k	the size of a sample from the kth haul, usually a weight sample
n_t	the sum of all sample sizes in the tth trip, usually weight samples
n_{Ok}	the number of fish in the kth haul from which otoliths are retrieved
n_{Ak}	the number of sampled otoliths from the kth haul which are aged
N_{Om}	the number of otoliths sampled from hauls within the mth management period/region
n_{Am}	the number of otoliths sampled from hauls within the mth management period/region that are aged
n_{am}	total number of fish sampled from hauls in the mth management region/period where the given species is prevalent (s-prevalent)
N_k	number caught in the kth haul for a given species
N_t	number caught in the tth trip for a given species
$\overline{N}_t$	average number caught per haul in the tth trip for a given species
ν_m	mean parameter for a super-population model that describes the number of interactions of marine mammals (of a given species) per hook in the mth management period/region
$\mathbf{p}_{\lambda,k}$	the mean vector of a superpopulation model that describes the probability a sampled fish in the kth haul is in one of L length classes
$\mathbf{p}_{\psi,k}$	the mean vector of a superpopulation model that describes the probability a sampled fish in the kth haul is in one of A age classes
$\phi_{\theta,t}$	a mean parameter that describes the rate of increase in the catch parameter (Θ) for the tth trip with respect to a given measured covariate (usually catch weight υ or volume v for trawlers and pots κ for pot vessels)

Table 5.1. (Continued)

$\phi_{\theta,at}$	a mean parameter that describes the rate of increase in the catch parameter (Θ) for the tth trip with respect to a given measured covariate (usually catch weight υ or volume v for trawlers and pots κ for pot vessels) among s-prevalent hauls (i.e., the subset a)
$\phi_{\theta,m}$	a mean parameter that describes the rate of increase in the catch parameter (Θ) in the mth management region/period with respect to a given measured covariate (usually catch weight υ or volume v for trawlers and pots κ for pot vessels)
π_m	probability of capture on a hook in the mth management region/period
π_{am}	probability of capture on a hook in a haul where the species is prevalent (s-prevalent) in the mth management region/period
$\boldsymbol{\psi}_k$	is the vector of numbers in A age classes among aged otoliths from the kth haul
$\boldsymbol{\Psi}_k$	vector of numbers in A age classes caught in the kth haul for a given species
$\boldsymbol{\Psi}_{at}$	vector of numbers in A age classes caught in the tth trip for a given species where the species is s-prevalent
$\overline{\boldsymbol{\Psi}}_{at}$	vector of average numbers in A age classes caught per haul in the tth trip for a given species where the species is s-prevalent
S_t^2	variance among haul totals in the tth trip for a given catch parameter
$S_{\Theta,t}^2$	variance among haul totals of the catch parameter, Θ, in the tth trip
σ_c^2	a variance parameter for a model that describes the number of trips that are sampled for a given vessel in the medium size class
σ_{Wt}^2	variance parameter for a super-population model that describes the weights that fish (of a given species) in the tth trip take on
σ_{Wm}^2	variance parameter for a super-population model that describes the weights that fish (of a given species) in the mth management period/region take on
$\sigma_{\theta,t}^2$	a variance parameter for a super-population model that describes the values the parameter θ takes on in the tth trip

Table 5.1. (Continued)

$\sigma^2_{\theta,m}$	a variance parameter for a super-population model that describes the values the parameter θ takes on in the mth management region/period
$\sigma^2_{\theta,am}$	a variance parameter for a super-population model that describes the values the parameter θ takes on in the mth management region/period where the given species is prevalent (s-prevalent)
τ_m	dispersion parameter for a mixture model describing the probability of capture on a hook in the mth management region/period
τ_{am}	dispersion parameter for a mixture model describing the probability of capture on a hook in a haul where the species is prevalent (s-prevalent) in the mth management region/period
$\tau_{\alpha m}$	dispersion parameter for a mixture model describing the probability a hook in the mth management period/region is from a s-prevalent haul
$\tau_{\nu m}$	dispersion parameter for a mixture model describing the number of marine mammal interactions per hook (for a given species) in the mth management region/period
Θ_k	generic catch parameter total in the kth haul, usually for numbers or weight
$\boldsymbol{\Theta}_k$	generic vector-valued catch parameter for totals in the kth haul, usually for numbers in length or age classes
$\boldsymbol{\Theta}_t$	generic vector-valued catch parameter for totals in the tth trip, usually for numbers in length or age classes
$\overline{\boldsymbol{\Theta}}_t$	average generic vector-valued catch parameter total per haul in the tth trip, usually for numbers in length or age classes
$\overline{\boldsymbol{\Theta}}_t$	generic vector-valued catch parameter for the average totals per haul in the tth trip, usually for numbers in length or age classes
$\boldsymbol{\Theta}_{mt}$	generic vector-valued catch parameter for totals in the tth trip occuring in the mth management period/region
υ_k	total weight of the sampled portion of catch made in the kth haul, particularly when trawl gear is used
Υ_k	total weight of the catch made in the kth haul, particularly when trawl is used

Table 5.1. (Continued)

υ'_k	total weight of the unsampled portion of catch made in the kth haul, particularly when trawl gear is used
V_k	total volume of the sampled portion of catch made in the kth haul, particularly when trawl gear is used
v_k	total volume of the catch made in the kth haul, particularly when trawl gear is used
v'_k	total volume of the unsampled portion of catch made in the kth haul, particularly when trawl gear is used
V	the total number of vessels in an arbitrarily defined set
V_M	the total number of vessels in the medium size class in a given quarter
W_k	weight of catch in the kth haul for a given species
$\overline{W}_k$	average weight of sampled fish in the kth haul for a given species
W_t	weight of catch in the tth trip for a given species
$\overline{W}_t$	average weight of catch per haul in the tth trip for a given species
x_k	a covariate total for the sampled portion of the catch made in the kth haul, usually total weight (υ_k) or total volume (v_k) of the catch made in the kth haul for trawl hauls or total number of pots (κ_k) for pot hauls
X_k	a covariate total for the kth haul, usually total weight (Υ_k) or total volume (V_k) of the catch made in the kth haul for trawl hauls or total number of pots (K_k) for pot hauls
x_t	a covariate total for the sampled portions of catches made on the tth fishing trip, usually total weight (υ_t) or total volume (v_t) of the catch made on the tth trip for trawlers or total number of pots (κ_t) when pots are fished
X_t	a covariate total for catches made on the tth fishing trip, usually total weight (Υ_t) or total volume (V_t) of the catch made on the tth trip for trawlers or total number of pots (K_t) when pots are fished
x'_t	a covariate total for the unsampled portions of catches made on the tth fishing trip, usually total weight (υ'_t) or total volume (v'_t) of the unsampled portions made on the tth trip for trawlers or total number of unsampled pots (κ'_t) when pots are fished

Table 5.1. (Continued)

x_m	a covariate total for the sampled portions of catches made in the mth managment region/period, usually total weight (υ_m) or total volume (v_m) of the catch made in the mth period/region for trawlers or total number of pots (κ_m) when pots are fished
X_{mt}	a covariate total for catches made in the tth trip occuring in the mth management region/period, usually total weight (Υ_m) or total volume (V_{mt}) of the catches for trawlers or total number of pots (K_{mt}) when pots are fished
ξ_v	a scalar defined as the inverse of the expected proportion of trips sampled for the vth vessel

5.1 Within a Trip

Recall from Section 1.1.1 that most fishing activities require observer coverage at all times aboard vessels greater than 125 ft in length (large)and vessels less than 125 ft but greater than 60 ft in length (medium) require observer coverage 30% of all fishing days within each quarter. More recently, medium pot vessels have been regulated to have observer coverage for 30% of all pots fished. Regardless, there is no distinction of observed fishing trips for each vessel in archived data, but there is distinction of effort made by each vessel and each observer contract.

The most obvious way to define a fishing trip is as a distinct group of hauls made by a vessel, however, what makes the group distinct can be unclear and there is no set criteria for data reported in Alaskan groundfish fisheries. As such, arbtitrary (but hopefully objective) methods to define trips are necessary. For the purposes of my applications in the following chapter, I will define the group by consecutive hauls that are made with less than D days between them. A period of D or more days between hauls separates two trips made by a vessel and trips cannot overlap consecutive quarters. Furthermore, hauls attributed to Community Development Quotas (CDQ) are separated from non-CDQ hauls.

During each trip, the observer follows one of several different randomization procedures to determine which hauls are sampled within a trip and MRAG (2003) notes that none of these procedures is exactly SRS. In fact, because of realities such as physical exhaustion due to sampling several consecutive hauls and because the total number of hauls is not known prior to sampling, it is not possible to obtain a SRS of hauls within a trip. However, when there is no relationship between the randomization procedures for haul selection and the contents of the hauls, a SRS assumption will yield appropriate inferences. Furthermore, if there is no correlation of randomization procedures used to collect data within different hauls (e.g., subsampling of collected otoliths for ageing), then the properties of all estimators for the tth trip are based on some type of a two-stage process: random haul selection and random sampling within hauls.

In Section 5.1.1, I will present trip-specific estimators for all catch parameters except numbers-at-age when design-based estimation is possible within hauls. The estimators will be general enough in form to cover all gear types. Numbers-at-age are not discussed because under the likely sampling scenarios for otoliths it is doubtful that design-based variance estimation within hauls is possible. There are just too few otoliths collected in a haul and too many stages of sampling within hauls to obtain them. Furthermore, subsampling of otoliths for ageing results in correlation across hauls from different trips and vessels.

In Section 5.1.2, I will present estimators appropriate when model-based estimation within hauls is necessary. These will be gear-specific in many cases because of the type of models I use. The model-based methods needed for data collected within hauls and the nature of the various proposed estimators require that correlation across hauls be taken into account for many trip-specific estimators. Estimators of total numbers for fish, seabirds, marine mammal interactions and numbers in length and age classes for trips made using longline gear will have no correlation among hauls.

5.1.1 When Design-based Estimation is Used within Hauls

5.1.1.1 Total Number for a Fish or Seabird Species

When design-based methods are possible within trips and a SRS of hauls within a trip is assumed, the natural unbiased estimator of the total number caught on the tth trip for a given species is

$$\widehat{N}_t = \frac{G_t}{g_t} \sum_{k=1}^{g_t} \widehat{N}_k \tag{5.1}$$

where G_t, g_t and $\widehat{N}_k$ are the total number of hauls made and sampled during the tth trip and the estimated total number in the kth haul on the tth trip, respectively . This is the expanded sum of the within-haul estimates of the total and the Horvitz-Thompson estimator (Horvitz and Thompson 1952) of the total in a two-stage sampling design. The method of estimation within each haul will depend on the sampling procedure and the gear type. See Sections 2.1, 3.2 and 4.1 for estimators appropriate when longline, trawl or pot gear, respectively, is used. The variance is

$$V\left(\widehat{N}_t\right) = G_t\left(\frac{G_t}{g_t} - 1\right)S_t^2 + \frac{G_t}{g_t}\sum_{k=1}^{G_t} V\left(\widehat{N}_k\right) \tag{5.2}$$

where

$$S_t^2 = \frac{\sum_{k=1}^{G_t}\left(N_k - \overline{N}_t\right)^2}{G_t - 1},$$

$\overline{N}_t = \sum_{k=1}^{G_t} N_k/G_t$ and $V\left(\widehat{N}_k\right)$ is the within-haul variance which, of course, also depends on gear type. The variance estimator is

$$\widehat{V}\left(\widehat{N}_t\right) = G_t\left(\frac{G_t}{g_t} - 1\right)\widetilde{S}_t^2 + \frac{G_t}{g_t}\sum_{k=1}^{g_t} \widehat{V}\left(\widehat{N}_k\right) \tag{5.3}$$

where

$$\widetilde{S}_t^2 = \frac{\sum_{k=1}^{g_t}\left(\widehat{N}_k - \widehat{\overline{N}}_t\right)^2}{g_t - 1}$$

and $\widehat{\overline{N}}_t = \sum_{k=1}^{g_t} \widehat{N}_k/g_t$. The form of the within-haul variance estimators, $\widehat{V}\left(\widehat{N}_k\right)$, corresponds to the estimator of the total in the kth haul. The derivation of this estimator and properties in general for a two-stage design are given in Sections A.3 and A.3.1.

5.1.1.2 Total Weight for a Fish Species

Because the same SRS assumption is made, the form of the estimator for total catch weight for a species is identical to that for total number (Eq. 5.1). The unbiased total weight estimator for the tth trip is

$$\widehat{W}_t = \frac{G_t}{g_t}\sum_{k=1}^{g_t}\widehat{W}_k \tag{5.4}$$

where $\widehat{W}_k$ depends on the type of gear deployed and the within-haul sampling procedure. For longline hauls, estimators appropriate for a variety of sampling scenarios were presented in Section 2.2.1 and various estimators were also presented for trawl and pot vessels in Sections 3.1 and 4.2, respectively. The variance is analogous to Eq. 5.2 as is the variance estimator, however, I present the variance estimator for completeness:

$$\widehat{V}\left({}_1\widehat{W}_t\right) = G_t\left(\frac{G_t}{g_t}-1\right)\widetilde{S}^2_{W_t} + \frac{G_t}{g_t}\sum_{k=1}^{g_t}\widehat{V}\left(\widehat{W}_k\right) \tag{5.5}$$

where

$$\widetilde{S}^2_{W_t} = \frac{\sum_{k=1}^{g_t}\left(\widehat{W}_k - \widehat{\overline{W}}_t\right)^2}{g_t - 1}$$

and $\widehat{\overline{W}}_t = \sum_{k=1}^{g_t}\widehat{W}_k/g_t$. Like the variance of the estimator, the appropriate estimator of within-haul variance, $\widehat{V}\left(\widehat{W}_k\right)$, depends on the within-haul estimator of total biomass. For longline hauls, many of the within-haul estimators of total biomass are asymptotically unbiased and the variance presented are approximate results (asymptotically unbiased). In these cases, the $AV\left(\widehat{W}_k\right)$ and $\widehat{AV}\left(\widehat{W}_k\right)$ are substituted for $V\left(\widehat{W}_k\right)$ and $\widehat{V}\left(\widehat{W}_k\right)$, respectively (AV = Asymptotic Variance).

5.1.1.3 Marine Mammal Interactions

The form of the estimator of the number of marine mammal interactions for the tth trip (and the corresponding variance estimator) is the same as that for total number of a fish or seabird species and the total weight of a fish species because of the SRS assumption for hauls within the trip. The estimator for the total number of interactions and corresponding variance estimator are

$$\widehat{\Gamma}_t = \frac{G_t}{g_t}\sum_{k=1}^{g_t}\widehat{\Gamma}_k \tag{5.6}$$

and

$$\widehat{V}\left(\widehat{\Gamma}_t\right) = G_t\left(\frac{G_t}{g_t} - 1\right)\widetilde{S}^2_{\Gamma,t} + \frac{G_t}{g_t}\sum_{k=1}^{g_t}\widehat{V}\left(\widehat{\Gamma}_k\right) \tag{5.7}$$

where

$$\widetilde{S}^2_{\Gamma,t} = \frac{\sum_{k=1}^{g_t}\left(\widehat{\Gamma}_k - \widehat{\overline{\Gamma}}_t\right)^2}{g_t - 1}$$

and $\widehat{\overline{\Gamma}}_t = \sum_{k=1}^{g_t}\widehat{\Gamma}_k/g_t$. The form of the within-haul estimator, $\widehat{\Gamma}_k$, depends on the gear type.

When trawl gear is used, it was noted in Section 3.5 that the entire haul is observed for marine mammals. Therefore, there is no uncertainty in the number within the sampled hauls and the estimator reduces to

$$\widehat{\Gamma}_t = \frac{G_t}{g_t}\sum_{k=1}^{g_t}\Gamma_k \tag{5.8}$$

and the variance estimator reduces to

$$\widehat{V}\left(\widehat{\Gamma}_t\right) = G_t\left(\frac{G_t}{g_t} - 1\right)\widehat{S}^2_{\Gamma,t} \tag{5.9}$$

where

$$\widehat{S}^2_{\Gamma,t} = \frac{\sum_{k=1}^{g_t}\left(\Gamma_k - \widehat{\overline{\Gamma}}_t\right)^2}{g_t - 1}$$

and $\widehat{\overline{\Gamma}}_t = \sum_{k=1}^{g_t}\Gamma_k/g_t$. The estimator (Eq. 5.8) and variance estimator (Eq. 5.9) are the well-known results appropriate for single-stage SRS presented in all introductory sampling texts.

5.1.1.4 Total Numbers in Length Classes

Estimation of the trip-specific total numbers in each length class can be achieved, but because length samples are only taken from hauls where the species is prevalent, estimates will pertain only to hauls made in the trip with this characteristic. Thus, the set of hauls where s is prevalent is a domain, a, within the set of all hauls that are sampled, hereafter referred to as s-prevalent. Furthermore, the randomization scheme that an observer follows is not dictated by s-prevalence and so, under the SRS assumption for haul sampling, the number of hauls sampled in the s-prevalent subset is random and the total number of s-prevalent hauls is unknown. Also, it is assumed that whether a sampled haul is s-prevalent is known without error for sampled hauls and is indicated by the decision of the observer to take length measurements for species s. Therefore, for whatever species the observer takes length measurements in the kth haul, that haul is s-prevalent for all of those species.

The estimator of the vector of total numbers-at-length in hauls prevalent in the species of interest during the tth trip would be

$$\widehat{\mathbf{\Lambda}}_{at} = \frac{G_t}{g_t} \sum_{k=1}^{g_{at}} \widehat{\mathbf{\Lambda}}_k \tag{5.10}$$

where g_{at} denotes the number of hauls sampled that are s-prevalent and $\widehat{\mathbf{\Lambda}}_k$ depends on the gear type and within-haul sampling procedure. The variance-covariance matrix (VCM) is

$$\begin{aligned} V\left(\widehat{\mathbf{\Lambda}}_{at}\right) =& G_t \left(\frac{G_t}{g_t} - 1\right) \left[\frac{\sum_{k=1}^{G_{at}} \mathbf{\Lambda}_k \mathbf{\Lambda}_k^T - \left(\sum_{k=1}^{G_{at}} \mathbf{\Lambda}_k\right) \left(\sum_{k=1}^{G_{at}} \mathbf{\Lambda}_k\right)^T / G_t}{G_t - 1}\right] \\ &+ \frac{G_t}{g_t} \sum_{k=1}^{G_{at}} V\left(\widehat{\mathbf{\Lambda}}_k\right) \end{aligned}$$

where $V\left(\widehat{\mathbf{\Lambda}}_k\right)$ is the VCM of the kth within-haul estimator. The corresponding VCM estimator is

$$\begin{aligned} \widehat{V}\left(\widehat{\mathbf{\Lambda}}_{at}\right) =& G_t \left(\frac{G_t}{g_t} - 1\right) \left[\frac{\sum_{k=1}^{g_{at}} \widehat{\mathbf{\Lambda}}_k \widehat{\mathbf{\Lambda}}_k^T - \left(\sum_{k=1}^{g_{at}} \widehat{\mathbf{\Lambda}}_k\right) \left(\sum_{k=1}^{g_{at}} \widehat{\mathbf{\Lambda}}_k\right)^T / g_t}{g_t - 1}\right] \\ &+ \frac{G_t}{g_t} \sum_{k=1}^{g_{at}} \widehat{V}\left(\widehat{\mathbf{\Lambda}}_k\right) \end{aligned} \tag{5.11}$$

where $\widehat{V}\left(\widehat{\mathbf{\Lambda}}_k\right)$ is the appropriate within-haul VCM estimator. Notice that for unbiased, design-based, variance estimation, $g_t > 1$ is necessary rather than $g_{at} > 1$.

5.1.1.5 When Only One Haul is Made in the Trip

For completeness, I relate here the general form for estimators of the various catch parameters when only one haul happens to be made during a pre-defined trip. Using the generic parameter, $\boldsymbol{\Theta}$ the total for the tth trip is just the total for the single haul, $\boldsymbol{\Theta}_t = \boldsymbol{\Theta}_k$, and the estimator and VCM estimator are just

$$\widehat{\boldsymbol{\Theta}}_t = \widehat{\boldsymbol{\Theta}}_k \tag{5.12}$$

and

$$\widehat{V}\left(\widehat{\boldsymbol{\Theta}}_t\right) = \widehat{V}\left(\widehat{\boldsymbol{\Theta}}_k\right). \tag{5.13}$$

It is, of course, necessary that the single haul is sampled for estimation of the trip total. If the single haul is not sampled, then model-based methods are necessary to predict the total for the haul. I deal with this problem in Section 5.1.4.

5.1.2 When Model-based Estimation is Necessary within Hauls

5.1.2.1 Total Number of Animals or Mammal Interactions for Catches Made in a Longline Trip

When the model-based estimators for total number of fish or seabirds (Sections 2.1.2) or marine mammal interactions (Section 2.5.2) are used within longline hauls, the model parameters are unique to each haul and, thus, within-haul estimators are independent from one another. Therefore, the within-trip estimator of total number and the corresponding variance have the same form as Eq. 5.1 and Eq. 5.2, in the case of fish and seabirds, or Eq. 5.8 and Eq. 5.9, in the case of marine mammal interactions, whether or not model-based estimators are used within hauls. The difference is in the form of each within-haul estimator, $\widehat{N}_k$ or $\widehat{\Gamma}_k$ and variance, $V\left(\widehat{N}_k\right)$ or $V\left(\widehat{\Gamma}_k\right)$. Letting N_k denote either fish and seabird species or marine mammal interactions, the estimator and variance estimator are

$$\widehat{N}_t = \frac{G_t}{g_t}\sum_{k=1}^{g_t}\widehat{N}_k \tag{5.14}$$

and

$$\widehat{V}\left({}_{2}\widehat{N}_t\right) = G_t\left(\frac{G_t}{g_t} - 1\right)\widetilde{S}_t^2 + \frac{G_t}{g_t}\sum_{k=1}^{g_t}\widehat{V}\left(\widehat{N}_k\right) \tag{5.15}$$

where

$$\widetilde{S}_t^2 = \frac{\sum_{k=1}^{g_t}\left(\widehat{N}_k - \overline{\widehat{N}}_t\right)^2}{g_t - 1}$$

and $\overline{\widehat{N}}_t = \sum_{k=1}^{g_t}\widehat{N}_k/g_t$.

5.1.2.2 *Total Weight for Longline Trips*

Under the super-population model assumed in Section 2.2.2 and given an unbiased and independent estimator of the total number, an unbiased estimator for the total weight of a species in the tth trip is

$${}_{2}\widehat{W}_t = \widehat{N}_t\frac{\sum_{k=1}^{g_t} n_k\overline{W}_k}{n_t} = \widehat{N}_t\widehat{\mu}_{Wt} \tag{5.16}$$

where $n_t = \sum_{k=1}^{g_t} n_k$ is the total number of fish weighed during the trip and $\overline{W}_k$ is the average weight of the sampled fish in the kth haul. The corresponding variance and unbiased variance estimator are

$$\begin{aligned} V\left({}_{2}\widehat{W}_t\right) =& N_t\left(\frac{N_t}{n_t} - 1\right)\sigma_{Wt}^2 + \left(\mu_{Wt}^2 + \frac{\sigma_{Wt}^2}{n_t}\right)V\left(\widehat{N}_t\right) \\ =& N_t\left(\frac{N_t}{n_t} - 1\right)\sigma_{Wt}^2 + \mu_{Wt}^2\widehat{V}\left(\widehat{N}_t\right) + V\left(\widehat{N}_t\right)\frac{\sigma_{Wt}^2}{n_t} \end{aligned}$$

and

$$\begin{aligned} \widehat{V}_1\left({}_{2}\widehat{W}_t\right) =& \left[\frac{\widehat{N}_t^2 - \widehat{V}\left(\widehat{N}_t\right)}{n_t} - \widehat{N}_t\right]\widehat{\sigma}_{Wt}^2 + \widehat{\mu}_{Wt}^2\widehat{V}\left(\widehat{N}_t\right) \\ =& \widehat{N}_t\left(\frac{\widehat{N}_t}{n_t} - 1\right)\widehat{\sigma}_{Wt}^2 + \widehat{\mu}_{Wt}^2\widehat{V}\left(\widehat{N}_t\right) - \widehat{V}\left(\widehat{N}_t\right)\frac{\widehat{\sigma}_{Wt}^2}{n_t} \end{aligned} \tag{5.17}$$

where

$$\widehat{\sigma}_{Wt}^2 = \frac{g_t}{\sum_{k=1}^{g_t}\frac{1}{n_k}}\frac{\sum_{k=1}^{g_t}\left(\overline{W}_k - \overline{\overline{W}}_t\right)^2}{g_t - 1},$$

$\overline{\overline{W}}_t = \sum_{k=1}^{g_t} \overline{W}_k / g_t$ and $\widehat{N}_t$, $V\left(\widehat{N}_t\right)$ and $\widehat{V}\left(\widehat{N}_t\right)$ are the estimator, variance and variance estimator, respectively, for the total number of the species in the trip. Design- or model-based methods may be possible for the total number and the corresponding estimators should be used. Derivations of the estimator and variance estimator can be found in Section A.8 of the appendix.

5.1.2.3 *Total Number of Animals or Mammal Interactions for Catches Made in a Trawl or Pot Trip*

The nature of data collection from trawl and pot hauls is such that trip-specific, model-based estimation of total numbers caught of fish, invertebrate or seabird species and weight caught of fish or invertebrate species is different from trips where longline gear is used. Recall from Sections 3.6 and 4.6 that the model-based methods I developed for estimation of total numbers and weight were identical whether pots or trawl gear is used, however, the covariates used are numbers of pots for the former and weight or volume for the latter. For marine mammal interactions model-based methods should only be necessary for pot gear because when a trawl haul is selected for sampling the observer censuses (samples the entire haul) for any marine mammals.

Letting Θ_t be either total number, N_t or total weight, W_t, the best linear unbiased predictor of the total under the constant variance model (Eq. 3.29) is

$${}_1\widehat{\Theta}_t = \sum_{k=1}^{g_t} \theta_k + {}_1\widehat{\phi}_{\theta,t} \left(\sum_{k=1}^{g_t} x'_k + \sum_{l=1}^{g'_t} X_l \right) = \theta_t + {}_1\widehat{\phi}_{\theta,t} x'_t$$

where ${}_1\widehat{\phi}_{\theta,t}$ is the scalar form of the slope estimator defined in Eq. 3.31, x'_k is the covariate total for the unsampled portion of the kth haul (total weight, υ, or volume, v for trawlers or pots κ for pot gear), g'_t is the the number of unsampled hauls in the tth trip and X_k is the covariate total for the kth haul. The prediction error variance estimator is

$$\widehat{V}_M\left({}_1\widehat{\Theta}_t\right) = \widehat{\sigma}^2_{\theta,t} \left[\frac{(x'_t)^2}{\sum_{k=1}^{g_t} x_k^2} + m'_t \right]$$

where $m'_t = \sum_{k=1}^{g_t} m'_k + \sum_{k=1}^{g_t'} M_k$, m'_k is the number of unsampled sampling units in the k'th haul, M_k is the total number of sampling units for the kth haul and

$$\widehat{\sigma}^2_{\theta,t} = \frac{\sum_{k=1}^{g_t}\left(\theta_k - {}_1\widehat{\phi}_{\theta,t}x_k\right)^2}{g_t - 1}$$

is the scalar form of Eq. 3.32. The definition of sampling units was different for trawlers and pots in the respective sections on within-haul estimation. Even for trawlers, there were multiple definitions depending on the way hauls were sampled for species composition. Notice that the variance estimator requires M_k, the total number of sampled units that would have been sampled in unsampled hauls. This information is not known because the fraction of the haul an observer samples is not consistent from haul to haul. Fortunately, the alternative model in Eq. 3.33 that assumes variance proportional to the covariate yields a more useful variance estimator. Furthermore, the proportional variance model is perhaps more plausible due to the relationship with binomial and Poisson models.

When the alternative model is assumed, the best linear unbiased estimator is

$$ {}_2\widehat{\Theta}_t = \sum_{k=1}^{g_t}\theta_k + {}_2\widehat{\phi}_{\theta,t}\left(\sum_{k=1}^{g_t}x'_k + \sum^{g'_t}X_k\right) = \theta_t + {}_2\widehat{\phi}_{\theta,t}x'_t \tag{5.18}$$

where ${}_2\widehat{\phi}_{\theta,t}$ is the scalar form of Eq. 3.34 and the prediction error variance estimator is

$$\widehat{V}_M\left({}_2\widehat{\Theta}_t\right) = \widehat{\sigma}^2_{\theta,t}\frac{x'_t}{x_t}X_t \tag{5.19}$$

where $x_t = \sum_{k=1}^{g_t}x_k$, $X_t = \sum_{k=1}^{G_t}X_k$ and

$$\widehat{\sigma}^2_{\theta,t} = \frac{x_t}{x_t^2 - \sum_{k=1}^{g_t}x_k^2}\sum_{k=1}^{g_t}\left(\theta_k - {}_2\widehat{\phi}_{\theta,t}x_k\right)^2.$$

As I alluded, only the covariate totals for unsampled hauls, X_k, and unsampled portions of sampled hauls, x'_k, are needed for variance estimation with this model. Also, notice that $x_t^2 - \sum_{k=1}^{g_t}x_k^2$ implies the same requirement as $g_t - 1$ that more than one haul must be sampled for variance estimation. In fact, because, in general, we have only one unit sampled within each trip other methods are necessary when there is only one haul made and sampled in a given trip.

For trawlers, I mentioned in Section 3.6 the use of weight or volume of sampling units for the covariate. For trawler effort already archived, total weight will presumably be the easiest covariate to use because some value is always present in the NPGOP database as the "Official Total Catch," however, as mentioned with respect to estimation for hauls, that this is known without error can be a rather strong assumption. For pot vessels, the sampling units are pot-groups within a string (haul) and because the number of pots is more well-defined, they will lend themselves to model-based estimation more easily.

5.1.2.4 Total Numbers in Length Classes for a Longline Trip

Recall from Section 5.1.1.4, that estimator of the total numbers in length classes pertain only to hauls in the trip where the given species, s, is deemed prevalent (s-prevalent) which was denoted as the domain, a. In the models I assumed for longline hauls in Section 2.3, all parameters were unique to each haul and there is no correlation of predictors across sampled hauls. If we assume the same SRS sampling procedures to determine hauls that are sampled as those in Section 5.1.1, then the estimator for the numbers-at-length and the corresponding VCM estimator are identical to Eq. 5.10 and Eq. 5.11, respectively, which are appropriate when design-based estimation is possible within hauls. Nevertheless, I present Eq. 5.20 and Eq. 5.21 for completeness.

The estimator of the vector of total numbers-at-length in s-prevalent hauls during the tth trip would be

$$\widehat{\mathbf{\Lambda}}_{at} = \frac{G_t}{g_t} \sum_{k=1}^{g_{at}} \widehat{\mathbf{\Lambda}}_k \tag{5.20}$$

where g_{at} denotes the number of s-prevalent hauls sampled and $\widehat{\mathbf{\Lambda}}_k$ is one of the model-based within-haul estimators presented in Section 2.3. The corresponding VCM estimator is

$$\begin{aligned}\widehat{V}\left(\widehat{\mathbf{\Lambda}}_{at}\right) =& G_t\left(\frac{G_t}{g_t}-1\right)\left[\frac{\sum_{k=1}^{g_{at}} \widehat{\mathbf{\Lambda}}_k \widehat{\mathbf{\Lambda}}_k^T - \left(\sum_{k=1}^{g_{at}} \widehat{\mathbf{\Lambda}}_k\right)\left(\sum_{k=1}^{g_{at}} \widehat{\mathbf{\Lambda}}_k\right)^T / g_t}{g_t - 1}\right] \\ &+ \frac{G_t}{g_t} \sum_{k=1}^{g_{at}} \widehat{V}\left(\widehat{\mathbf{\Lambda}}_k\right)\end{aligned} \tag{5.21}$$

where $\widehat{V}\left(\widehat{\mathbf{\Lambda}}_k\right)$ is the appropriate within-haul VCM estimator.

5.1.2.5 Total Numbers in Length Classes for Trawl and Pot Trips

Recall from Sections 3.6.4 and 4.6.3 that the model-based estimator for total numbers in length classes within hauls, Eq. 3.43 or Eq. 4.25, is a product of the model-based estimators of total number and proportions-at-length. This is also the case for the estimator developed for longline hauls, but for trawl and pot gear, the model I used for total numbers had parameters common to the trip where the hauls are made whereas for longline gear it is possible to have models specific to each haul. Therefore, although the numbers-at-length estimator for a pot/trawl trip,

$$\widehat{\mathbf{\Lambda}}_{at} = \frac{G_t}{g_t}\sum_{k=1}^{g_{at}}\widehat{\mathbf{\Lambda}}_k = \frac{G_t}{g_t}\sum_{k=1}^{g_{at}}\widehat{N}_k\widehat{\mathbf{p}}_{\lambda,k}, \tag{5.22}$$

is the same in form as that for a longline trip, the variance estimator,

$$\begin{aligned}\widehat{V}\left(\widehat{\mathbf{\Lambda}}_{at}\right) =& G_t\left(\frac{G_t}{g_t}-1\right)\left[\frac{\sum_{k=1}^{g_{at}}\widehat{\mathbf{\Lambda}}_k\widehat{\mathbf{\Lambda}}_k^T - \left(\sum_{k=1}^{g_{at}}\widehat{\mathbf{\Lambda}}_k\right)\left(\sum_{k=1}^{g_{at}}\widehat{\mathbf{\Lambda}}_k\right)^T/g_t}{g_t-1}\right] \\ &+ \frac{G_t}{g_t}\left[\sum_{k=1}^{g_{at}}\widehat{V}\left(\widehat{\mathbf{\Lambda}}_k\right) + \frac{G_t-1}{g_t-1}\sum\sum_{k\neq l}^{g_{at}}\widehat{Cov}\left(\widehat{\mathbf{\Lambda}}_k, \widehat{\mathbf{\Lambda}}_l\right)\right],\end{aligned} \tag{5.23}$$

must account for covariance of haul-specific estimates. Moreover, for numbers in length classes it seems appropriate to make the parameters of the model used to estimate total numbers specific to s-prevalent hauls. That is, instead of the trip-specific parameters $\phi_{n,t}$ and $\sigma^2_{n,t}$ I use $\phi_{n,at}$ and $\sigma^2_{n,at}$ which are specific to s-prevalent hauls in the tth trip. The two modified models are

$$\begin{aligned}E_M\left(n_k\right) =& x_k\phi_{n,at} \\ V_M\left(n_k\right) =& \sigma^2_{n,at} \\ Cov_M\left(n_k, n_l\right) =& 0 \quad \text{for} \quad k\neq l.\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}E_M\left(n_k\right) =& x_k\phi_{n,at}\\V_M\left(n_k\right) =& x_k\sigma^2_{n,at}\\Cov_M\left(n_k, n_l\right) =& 0 \quad \text{for} \quad k \neq l..\end{aligned} \tag{5.25}$$

The within-haul predictor of total number specific to s-prevalent hauls is

$$\widehat{N}_k = n_k + \widehat{\phi}_{n,at}x'_k$$

where

$$\widehat{\phi}_{n,at} = \frac{\sum_{k=1}^{g_{at}} n_k x_k}{\sum_{k=1}^{g_{at}} x_k^2} \equiv {}_1\widehat{\phi}_{n,at}$$

when the constant variance model in Eq. 5.24 is used and

$$\widehat{\phi}_{n,at} = \frac{\sum_{k=1}^{g_{at}} n_k}{\sum_{k=1}^{g_{at}} x_k} \equiv {}_2\widehat{\phi}_{n,at}$$

when the proportional variance model in Eq. 5.25 is used. The respective within-haul variance estimators, are

$$\widehat{V}_M\left(\widehat{N}_k\right) = \left[\frac{x^2_{k'}}{\sum_{k=1}^{g_{at}} x_k^2} + (M_k - 1)\right]\widehat{\sigma}^2_{n,at}. \tag{5.26}$$

where

$$\widehat{\sigma}^2_{n,at} = \frac{\sum_{k=1}^{g_{at}}\left(n_k - {}_1\widehat{\phi}_{n,at}x_k\right)^2}{g_{at} - 1}.$$

and

$$\widehat{V}_M\left(\widehat{N}_k\right) = \left(\frac{x_{k'}}{x_{at}} + 1\right) x_{k'}\widehat{\sigma}^2_{n,at} \tag{5.27}$$

where

$$\widehat{\sigma}^2_{n,at} = \frac{x_{at}}{x^2_{at} - \sum_{k=1}^{g_{at}} x_k^2}\sum_{k=1}^{g_{at}}\left(n_k - {}_2\widehat{\phi}_{n,at}x_k\right)^2.$$

and $x_{at} = \sum_{k=1}^{g_{at}} x_k$. The within-haul VCM estimator, $\widehat{V}\left(\widehat{\mathbf{\Lambda}}_k\right)$ is given by Eq. 3.44 or Eq. 4.26 depending on the gear and the covariance estimator is given by

$$\widehat{Cov}\left(\widehat{\mathbf{\Lambda}}_k, \widehat{\mathbf{\Lambda}}_l\right) = x'_k x'_l \widehat{V}\left(\widehat{\phi}_{n,at}\right) \widehat{\mathbf{p}}_{\lambda,k} \widehat{\mathbf{p}}_{\lambda,l}^T.$$

where $\widehat{\mathbf{p}}_{\lambda,k} = \boldsymbol{\lambda}_k / n_{Lk}$ and

$$\widehat{V}\left(\widehat{\phi}_{n,at}\right) = \frac{\widehat{\sigma}^2_{n,at}}{\sum_{k=1}^{g_{at}} x_k^2} \tag{5.28}$$

with $\widehat{\sigma}^2_{n,at}$ given in Eq. 5.26 or

$$\widehat{V}\left(\widehat{\phi}_{n,at}\right) = \frac{\widehat{\sigma}^2_{n,at}}{\sum_{k=1}^{g_{at}} x_k} \tag{5.29}$$

with $\widehat{\sigma}^2_{n,at}$ given in Eq. 5.27. The covariate total for the unsampled portion of the kth haul is given by x'_k and $x'_k = X_k$ is the total of the covariate for the entire haul when unsampled (See Table 5.1 for covariate definitions).

5.1.2.6 Total Numbers in Age Classes

The observers are instructed to take otolith samples from every zth length sample where z depends on the species of interest and a subsample of those otoliths are analyzed to determine age. The determination of hauls where otoliths are sampled implies a systematic sample of s-prevalent hauls and when there is no reason to believe a periodic pattern in the abundant hauls a SRS can safely be assumed for the (systematic) subsample of the s-prevalent hauls that were sampled for lengths (e.g., Särndal et al. 1992, Section 3.4).

If SRS is used to subsample all otoliths collected in the mth period/region for ageing, then there is covariance between hauls made in that period/region where otoliths are obtained. Covariance of hauls-specific estimators of total numbers-at-age due to otolith subsampling is common to all three gear types I have treated, but, as discussed in Section 5.1.2.5 for numbers-at-length estimators, the model-based methods specific to trawl and pot gear introduce additional covariance via the total numbers estimators.

Assuming simple random subsampling of otoliths for ageing, an estimator for the vector of total numbers-at-age within the tth trip for s-prevalent hauls in the mth subsampling time period/region is

$$\widehat{\Psi}_{amt} = \frac{G_t}{g_t}\frac{g_{at}}{g_{Ot}}\sum_{k=1}^{g_{Omt}}\widehat{\Psi}_k \tag{5.30}$$

where g_{Omt} is the number of hauls where otoliths are obtained in the mth period/region, g_{Ot} is the total number of hauls where otoliths are obtained and g_{at} is the number of s-prevalent hauls. I equate indication of s-prevalence with the determination of observers to obtain length measurements for the respective species. The VCM is

$$\begin{aligned} V\left(\widehat{\Psi}_{amt}\right) =& G_t\left(\frac{G_t}{g_t}-1\right)\frac{\sum_{k=1}^{G_{amt}}\Psi_k\Psi_k^T - \left(\sum_{k=1}^{G_{amt}}\Psi_k\right)\left(\sum_{k=1}^{G_{amt}}\Psi_k\right)^T/G_t}{G_t-1} \\ &+\left(\frac{G_t}{g_t}\right)^2 E_1\Bigg\{V\left(\widehat{\Psi}_{amt}|s_1\right) \\ &+\frac{g_{at}}{g_{Ot}}\left[\sum_{k=1}^{g_{amt}}V\left(\widehat{\Psi}_k\right)+\frac{g_{Ot}-1}{g_{at}-1}\sum_{k\neq l}^{g_{amt}}\sum Cov\left(\widehat{\Psi}_k,\widehat{\Psi}_l\right)\right]\Bigg\}. \end{aligned}$$

where

$$V\left(\widehat{\Psi}_{amt}|s_1\right) = g_{at}\left(\frac{g_{at}}{g_{Ot}}-1\right)\left[\frac{\sum_{k=1}^{g_{amt}}\Psi_k\Psi_k^T}{g_{at}-1} - \frac{\left(\sum_{k=1}^{g_{amt}}\Psi_k\right)\left(\sum_{k=1}^{g_{amt}}\Psi_k\right)^T}{g_{at}(g_{at}-1)}\right].$$

The expectation of the second term on the right hand side is not explicit because of the sample size, g_{Ot}, is conditional on the number of prevalent hauls sampled, g_{at}, which varies over the sampling distribution of total sampled hauls. The form of the variance and covariance within hauls depends on the within-haul sampling methodology or the assumed model. When longline gear is used and the within-haul estimator, Eq. 2.35, is used,

$$Cov\left(\widehat{\Psi}_k,\widehat{\Psi}_l\right) = \frac{Cov\left(\widehat{n}_{Ok},\widehat{n}_{Ol}\right)}{n_{Ok}n_{Ol}}\Psi_k\Psi_l^T$$

where

$$Cov\left(\widehat{n}_{Ok},\widehat{n}_{Ol}\right) = -N_{Om}^2\left(\frac{N_{Om}}{n_{Am}}-1\right)\frac{P_{Ok}P_{Ol}}{N_{Om}-1},$$

$$\widehat{n}_{Ok} = \frac{N_{Om}}{n_{Am}} n_{Ak},$$

$P_{Ok} = n_{Ok}/N_{Om}$, N_{Om} is the number of otoliths sampled within the mth management region/period, n_{Am} is the number of aged otoliths from the mth management region/period and n_{Ak} is the number of otoliths sampled from the kth haul that are aged. When trawl or pot gear are used, the covariance of different within-haul estimates is

$$Cov\left(\widehat{\mathbf{\Psi}}_k, \widehat{\mathbf{\Psi}}_l\right) = \left[1 + \frac{Cov\left(\widehat{n}_{Ok}, \widehat{n}_{Ol}\right)}{n_{Ok} n_{Ol}}\right] V\left(\widehat{\phi}_{n,at}\right) x'_k x'_l \mathbf{p}_{\psi,k} \mathbf{p}^T_{\psi,l} + \frac{Cov\left(\widehat{n}_{Ok}, \widehat{n}_{Ol}\right)}{n_{Ok} n_{Ol}} \mathbf{\Psi}_k \mathbf{\Psi}^T_l$$

where

$$V\left(\widehat{\phi}_{n,at}\right) = \frac{\sigma^2_{n,at}}{\sum_{k=1}^{g_{at}} x_k^2}$$

when the constant variance model in Eq. 5.24 is assumed for total numbers or

$$V\left(\widehat{\phi}_{n,at}\right) = \frac{\sigma^2_{n,at}}{\sum_{k=1}^{g_{at}} x_k}$$

when the proportional variance model in Eq. 5.25 is assumed (see Section 5.1.2.5).

The corresponding VCM estimator for the numbers-at-age vector is

$$\begin{aligned}
\widehat{V}\left(\widehat{\mathbf{\Psi}}_{amt}\right) =& \frac{G_t}{g_t}\left(\frac{G_t}{g_t} - 1\right)\left\{\frac{g_{at}}{g_{Ot}}\left[\sum_{k=1}^{g_{Omt}} \widehat{\mathbf{\Psi}}_k \widehat{\mathbf{\Psi}}^T_k - \frac{g_{at} - 1}{(g_t - 1)(g_{Ot} - 1)} \sum\sum_{k \neq l}^{g_{Omt}} \widehat{\mathbf{\Psi}}_k \widehat{\mathbf{\Psi}}^T_l\right]\right\} \\
&+ \left(\frac{G_t}{g_t}\right)^2 \left\{\frac{g_{at}}{g_{Ot}}\left(\frac{g_{at}}{g_{Ot}} - 1\right)\left[\sum_{k=1}^{g_{Omt}} \widehat{\mathbf{\Psi}}_k \widehat{\mathbf{\Psi}}^T_k - \frac{1}{(g_{Ot} - 1)} \sum\sum_{k \neq l}^{g_{Omt}} \widehat{\mathbf{\Psi}}_k \widehat{\mathbf{\Psi}}^T_l\right]\right\} \\
&+ \frac{G_t}{g_t}\frac{g_{at}}{g_{Ot}}\left[\sum_{k=1}^{g_{Omt}} \widehat{V}\left(\widehat{\mathbf{\Psi}}_k\right) + \frac{(G_t - 1)(g_{at} - 1)}{(g_t - 1)(g_{Ot} - 1)} \sum\sum_{k \neq l}^{g_{Omt}} \widehat{Cov}\left(\widehat{\mathbf{\Psi}}_k, \widehat{\mathbf{\Psi}}_l\right)\right]
\end{aligned} \tag{5.31}$$

where

$$\widehat{Cov}\left(\widehat{\mathbf{\Psi}}_k, \widehat{\mathbf{\Psi}}_l\right) = \frac{Cov\left(\widehat{n}_{Ok}, \widehat{n}_{Ol}\right)}{n_{Ok} n_{Ol}} \widehat{\mathbf{\Psi}}_k \widehat{\mathbf{\Psi}}^T_l \tag{5.32}$$

if longline gear is used. When trawl or pot gear is used,

$$\widehat{Cov}\left(\widehat{\mathbf{\Psi}}_k, \widehat{\mathbf{\Psi}}_l\right) = \widehat{V}\left(\widehat{\phi}_{n,at}\right) x'_k x'_l \widehat{\mathbf{p}}_{\psi,k} \widehat{\mathbf{p}}^T_{\psi,l} + \frac{Cov\left(\widehat{n}_{Ok}, \widehat{n}_{Ol}\right)}{n_{Ok} n_{Ol}} \widehat{\mathbf{\Psi}}_k \widehat{\mathbf{\Psi}}_l, \tag{5.33}$$

$$\widehat{\mathbf{p}}_{\psi,k} = \frac{N_{Om}}{n_{Am}} \frac{\psi_k}{n_{Ok}},$$

ψ_k is the vector of numbers-at-age among aged otoliths from the kth haul and the form of $\widehat{V}\left(\widehat{\phi}_{n,at}\right)$ is either Eq. 5.28 or Eq. 5.29 depending on the total numbers model assumed.

Because a random subsample of otoliths is aged, not every haul where otoliths are taken will have age information and estimates for these hauls will be zero. However, this just adds variability that is taken into accounted in the form of the VCM estimator. Notice that when otolith samples are taken with every length sample ($g_{at}/g_{Ot} = 1$) and there is no covariance between hauls, Eq. 5.31 reduces in form to Eq. 5.11. Furthermore, when the subset *am* equals the entire set of hauls Eq. 5.11 reduces to the more familiar two-stage variance estimator.

5.1.3 Within an Arbitrary Time Period and/or Region

For various analyses, estimation of catch parameters over specific regions or time periods is of interest. Regional specifications do not coincide with the controlled sampling design and neither do time periods that overlap or occur within quarterly subsets. For example, in-season managers must make estimates of catch parameters within specific regions and over short weekly time periods. Similarly, stock assessment scientists manage fish stocks by regions and time periods that may overlap two or more quarters. In a sampling theory context, these subsets of data or population subspaces are known as domains or sub-populations. Design-unbiased estimation within domains is possible, but when the sampling effort is not contolled within these domains the variability in sample sizes must be taken into account. When the number of elements sampled is large, fixed sample size approximations may be appropriate, however, the number of sampled elements may be small in many cases.

Estimation pertaining to these subsets of prevalent hauls or hauls made during particular time periods/regions or a combination thereof is known as subpopulation (Cochran 1977) or domain (Särndal et al. 1992) estimation in the design-based paradigm. When sample sizes are controlled within these subsets we have the more familiar stratified sampling.

Although I did not elaborate in Section 5.1.1.4, the estimator of the numbers-at-length within trips Eq. 5.10 is an example of a domain estimator because estimation is specifically in regard to s-prevalent hauls (i.e., the subset, a). These hauls are not made in particular time periods/regions, but they do constitute a subset of hauls that are not controlled with respect to sampling effort.

It is important to also realize that the haul subset of interest can be defined by any combination of uncontrolled haul characteristics (e.g., time period, region, prevalence in a species, bottom depth, haul duration, etc.) simultaneously. The estimator of numbers-at-age (Eq. 5.30) is likewise a domain estimator in this regard because it is specific to s-prevalent hauls (a) and the subset of hauls where subsampling of otoliths for ageing is independent (m). However, the form of the numbers-at-age estimator is complicated due to possible subsampling of hauls within the uncontrolled subset of s-prevalent hauls.

In general, we might be interested in the vector-valued total, $\boldsymbol{\Theta}_{dt}$, in some uncontrolled subset d of hauls made during the tth trip which, under SRS, is unbiasedly estimated as

$$\widehat{\boldsymbol{\Theta}}_{dt} = \frac{G_t}{g_t} \sum_{k=1}^{g_{dt}} \widehat{\boldsymbol{\Theta}}_k \tag{5.34}$$

where g_{dt} is the number of hauls made in the dth subset and $\widehat{\boldsymbol{\Theta}}_k$ is the estimated vector in the kth haul. In the case of the numbers-at-length estimator (Eq. 5.10), $d = a$. A general, unbiased VCM estimator is

$$\begin{aligned}\widehat{V}\left(\widehat{\boldsymbol{\Theta}}_{dt}\right) =& G_t\left(\frac{G_t}{g_t} - 1\right)\left[\frac{\sum_{k=1}^{g_{dt}} \widehat{\boldsymbol{\Theta}}_k \widehat{\boldsymbol{\Theta}}_k^T - \left(\sum_{k=1}^{g_{dt}} \widehat{\boldsymbol{\Theta}}_k\right)\left(\sum_{k=1}^{g_{dt}} \widehat{\boldsymbol{\Theta}}_k\right)^T / g_t}{g_t - 1}\right] \\ &+ \frac{G_t}{g_t}\left[\sum_{k=1}^{g_{dt}} \widehat{V}\left(\widehat{\boldsymbol{\Theta}}_k\right) + \frac{G_t - 1}{g_t - 1} \sum\sum_{k \neq l}^{g_{dt}} \widehat{Cov}\left(\widehat{\boldsymbol{\Theta}}_k, \widehat{\boldsymbol{\Theta}}_l\right)\right]\end{aligned} \tag{5.35}$$

where $\widehat{V}\left(\widehat{\boldsymbol{\Theta}}_k\right)$ is the VCM estimator for the kth haul. The exact forms of the within-haul estimator and $\widehat{Cov}\left(\widehat{\boldsymbol{\Theta}}_k, \widehat{\boldsymbol{\Theta}}_l\right)$ depends on the parameter of interest and whether model- or design-based approaches (or both) are used and will be a 1×1 matrix (i.e., scalar) for such parameters as total numbers or weight of a given species. Covariance of different hauls may occur if model-based estimates for different hauls use common information. Notice that the unbiased estimators and variance estimators appropriate when domains are not of interest are special cases of the domain-specific results presented here. That is, when $g_t = g_{dt}$,

$$\frac{\sum_{k=1}^{g_{dt}} \widehat{\Theta}_k \widehat{\Theta}_k^T - \left(\sum_{k=1}^{g_{dt}} \widehat{\Theta}_k\right) \left(\sum_{k=1}^{g_{dt}} \widehat{\Theta}_k\right)^T / g_t}{g_t - 1} = \frac{\sum_{k=1}^{g_t} \left(\widehat{\Theta}_k - \widehat{\overline{\Theta}}_t\right) \left(\widehat{\Theta}_k - \widehat{\overline{\Theta}}_t\right)^T}{g_t - 1}.$$

For fish and seabirds on pot and trawl vessels or marine mammals on pot vessels, completely model-based methods are used within trips. When the covariate total (total catch weight or volume for trawlers and total pots for pot gear) for the domain d is known in these cases, an alternative approach uses models identical to those in Eq. 5.24 and Eq. 5.25 that have domain-specific parameters:

$$\begin{aligned} E_M\left(\boldsymbol{\theta}_k\right) =& x_k \boldsymbol{\phi}_{\theta,dt} \\ V_M\left(\boldsymbol{\theta}_k\right) =& \boldsymbol{\Sigma}_{\theta,dt} \\ Cov_M\left(\boldsymbol{\theta}_k, \boldsymbol{\theta}_l\right) =& \mathbf{0} \quad \text{for} \quad k \neq l. \end{aligned} \tag{5.36}$$

or

$$\begin{aligned} E_M\left(\boldsymbol{\theta}_k\right) =& x_k \boldsymbol{\phi}_{\theta,dt} \\ V_M\left(\boldsymbol{\theta}_k\right) =& x_k \boldsymbol{\Sigma}_{\theta,dt} \\ Cov_M\left(\boldsymbol{\theta}_k, \boldsymbol{\theta}_l\right) =& \mathbf{0} \quad \text{for} \quad k \neq l \end{aligned} \tag{5.37}$$

where $\boldsymbol{\phi}_{\theta,dt}$ is a $p \times 1$ vector, $\boldsymbol{\Sigma}_{\theta,dt}$ is a corresponding $p \times p$ symmetric VCM and x_k is a measured covariate (sample weight υ or sample volume v for trawl gear or sampled pots κ for pot gear). Because the covariate total over the domain is known, we can predict the parameter value for any unsampled portions. Thus, the predictor of the domain total is

$$\widehat{\boldsymbol{\Theta}}_{dt} = \sum_{k=1}^{g_{dt}} \boldsymbol{\theta}_k + \widehat{\boldsymbol{\phi}}_{\theta,dt} \left(\sum_{k=1}^{g_{dt}} x_{k'} + \sum_{l=1}^{g_{dt'}} X_l \right) = \boldsymbol{\theta}_{dt} + \widehat{\boldsymbol{\phi}}_{\theta,dt} x_{dt'} \tag{5.38}$$

where $g_{dt'} = G_{dt} - g_{dt}$ is the number of unsampled hauls in the domain and X_k is the covariate total for the kth haul. The best linear unbiased estimators of $\widehat{\boldsymbol{\phi}}_{\theta,dt}$ under models in Eq. 5.36 and Eq. 5.37 are

$$_1\widehat{\phi}_{dt} = \frac{\sum_{k=1}^{g_{dt}} \boldsymbol{\theta}_k x_k}{\sum_{k=1}^{g_{dt}} x_k^2}$$

and

$$_2\widehat{\phi}_{dt} = \frac{\sum_{k=1}^{g_{dt}} \boldsymbol{\theta}_k}{\sum_{k=1}^{g_{dt}} x_k},$$

respectively. Variance estimators under the models in Eq. 5.36 and Eq. 5.37 are

$$\widehat{V}_M\left({}_1\widehat{\boldsymbol{\Theta}}_{dt}\right) = {}_1\widehat{\boldsymbol{\Sigma}}_{\theta,dt}\left[\frac{(x_{dt'})^2}{\sum_{k=1}^{g_{dt}} x_k^2} + m'_{dt}\right]$$

and

$$\widehat{V}_M\left({}_2\widehat{\boldsymbol{\Theta}}_{dt}\right) = {}_2\widehat{\boldsymbol{\Sigma}}_{\theta,dt}\frac{x_{dt'}}{x_{dt}}X_{dt}, \tag{5.39}$$

respectively, where

$${}_1\widehat{\boldsymbol{\Sigma}}_{\theta,dt} = \frac{\sum_{k=1}^{g_{dt}}\left(\boldsymbol{\theta}_k - {}_1\widehat{\phi}_{\theta,dt}x_k\right)\left(\boldsymbol{\theta}_k - {}_1\widehat{\phi}_{\theta,dt}x_k\right)^T}{g_{dt}-1},$$

$${}_2\widehat{\boldsymbol{\Sigma}}_{\theta,dt} = \frac{x_{dt}}{x_{dt}^2 - \sum_{k=1}^{g_{dt}} x_k^2}\sum_{k=1}^{g_{dt}}\left(\boldsymbol{\theta}_k - {}_2\widehat{\phi}_{\theta,dt}x_k\right)\left(\boldsymbol{\theta}_k - {}_2\widehat{\phi}_{\theta,dt}x_k\right)^T,$$

$x_{dt} = \sum_{k=1}^{g_{dt}} x_k$, $X_{dt} = \sum_{k=1}^{G_{dt}} X_k$ and $m_{dt'} = \sum_{k=1}^{g_{dt}} m'_k + \sum_{k=1}^{g_{dt'}} M_k$.

In Section 5.1.3.1, I present domain-specific estimators for total numbers of either fish, seabird or marine mammal interactions and in Section 5.1.3.2, I present an estimator for total weight of fish catches, but I do not give results for total number in length or age classes because the results already given for trip-specific estimation of these parameters are appropriate (i.e., these estimators are already specific to the domain, a). The only difference is whether the domain becomes even more specific than s-prevalent hauls (for numbers in length classes) or s-prevalent hauls in the mth management region (for numbers in age classes). As I will explain later in Section 5.2, the domain-specific estimates of catch parameters can be treated in the same way as estimates pertaining to all hauls in a trip when interest is in vessel-specific or fleet-wide estimation.

5.1.3.1 *Total Number*

Suppose we are interested in estimating, within a particular region and/or time interval (d), either the total number of a fish or seabird species in catches or the total number of marine mammal interactions with vessels. The time interval must be within a quarter of fixed observer coverage (or split the interval into the pieces that occur in different quarters). For a given trip, each of the hauls are either in or out of the spatial/temporal domain, d, of interest. Letting N denote numbers of fish and seabirds or numbers of marine mammal interactions, the design-unbiased estimator, based on Eq. 5.34, for the total number in the dth domain for the tth trip is

$$\widehat{N}_{dt} = \frac{G_t}{g_t} \sum_{k=1}^{g_{dt}} \widehat{N}_k \tag{5.40}$$

and the design-unbiased variance estimator is

$$\widehat{V}\left(\widehat{N}_{dt}\right) = G_t \left(\frac{G_t}{g_t} - 1\right) \left[\frac{\sum_{k=1}^{g_{dt}} \widehat{N}_k^2 - \left(\sum_{k=1}^{g_{dt}} \widehat{N}_k\right)^2 / g_t}{g_t - 1}\right] + \frac{G_t}{g_t} \sum_{k=1}^{g_{dt}} \widehat{V}\left(\widehat{N}_k\right). \tag{5.41}$$

For longline gear, the model-based estimators for total numbers within each haul were independent and Eq. 5.40 and Eq. 5.41 are thus appropriate for that situation. The domain-based estimators are also appropriate for other gear types when design-based methods are possible within hauls. Specifically, recall that there is no sampling variability for marine mammal mortalities within trawler hauls and the estimator and variance estimator reduce to

$$\widehat{N}_{dt} = \frac{G_t}{g_t} \sum_{k=1}^{g_{dt}} N_k \tag{5.42}$$

and

$$\widehat{V}\left(\widehat{N}_{dt}\right) = G_t \left(\frac{G_t}{g_t} - 1\right) \frac{\sum_{k=1}^{g_{dt}} N_k^2 - \left(\sum_{k=1}^{g_{dt}} N_k\right)^2 / g_t}{g_t - 1}, \tag{5.43}$$

respectively. Furthermore, trip-specific models are necessary in the absence of design-based methods for total numbers of fish and seabirds when trawl or pot gear is used or total mammal interactions when pot gear is used. In such cases, the estimator of the total is still Eq. 5.40, but the variance estimator must account for covariance of different hauls so that the full form of the general result, Eq. 5.35, is used and

$$\begin{aligned}\widehat{V}\left(\widehat{N}_{dt}\right) =& G_t\left(\frac{G_t}{g_t}-1\right)\left[\frac{\sum_{k=1}^{g_{dt}}\widehat{N}_k^2-\left(\sum_{k=1}^{g_{dt}}\widehat{N}_k\right)^2/g_t}{g_t-1}\right]\\ &+\frac{G_t}{g_t}\left[\sum_{k=1}^{g_{dt}}\widehat{V}\left(\widehat{N}_k\right)+\frac{G_t-1}{g_t-1}\sum_{k\neq l}^{g_{dt}}\sum\widehat{Cov}\left(\widehat{N}_k,\widehat{N}_l\right)\right].\end{aligned} \quad (5.44)$$

where

$$\widehat{Cov}\left(\widehat{N}_k,\widehat{N}_l\right)=\widehat{V}\left(\widehat{\phi}_{n,t}\right)x_k'x_l'$$

and $\widehat{V}\left(\widehat{\phi}_{n,t}\right)$ depends on the variance model assumed (see Section 3.6). When the covariate value (See Table 5.1 for covariate definitions) is known for all unsampled portions of hauls and unsampled hauls, estimators and variance estimators based on the models in Eq. 5.36 or Eq. 5.37 make better use of available information.

5.1.3.2 Total Weight

When design-based estimation is possible within each haul, the estimator for total weight for a subset of hauls and the corresponding variance estimator are analogous to those of total number, Eq. 5.40 and Eq. 5.41. However, there is covariance of total weight estimates for all gear types when models-based methods are required within hauls. The estimator and variance estimator for total weight when models are necessary are

$$\widehat{W}_{dt}=\frac{G_t}{g_t}\sum_{k=1}^{g_{dt}}\widehat{W}_k \quad (5.45)$$

and

$$\begin{aligned}\widehat{V}\left(\widehat{W}_{dt}\right) =& G_t\left(\frac{G_t}{g_t}-1\right)\left[\frac{\sum_{k=1}^{g_{dt}}\widehat{W}_k^2-\left(\sum_{k=1}^{g_{dt}}\widehat{W}_k\right)^2/g_t}{g_t-1}\right]\\ &+\frac{G_t}{g_t}\left[\sum_{k=1}^{g_{dt}}\widehat{V}\left(\widehat{W}_k\right)+\frac{G_t-1}{g_t-1}\sum_{k\neq l}^{g_{dt}}\sum\widehat{Cov}\left(\widehat{W}_k,\widehat{W}_l\right)\right].\end{aligned} \quad (5.46)$$

The only difference across gear types and models is the form of the within-haul estimator and variance estimator and the estimator of covariance between hauls. For longline gear, the covariance estimator is

$$\widehat{Cov}\left(\widehat{W}_k, \widehat{W}_l\right) = \widehat{N}_k \widehat{N}_l \widehat{\sigma}^2_{W,t}$$

for trawl and pot vessels, the covariance is

$$\widehat{Cov}\left(\widehat{N}_k, \widehat{N}_l\right) = \widehat{V}\left(\widehat{\phi}_{W,t}\right) x'_k x'_l.$$

where $\widehat{V}\left(\widehat{\phi}_{W,t}\right)$ depends on the model assumed. As I noted for domain-specific estimation of total numbers, estimators and variance estimators based on the models in Eq. 5.36 or Eq. 5.37 make better use of available information when the covariate value is known for all unsampled portions of hauls and unsampled hauls.

5.1.4 Model-based Estimation for Undersampled Trips

Inevitably, there will be trips where an observer is present on a vessel, but for one reason or another, samples an insufficient number of hauls for variance estimation. In these cases, useful information about hauls, including the number of hauls made during the trip and the location of each haul, is known. Furthermore, for longline sets, the number of hooks deployed for each haul is known, for pot gear, the number of pots deployed for each string is known and for trawl gear, an the total catch weight is usually known (with error). Models based on observed effort during other trips can be used to predict catch parameters for the undersampled trips. Because of possible differences in gear selectivity, regional/seasonal fishing practices and regional/seasonal fish populations attributes, models I use for prediciton of catch parameters have time period/region/gear-specific parameters (i.e., fine-scale).

5.1.4.1 Longline effort

A model that assumes a region/period-specific probability of capture for a particular species of fish or seabird on a given hook can be used to predict the number captured for the hauls in an undersampled trip. However, there is surely greater variability in the number caught on sampled hooks, than expected under simple binomial conditions and uncertainty in the total number estimate is better treated using a hierarchical or mixture model that allows different conditional capture probabilities among hauls. Thus, given a haul, we assume the presence of a fish on a hook is a Bernoulli random variable, but that this probability varies between hauls. Under this hierarchical model, the marginal mean and variance for the random variable that indicates capture of an individual on a given hook in the mth region/period are $E(x_i) = \pi_m$ and $V(x_i) = \pi_m(1-\pi_m)\tau_m$, respectively. The dispersion parameter, τ_m, accounts for departure from the simpler Bernoulli model. The best linear unbiased estimator of the marginal capture probability in region/period m is

$$\widehat{\pi}_m = \frac{\sum_{v=1}^{V} \sum_{t=1}^{c_v} \sum_{k=1}^{g_{mt}} n_k}{\sum_{t=1}^{V} \sum_{t=1}^{c_v} \sum_{k=1}^{g_{mt}} h_k} \equiv \frac{n_m}{h_m} \tag{5.47}$$

where V is the total number of vessels observed during a given quarter and the predictor of the total number caught in a haul occuring in region m during the undersampled trip is $\widehat{N}_k^* = H_k \widehat{\pi}_m$. For the predicted catch in the mth region in the entire trip we simply sum all predictors for hauls made in the region during the trip so that

$$\widehat{N}_{mt}^* = \sum_{k=1}^{G_{mt}} \widehat{N}_k^* = \widehat{\pi}_m H_{mt} \tag{5.48}$$

where $H_{mt} = \sum_{k=1}^{G_{mt}} H_{tk}$ and the trip prediction error variance estimator is

$$\widehat{V}\left(\widehat{N}_{mt}^*\right) = H_{mt}\widehat{\pi}_m(1-\widehat{\pi}_m)\widehat{\tau}_m \left(\frac{H_{mt}}{h_m} + 1\right) \tag{5.49}$$

where

$$\widehat{\tau}_m = \frac{1}{g_m - 1} \sum_{v=1}^{V} \sum_{t=1}^{c_v} \sum_{k=1}^{g_t} \frac{(n_k - h_k\widehat{\pi}_m)^2}{h_k\widehat{\pi}_m(1-\widehat{\pi}_m)} \tag{5.50}$$

and $g_m = \sum_{v=1}^{V} \sum_{t=1}^{c_v} g_{mt}$ is the number of hauls made in the mth region/period by all longline vessels in which the species of interest was caught. Because the subsets of hauls occuring in different periods/regions and model parameters are mutually exclusive, the estimator of the total number over multiple or all subsets is just the sum of period/region specific estimators. Specifically, the estimate of the total over

$M = \bigcup_{i=1}^{D} m_i$ is

$$\widehat{N}^*_{Mt} = \sum_{i=1}^{D} \widehat{N}^*_{it} \tag{5.51}$$

and the variance estimator is

$$\widehat{V}\left(\widehat{N}^*_{Mt}\right) = \sum_{i=1}^{D} \widehat{V}\left(\widehat{N}^*_{it}\right) \tag{5.52}$$

where D is the number of period/regions in the larger domain, M. These estimators are useful when interest is in estimation for a large domain, but more specific models are appropriate for predicting numbers in unique subdomains that make up the larger domain.

To predict total weight of captured fish for undersampled trips, an additional and similar model for average weight is necessary. A region/period-specific average weight where the region/period is the same as that used for probability of capture can be multiplied by the predicted total number of fish to obtain a prediction of tota weight. Assuming a model for the weights of all fish captured in the mth region/period with mean, μ_{Wm}, an estimator of the mean weight is

$$\widehat{\mu}_{Wm} = \frac{\sum_{k=1}^{g_m} n_k \overline{W}_k}{\sum_{k=1}^{g_m} n_k}$$

where n_k is the number weighed in the kth haul and $\overline{W}_k$ is the average weight of the n_k fish. Let the model variance be σ^2_{Wm} and I assume no correlation of individual fish because of random sampling by observers. An estimator of the model variance is

$$\widehat{\sigma}^2_{Wm} = \frac{g_m}{\sum_{k=1}^{g_m} 1/n_k} \frac{\sum_{k=1}^{g_m}\left(\overline{W}_k - \overline{\overline{W}}_m\right)^2}{g_m - 1} \tag{5.53}$$

where $\overline{\overline{W}}_m = \sum_{k=1}^{g_m} \overline{W}_k / g_m$. Under the model the expected weight of fish captured in hauls made in the mth region/period is $H_{mt}\pi_m\mu_{Wm}$ where H_{mt} is the number of hooks deployed in the region/period during the trip and $H_{mt}\pi_m$ is the expected number of fish caught. The obvious predictor that is unbiased when the estimators for probability of capture and average weight are indpendent is

$$\widehat{W}^*_{mt} = H_{mt}\widehat{\pi}_m\widehat{\mu}_{Wm}. \tag{5.54}$$

The prediction error variance estimator is

$$\begin{aligned}\widehat{V}\left(\widehat{W}^*_{mt}\right) =& H^2_{mt}\left(\widehat{\pi}^2_m\frac{\widehat{\sigma}^2_{Wm}}{n_m} + \widehat{\mu}^2_{Wm}\frac{\widehat{\sigma}^2_{\pi m}}{h_m} - \frac{\widehat{\sigma}^2_{Wm}}{n_m}\frac{\widehat{\sigma}^2_{\pi m}}{h_m}\right)\\ &+ H_{mt}\left(\widehat{\mu}^2_{Wm}\widehat{\sigma}^2_{\pi m} + \widehat{\pi}_m\widehat{\sigma}^2_{Wm} - \widehat{\sigma}^2_{\pi m}\frac{\widehat{\sigma}^2_{Wm}}{n_m}\right)\end{aligned} \tag{5.55}$$

where $\widehat{\sigma}^2_{\pi m} = \widehat{\pi}_m\left(1 - \widehat{\pi}_m\right)\widehat{\tau}_m$ and $\widehat{\tau}_m$ is estimated by Eq. 5.50. The derivation of the variance that Eq. 5.55 estimates is a simpler, two-random variable version of that presented in Section 5.A.4.

For estimation over multiple period/regions the same procedure as defined for total numbers is appropriate. Specifically, the estimator for the total weight over the larger set of period/regions, M, is

$$\widehat{W}^*_{Mt} = \sum_{i=1}^{D}\widehat{W}^*_{it}$$

and the variance estimator is

$$\widehat{V}\left(\widehat{W}^*_{Mt}\right) = \sum_{i=1}^{D}\widehat{V}\left(\widehat{W}^*_{it}\right).$$

To predict total numbers of marine mammal interactions in undersampled trips an approach analogous to that used for fish and seabirds seems appropriate. However, the hierarchical model here is a mixture of Poisson random variables in the mth region. So that the mean capture rate per hook for marine mammals is ν_m and the variance is $\nu_m\tau_{\nu m}$ where $\tau_{\nu m}$ is a dispersion parameter that allows variance to be different than that under the simple Poisson assumption. The unbiased estimator of the capture rate is

$$\widehat{\nu}_m = \frac{\sum_{v=1}^{V}\sum_{t=1}^{c_v}\sum^{g_{mt}}\gamma_k}{\sum_{t=1}^{V}\sum_{t=1}^{c_v}\sum^{g_{mt}}h_k} \equiv \frac{\gamma_m}{h_m}$$

where γ_k is the number of interactions in sampled portion of the kth haul. The estimator for the number of interactions in the mth period/region during the undersampled trip is

$$\widehat{\Gamma}^*_{mt} = H_{mt}\widehat{\nu}_m \tag{5.56}$$

and the prediction error variance estimator is

$$\widehat{V}\left(\widehat{\Gamma}^*_{mt}\right) = H_{mt}\left(\frac{H_{mt}}{h_m}+1\right)\widehat{\nu}_m\widehat{\tau}_{\nu m} \tag{5.57}$$

where

$$\widehat{\tau}_{\nu m} = \frac{1}{g_m - 1}\sum_{v=1}^{V}\sum_{t=1}^{c_v}\sum_{k=1}^{g_{mt}}\frac{\left(\gamma_k - h_k\widehat{\nu}_m\right)^2}{h_k\widehat{\nu}_m} \tag{5.58}$$

To predict total numbers of fish in s-prevalent hauls (i.e., the domain a), hierarchical models are needed for the probability a hook is from a s-prevalent haul in a region (α_m) and the probability of capture on a hook in the region given the fish is from an s-prevalent haul (π_{am}). A model analogous to the capture probability model supposes a marginal probability that a hook is from a prevalent haul within a region, α_m, with variance $\alpha_m(1-\alpha_m)\tau_{am}$. An estimator of the s-prevalence probability in the mth region is

$$\widehat{\alpha}_m = \frac{\sum_{v=1}^{V}\sum_{t=1}^{c_v}\sum_{k=1}^{g_{amt}} H_k}{\sum_{v=1}^{V}\sum_{t=1}^{c_v}\sum_{k=1}^{g_{mt}} H_k} = \frac{H_{am}}{H_m}$$

and the variance is estimated by

$$\widehat{V}\left(\widehat{\alpha}_m\right) = \frac{\widehat{\alpha}_m\left(1-\widehat{\alpha}_m\right)\widehat{\tau}_{am}}{H_m - 1}$$

where

$$\widehat{\tau}_{am} = \frac{1}{c-1}\sum_{v=1}^{V}\sum_{t=1}^{c_v}\frac{\left(H_{amt} - H_{mt}\widehat{\alpha}_m\right)^2}{H_{mt}\widehat{\alpha}_m\left(1-\widehat{\alpha}_m\right)}$$

and $c = \sum_{v=1}^{V} c_v$.

An alternative estimator of the probability of s-prevalence that uses numbers of hauls rather than numbers of hooks is

$$\widehat{\alpha}_m = \frac{\sum_{v=1}^{V}\sum_{t=1}^{c_v} g_{amt}}{\sum_{v=1}^{V}\sum_{t=1}^{c_v} g_{mt}} = \frac{g_{am}}{g_m} \tag{5.59}$$

with variance estimated by

$$\widehat{V}\left(\widehat{\alpha}_m\right)=\frac{\widehat{\alpha}_m\left(1-\widehat{\alpha}_m\right)\widehat{\tau}_{\alpha m}}{g_m-1} \tag{5.60}$$

where

$$\widehat{\tau}_{\alpha m}=\frac{1}{c-1}\sum_{v=1}^{V}\sum_{t=1}^{c_v}\frac{\left(g_{amt}-g_{mt}\widehat{\alpha}_m\right)^2}{g_{mt}\widehat{\alpha}_m\left(1-\widehat{\alpha}_m\right)}.$$

This second model actually considers the probability that a haul is s-prevalent rather than a hook, but it is robust to any relationship of total numbers of hooks per haul and s-prevalence.

For the capture probability model in s-prevalent hauls, with mean, π_{am}, and variance, $\pi_{am}(1-\pi_{am})\tau_{am}$, the estimator is

$$\widehat{\pi}_{am}=\frac{\sum_{v=1}^{V}\sum_{t=1}^{c_v}\sum_{k=1}^{g_{amt}}n_k}{\sum_{v=1}^{V}\sum_{t=1}^{c_v}\sum_{k=1}^{g_{amt}}h_k}\equiv\frac{n_{am}}{h_{am}}$$

and the variance estimator is

$$\widehat{V}\left(\widehat{\pi}_{am}\right)=\frac{\widehat{\pi}_{am}\left(1-\widehat{\pi}_{am}\right)\widehat{\tau}_{am}}{h_{am}-1}$$

where

$$\widehat{\tau}_{am}=\frac{1}{g_{am}-1}\sum_{v=1}^{V}\sum_{t=1}^{c_v}\sum_{k=1}^{g_{am}}\frac{\left(n_k-h_k\widehat{\pi}_{am}\right)^2}{h_k\widehat{\pi}_{am}\left(1-\widehat{\pi}_{am}\right)}$$

and $g_{am}=\sum_{v=1}^{V}\sum_{t=1}^{c_v}g_{amt}$.

To predict the total numbers-at-length or -age another model for the vector of probabilities that a fish caught in an s-prevalent haul and mth region/period is in a given length or age class ($\mathbf{p}_{\Theta,am}$) where $\mathbf{\Theta}$ is generic for either lengths ($\mathbf{\Lambda}$) or ages ($\mathbf{\Psi}$). A vector-generalization of the models for capture and s-prevalence probabilities is used for proportions-at-length and -age. The model has a common mean and overdispersed VCM in the mth region/period so that the random indicator vector for the length or age class of the ith fish has $E\left(\mathbf{x}_i\right)=\mathbf{p}_{\lambda,am}$ and $V\left(\mathbf{x}_i\right)=\left[\operatorname{diag}\left(\mathbf{p}_{\lambda,am}\right)-\mathbf{p}_{\lambda,am}\mathbf{p}_{\lambda,am}^T\right]\tau_{\lambda_{am}}$. The estimator of the probability vector is

$$\widehat{\mathbf{p}}_{\lambda,am}=\frac{\sum_{v=1}^{V}\sum_{t=1}^{c_v}\sum_{k=1}^{g_{tam}}\boldsymbol{\lambda}_k}{\sum_{v=1}^{V}\sum_{t=1}^{c_v}\sum_{k=1}^{g_{tam}}n_{\lambda k}}\equiv\frac{\boldsymbol{\lambda}_{am}}{n_{\lambda_{am}}}.$$

and the VCM estimator is

$$\widehat{V}\left(\widehat{\mathbf{p}}_{\lambda,am}\right) = \frac{\widehat{\tau}_{\lambda_{am}}}{n_{\lambda_{am}} - 1}\left[\operatorname{diag}\{\widehat{\mathbf{p}}_{\lambda,am}\} - \widehat{\mathbf{p}}_{\lambda,am}\widehat{\mathbf{p}}_{\lambda,am}^{T}\right]$$

where

$$\widehat{\tau}_{\lambda_{am}} = \frac{\sum_{v=1}^{V}\sum_{t=1}^{c_v}\sum_{k=1}^{g_{tam}}\sum_{i=1}^{L_m}\left(\lambda_i - n_{\lambda k}\widehat{p}_{i\lambda_{am}}\right)^2 / n_{\lambda k}\widehat{p}_{i\lambda_{am}}}{\left(L_m - 1\right)\sum_{v=1}^{V}\sum_{t=1}^{c_v} g_{amt} - 1}$$

and L_m is the number of estimable length classes in the region. Here we use the decision of the observer to take length measurements for a haul as an indicator of s-prevalent.

Assuming independence of each of the models, a predictor of the total number in s-prevalent hauls made in the mth region is

$$\widehat{N}^*_{amt} = H_{mt}\widehat{\alpha}_m\widehat{\pi}_{am}$$

where $H_{mt} = \sum_{k=1}^{G_{mt}} H_k$. The corresponding prediction error variance estimator,

$$\widehat{V}\left(\widehat{N}^*_{amt}\right) = H_{mt}\left\{H_{mt}\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\right) + \widehat{\alpha}_m\widehat{\pi}_{am}\left[\widehat{\pi}_{am}\left(1 - \widehat{\alpha}_m\right)\widehat{\tau}_{\alpha_m} + \left(1 - \widehat{\pi}_{am}\right)\widehat{\tau}_{\pi_{am}}\right]\right\}$$

where

$$\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\right) = \widehat{\alpha}^2_m\widehat{\pi}^2_{am} - \left[\widehat{\alpha}^2_m - \widehat{V}\left(\widehat{\alpha}_m\right)\right]\left[\widehat{\pi}^2_{am} - \widehat{V}\left(\widehat{\pi}_{am}\right)\right] \tag{5.61}$$

The predictor of the numbers in length classes in s-prevalent hauls in the mth region for the undersampled trip is

$$\widehat{\mathbf{\Lambda}}^*_{amt} = \widehat{N}^*_{amt}\widehat{\mathbf{p}}_{\lambda,am} = H_{mt}\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\lambda,am} \tag{5.62}$$

and the prediction error VCM estimator is

$$\begin{aligned}\widehat{V}\left(\widehat{\mathbf{\Lambda}}^*_{amt}\right) = &H_{mt}\Bigg\{H_{mt}\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\lambda,am}\right) + \widehat{\alpha}_m\widehat{\pi}_{am}\Big\{\operatorname{diag}\{\widehat{\mathbf{p}}_{\lambda,am}\}\widehat{\tau}_{\lambda_{am}} \\ &+ \left[\left(1 - \widehat{\pi}_{am}\right)\widehat{\tau}_{\pi_{am}} + \widehat{\pi}_{am}\left(1 - \widehat{\alpha}_m\right)\widehat{\tau}_{\alpha_m} - \widehat{\tau}_{\lambda_{am}}\right]\widehat{\mathbf{p}}_{\lambda,am}\widehat{\mathbf{p}}_{\lambda,am}^{T}\Big\}\Bigg\}\end{aligned} \tag{5.63}$$

where

$$\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\lambda,am}\right) = \widehat{\alpha}_m^2\widehat{\pi}_{am}^2\widehat{\mathbf{p}}_{\lambda,am}\widehat{\mathbf{p}}_{\lambda,am}^T - \left[\widehat{\alpha}_m^2 - \widehat{V}\left(\widehat{\alpha}_m\right)\right]\left[\widehat{\pi}_{am}^2 - \widehat{V}\left(\widehat{\pi}_{am}\right)\right]\left[\widehat{\mathbf{p}}_{\lambda,am}\widehat{\mathbf{p}}_{\lambda,am}^T - \widehat{V}\left(\widehat{\mathbf{p}}_{\lambda,am}\right)\right]. \tag{5.64}$$

Analogous models can be used to predict numbers-at-age for undersampled trips. The only difference would be the model for the probability vector for age classes. For a motivation of the overdispersion parameter estimators see, for example, McCullagh and Nelder (1989) and for a derivation of the prediction error VCM estimator see Section 5.A.4.

Estimators identical to Eq. 5.51 and Eq. 5.52 are appropriate for predicting total numbers or total numbers in length or age classes in abundant hauls over multiple period/regions.

5.1.4.2 Trawl and Pot effort

The approach I propose for predicting total numbers of fish, seabirds or marine mammal interactions and total weight of a fish species in trawl or pot catches of an undersampled trip is analogous to one I proposed for domain-specific estimation within trips (Section 5.1.3). Consider the closely related model:

$$\begin{aligned} E_M\left(\boldsymbol{\theta}_k\right) &= x_k\boldsymbol{\phi}_{\theta,m} \\ V_M\left(\boldsymbol{\theta}_k\right) &= x_k\boldsymbol{\Sigma}_{\theta,m} \\ Cov_M\left(\boldsymbol{\theta}_k,\boldsymbol{\theta}_l\right) &= \mathbf{0} \quad \text{for} \quad k \neq l. \end{aligned} \tag{5.65}$$

The only difference between this model and that in Eq. 5.37 presented in Section 5.1.3 is that it is not trip and s-prevalent specific. The general predictor for the parameter total in the mth period/region of an undersampled trip is

$$\widehat{\boldsymbol{\Theta}}_{mt} = X_{mt}\widehat{\boldsymbol{\phi}}_{\theta,m} \tag{5.66}$$

where

$$\widehat{\boldsymbol{\phi}}_{\theta,m} = \frac{\sum_{k=1}^{g_m}\boldsymbol{\theta}_k}{\sum_{k=1}^{g_m}x_k}.$$

The corresponding variance estimator is

$$\widehat{V}_M\left(\widehat{\boldsymbol{\Theta}}^*_{mt}\right) = \widehat{\boldsymbol{\Sigma}}_{\theta,m} X_{mt}\left(1 + \frac{X_{mt}}{x_m}\right), \tag{5.67}$$

where

$$\widehat{\boldsymbol{\Sigma}}_{\theta,m} = \frac{x_m}{x_m^2 - \sum_{k=1}^{g_m} x_k^2} \sum_{k=1}^{g_m} \left(\boldsymbol{\theta}_k - \widehat{\boldsymbol{\phi}}_{\theta,m} x_k\right)\left(\boldsymbol{\theta}_k - \widehat{\boldsymbol{\phi}}_{\theta,m} x_k\right)^T$$

and $x_m = \sum_{k=1}^{g_m} x_k$. The same type of model and corresponding estimators can be used to predict the total numbers or weight in s-prevalent hauls in the mth period/region by simply limiting the data used in the model parameter estimators to that obtained from s-prevalent hauls in the appropriate period/region.

For predicting the total numbers in s-prevalent hauls, I employ a model for probability of s-prevalence similar to that I used for longline hauls. Here, however, we do not have hooks to model, but we can think of the probability a whole haul is s-prevalent α_m and the estimator I use here is identical to Eq. 5.59 and the variance of the estimator is estimated by Eq. 5.60. The predictor for the total number in s-prevalent hauls is

$$\widehat{N}^*_{amt} = \widehat{\alpha}_m \widehat{\phi}_{n,am} X_{mt} \tag{5.68}$$

where $\widehat{\phi}_{n,am}$ is analogous to Eq. 5.65. The prediction error variance estimator is

$$\widehat{V}\left(\widehat{N}^*_{amt}\right) = \widehat{V}_1 + \widehat{V}_2$$

where

$$\begin{aligned}
\widehat{V}_1 =& X_{mt}^2 \left[\widehat{\alpha}_m^2 \widehat{V}\left(\widehat{\phi}_{n,am}\right) + \widehat{\phi}_{n,am}^2 \widehat{V}\left(\widehat{\alpha}_m\right) - \widehat{V}\left(\widehat{\alpha}_m\right)\widehat{V}\left(\widehat{\phi}_{n,am}\right)\right], \\
\widehat{V}_2 =& X_{mt}\left[\widehat{\alpha}_m \widehat{\sigma}^2_{n,am} + X_{mt}\widehat{\phi}^2_{n,am}\widehat{\sigma}^2_{\alpha m}\right],
\end{aligned}$$

$$\widehat{V}\left(\widehat{\phi}_{n,am}\right) = \frac{\widehat{\sigma}^2_{n,am}}{x_m},$$

$$\widehat{\sigma}^2_{n,am} = \frac{x_{am}}{x_{am}^2 - \sum_{k=1}^{g_{am}} x_k^2} \sum_{k=1}^{g_{am}} \left(n_k - \widehat{\phi}_{n,am} x_k\right)^2$$

and $\widehat{\sigma}^2_{\alpha m} = \widehat{\alpha}_m\left(1 - \widehat{\alpha}_m\right)\widehat{\tau}_{am}$.

To predict the numbers in length or age classes the same approach as that used for longline hauls is used here for pots and trawls. The predictor of the total numbers in length or age classes in the mth period/region is

$$\widehat{\boldsymbol{\Theta}}^*_{amt} = \widehat{N}^*_{amt}\widehat{\mathbf{p}}_{\Theta,am} \tag{5.69}$$

where $\widehat{N}^*_{amt}$ is given in Eq. 5.68 and

$$\widehat{\mathbf{p}}_{\Theta,am} = \frac{\sum_{k=1}^{g_{am}} \boldsymbol{\lambda}_k}{\sum_{k=1}^{g_{am}} n_{\lambda,k}}.$$

The prediction error VCM estimator is

$$\widehat{V}\left(\widehat{\boldsymbol{\Theta}}^*_{amt}\right) = \widehat{V}\left(\widehat{N}^*_{amt}\right)\widehat{\mathbf{p}}_{\Theta,am}\widehat{\mathbf{p}}^T_{\Theta,am} + \widehat{N}^{*2}_{amt}\widehat{V}\left(\widehat{\mathbf{p}}_{\Theta,am}\right) - \widehat{V}_1\widehat{V}\left(\widehat{\mathbf{p}}_{\Theta,am}\right) + \widehat{N}^*_{amt}\widehat{\boldsymbol{\Sigma}}_{\Theta,am} \tag{5.70}$$

where

$$\widehat{\boldsymbol{\Sigma}}_{\Theta,am} = \left[\operatorname{diag}\left(\widehat{\mathbf{p}}_{\Theta,am}\right) - \widehat{\mathbf{p}}_{\Theta,am}\widehat{\mathbf{p}}^T_{\Theta,am}\right]\widehat{\tau}_{\Theta,am}$$

and

$$\widehat{V}\left(\widehat{\mathbf{p}}_{\Theta,am}\right) = \frac{\widehat{\boldsymbol{\Sigma}}_{\Theta,am}}{n_{\lambda,m}}.$$

See Section 5.A.3 for a derivation of the prediction error VCM.

5.2 *Within a Vessel*

For estimation at the vessel level and higher, parameter estimators, variances and variance estimators can be viewed in a generalized form for a generic vector-valued parameter, $\boldsymbol{\Theta}$, and those results are presented. Because the form of some generalized terms depends on the catch parameter being estimated or model used for prediction in undersampled trips, however, results specific to all the catch parameters I am addressing will also be presented.

Under SRS assumptions and given the total number trips made by the vessel and the number of observed trips, the general design-unbiased within-vessel estimator for the generic catch parameter is

$$\widehat{\boldsymbol{\Theta}}_v = \frac{C_v}{c_v}\sum_{t=1}^{c_v}\widehat{\boldsymbol{\Theta}}_t \tag{5.71}$$

where C_v and c_v are the total and observed number of trips made by the vessel. The general VCM estimator is

$$\widehat{V}\left(\widehat{\Theta}_v\right) = C_v\left(\frac{C_v}{c_v} - 1\right)\widetilde{\mathbf{S}}^2_{\Theta,v} + \frac{C_v}{c_v}\left[\sum_{t=1}^{c_v}\widehat{V}\left(\widehat{\Theta}_t\right) + \frac{C_v - 1}{c_v - 1}\sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right)\right]. \tag{5.72}$$

where

$$\widetilde{\mathbf{S}}^2_{\Theta,v} = \frac{\sum_{t=1}^{c_v}\left(\widehat{\Theta}_t - \widehat{\widehat{\Theta}}_v\right)\left(\widehat{\Theta}_{vt} - \widehat{\widehat{\Theta}}_v\right)^T}{c_v - 1}$$

and $\widehat{\widehat{\Theta}}_v = \sum_{t=1}^{c_v}\widehat{\Theta}_t/c_v$. Of course, for scalar parameters (i.e., vectors that are 1×1) the VCM and estimator are themselves scalars and the measure of between trip variability reduces to

$$\widetilde{S}^2_{\Theta,v} = \frac{\sum_{t=1}^{c_v}\left(\widehat{\Theta}_t - \widehat{\widehat{\Theta}}_v\right)^2}{c_v - 1}.$$

For all parameters I address other than numbers in age classes, the covariance of estimators for tth and uth trips, $Cov\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right)$, will be non-zero only when both trips are undersampled . The estimators of numbers-at-age will have covariance between different trips when a simple random subsample of collected otoliths is aged or when both trips are undersampled and model-based methods are used. The form of the vessel-specific estimators do not change when estimation is with regard to particular time periods/regions because all corresponding uncertainty occurs at the trip level. When vessels have 100% observer coverage, $C_v = c_v$ and the within-vessel estimator reduces to the sum of the trip-specific estimates and the variance estimator for large vessels reduces to the sum of the within-trip variances and between-trip covariances. However, the information deficiencies described in the following section inhibit the use of the design-based estimators for medium vessels.

5.2.1 Addressing Unknown Total Number of Trips

A further complication with current NPGOP sampling design is the lack of information on unobserved fishing effort for vessels in the medium vessel size class. The total number of trips made by each medium vessel is unknown, but required for design-based estimation of catch parameters for these vessels. What is known, however, is that medium longline vessels are regulated to have at least 30% of their fishing days covered by observers. Since it is not in the financial interest of the vessels to have observers aboard fewer or greater than the necessary number of fishing days an assumption that the number of observed fishing days is 30% of the total number spent fishing is not unreasonable.

I have already mentioned that inference of catch attributes over all fishing trips requires the assumption that fishing trip activities are independent of the presence of an observer. If we further assume the number of days per trip and observer presence are independent, then with respect to the process that determines the observed trips, 30% of fishing trips are observed for a given vessel, on average. This equates to a model where $E_m(c_v) = 3C_v/10$ for all medium vessels. A more general model that treats any specific coverage level has $E_m(c_v) = C_v/\xi_v$ where ξ_v is the inverse of the proportion of trips sampled on average. That is, $100/\xi_v$ equates to percent observer coverage. In the present context, $\xi_v = 10/3$ for all vessels in the medium size class and $\xi_v = 1$ for all vessels in the large size class. If we assume a constant model variance for the number of observed trips across all vessels, then $V_m(c_v) = \sigma_c^2$ for all vessels. Therefore,

$$\widehat{\Theta}_v = \xi_v \sum_{t=1}^{c_v} \widehat{\Theta}_t \tag{5.73}$$

is unbiased with respect to the joint distribution of the number of sampled trips and the sampling of trips for medium vessels. A conservative VCM estimator is

$$\begin{aligned}\widehat{V}\left(\widehat{\Theta}_v\right) =& \xi_v^2 \left[\widehat{\sigma}_c^2 \widehat{\overline{\Theta}}_v \widehat{\overline{\Theta}}_v^T + \sum_{t=1}^{c_v} \widehat{V}\left(\widehat{\Theta}_t\right) + \sum\sum_{t \neq u}^{c_v} \widehat{Cov}\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right)\right] \\ &+ \xi_v \left[(\xi_v - 1)\, c_v - \frac{\widehat{\sigma}_c^2}{c_v}\right] \widehat{\mathbf{S}}^2_{\Theta,v}\end{aligned} \tag{5.74}$$

where

$$\widehat{\mathbf{S}}^2_{\Theta,v} = \widetilde{\mathbf{S}}_{\Theta,v} - \frac{1}{c_v}\left[\sum_{t=1}^{c_v} \widehat{V}\left(\widehat{\Theta}_t\right) - \frac{1}{c_v - 1} \sum\sum_{t \neq u}^{c_v} \widehat{Cov}\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right)\right],$$

$$\widehat{\sigma}_c^2 = \frac{\sum_{v=1}^{V_M} \left(c_v - \overline{c}_M\right)^2}{V_M - 1},$$

V_M is the number of vessels in the medium size class for a given quarter and $\overline{c}_M = \sum_{v=1}^{V_M} c_v / V_M$ is the average number of sampled trips per medium vessels in the given quarter. The VCM estimator is conservative along the diagonal because the estimator,

$\widehat{\sigma}_c^2$, will overestimate the variance of the number of sampled trips unless the mean number of sampled trips is also constant across vessels. Another reason the variance estimator is conservative is the that $\widehat{\Theta}_v\widehat{\Theta}_v^T$ is positively biased for $\Theta_v\Theta_v^T$. A more extensive unbiased variance estimator is presented in Section 5.A.5.

5.2.2 Total Number

As I have for some within-trip estimators, here I will treat total numbers of marine mammal interactions with fish and seabird species. Let N denote either numbers of fish, seabirds or marine mammal interactions of a given species. When the total number of trips made by the vessel is known, the estimator of the total number attributed to the vth vessel is

$$_1\widehat{N}_v = \frac{C_v}{c_v}\sum_{t=1}^{c_v}\widehat{N}_t \tag{5.75}$$

and the corresponding variance estimator is

$$\widehat{V}\left(\widehat{N}_v\right) = C_v\left(\frac{C_v}{c_v}-1\right)\widetilde{S}^2_{N,v} + \frac{C_v}{c_v}\left[\sum_{t=1}^{c_v}\widehat{V}\left(\widehat{N}_t\right) + \frac{C_v-1}{c_v-1}\sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{N}_t,\widehat{N}_u\right)\right]. \tag{5.76}$$

where

$$\widetilde{S}^2_{N,v} = \frac{\sum_{t=1}^{c_v}\left(\widehat{N}_t - \widehat{\overline{N}}_v\right)^2}{c_v-1}$$

and $\widehat{\overline{N}}_v = \widehat{N}_v/C_v$ and $\widehat{N}_t$ is an appropriate within-trip estimator of the total number caught.

For collected data, we do not know the total number of trips made by medium vessels. Assuming the model for sampled tris described in Section 5.2.1, an estimator for total number caught by the vth vessel in the large size class is

$$_2\widehat{N}_v = \xi_v\sum_{t=1}^{c_v}\widehat{N}_{vt} \tag{5.77}$$

where $\xi_v = 10/3$ in the case of 30% observer coverage. The corresponding variance estimator is

$$\widehat{V}\left(\widehat{N}_v\right) = \xi_v^2\left[\widehat{\sigma}_c^2 \widehat{\overline{N}}_v^2 + \sum_{t=1}^{c_v} \widehat{V}\left(\widehat{N}_t\right) + \sum\sum_{t \neq u}^{c_v} \widehat{Cov}\left(\widehat{N}_t, \widehat{N}_u\right)\right] + \xi_v\left[\left(\xi_v - 1\right) c_v - \frac{\widehat{\sigma}_c^2}{c_v}\right] \widehat{S}_{N,v}^2 \tag{5.78}$$

where

$$\widehat{S}_{N,v}^2 = \widetilde{S}_{N,v}^2 - \frac{1}{c_v}\left[\sum_{t=1}^{c_v} \widehat{V}\left(\widehat{N}_t\right) - \frac{1}{c_v - 1}\sum\sum_{t \neq u}^{c_v} \widehat{Cov}\left(\widehat{N}_t, \widehat{N}_u\right)\right]$$

and $\widehat{\overline{N}}_v = \sum_{t=1}^{c_v} \widehat{N}_t / c_t$. The forms for $\widehat{V}\left(\widehat{N}_t\right)$ and $\widehat{Cov}\left(\widehat{N}_t, \widehat{N}_u\right)$ depend on the sampling or modeling procedures used within respective trips.

When two or more of the trips made by a longline vessel are undersampled and model-based methods described in Section 5.1.4.1 are used, the covariance estimate for predicted total numbers of fish or seabirds caught in the tth and uth trips in the mth management region is

$$\widehat{Cov}\left(\widehat{N}_{mt}^*, \widehat{N}_{mu}^*\right) = H_{mt} H_{mu} \frac{\widehat{\pi}_m \left(1 - \widehat{\pi}_m\right) \widehat{\tau}_m}{h_m} \tag{5.79}$$

where $\widehat{\pi}_m$ and $\widehat{\tau}_m$ are given in Eq. 5.47 and Eq. 5.50, respectively. For marine mammal interactions, the covariance estimator is

$$\widehat{Cov}\left(\widehat{N}_{mt}^*, \widehat{N}_{mu}^*\right) = H_{mt} H_{mu} \frac{\widehat{\nu}_m \widehat{\tau}_{\nu m}}{h_m}.$$

When two or more undersampled trips are made by a pot or trawl vessel and Eq. 5.65 is assumed, the covariance estimator is

$$\widehat{Cov}\left(\widehat{N}_{mt}^*, \widehat{N}_{mu}^*\right) = X_{mt} X_{mu} \frac{\widehat{\sigma}_{n,m}^2}{x_m}$$

where

$$\widehat{\sigma}_{n,m}^2 = \frac{x_m}{x_m^2 - \sum_{k=1}^{g_m} x_m^2} \sum_{k=1}^{g_m} \left(n_k - \widehat{\phi}_{n,m} x_k\right)^2.$$

To obtain the covariance estimates for predicted numbers caught in multiple modeled periods/regions we just sum the covariance estimate for the individual periods/regions. See Section 5.1.4.2 for more details on the estimators for the model parameters in the covariance estimators.

5.2.3 Total Weight

When a SRS of trips is assumed and the total number trips made be the vth vessel is known, the total weight of catch for a given species is

$$ {}_1\widehat{W}_v = \frac{C_v}{c_v} \sum_{t=1}^{c_v} \widehat{W}_t \tag{5.80} $$

where $\widehat{W}_t$ is the within-trip catch weight estimate for the tth trip, C_v is the total number of trips made by the vth vessel and c_v is the number of trips observed. The variance estimator is

$$ \widehat{V}\left({}_1\widehat{W}_v\right) = C_v \left(\frac{C_v}{c_v} - 1\right) \widetilde{S}^2_{W,v} + \frac{C_v}{c_v} \sum_{t=1}^{c_v} \widehat{V}\left(\widehat{W}_t\right) \tag{5.81} $$

where $\widehat{V}\left(\widehat{W}_t\right)$ is the appropriate within-trip variance estimate for the tth trip and

$$ \widetilde{S}^2_{W,v} = \frac{\sum_{t=1}^{c_v} \left(\widehat{W}_t - \overline{\widehat{W}}_v\right)^2}{c_v - 1}. $$

In the case that the total number of trips is unknown and there is less than 100% observer coverage, the approach described in Section 5.2.1 yields the total weight estimator,

$$ {}_2\widehat{W}_v = \xi_v \sum_{t=1}^{c_v} \widehat{W}_t \tag{5.82} $$

and corresponding variance estimator,

$$ \begin{aligned} \widehat{V}\left(\widehat{W}_v\right) =& \xi_v^2 \left[\widehat{\sigma}^2_c \widehat{\overline{W}}_v^2 + \sum_{t=1}^{c_v} \widehat{V}\left(\widehat{W}_t\right) + \sum_{t \neq u}^{c_v}\sum \widehat{Cov}\left(\widehat{W}_t, \widehat{W}_u\right)\right] \\ &+ \xi_v \left[(\xi_v - 1) c_v - \frac{\widehat{\sigma}^2_c}{c_v}\right] \widetilde{S}^2_{W,v} \end{aligned} \tag{5.83} $$

where

$$\widehat{S}^2_{W,v} = \widetilde{S}^2_{W,v} - \frac{1}{c_v}\left[\sum_{t=1}^{c_v} \widehat{V}\left(\widehat{W}_t\right) - \frac{1}{c_v - 1}\sum\sum_{t \neq u}^{c_v} \widehat{Cov}\left(\widehat{W}_t, \widehat{W}_u\right)\right].$$

When more than one trip is undersampled for a longliner, the covariance estimator for the tth and uth trips is

$$\widehat{Cov}\left(\widehat{W}^*_{mt}, \widehat{W}^*_{mu}\right) = H_{mt}H_{mu}\widehat{V}\left(\widehat{\pi}_m \widehat{\mu}_{Wm}\right)$$

where

$$\widehat{V}\left(\widehat{\pi}_m \widehat{\mu}_{Wm}\right) = \widehat{\pi}^2_m \frac{\widehat{\sigma}^2_{Wm}}{n_m} + \widehat{\mu}^2_{Wm}\widehat{V}\left(\widehat{\pi}_m\right) - \widehat{V}\left(\widehat{\pi}_m\right)\frac{\widehat{\sigma}^2_{Wm}}{n_m}$$

where

$$\widehat{V}\left(\widehat{\pi}_m\right) = \frac{\widehat{\pi}_m\left(1 - \widehat{\pi}_m\right)\widehat{\tau}_m}{h_m}. \tag{5.84}$$

When more than one trip is undersampled for a pot or trawl vessel and the model in Eq. 5.65 is assumed, the covariance estimator for the tth and uth trips is

$$\widehat{Cov}\left(\widehat{W}^*_{mt}, \widehat{W}^*_{mu}\right) = X_{mt}X_{mu}\frac{\widehat{\sigma}^2_{w,m}}{x_m}$$

where

$$\widehat{\sigma}^2_{w,m} = \frac{x_m}{x^2_m - \sum_{k=1}^{g_m} x^2_k}\sum_{k=1}^{g_m}\left(w_k - \widehat{\phi}_{w,m}x_k\right)^2,$$

$$\widehat{\phi}_{w,m} = \frac{\sum_{k=1}^{g_m} w_k}{\sum_{k=1}^{g_m} x_k}$$

and w_k is the sample weight in the kth haul for the given species. To obtain the covariance estimates for predicted numbers caught in multiple modeled periods/regions we just sum the covariance estimate for the individual periods/regions. See Section 5.1.4.2 for more details on estimators for the model parameters.

5.2.4 Total Numbers in Length Classes

In general, it is important to remember that the estimation of total numbers in length classes for all effort made by a vessel is limited to s-prevalent catches. However, I omit the a subscript in the presenting following estimators because, at the vessel-level, any of the catch parameter estimators could be domain-specific and should be viewed as such. Under SRS assumptions and given the total number of observed trips, the general design-unbiased within-vessel estimator for a numbers-at-length is

$$\widehat{\mathbf{\Lambda}}_v = \frac{C_v}{c_v}\sum_{t=1}^{c_v}\widehat{\mathbf{\Lambda}}_t \tag{5.85}$$

where C_v and c_v are the total and observed number of trips made by the vessel. The general variance estimator is

$$\widehat{V}\left(\widehat{\mathbf{\Lambda}}_v\right) = C_v\left(\frac{C_v}{c_v}-1\right)\widetilde{\mathbf{S}}^2_{\Lambda,v} + \frac{C_v}{c_v}\left[\sum_{t=1}^{c_v}\widehat{V}\left(\widehat{\mathbf{\Lambda}}_t\right) + \frac{C_v-1}{c_v-1}\sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{\mathbf{\Lambda}}_t,\widehat{\mathbf{\Lambda}}_u\right)\right]. \tag{5.86}$$

where

$$\widetilde{\mathbf{S}}^2_{\Lambda,v} = \frac{\sum_{t=1}^{c_v}\left(\widehat{\mathbf{\Lambda}}_{vt}-\widehat{\overline{\mathbf{\Lambda}}}_v\right)\left(\widehat{\mathbf{\Lambda}}_t-\widehat{\overline{\mathbf{\Lambda}}}_v\right)^T}{c_v-1}$$

and $\widehat{\overline{\mathbf{\Lambda}}}_v = \sum_{t=1}^{c_v}\widehat{\mathbf{\Lambda}}_t/c_v$.

When the total number of trips is unknown and there is less than 100% observer coverage, the approach described in Section 5.2.1 yields,

$$\widehat{\mathbf{\Lambda}}_v = \xi_v\sum_{t=1}^{c_v}\widehat{\mathbf{\Lambda}}_t \tag{5.87}$$

where $\xi_v = 10/3$ in the present case. The corresponding VCM estimator is

$$\begin{aligned}\widehat{V}\left(\widehat{\mathbf{\Lambda}}_v\right) =& \xi_v^2\left[\widehat{\sigma}^2_c\widehat{\overline{\mathbf{\Lambda}}}_v\widehat{\overline{\mathbf{\Lambda}}}_v^T + \sum_{t=1}^{c_v}\widehat{V}\left(\widehat{\mathbf{\Lambda}}_t\right) + \sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{\mathbf{\Lambda}}_t,\widehat{\mathbf{\Lambda}}_u\right)\right] \\ &+ \xi_v\left[\left(\xi_v-1\right)c_v - \frac{\widehat{\sigma}^2_c}{c_v}\right]\widehat{\mathbf{S}}^2_{\Lambda,v}\end{aligned} \tag{5.88}$$

where

$$\widehat{\mathbf{S}}^2_{\Lambda,v} = \widetilde{\mathbf{S}}^2_{\Lambda,v} - \frac{1}{c_v}\left[\sum_{t=1}^{c_v}\widehat{V}\left(\widehat{\mathbf{\Lambda}}_t\right) - \frac{1}{c_v-1}\sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{\mathbf{\Lambda}}_t,\widehat{\mathbf{\Lambda}}_u\right)\right].$$

When both the tth and uth trips are undersampled on a longliner, the covariance estimator for numbers-at-length predictions takes the form

$$\widehat{Cov}\left(\widehat{\mathbf{\Lambda}}^{*}_{amt}, \widehat{\mathbf{\Lambda}}^{*}_{amu}\right) = H_{mt}H_{mu}\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Lambda am}\right)$$

where $\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Lambda am}\right)$ is given in Eq. 5.64. The prediction error covariance is derived where in Section 5.A.4

When the tth and uth trips are undersampled for pot or trawl vessels, the covariance estimator for the numbers-at-length predictions in the mth period/region is

$$\widehat{Cov}\left(\widehat{\mathbf{\Lambda}}^{*}_{amt}, \widehat{\mathbf{\Lambda}}^{*}_{amu}\right) = X_{mt}X_{mu}\widehat{V}\left(\widehat{\alpha}_m\widehat{\phi}_{n,am}\widehat{\mathbf{p}}_{\lambda,am}\right)$$

where

$$\begin{aligned}\widehat{V}\left(\widehat{\alpha}_m\widehat{\phi}_{n,am}\widehat{\mathbf{p}}_{\lambda,am}\right) =& \widehat{\alpha}^2_m\widehat{\phi}^2_{n,am}\widehat{\mathbf{p}}_{\lambda,am}\widehat{\mathbf{p}}^T_{\lambda,am} - \\ & \left[\widehat{\alpha}^2_m - \widehat{V}\left(\widehat{\alpha}_m\right)\right]\left[\widehat{\phi}^2_{n,am} - \widehat{V}\left(\widehat{\phi}_{n,am}\right)\right]\left[\widehat{\mathbf{p}}_{\lambda,am}\widehat{\mathbf{p}}^T_{\lambda,am} - \widehat{V}\left(\widehat{\mathbf{p}}_{\lambda,am}\right)\right].\end{aligned} \tag{5.89}$$

For details on the model-parameter estimators in any of the covariance estimators see Section 5.1.4.2.

5.2.5 *Total Numbers in Age Classes*

As I do for total numbers in length classes, I omit here any subscript denoting estimation for particular domains unless necessary. Nonetheless, the estimators for total numbers in age classes are specific to period/regions where otolith subsamples are made for ageing. Under the assumption of a SRS of trips and given the total number of fishing trips made by the vessel, the estimator for the total numbers-at-age is

$$\widehat{\mathbf{\Psi}}_v = \frac{C_v}{c_v}\sum_{t=1}^{c_v}\widehat{\mathbf{\Psi}}_t \tag{5.90}$$

where C_v and c_v are the total and observed number of trips made by the vessel. The general VCM estimator is

$$\begin{aligned}\widehat{V}\left(\widehat{\mathbf{\Psi}}_v\right) =& C_v\left(\frac{C_v}{c_v}-1\right)\widetilde{\mathbf{S}}^2_{\Psi,v} \\ &+\frac{C_v}{c_v}\left[\sum_{t=1}^{c_v}\widehat{V}\left(\widehat{\mathbf{\Psi}}_t\right)+\frac{C_v-1}{c_v-1}\sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{\mathbf{\Psi}}_t,\widehat{\mathbf{\Psi}}_u\right)\right].\end{aligned} \tag{5.91}$$

where

$$\widetilde{\mathbf{S}}^2_{\Psi,v}=\frac{\sum_{t=1}^{c_v}\left(\widehat{\mathbf{\Psi}}_{vt}-\widehat{\overline{\mathbf{\Psi}}}_v\right)\left(\widehat{\mathbf{\Psi}}_{vt}-\widehat{\overline{\mathbf{\Psi}}}_v\right)^T}{c_v-1}$$

and $\widehat{\overline{\mathbf{\Psi}}}_v=\sum_{t=1}^{c_v}\widehat{\mathbf{\Psi}}_t/c_v$.

The total number of trips is unknown and for vessels with less than 100% observer coverage, the estimator based on the approach described in Section 5.2.1 is

$$\widehat{\mathbf{\Psi}}_v=\xi_v\sum_{t=1}^{c_v}\widehat{\mathbf{\Psi}}_t \tag{5.92}$$

where $\xi_v=10/3$ in the present case. The corresponding VCM estimator is

$$\begin{aligned}\widehat{V}\left(\widehat{\mathbf{\Psi}}_v\right) =& \xi_v^2\left[\widehat{\sigma}_c^2\widehat{\overline{\mathbf{\Psi}}}_v\widehat{\overline{\mathbf{\Psi}}}_v^T+\sum_{t=1}^{c_v}\widehat{V}\left(\widehat{\mathbf{\Psi}}_t\right)+\sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{\mathbf{\Psi}}_t,\widehat{\mathbf{\Psi}}_u\right)\right] \\ &+\xi_v\left[\left(\xi_v-1\right)c_v-\frac{\widehat{\sigma}_c^2}{c_v}\right]\widehat{\mathbf{S}}^2_{\Psi,v}\end{aligned} \tag{5.93}$$

where

$$\widehat{\mathbf{S}}^2_{\Psi,v}=\widetilde{\mathbf{S}}^2_{\Psi,v}-\frac{1}{c_v}\left[\sum_{t=1}^{c_v}\widehat{V}\left(\widehat{\mathbf{\Psi}}_t\right)-\frac{1}{c_v-1}\sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{\mathbf{\Psi}}_t,\widehat{\mathbf{\Psi}}_u\right)\right].$$

When both the tth and uth trips are not undersampled, the between-trip covariance matrix estimator is

$$\begin{aligned}\widehat{Cov}\left(\widehat{\mathbf{\Psi}}_{amt},\widehat{\mathbf{\Psi}}_{amu}\right) =& \frac{G_t g_{at} G_u g_{au}}{g_t g_{Ot} g_u g_{Ou}}\sum_{k=1}^{g_{Omt}}\sum_{l=1}^{g_{Omu}}\widehat{Cov}\left(\widehat{\mathbf{\Psi}}_k,\widehat{\mathbf{\Psi}}_l\right) \\ =& -\left(\frac{1}{n_{Am}}-\frac{1}{N_{Om}}\right)\frac{\widehat{\mathbf{\Psi}}_{amt}\widehat{\mathbf{\Psi}}_{amu}^T}{N_{Om}-1}\end{aligned} \tag{5.94}$$

where $\widehat{Cov}\left(\widehat{\boldsymbol{\Psi}}_k, \widehat{\boldsymbol{\Psi}}_l\right)$ is defined in Eq. 5.32 for any k and l. When both the tth and uth trips are undersampled for a longliner, the covariance estimator for numbers-at-age predictions in the mth region take the form

$$\widehat{Cov}\left(\widehat{\boldsymbol{\Psi}}_{amt}, \widehat{\boldsymbol{\Psi}}_{amu}\right) = H_{mt}H_{mu}\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\psi,am}\right)$$

where $\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\psi,am}\right)$ is analogous to Eq. 5.64. The prediction error covariance for longliners is derived in Section 5.A.4.

When both trips are undersampled for a pot or trawl vessel the covariance estimator is

$$\widehat{Cov}\left(\widehat{\boldsymbol{\Psi}}_{amt}, \widehat{\boldsymbol{\Psi}}_{amu}\right) = X_{mt}X_{mu}\widehat{V}\left(\widehat{\alpha}_m\widehat{\phi}_{n,am}\widehat{\mathbf{p}}_{\psi,am}\right)$$

where $\widehat{V}\left(\widehat{\alpha}_m\widehat{\phi}_{n,am}\widehat{\mathbf{p}}_{\psi,am}\right)$ is analogous to Eq. 5.89.

5.3 *Within a Fleet*

Estimation of catch parameters over any arbitrary group of vessels within a given quarter that require observer coverage is analogous to vessel-specific estimation for vessels in the large size class. The general estimator and variance estimator for the fth fleet are

$$\widehat{\boldsymbol{\Theta}}_f = \sum_{v=1}^{V_f}\widehat{\boldsymbol{\Theta}}_v \tag{5.95}$$

and

$$\widehat{V}\left(\widehat{\boldsymbol{\Theta}}_f\right) = \sum_{v=1}^{V_f}\widehat{V}\left(\widehat{\boldsymbol{\Theta}}_v\right) + \sum\sum_{v\neq w}^{V_f}\widehat{Cov}\left(\widehat{\boldsymbol{\Theta}}_v, \widehat{\boldsymbol{\Theta}}_w\right) \tag{5.96}$$

where V_f is the total number of vessels in the fleet and the within-vessel estimates and corresponding variance estimates depend on the parameter of interest. The covariance between vessel estimates will be non-zero for numbers in age classes and when both the vth and wth vessel have some model-based predictions for some undersampled trips. Models I have developed for predicting catch parameters are gear-specific and, therefore, the covariances between vessels will only occur for those fishing the same gear and with undersampled trips.

5.3.1 Covariance Estimation

5.3.1.1 Total Number

When total numbers are being estimated for longliners, there are undersampled trips within the vth and wth vessels and the total number of trips made by each vessel is known, the covariance estimator for the totals in the mth period/region is

$$\begin{aligned}\widehat{Cov}\left(\widehat{N}_{mv}, \widehat{N}_{mw}\right) =& \frac{C_v}{c_v}\frac{C_w}{c_w}\sum_{t=1}^{c_v^*}\sum_{u=1}^{c_w^*}\widehat{Cov}\left(\widehat{N}_{mt}^*, \widehat{N}_{mu}^*\right)\\ =& \frac{\widehat{\pi}_m\left(1-\widehat{\pi}_m\right)\widehat{\tau}_m}{h_m - 1}\left(\frac{C_v}{c_v}\sum_{t=1}^{c_v^*}H_{mt}\right)\left(\frac{C_w}{c_w}\sum_{u=1}^{c_w^*}H_{mu}\right)\end{aligned}$$

where c_v^* is the number of undersampled trips made by the vth vessel. When the total number of trips is unknown and the approach described in Section 5.2.1 is used the coviance estimator is

$$\widehat{Cov}\left(\widehat{N}_{mv}, \widehat{N}_{mw}\right) = \frac{\widehat{\pi}_m\left(1-\widehat{\pi}_m\right)\widehat{\tau}_m}{h_m - 1}\left(\xi_v\sum_{t=1}^{c_v^*}H_{mt}\right)\left(\xi_w\sum_{u=1}^{c_w^*}H_{mu}\right)$$

where, in the current design, $\xi_v = 10/3$ for vessels in the medium size class and $\xi_v = 1$ for vessels in the large size class. When total numbers are being estimated for trawlers or pot vessels, the covariance estimator for the predicted totals in the mth period/region is

$$\begin{aligned}\widehat{Cov}\left(\widehat{N}_{mv}, \widehat{N}_{mw}\right) =& \frac{C_v}{c_v}\frac{C_w}{c_w}\sum_{t=1}^{c_v^*}\sum_{u=1}^{c_w^*}\widehat{Cov}\left(\widehat{N}_{mt}^*, \widehat{N}_{mu}^*\right)\\ =& \frac{\widehat{\sigma}_{n,m}^2}{x_m}\left(\frac{C_v}{c_v}\sum_{t=1}^{c_v^*}X_{mt}\right)\left(\frac{C_w}{c_w}\sum_{u=1}^{c_w^*}X_{mu}\right).\end{aligned}$$

When the total number of trips is unknown and the approach described in Section 5.2.1 is used the coviance estimator is

$$\widehat{Cov}\left(\widehat{N}_{mv}, \widehat{N}_{mw}\right) = \frac{\widehat{\pi}_m\left(1-\widehat{\pi}_m\right)\widehat{\tau}_m}{h_m - 1}\left(\xi_v\sum_{t=1}^{c_v^*}X_{mt}\right)\left(\xi_w\sum_{u=1}^{c_w^*}X_{mu}\right).$$

5.3.1.2 Total Weight

For total weight, the covariance estimator appropriate to longline vessels within undersampled trips is

$$\widehat{Cov}\left(\widehat{W}_{mv}, \widehat{W}_{mw}\right) = \widehat{V}\left(\widehat{\pi}_m \widehat{\mu}_{Wm}\right)\left(\frac{C_v}{c_v}\sum_{t=1}^{c_v^*} H_{mt}\right)\left(\frac{C_w}{c_w}\sum_{u=1}^{c_w^*} H_{mu}\right)$$

where $\widehat{V}\left(\widehat{\pi}_m \widehat{\mu}_{Wm}\right)$ is given in Eq. 5.84. For trawl and pot vessels the appropriate covariance estimator is

$$\widehat{Cov}\left(\widehat{W}_{mv}, \widehat{W}_{mw}\right) = \frac{\widehat{\sigma}^2_{w,t}}{x_m}\left(\frac{C_v}{c_v}\sum_{t=1}^{c_v^*} X_{mt}\right)\left(\frac{C_w}{c_w}\sum_{u=1}^{c_w^*} X_{mu}\right).$$

Again, when the total numbers of trips made by the vessels are unknown, ξ_v replaces C_v/c_v.

When interest is in estimation over multiple separately modeled period/regions the covariance is just the sum of the period/region-specific covariances. See Section 5.1.4.1 for more details on estimation of the model parameters.

5.3.1.3 Total Numbers in Length or Age Classes

When there is model-based prediction of numbers in length or age classes in undersampled trips, the covariance estimator for longline vessels is

$$\widehat{Cov}\left(\widehat{\Theta}_{mv}, \widehat{\Theta}_{mw}\right) = \widehat{V}\left(\widehat{\alpha}_m \widehat{\pi}_{am} \widehat{\mathbf{p}}_{\theta,am}\right)\left(\frac{C_v}{c_v}\sum_{t=1}^{c_v^*} H_{mt}\right)\left(\frac{C_w}{c_w}\sum_{u=1}^{c_w^*} H_{mu}\right) \qquad (5.97)$$

where $\widehat{V}\left(\widehat{\alpha}_m \widehat{\pi}_{am} \widehat{\mathbf{p}}_{\theta,am}\right)$ is given in (Eq. 5.64) for numbers in length classes, but an analogous form uses the model for proportions in age classes. When the vessels are fishing pot or trawl gear,

$$\widehat{Cov}\left(\widehat{\Theta}_{mv}, \widehat{\Theta}_{mw}\right) = \widehat{V}\left(\widehat{\alpha}_m \widehat{\phi}_{n,am} \widehat{\mathbf{p}}_{\theta,am}\right)\left(\frac{C_v}{c_v}\sum_{t=1}^{c_v^*} X_{mt}\right)\left(\frac{C_w}{c_w}\sum_{u=1}^{c_w^*} X_{mu}\right)$$

where the form of $\widehat{V}\left(\widehat{\alpha}_m \widehat{\phi}_{n,am} \widehat{\mathbf{p}}_{\theta,am}\right)$ is the same as that given in (Eq. 5.89).

Covariance of sufficiently sampled trips also occurs for estimation of numbers in age classes and the covariance estimator specific to longliners for numbers-at-age in the mth period/region is

$$\widehat{Cov}\left(\widehat{\mathbf{\Psi}}_{amv}, \widehat{\mathbf{\Psi}}_{amw}\right) = -\frac{\frac{1}{n_{Am}} - \frac{1}{N_{Om}}}{N_{Om} - 1}\left(\frac{C_v}{c_v}\sum_{t=1}^{c_v^{**}} \widehat{\mathbf{\Psi}}_{amt}\right)\left(\frac{C_w}{c_w}\sum_{u=1}^{c_w^{**}} \widehat{\mathbf{\Psi}}_{amu}\right)^T \tag{5.98}$$

where $c_v^{**} = c_v - c_v^*$ is the number of sufficiently sampled trips made by the vth vessel. Again, when the total numbers of trips made by the vessels are unknown, ξ_v replaces C_v/c_v.

5.3.2 Conservative variance estimation for undersampled vessels

In Section 5.2.1 I give a conservative variance estimator for medium size vessels, but often several medium vessels are undersampled, that is, only one trip is sampled for those vessels during a given quarter. This inhibits the use of Eq. 5.74 and an alternative conservative estimator for the variance over all medium vessels where only one trip is sampled is,

$$\widehat{V}\left(\widehat{\mathbf{\Theta}}_U\right) = \frac{V_U}{V_U - 1}\left[\sum_{v=1}^{V_U}\left(\widehat{\mathbf{\Theta}}_v - \overline{\widehat{\mathbf{\Theta}}}_U\right)\left(\widehat{\mathbf{\Theta}}_v - \overline{\widehat{\mathbf{\Theta}}}_U\right)^T + \sum\sum_{v \neq w}^{V_U} \widehat{Cov}\left(\widehat{\mathbf{\Theta}}_v, \widehat{\mathbf{\Theta}}_w\right)\right] \tag{5.99}$$

where $\overline{\widehat{\mathbf{\Theta}}}_U = \sum_{v=1}^{V_U} \widehat{\mathbf{\Theta}}_v / V_U$ and V_U is the number of undersampled vessels. This variance estimator is similar to the collapsed-strata variance estimators given by Cochran (1977) and others except that here we treat the presence of covariance between elements in different strata.

5.4 Estimation Over Quarters

Estimation over an arbitrary set of quarters is analogous to fleet-wide estimation, so that the generic estimator over Q quarters is

$$\widehat{\mathbf{\Theta}}_{Qf} = \sum_{q=1}^{Q} \widehat{\mathbf{\Theta}}_{qf} \tag{5.100}$$

where Q is the number of quarters in the time period of interest and $\widehat{\Theta}_{qf}$ is the estimator for the fth fleet in the qth quarter. Usually, interest is in estimation over all vessels rather than a specific fleet so the f subscript is dropped hereafter. The general variance estimator is

$$\widehat{V}\left(\widehat{\Theta}_Q\right) = \sum_{q=1}^{Q} \widehat{V}\left(\widehat{\Theta}_q\right) + \sum_{q \neq r}^{Q}\sum \widehat{Cov}\left(\widehat{\Theta}_q, \widehat{\Theta}_r\right) \tag{5.101}$$

where covariance of quarter-specific estimates again only arises when the models used for predicting undersampled trips have parameters that are not specific to each quarter or when the sample sizes of otoliths for ageing are not quarter-specific.

I intend that models for predicting catch parameters for undersampled trips are quarter-specific because of such realities as seasonal changes in the system where catches are made. With these quarter-specific models, there is no model-based covariance of quarterly catch parameter estimates, but for numbers-at-age, there is design-based covariance between quarter-specific estimates of numbers-at-age due to subsampling all otoliths collected throughout the year for ageing. The covariance estimator for numbers in age classes in different quarters is

$$\begin{aligned}\widehat{Cov}\left(\widehat{\Psi}_{amq}, \widehat{\Psi}_{amr}\right) &= \sum_{v=1}^{V_q}\sum_{w=1}^{V_r} \widehat{Cov}\left(\widehat{\Psi}_v, \widehat{\Psi}_w\right) \\ &= -\frac{\frac{1}{n_{Am}} - \frac{1}{N_{Om}}}{N_{Om} - 1}\left(\sum_{v=1}^{V_r} \frac{C_v}{c_v} \sum_{t=1}^{c_v^{**}} \widehat{\Psi}_{amt}\right)\left(\sum_{w=1}^{V_r} \frac{C_w}{c_w} \sum_{u=1}^{c_w^{**}} \widehat{\Psi}_{amu}\right)^T\end{aligned} \tag{5.102}$$

where $\xi = 1$ for large vessels or $\xi = 10/3$ for medium vessels replace C_v/c_v when the total number of trips are not known.

5.5 *Seabird Catch Rates*

Many investigators are interested in estimating the bycatch rates with respect to fishing effort (i.e., catch-per-unit-effort). Bycatch rates can be more important than total bycatch for analyzing changes in mortality over time or as a function of implementated deterrents because changes in amounts of effort are controlled. As such, a mortality rate estimator as a function of deployed hooks is presented here.

The true measurable mortality rate is a function of all fishing effort for both large and medium vessels. When we measure rates for just the trips that were observed, insufficient weight is given to fishing effort made aboard medium vessels.

For large vessels with complete observer coverage, the total number of hooks can be assumed known because, even for unsampled hauls, the observer records the number of hooks deployed. However, there is uncertainty in the number of deployed hooks for medium vessels because of unobserved fishing trips. Under the same randomization assumptions made for catch parameters, there is a SRS within each vessel each quarter for trip totals of deployed hooks, H_t. Using the approach described in Section 5.2.1 for medium sized vessels, the estimator for the total number of hooks within a medium size vessel is $\widehat{H}_v = 10\sum_{t=1}^{c_v} H_t/3$ and the variance estimator is analogous to Eq. 5.74, but because the total number of hooks is known for each trip there is no variance estimation within trips (or covariance between trips). Moreover, the model-based approach for undersampled vessels can be used to provide conservative variance estimation of the total hooks estimator over the set of undersampled medium vessels. Over all vessels in a given quarter or larger aggregations of quarters, the estimator of seabird mortality rate is

$$\widehat{R} = \frac{\sum_{v=1}^{V} \widehat{N}_v}{\sum_{v=1}^{V} \widehat{H}_v} = \frac{\widehat{N}}{\widehat{H}} \tag{5.103}$$

where V is the number of vessels and, by Taylor Series approximation (Delta Method), the corresponding variance estimator is

$$\widehat{V}\left(\widehat{R}\right) = \widehat{R}^2 \left[\frac{\widehat{V}\left(\widehat{N}\right)}{\widehat{N}^2} + \frac{\widehat{V}\left(\widehat{H}\right)}{\widehat{H}^2} - 2\frac{\widehat{Cov}\left(\widehat{N}, \widehat{H}\right)}{\widehat{H}\widehat{N}} \right] \tag{5.104}$$

where $\widehat{Cov}\left(\widehat{N}, \widehat{H}\right)$ is the sum of all within-vessel estimates of covariance which are zero for all large vessels because the total number of hooks is known. For sufficiently sampled medium vessels, the approximate covariance estimator is

$$\widehat{Cov}\left(\widehat{N}_v, \widehat{H}_v\right) = \left(\frac{10}{3}\right)^2 \widehat{\sigma}_c^2 \widehat{\widehat{N}}_v \widehat{\widehat{H}}_v + \frac{10}{3}\left[\frac{7}{3}c_v - \frac{\widehat{\sigma}_c^2}{c_v}\right] \widehat{S}_{NH,v} \tag{5.105}$$

where

$$\widehat{S}_{NH,v} = \frac{\sum_{t=1}^{c_v} \left(\widehat{N}_t - \overline{\widehat{N}}_v\right)\left(H_t - \overline{\widehat{H}}_v\right)}{c_v - 1}$$

and $\overline{\widehat{H}}_v = \sum_{t=1}^{c_v} H_t / c_v$.

For undersampled medium vessels, the covariance estimator analogous to Eq. 5.99 is

$$\widehat{C}\left(\widehat{N}_U, \widehat{H}_U\right) = \frac{V_U}{V_U - 1} \sum_{v=1}^{V_U} \left(\widehat{N}_v - \overline{\widehat{N}}_U\right)\left(\widehat{H}_v - \overline{\widehat{H}}_U\right).$$

5.6 *Estimating Overall Length or Age Composition*

Because stock assessment scientists must account for all catches rather than catches in s-prevalent hauls, a more useful set of estimators are the numbers in age or length classes over all catches in a period/region. This can only be achieved by assuming that the proportions-at-age or -length are the same in catches whether or not the species is prevalent. With this assumption, an estimator of the numbers in length or age classes for all catches in the mth period/region is

$$\widehat{\boldsymbol{\Theta}}_m = \widehat{N}_m \frac{\widehat{\boldsymbol{\Theta}}_{am}}{\widehat{N}_{am}} = \left(\frac{\sum_{v=1}^{V} \widehat{\boldsymbol{\Theta}}_{amv}}{\sum_{v=1}^{V} \widehat{N}_{amv}}\right) \sum_{v=1}^{V} \widehat{N}_{mv} \tag{5.106}$$

where V is the number of observed vessels and, by Taylor Series approximation (Delta Method), the VCM estimator is

$$\begin{aligned}
\widehat{V}\left(\widehat{\boldsymbol{\Theta}}_m\right) = \left(\frac{\widehat{N}_m}{\widehat{N}_{am}}\right)^2 \Bigg\{ & \widehat{\boldsymbol{\Theta}}_m \widehat{\boldsymbol{\Theta}}_m^T \left[\frac{\widehat{V}\left(\widehat{N}_m\right)}{N_m^2} + \frac{\widehat{V}\left(\widehat{N}_{am}\right)}{\widehat{N}_{am}^2} - 2\frac{\widehat{Cov}\left(\widehat{N}_m, \widehat{N}_{am}\right)}{N_m N_{am}}\right] \\
& + \widehat{V}\left(\widehat{\boldsymbol{\Theta}}_{am}\right) + \frac{1}{\widehat{N}_m}\left[\widehat{\boldsymbol{\Theta}}_{am} \widehat{Cov}\left(\widehat{N}_m, \widehat{\boldsymbol{\Theta}}_{am}^T\right) + \widehat{Cov}\left(\widehat{\boldsymbol{\Theta}}_{am}, \widehat{N}_m\right) \widehat{\boldsymbol{\Theta}}_{am}^T\right] \\
& - \frac{1}{\widehat{N}_{am}}\left[\widehat{\boldsymbol{\Theta}}_{am} \widehat{Cov}\left(\widehat{N}_{am}, \widehat{\boldsymbol{\Theta}}_{am}^T\right) + \widehat{Cov}\left(\widehat{\boldsymbol{\Theta}}_{am}, \widehat{N}_{am}\right) \widehat{\boldsymbol{\Theta}}_{am}^T\right]\Bigg\},
\end{aligned}$$

where, in general, the component variance and covariance estimators each have components within vessels and between vessels so that

$$\widehat{Cov}\left(\widehat{\Theta}_1, \widehat{\Theta}_2\right) = \sum_{v=1}^{V} \widehat{Cov}\left(\widehat{\Theta}_{1v}, \widehat{\Theta}_{2v}\right) + \sum\sum_{v \neq w}^{V} \widehat{Cov}\left(\widehat{\Theta}_{1v}, \widehat{\Theta}_{2w}\right).$$

The within-vessel estimators for total numbers in the mth period/region or total numbers in s-prevalent hauls made in the mth period/region follow forms presented in Section 5.2.2 where trip-specific components follow the forms in Section 5.1.3. For within-vessel variance components regarding numbers-at-length we use estimators provided in Section 5.2.4 and for numbers-at-age we use those provided in Section 5.2.5. The within-vessel covariance estimators, $\widehat{Cov}\left(\widehat{N}_{mv}, \widehat{N}_{amv}\right)$, $\widehat{Cov}\left(\widehat{N}_{mv}, \widehat{\mathbf{\Lambda}}_{amv}\right)$ and $\widehat{Cov}\left(\widehat{N}_{amv}, \widehat{\mathbf{\Lambda}}_{amv}\right)$, take the same form as the within-vessel variance estimators in Section 5.2. That is, the covariance estimator for $\widehat{\Theta}_{1v}$ and $\widehat{\Theta}_{2v}$ for large vessels is

$$\widehat{Cov}\left(\widehat{\Theta}_{1v}, \widehat{\Theta}_{2v}\right) = \sum_{t=1}^{c_v} \widehat{Cov}\left(\widehat{\Theta}_{1t}, \widehat{\Theta}_{2t}\right) + \sum\sum_{t \neq u}^{c_v} \widehat{Cov}\left(\widehat{\Theta}_{1t}, \widehat{\Theta}_{2u}\right)$$

and for medium vessels,

$$\widehat{Cov}\left(\widehat{\Theta}_{1v}, \widehat{\Theta}_{2v}\right) = \left(\frac{10}{3}\right)^2 \left[\widehat{\sigma}_c^2 \widehat{\overline{\Theta}}_{1v} \widehat{\overline{\Theta}}_{2v}^T + \sum_{t=1}^{c_v} \sum_{u=1}^{c_v} \widehat{Cov}\left(\widehat{\Theta}_{1t}, \widehat{\Theta}_{2u}\right)\right]$$
$$+ \frac{10}{3}\left[\frac{7}{3}c_v - \frac{\widehat{\sigma}_c^2}{c_v}\right] \widehat{\mathbf{S}}_{\Theta_1\Theta_2,v}$$

where

$$\widehat{\mathbf{S}}_{\Theta_1\Theta_2,v} = \widetilde{\mathbf{S}}_{\Theta_1\Theta_2,v} - \frac{1}{c_v}\left[\sum_{t=1}^{c_v} \widehat{Cov}\left(\widehat{\Theta}_{1t}, \widehat{\Theta}_{2t}\right) - \frac{1}{c_v - 1} \sum\sum_{t \neq u}^{c_v} \widehat{Cov}\left(\widehat{\Theta}_{1t}, \widehat{\Theta}_{2u}\right)\right]$$

and

$$\widetilde{\mathbf{S}}_{\Theta_1\Theta_2,v} = \frac{\sum_{t=1}^{c_v}\left(\widehat{\Theta}_{1t} - \widehat{\overline{\Theta}}_{1v}\right)\left(\widehat{\Theta}_{2t} - \widehat{\overline{\Theta}}_{2v}\right)^T}{c_v - 1}.$$

The differences in form between the various combination of parameters occur in the within- and between-trip covariance terms. For total numbers in s-prevalent hauls and numbers-at-length the within-trip covariance estimator is

$$\widehat{Cov}\left(\widehat{N}_{amt}, \widehat{\mathbf{\Lambda}}_{amt}\right) = G_t\left(\frac{G_t}{g_t}-1\right)\frac{\sum_{k=1}^{g_{tam}}\widehat{N}_k\widehat{\mathbf{\Lambda}}_k-\left(\sum_{k=1}^{g_{amt}}\widehat{N}_k\right)\left(\sum_{k=1}^{g_{amt}}\widehat{\mathbf{\Lambda}}_k\right)/g_t}{g_t-1} + \frac{G_t}{g_t}\sum_{k=1}^{g_{amt}}\widehat{V}\left(\widehat{N}_k\right)\widehat{\mathbf{p}}_{\lambda k}$$

and for total numbers and the numbers-at-length,

$$\widehat{Cov}\left(\widehat{N}_{mt}, \widehat{\mathbf{\Lambda}}_{amt}\right) = G_t\left(\frac{G_t}{g_t}-1\right)\frac{\sum_{k=1}^{g_{amt}}\widehat{N}_k\widehat{\mathbf{\Lambda}}_k-\left(\sum_{k=1}^{g_{mt}}\widehat{N}_k\right)\left(\sum_{k=1}^{g_{amt}}\widehat{\mathbf{\Lambda}}_k\right)/g_t}{g_t-1} + \frac{G_t}{g_t}\sum_{k=1}^{g_{amt}}\widehat{V}\left(\widehat{N}_k\right)\widehat{\mathbf{p}}_{\lambda k}, \tag{5.107}$$

Similar to Eq. 5.107, the covariance estimator for the total number in the region/period and the total number in the prevalent hauls in the region/period for the tth trip is

$$\widehat{Cov}\left(\widehat{N}_{mt}, \widehat{N}_{amt}\right) = G_t\left(\frac{G_t}{g_t}-1\right)\frac{\sum_{k=1}^{g_{amt}}\widehat{N}_k^2-\left(\sum_{k=1}^{g_{mt}}\widehat{N}_k\right)\left(\sum_{k=1}^{g_{amt}}\widehat{N}_k\right)/g_t}{g_t-1} + \frac{G_t}{g_t}\sum_{k=1}^{g_{amt}}\widehat{V}\left(\widehat{N}_k\right).$$

For numbers-at-age,

$$\widehat{Cov}\left(\widehat{N}_{amt}, \widehat{\mathbf{\Psi}}_{amt}\right) = \frac{G_t}{g_t}\left(\frac{G_t}{g_t}-1\right)\frac{g_{at}}{g_{Ot}}\left\{\sum_{k=1}^{g_{Omt}}\widehat{N}_k\widehat{\mathbf{\Psi}}_k - \frac{(g_{at}-1)\sum\sum_{k\neq l}^{g_{Omt}}\widehat{N}_k\widehat{\mathbf{\Psi}}_l}{(g_t-1)(g_{Ot}-1)}\right\} + \frac{G_t}{g_t}\frac{g_{at}}{g_{Ot}}\sum_{k=1}^{g_{Omt}}\widehat{V}\left(\widehat{N}_k\right)\widehat{\mathbf{p}}_{\psi k},$$

and

$$\widehat{Cov}\left(\widehat{N}_{mt}, \widehat{\Psi}_{amt}\right) = \widehat{Cov}\left(\widehat{N}_{amt}, \widehat{\Psi}_{amt}\right) - G_t\left(\frac{G_t}{g_t} - 1\right)\frac{g_{at}}{g_{Ot}}\frac{\left(\sum_{k=1}^{g_{a'mt}} \widehat{N}_k\right)\left(\sum_{k=1}^{g_{Omt}} \widehat{\Psi}_k\right)}{g_t\left(g_t - 1\right)}$$

where $g_{a'mt}$ is the number of hauls where the species is not prevalent in the region/period.

There is also covariance between model-based predictors corresponding to two undersampled trips. For either numbers-at-length or numbers-at-age and total numbers in s-prevalent hauls,

$$\widehat{Cov}\left(\widehat{N}^*_{amt}, \widehat{\Theta}^*_{amu}\right) = H_{mt}H_{mu}\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\right)\widehat{\mathbf{p}}_{\theta,am}.$$

where $\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\right)$ is estimated by Eq. 5.61. However, the predictor for $\widehat{N}^*_{mt}$ is assumed independent of those for $\widehat{N}^*_{amt}$, $\widehat{\mathbf{\Lambda}}^*_{amt}$ and $\widehat{\Psi}^*_{amt}$ and there is no covariance to estimate in these cases.

The covariance of different estimators across vessels must also be considered. For two different vessels,

$$\widehat{Cov}\left(\widehat{N}^*_{amv}, \widehat{\Theta}^*_{amw}\right) = \left(\xi_v\sum_{t=1}^{c_v^*} H_{mt}\right)\left(\xi_w\sum_{u=1}^{c_w^*} H_{mu}\right)\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\right)\widehat{\mathbf{p}}_{\theta,am}.$$

A perhaps more intuitive estimator of the overall numbers-at-length or -age is obtained by dividing the vector of numbers-at-length or -age by the respective sum which yields

$$_2\widehat{\Theta}_m = \widehat{N}_m\frac{\widehat{\Theta}_{am}}{\sum_{i=1}^{I}\widehat{\Theta}_{ami}} \equiv \widehat{N}_m\frac{\widehat{\Theta}_{am}}{_2\widehat{N}_{am}} \tag{5.108}$$

with covariance matrix estimator,

$$\widehat{V}\left({}_2\widehat{\Theta}_m\right) = \left(\frac{\widehat{N}_m}{{}_2\widehat{N}_{am}}\right)^2 \left\{ \left[\frac{\widehat{V}\left(\widehat{N}_m\right)}{\widehat{N}_m^2} + \frac{\widehat{V}\left({}_2\widehat{N}_{am}\right)}{{}_2\widehat{N}_{am}^2} - 2\frac{\widehat{Cov}\left({}_2\widehat{N}_{am}, \widehat{N}_m\right)}{{}_2\widehat{N}_{am}\widehat{N}_m}\right] \widehat{\Theta}_{am}\widehat{\Theta}_{am}^T \right.$$
$$+ \widehat{V}\left(\widehat{\Theta}_{am}\right) + \frac{1}{\widehat{N}_m}\left[\widehat{\Theta}_{am}\widehat{Cov}\left(\widehat{N}_m, \widehat{\Theta}_{am}^T\right) + \widehat{Cov}\left(\widehat{\Theta}_{am}, \widehat{N}_m\right)\widehat{\Theta}_{am}^T\right]$$
$$\left. - \frac{1}{{}_2\widehat{N}_{am}}\left[\widehat{\Theta}_{am}\widehat{Cov}\left({}_2\widehat{N}_{am}, \widehat{\Theta}_{am}^T\right) + \widehat{Cov}\left(\widehat{\Theta}_{am}, {}_2\widehat{N}_{am}\right)\widehat{\Theta}_{am}^T\right] \right\}$$

where

$$\widehat{V}\left({}_2\widehat{N}_{am}\right) = \sum_{i=1}^{L}\sum_{j=1}^{L}\widehat{Cov}\left(\widehat{\Theta}_{ami}, \widehat{\Theta}_{amj}\right), \tag{5.109}$$

$$\widehat{Cov}\left(\widehat{\Theta}_{am}, {}_2\widehat{N}_{am}\right) = \sum_{j=1}^{L}\widehat{Cov}\left(\widehat{\Theta}_{am}, \widehat{\Theta}_{amj}\right) \tag{5.110}$$

and

$$\widehat{Cov}\left({}_2\widehat{N}_{am}, \widehat{\Theta}_{am}^T\right) = \sum_{i=1}^{L}\widehat{Cov}\left(\widehat{\Theta}_{ami}, \widehat{\Theta}_{am}^T\right). \tag{5.111}$$

In words, Eq. 5.109 is just the sum of all elements in the covariance matrix, $\widehat{V}\left(\widehat{\Theta}_{am}\right)$, Eq. 5.110 is the row-wise sum of elements and Eq. 5.111 is the column-wise sum of elements.

Estimation of proportions-at-length and -age follows the same methodology as that used to obtain estimators of overall numbers-at-length and -age. Specifically, we have

$$\widehat{\mathbf{p}}_{\Theta,am} = \frac{\widehat{\Theta}_{am}}{\widehat{N}_{am}} \tag{5.112}$$

and the corresponding approximate variance estimator is

$$\widehat{V}\left(\widehat{\mathbf{p}}_{\Theta,am}\right) = \left(\frac{1}{\widehat{N}_{am}}\right)^2 \left\{ \frac{\widehat{V}\left(\widehat{N}_{am}\right)}{\widehat{N}_{am}^2}\widehat{\Theta}_{am}\widehat{\Theta}_{am}^T + \widehat{V}\left(\widehat{\Theta}_{am}\right) \right.$$
$$\left. - \frac{1}{\widehat{N}_{am}}\left[\widehat{\Theta}_{am}\widehat{Cov}\left(\widehat{N}_{am}, \widehat{\Theta}_{am}^T\right) + \widehat{Cov}\left(\widehat{\Theta}_{am}, \widehat{N}_{am}\right)\widehat{\Theta}_{am}^T\right] \right\} \tag{5.113}$$

where the components are estimated using the estimators presented above.

5.7 Choosing an Estimator

In this section I explain, with the aid of nested decision trees, how to choose estimators appropriate for the various catch parameters . The decision trees are provided for estimation within vessels and are nested because of a multitude of options at the various level of the design based on the available information and the different gear types treated. I begin without any decisions about the catch parameter of interest and I assume that we are interested in yearly totals.

Regardless of the catch parameter, the yearly estimator is just the sum of the quarter-specific estimators, Eq. 5.100. In fact, we could just as well be interested in any set of quarters without regard to years. The corresponding variance estimator is Eq. 5.101. For catch parameters such as numbers-at-age, there may be covariance of estimates across quarters because of the subsampling of otoliths for ageing. This covariance is estiamted with Eq. 5.102.

Within a given quarter, we also just sum the vessel-specific estimates with Eq. 5.95 for any given set of vessels we are interested in. Generally, we will estimate over all vessels of a given gear-type or over all vessels without regard to gear type. The corresponding variance estimator is Eq. 5.96. Depending on the catch parameter and models for insufficient sampling of trips, there may be covariance between the vessel-specific estimates and these are estimated with results I provide in Section 5.3.1.

The within-vessel estimators used in the quarter-specific estimator, Eq. 5.95 can be determined by observing the decision tree in Figure 5.1 for each vessel. Given a particular catch parameter, the tree in Figure 5.1 tells us estimators appropriate when the vessel has either 100% observer coverage or 30% observer coverage and, given a 30% coverage vessel, we use different estimators when only one trip is sampled than when more than one trip is sampled. In the following sections I discuss choosing vessel-specific estimators for different catch parameters.

5.7.1 *Numbers in Catch*

When we are interested in estimation of total numbers caught for a given species we are directed by Figure 5.1 to the next decision tree in Figure 5.2 for choosing trip-specific estimators. In Figure 5.2, we now use different estimators depending on the

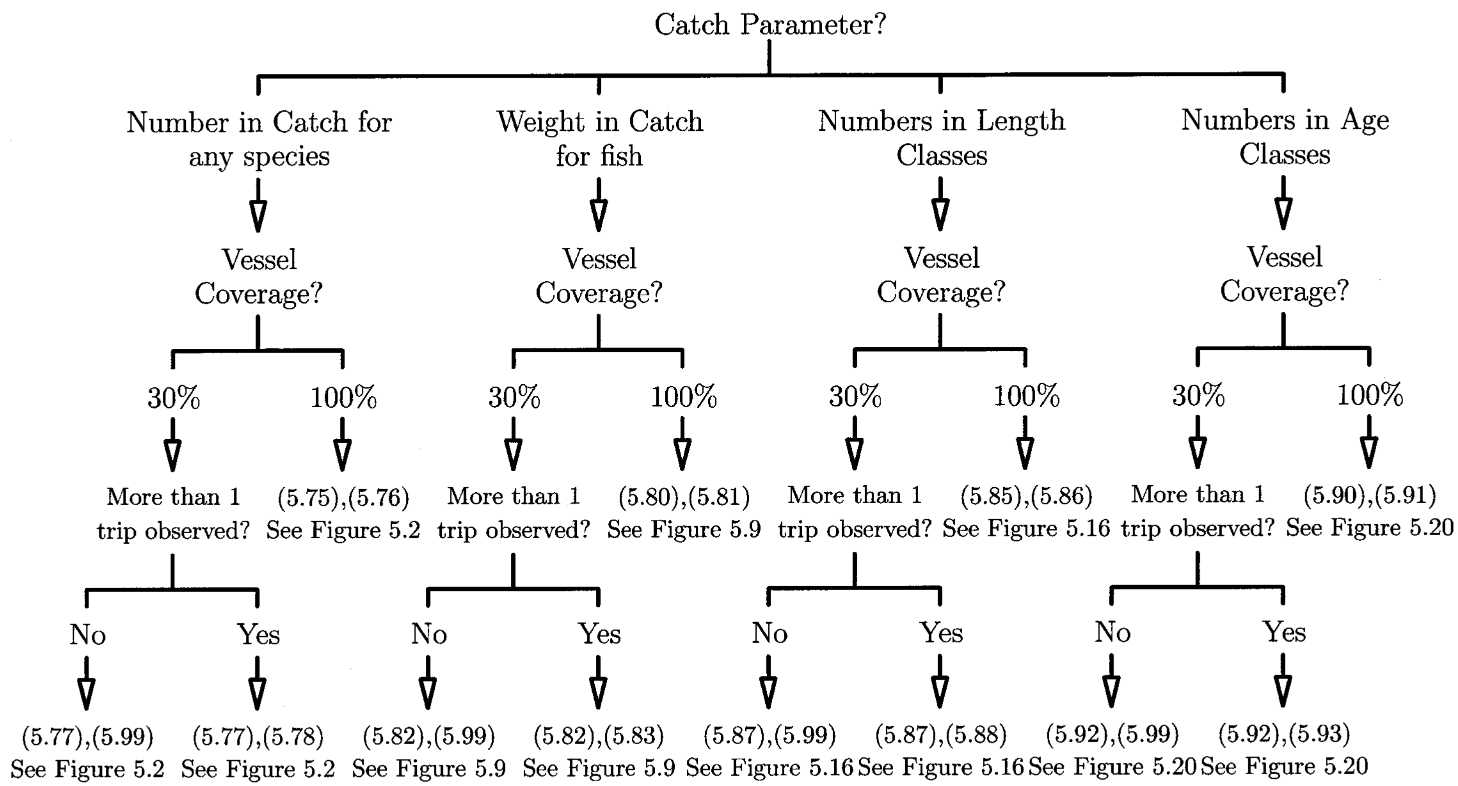

Figure 5.1. Decision tree for choosing an appropriate estimator for a catch parameter total on a vessel in a given quarter.

gear type used by the vessel and whether or not we are interested only in a particular time period or region.

When the vessel uses longline gear and we are not interested in a period/region, we are directed by Figure 5.2 to Figure 5.3 and within that tree we are directed to use different estimators depending on the number of sampled hauls, whether we are interested in marine mammals and whether we can use design-based methods within hauls. When the vessel uses longline gear and we are interested in a period/region, we are directed to Figure 5.4 and within that tree we are directed to use different estimators depending on similar criteria as Figure 5.3.

When the vessel uses trawl gear and we are not interested in a period/region, we are directed by Figure 5.2 to Figure 5.5 and within that tree we are directed to use different estimators depending on similar criteria as Figures 5.3 and 5.4. When the vessel uses trawl gear and we are interested in a period/region, we are directed to Figure 5.6.

When the vessel uses pot gear and we are not interested in a period/region, we are directed by Figure 5.2 to Figure 5.7 and within that tree we are directed to use different estimators depending on similar criteria as other gear types. However, I do not treat estimation of marine mammals for pot gear because marine mammals have not been caught in pot gear. When the vessel uses pot gear and we are interested in a period/region, we are directed to Figure 5.8.

5.7.2 Weight in Catch

When we are interested in estimation of total weight caught for a given species, we are directed by Figure 5.1 to Figure 5.9 for choosing trip-specific estimators. Like total numbers (Figure 5.2), in Figure 5.9, we use different estimators depending on the gear type used by the vessel and whether or not we are interested only in a particular time period or region.

When the vessel uses longline gear and we are not interested in a period/region, we are directed by Figure 5.9 to Figure 5.10 and within that tree we are directed to use different estimators depending on the number of sampled hauls and whether we can use design-based methods within hauls. When the vessel uses longline gear and we are interested in a period/region, we are directed to Figure 5.11 and within that tree we are directed to use different estimators depending on similar criteria as Figure

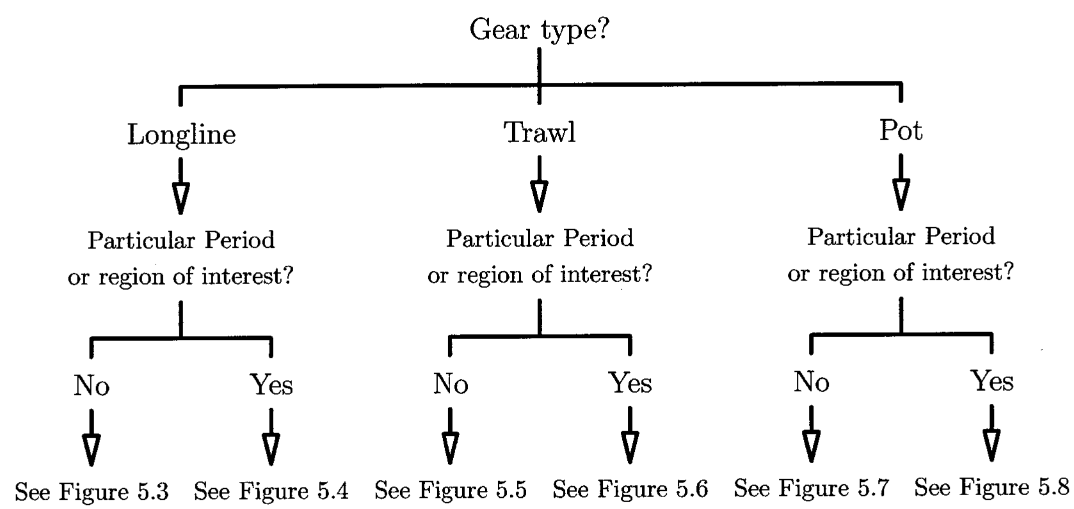

Figure 5.2. Decision tree for determining which estimators to use for total number in catch of any given species for a trip.

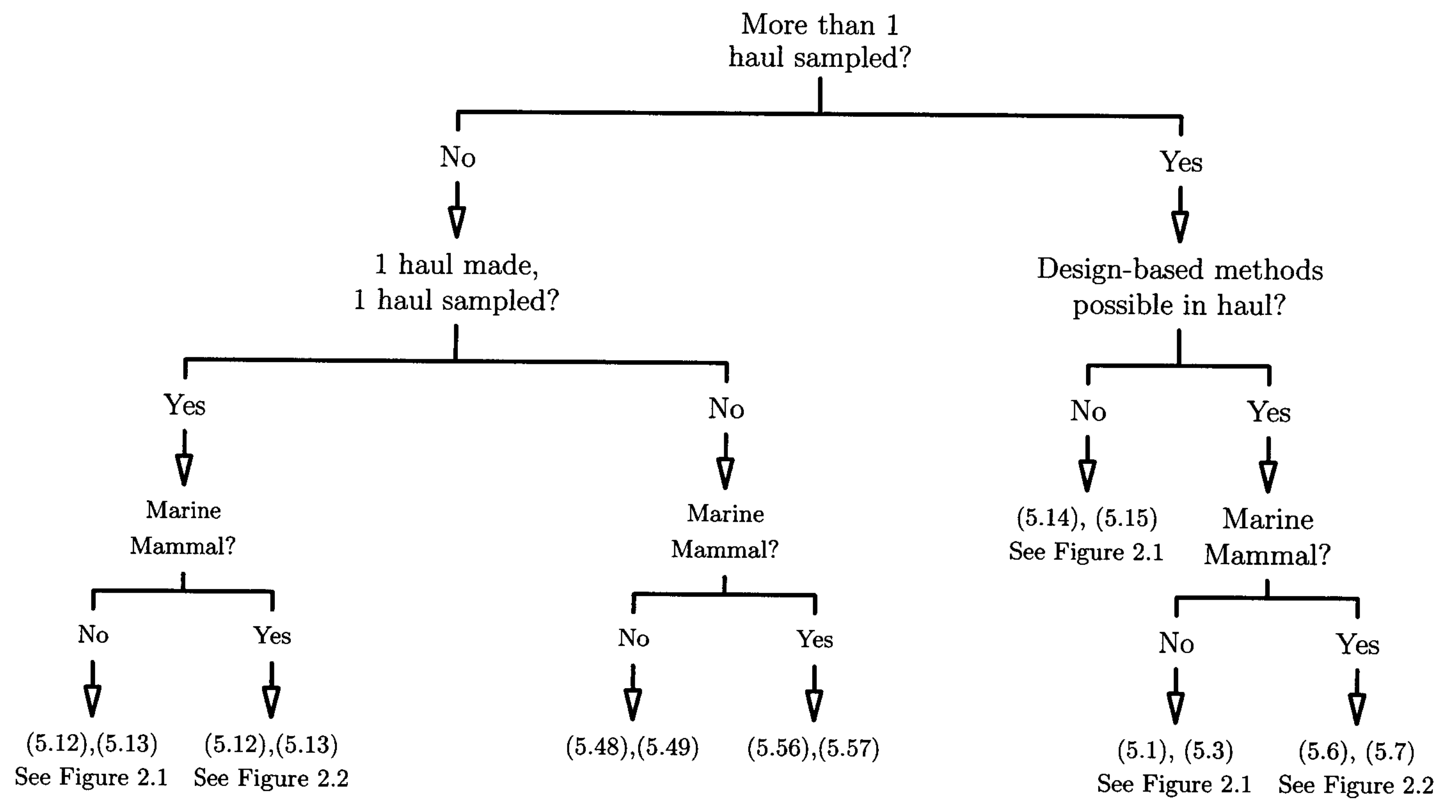

Figure 5.3. Decision tree for determining which estimator to use for total number in catch of any given species for a longline trip when no particular time period or region is of interest.

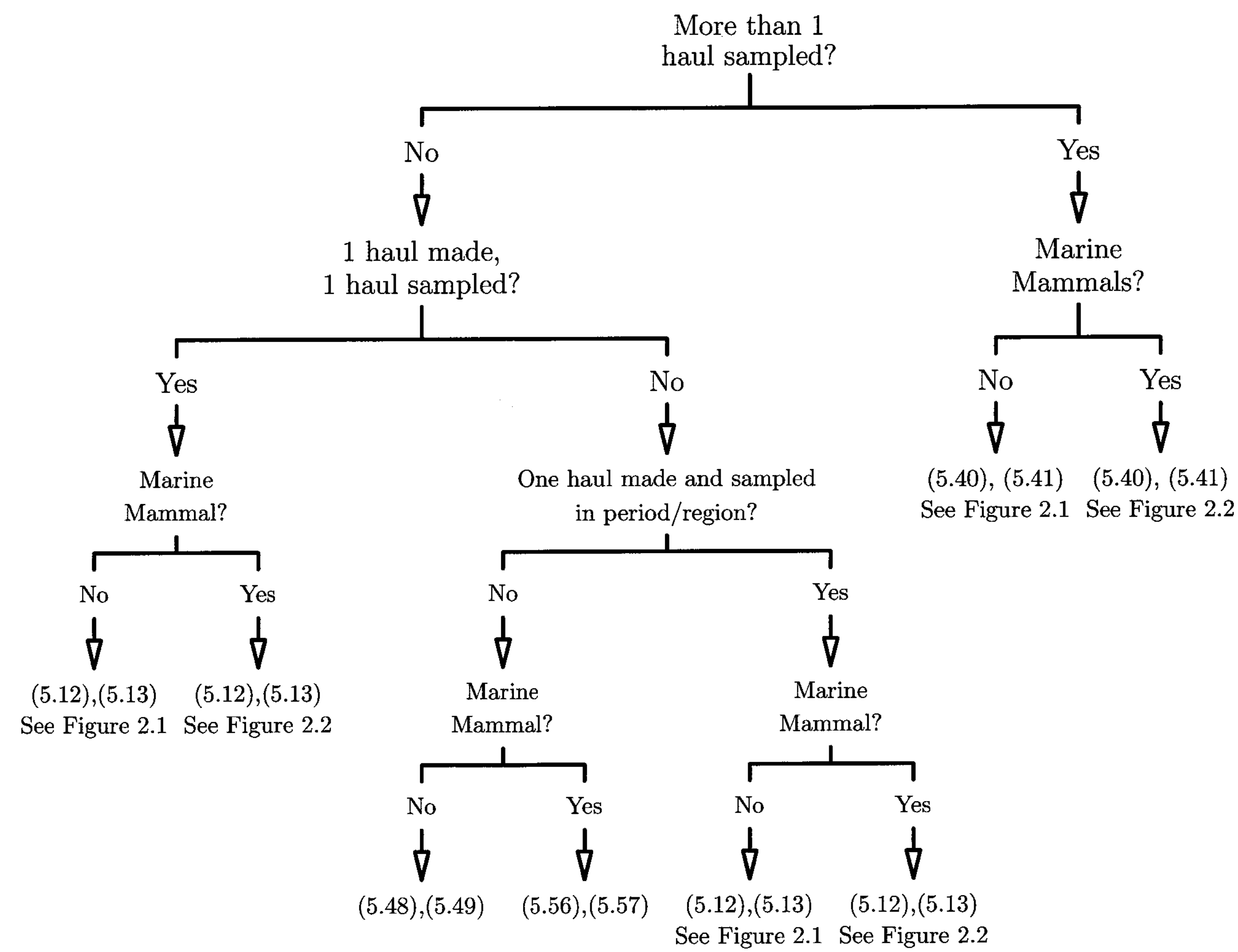

Figure 5.4. Decision tree for determining which estimator to use for total number in catch of any given species for a longline trip when a particular time period or region is of interest.

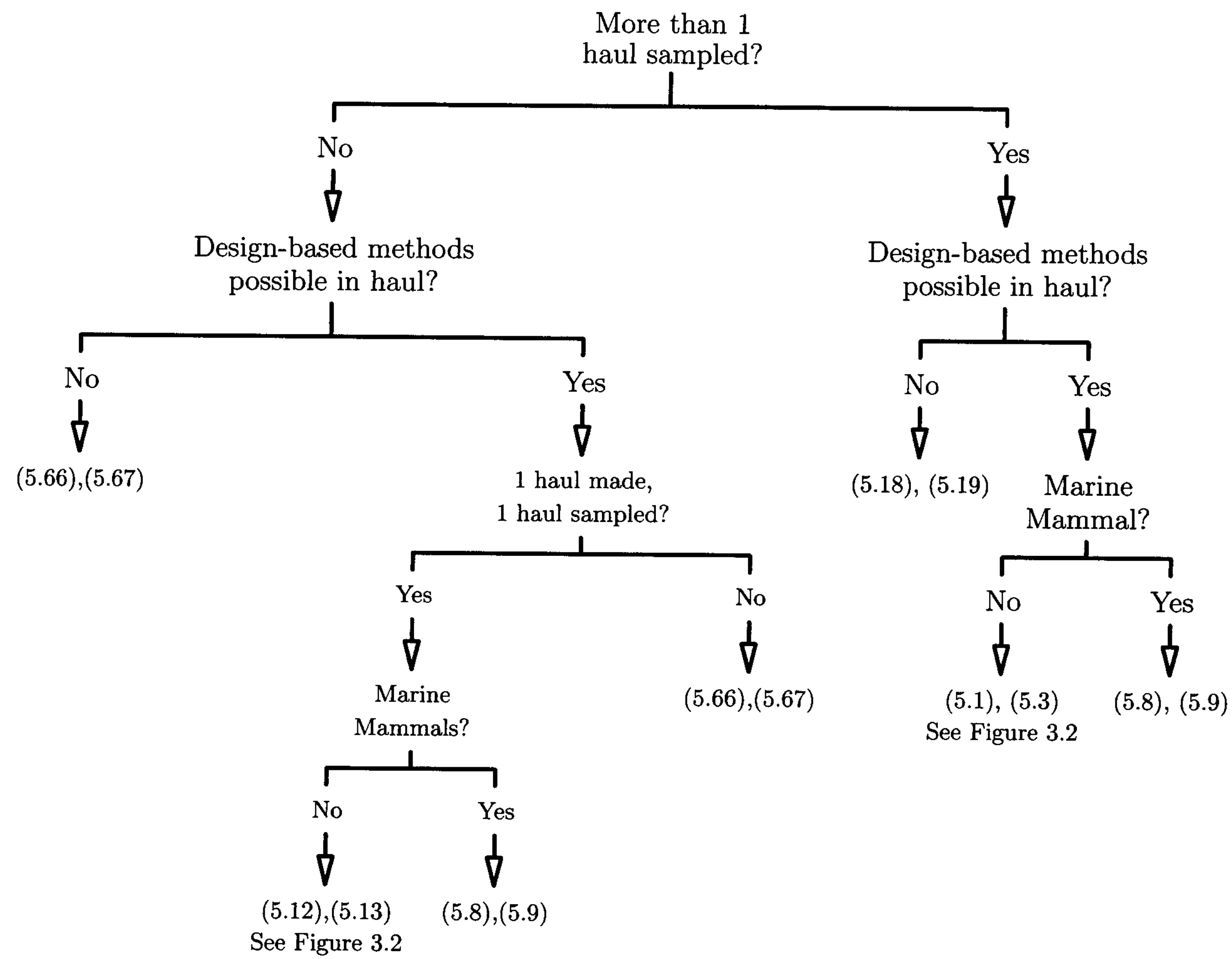

Figure 5.5. Decision tree for determining which estimator to use for total number in catch of any given species for a trawler trip when no particular time period or region is of interest.

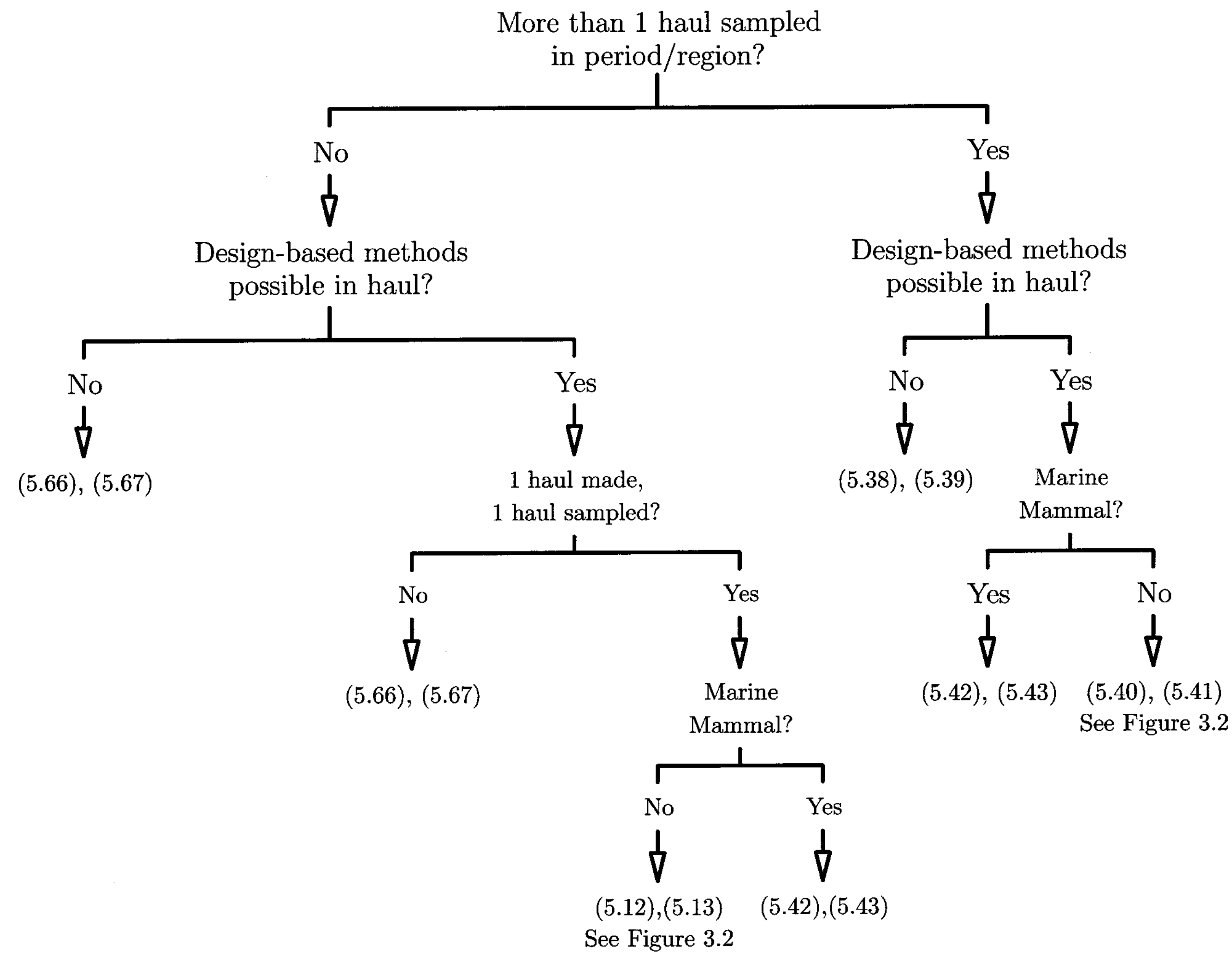

Figure 5.6. Decision tree for determining which estimator to use for total number in catch of any given species for a trawler trip when a particular time period or region is of interest.

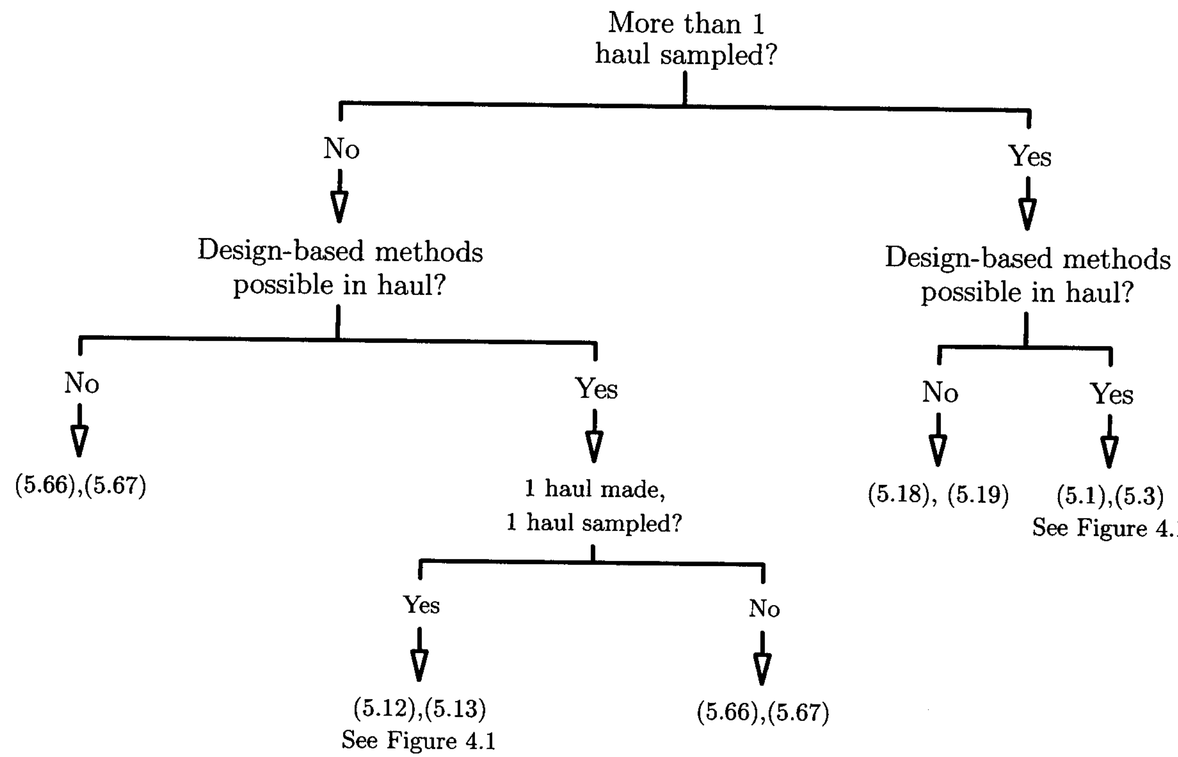

Figure 5.7. Decision tree for determining which estimator to use for total number in catch of any given species (excluding marine mammals) for a pot trip when no particular time period or region is of interest.

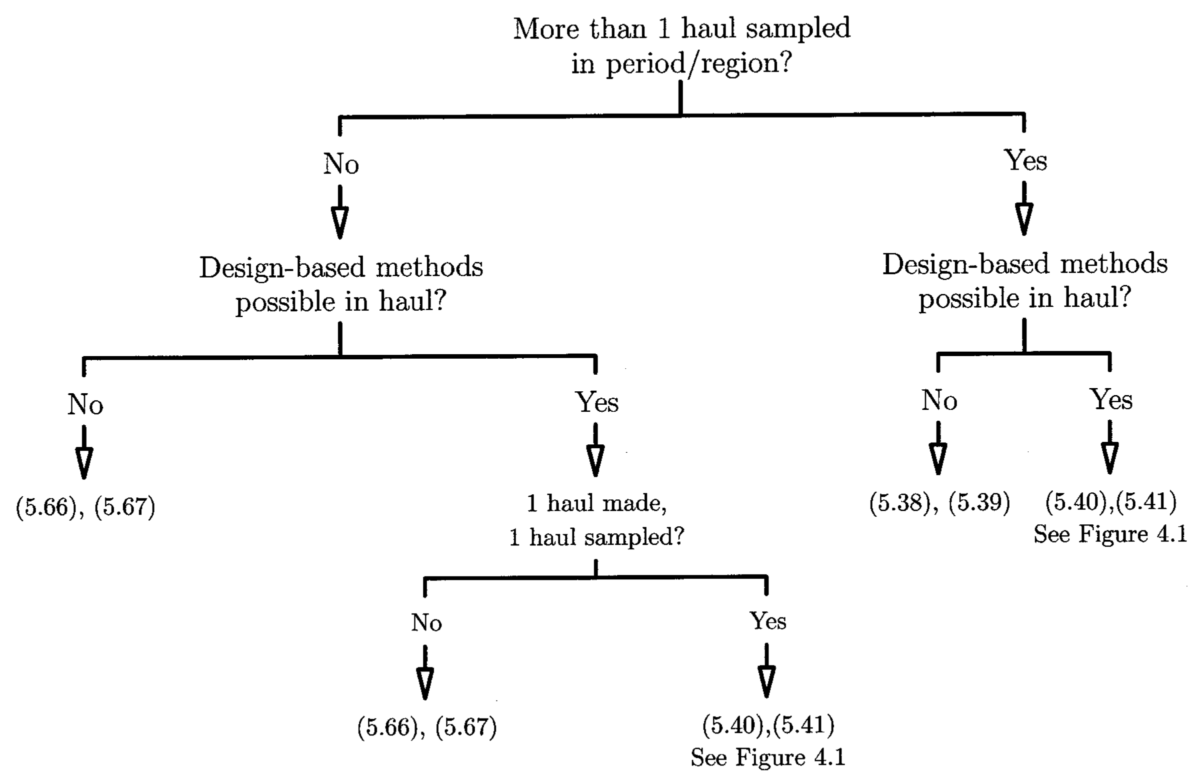

Figure 5.8. Decision tree for determining which estimator to use for total number in catch of any given species (excluding marine mammals) for a pot trip when a particular time period or region is of interest.

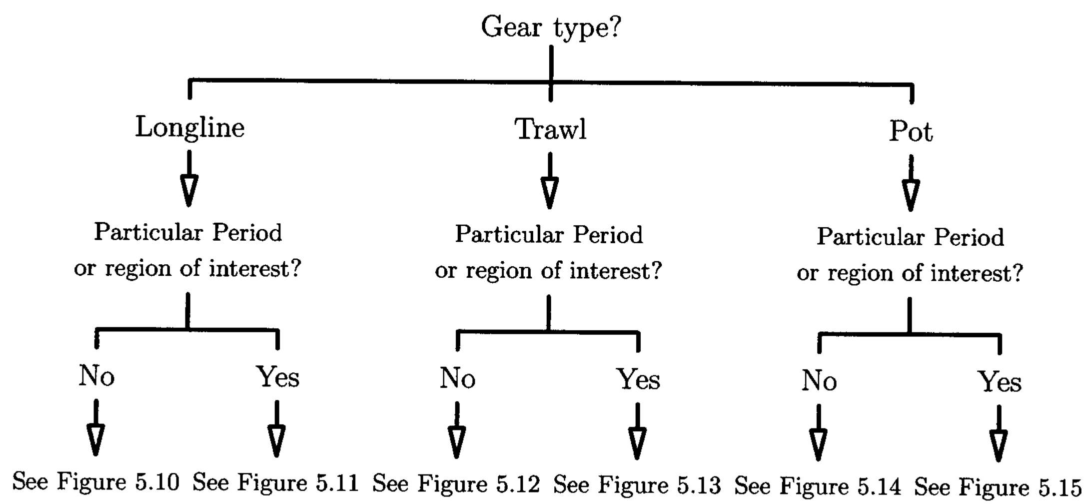

Figure 5.9. Decision tree for determining which estimators to use for total weight in catch of a given species for a trip.

5.10.

When the vessel uses trawl gear and we are not interested in a period/region, we are directed by Figure 5.9 to Figure 5.12 and within that tree we are directed to use different estimators depending on similar criteria as Figures 5.10 and 5.11. When the vessel uses trawl gear and we are interested in a period/region, we are directed to Figure 5.13.

When the vessel uses pot gear and we are not interested in a period/region, we are directed by Figure 5.9 to Figure 5.14 and within that tree we are directed to use different estimators depending on similar criteria as other gear types. When the vessel uses pot gear and we are interested in a period/region, we are directed to Figure 5.15.

5.7.3 Numbers in Length Classes

When we are interested in estimation of total numbers caught in specified length classes for a given species, we are directed by Figure 5.1 to Figure 5.16 for choosing trip-specific estimators. Like total numbers (Figure 5.2), in Figure 5.16, we use different estimators depending on the gear type used by the vessel, but we do not distinguish whether or not we are interested only in a particular time period or region because the estimators for numbers-at-length already only pertain to a similarly defined subset of fishing effort, namely the hauls where the species is prevalent (the domain a, s-prevalent). To estimate numbers-at-length over all fishing effort requires using the estimators treated here as components of the estimators discussed in Section 5.6.

When the vessel uses longline gear, we are directed by Figure 5.16 to Figure 5.17 and within that tree we are directed to use different estimators depending on the number of sampled hauls and whether we can use design-based methods within hauls. When the vessel uses trawl gear, we are directed by Figure 5.16 to Figure 5.18 and within that tree we are directed to use different estimators depending on similar criteria as Figure 5.17. When the vessel uses pot gear, we are directed by Figure 5.16 to Figure 5.19.

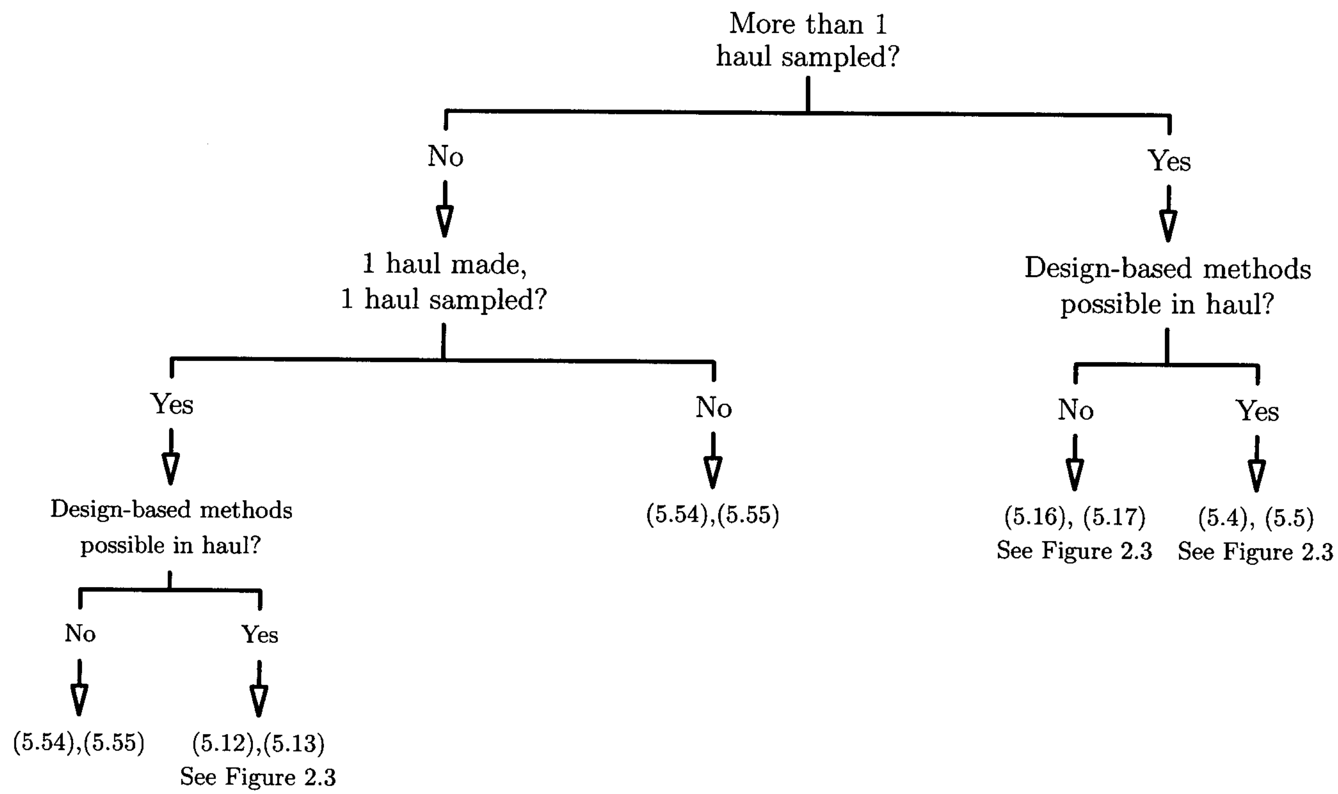

Figure 5.10. Decision tree for determining which estimator to use for total weight in catch of a given species for a longline trip when no particular time period or region is of interest.

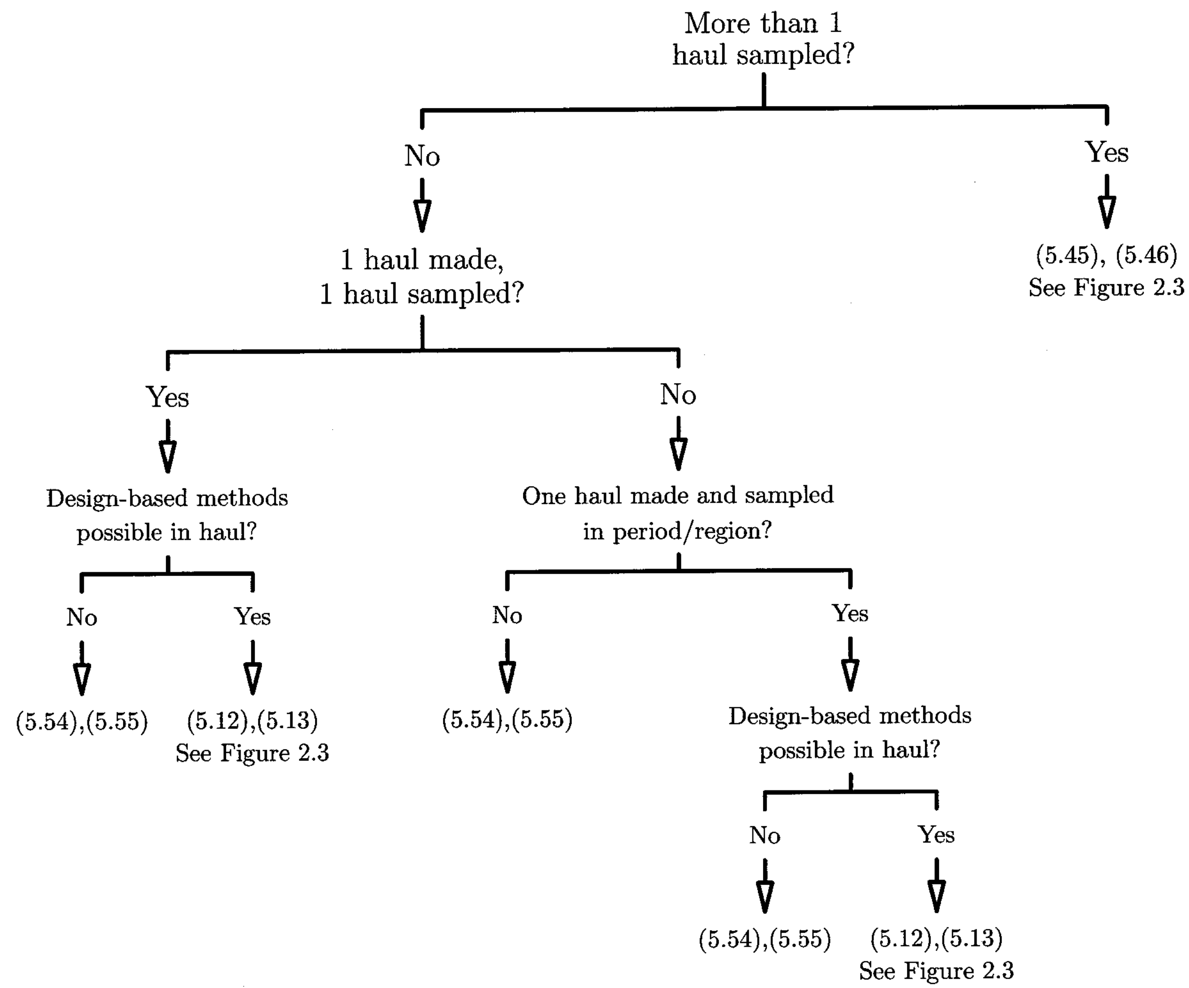

Figure 5.11. Decision tree for determining which estimator to use for total weight in catch of a given species for a longline trip when a particular time period or region is of interest.

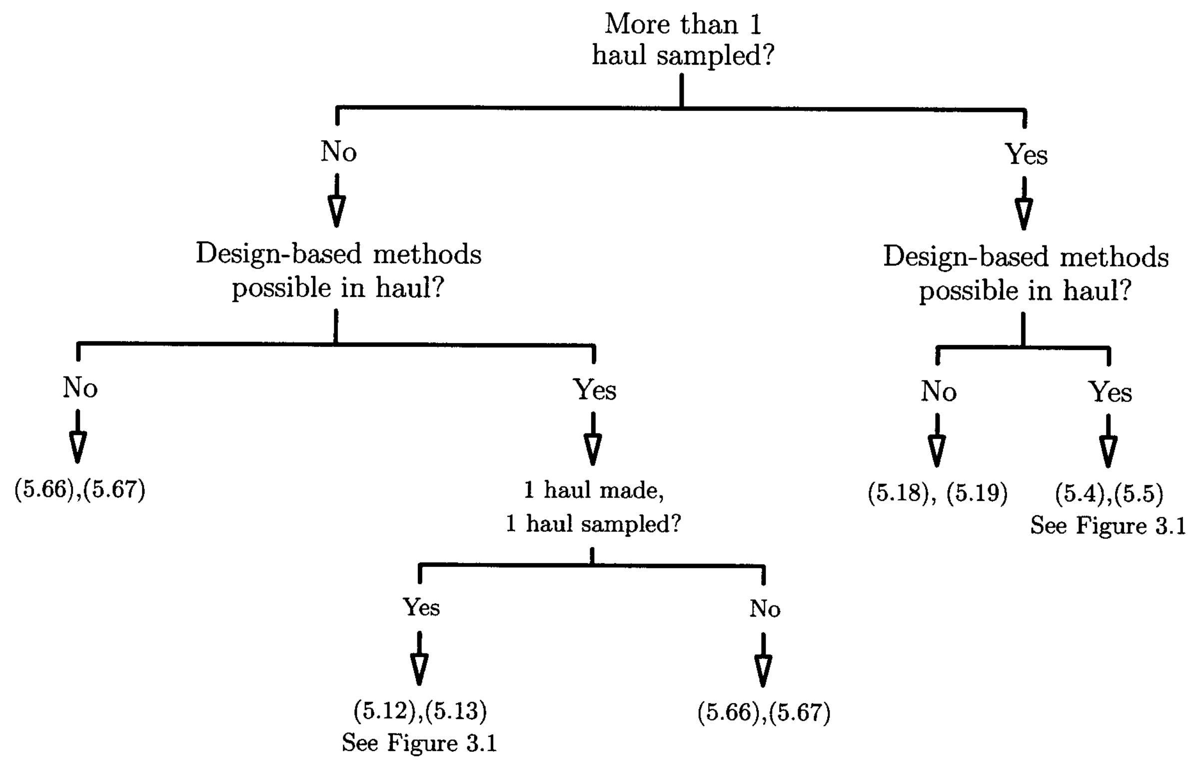

Figure 5.12. Decision tree for determining which estimator to use for total weight in catch of a given species for a trawler trip when no particular time period or region is of interest.

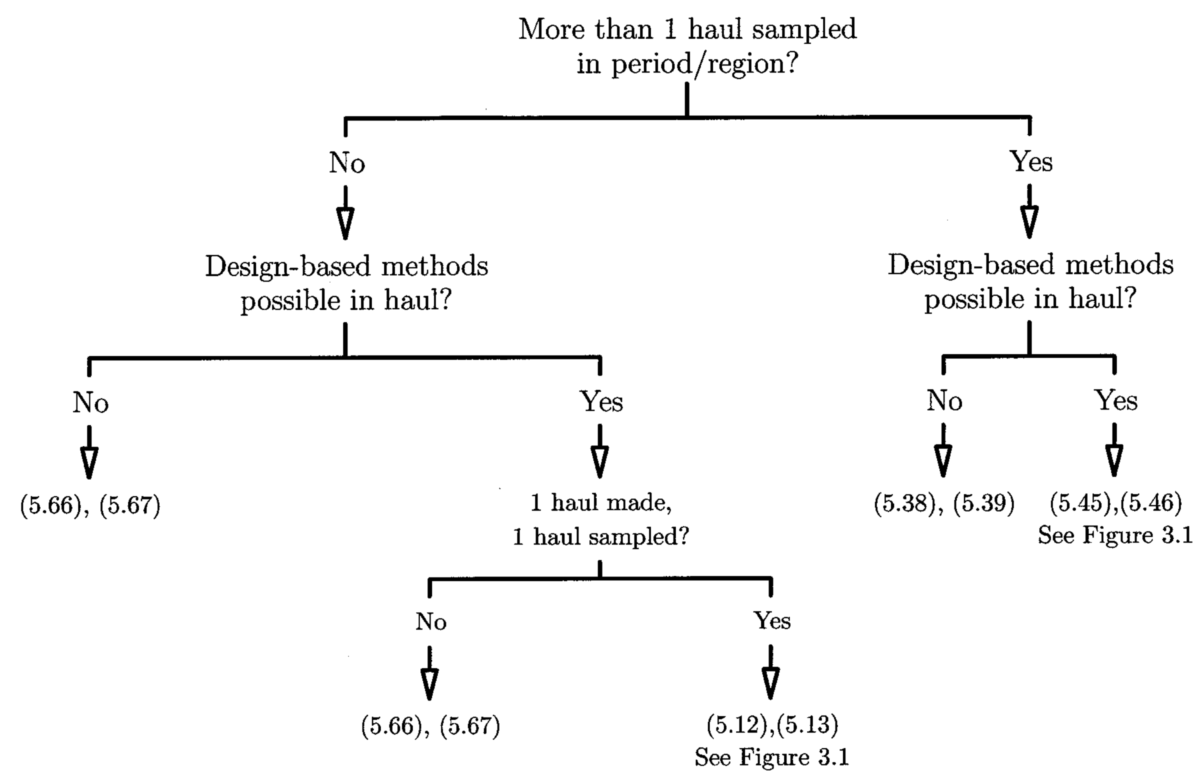

Figure 5.13. Decision tree for determining which estimator to use for total weight in catch of a given species for a trawler trip when a particular time period or region is of interest.

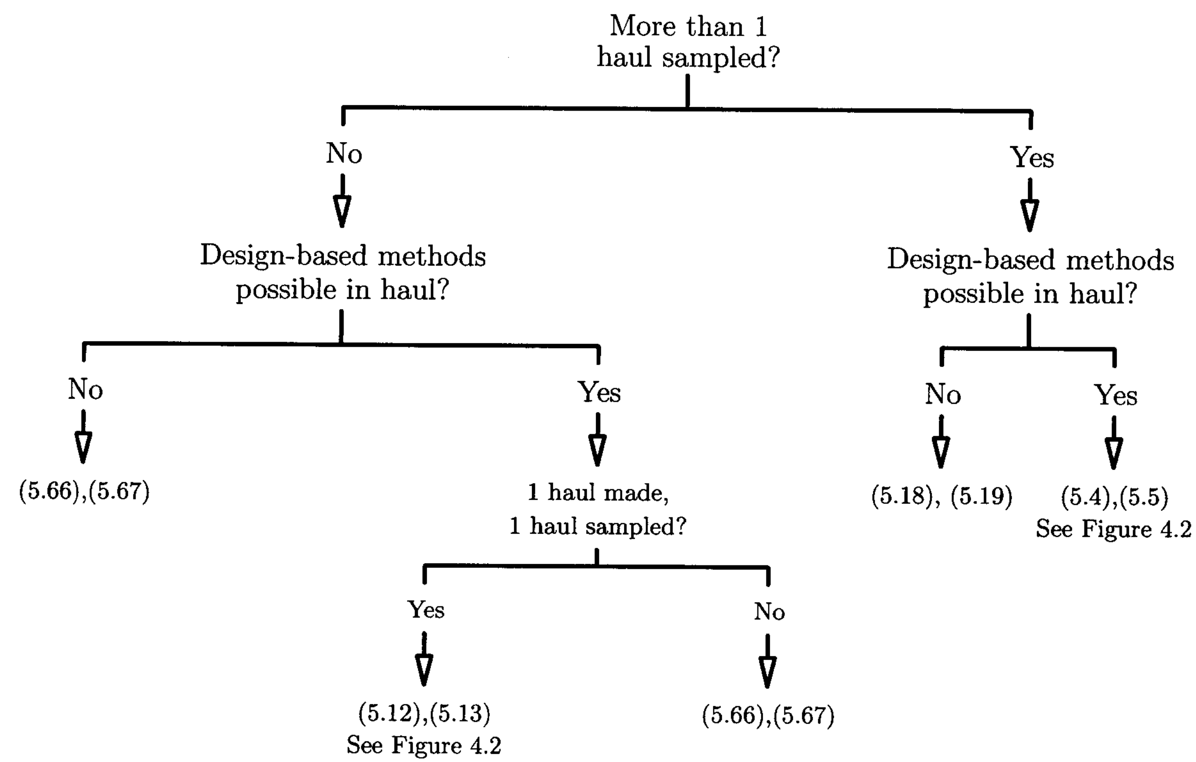

Figure 5.14. Decision tree for determining which estimator to use for total weight in catch of a given species for a pot trip when no particular time period or region is of interest.

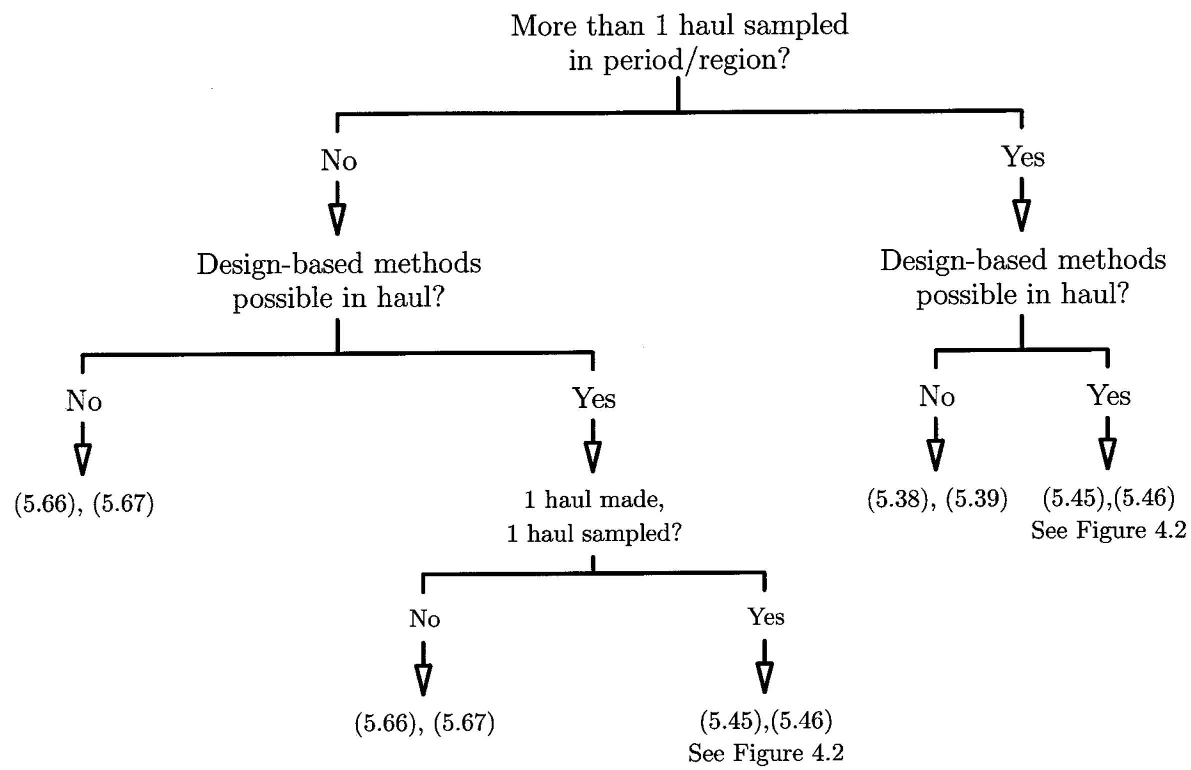

Figure 5.15. Decision tree for determining which estimator to use for total weight in catch of a given species for a pot trip when a particular time period or region is of interest.

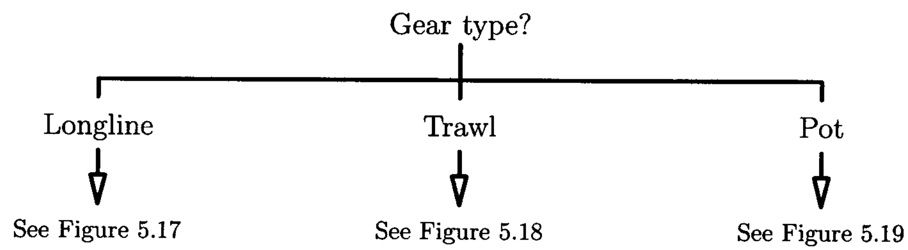

Figure 5.16. Decision tree for determining which estimators to use for total numbers in length classes in catch of a given species for a trip.

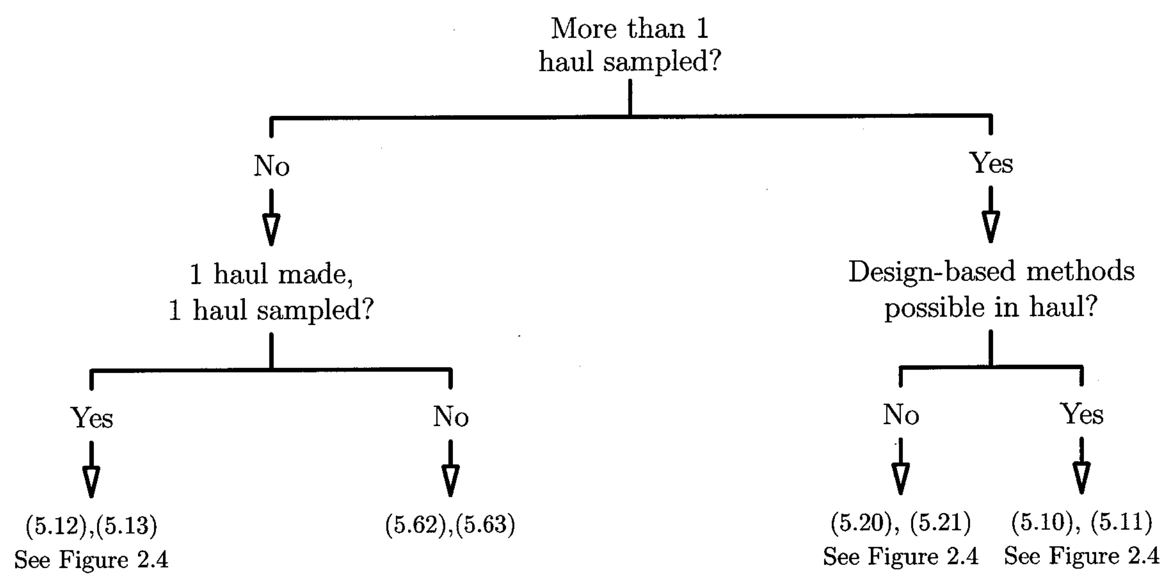

Figure 5.17. Decision tree for determining which estimator to use for total numbers in length classes in catch of a given targeted species for a longline trip.

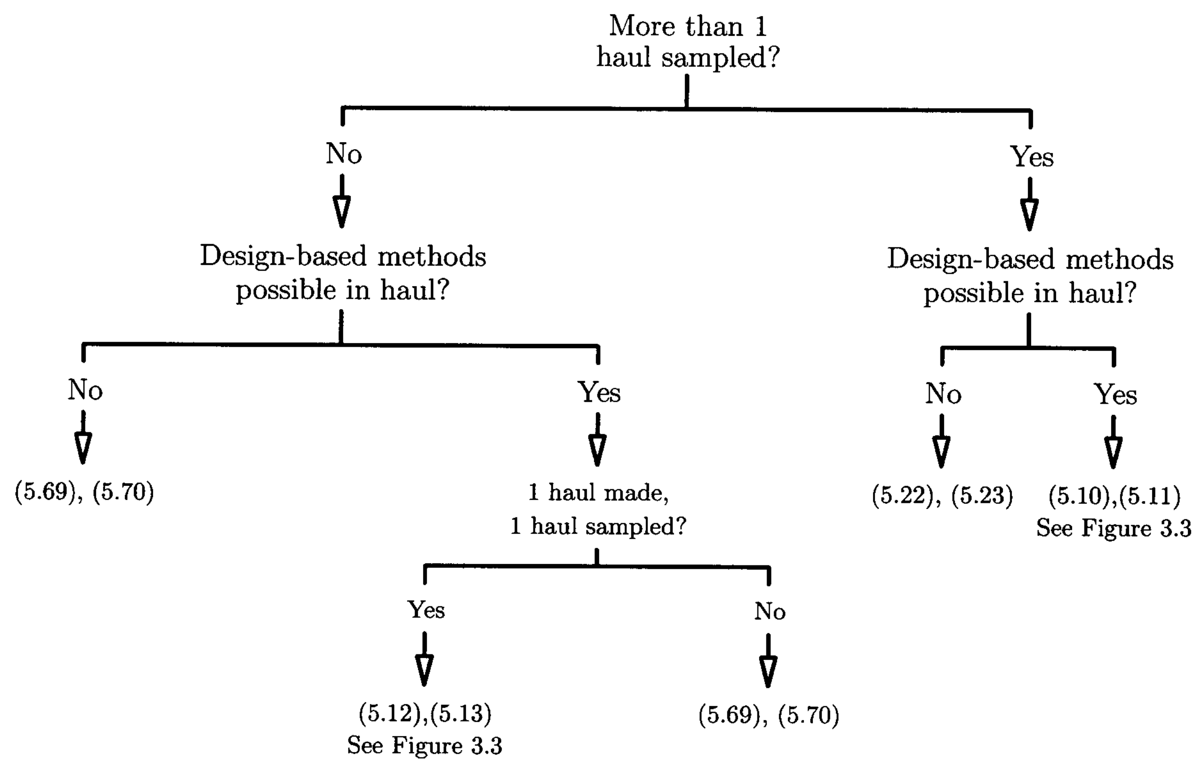

Figure 5.18. Decision tree for determining which estimator to use for total numbers in length classes in catch of a given targeted species for a trawler trip.

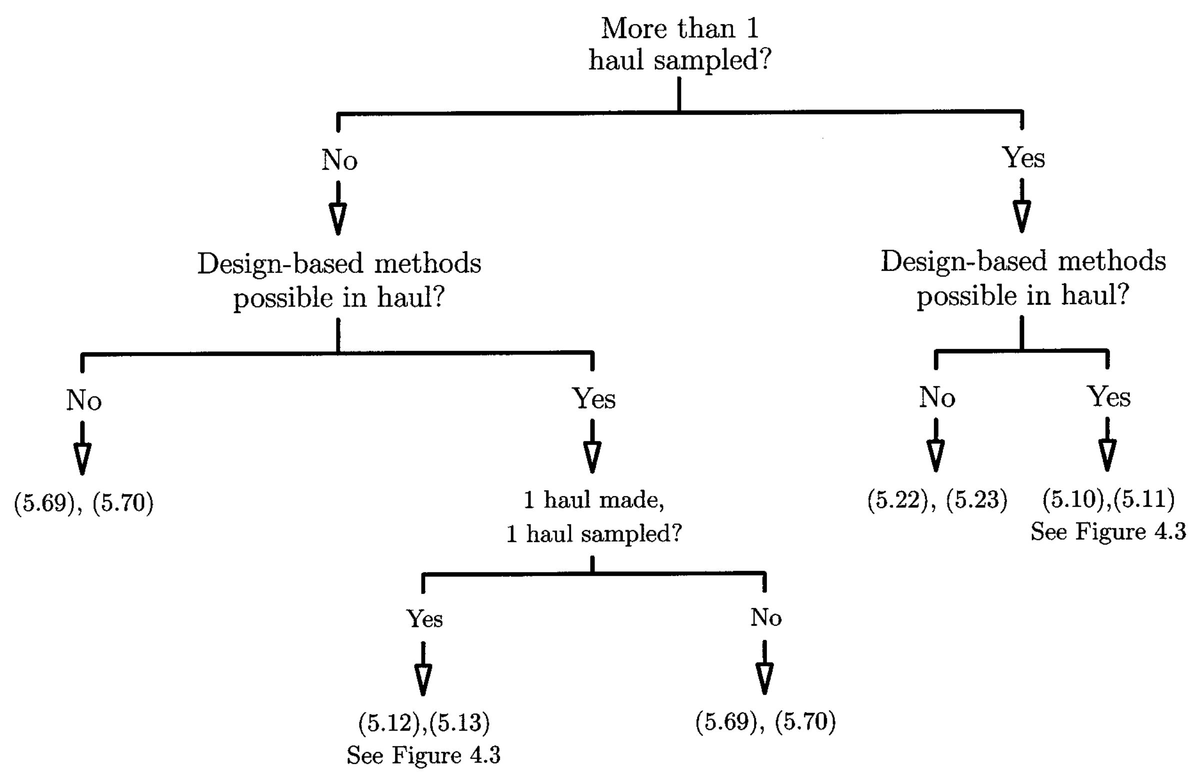

Figure 5.19. Decision tree for determining which estimator to use for total numbers in length classes in catch of a given targeted species for a pot trip.

5.7.4 Numbers in Age Classes

When we are interested in estimation of total numbers caught in specified age classes for a given species, we are directed by Figure 5.1 to Figure 5.20 for choosing trip-specific estimators. Like total numbers in length classes (Figure 5.16), in Figure 5.20, we use different estimators depending on the gear type used by the vessel and we do not distinguish whether or not we are interested only in a particular time period or region. To estimate numbers-at-age over all fishing effort requires using the estimators treated here as components of the estimators discussed in Section 5.6.

When the vessel uses longline gear, we are directed by Figure 5.20 to Figure 5.21 and within that tree we are directed to use different estimators depending on the number of sampled hauls. When the vessel uses trawl gear, we are directed by Figure 5.20 to Figure 5.22 and within that tree we are directed to use different estimators depending on the same criterion as Figure 5.21. When the vessel uses pot gear, we are directed by Figure 5.20 to Figure 5.23.

5.A Derivation of Estimators

5.A.1 Derivation of $\widehat{\Theta}_t$ for Pot and Trawl vessels

The population is defined as all clusters of captured animals in all hauls made during the tth trip where the number of clusters in each haul is M_k and the number for the entire trip is $\sum_{k=1}^{G_t} M_k = M_t$. Let $g_t' = G_t - g_t$ be the number of unsampled hauls, m_k be the number of clusters sampled in the kth haul, $m_k' = M_k - m_k$ be the number of unsampled hauls and $\sum_{i=1}^{g_t} m_k = m_t$ be the total number of sampled clusters.

Under the constant variance model, the best linear unbiased predictor of the total for the tth trip is

$$\begin{aligned}\widehat{\Theta}_t &= \sum_{k=1}^{g_t}\sum_{i=1}^{m_k}\theta_{ki} + {}_1\widehat{\phi}_{\theta,t}\left(\sum_{k=1}^{g_t}\sum_{i=1}^{m_k'}x_{ki} + \sum_{k=1}^{g_t'}\sum_{i=1}^{M_k}x_{ki}\right)\\ &= \sum_{k=1}^{g_t}\theta_k + {}_1\widehat{\phi}_{\theta,t}\left(\sum_{k=1}^{g_t}x_k' + \sum_{k=1}^{g_t'}X_k\right) = \theta_t + {}_1\widehat{\phi}_{\theta,t}x_t'\end{aligned}$$

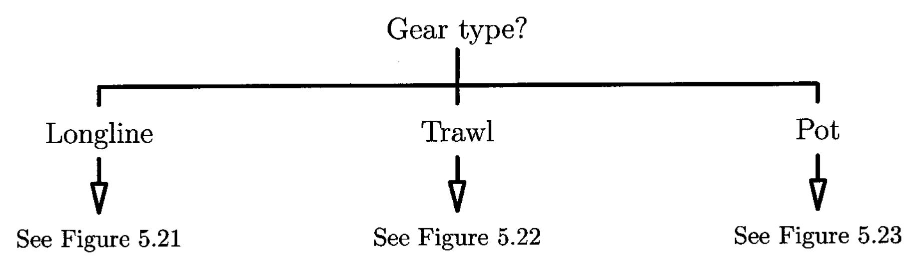

Figure 5.20. Decision tree for determining which estimators to use for total numbers in age classes in catch of a given species for a trip.

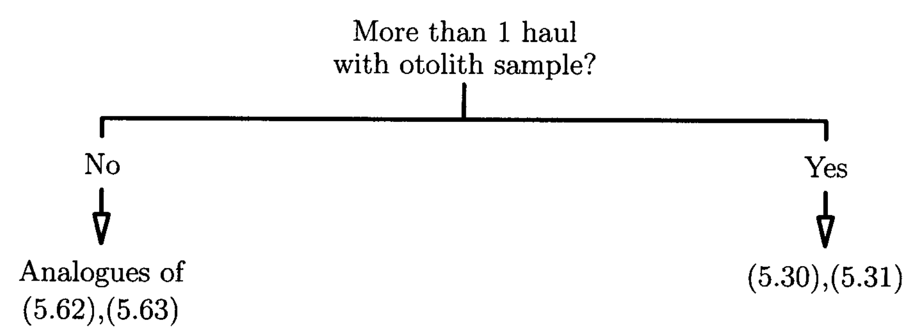

Figure 5.21. Decision tree for determining which estimator to use for total numbers in age classes in catch of a given targeted species for a longline trip.

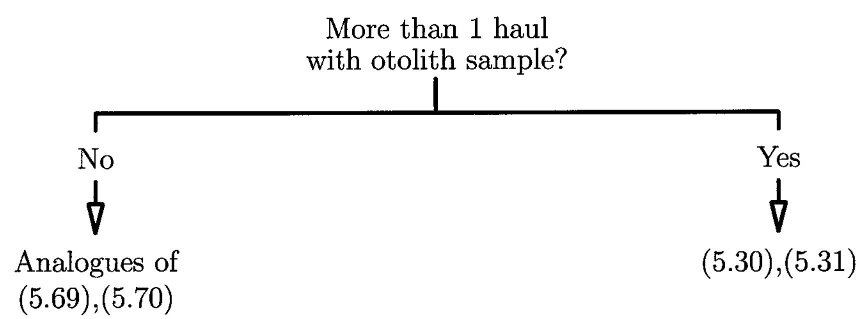

Figure 5.22. Decision tree for determining which estimator to use for total numbers in age classes in catch of a given targeted species for a trawler trip.

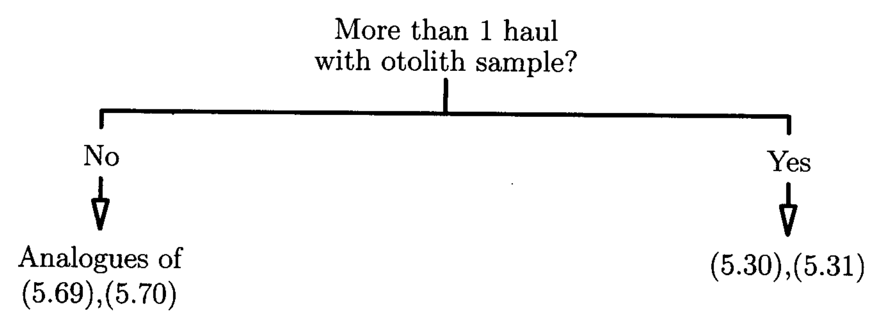

Figure 5.23. Decision tree for determining which estimator to use for total numbers in length classes in catch of a given targeted species for a pot trip.

where ${}_1\widehat{\phi}_{\theta,t}$ is the scalar form of Eq. 3.31 and the prediction error variance is

$$\begin{aligned}
V_M\left(\widehat{\Theta}_t\right) =& V\left[\widehat{{}_1\phi_{\theta,t}}\left(\sum_{k=1}^{g_t} x'_k + \sum_{k=1}^{g'_t} X_k\right) - \left(\sum_{k=1}^{g_t}\sum_{i=1}^{m'_k}\theta_{ki} + \sum_{k=1}^{g'_t}\sum_{i=1}^{M_k}\theta_{ki}\right)\right] \\
=& \left(\sum_{k=1}^{g_t} x'_k + \sum_{k=1}^{g'_t} X_k\right)^2 \frac{\sigma^2_{\theta,t}}{\sum_{k=1}^{g_t} x_k^2} + \sigma^2_{\theta,t}\left(\sum_{k=1}^{g_t} m'_k + \sum_{k=1}^{g'_t} M_k\right) \\
=& \sigma^2_{\theta,t}\left(\frac{(x'_t)^2}{\sum_{k=1}^{g_t} x_k^2} + m'_t\right)
\end{aligned}$$

When the alternative model with variance proportional to the covariate is assumed the estimator is

$$\widehat{\Theta}_t = \theta_t + {}_2\widehat{\phi}_{\theta,t} x'_t$$

where ${}_2\widehat{\phi}_{\theta,t}$ is the scalar form of Eq. 3.34 and the prediction error variance is

$$\begin{aligned}
V_M\left(\widehat{\Theta}_t\right) =& V\left[{}_2\widehat{\phi}_{\theta,t}\left(\sum_{k=1}^{g_t} x'_k + \sum_{k=1}^{g'_t} X_k\right) - \left(\sum_{k=1}^{g_t}\sum_{i=1}^{m'_k}\theta_{ki} + \sum_{k=1}^{g'_t}\sum_{i=1}^{M_k}\theta_{ki}\right)\right] \\
=& \sigma^2_{\theta,t}\left(\frac{(x'_t)^2}{x_t} + x'_t\right) = \sigma^2_{\theta,t}\frac{x'_t}{x_t}X_t
\end{aligned}$$

(See Table 5.1 for definition of x_k).

5.A.2 Derivation of $\widehat{\boldsymbol{\Psi}}_{amt}$

Let the sampling distributions of hauls within trips and hauls for otolith sampling among s-prevalent hauls be denoted as s_1 and s_2 and let the corresponding expectations be denoted as $E_1(\cdot)$, $E_2(\cdot)$. If model-based methods are used within hauls and the trip is made aboard a longliner, then the within-haul estimator of the numbers-at-age is Eq. 2.35 which was shown unbiased in Section 2.A.4. Therefore, the expected value of Eq. 5.30 is

$$E\left({}_1\widehat{\boldsymbol{\Psi}}_{amt}\right) = E_1\left[\frac{G_t}{g_t}E_2\left(\frac{g_{at}}{g_{Ot}}\sum_{k=1}^{g_{Omt}}\boldsymbol{\Psi}_k|s_1\right)\right] = E_1\left(\frac{G_t}{g_t}\sum_{k=1}^{g_{amt}}\boldsymbol{\Psi}_k|s_1\right)$$
$$= \sum_{k=1}^{G_{amt}}\boldsymbol{\Psi}_k = \boldsymbol{\Psi}_{amt}.$$

The above expectation follow directly from simple random sampling results for domain totals (see Section A.6).

There will be covariance between different within-haul estimators because of the way otoliths are sampled for ageing so, before the variance of Eq. 5.30 is derived, lets determine the covariance of haul-specific numbers-at-age estimators first. Using the same notation as Section 2.A.4 for different distributions (M_{N_k} or M_{N_l}, M_{ψ_k} or $M_{\psi_l}$and S_A). The covariance of the kth and lth haul is

$$\begin{aligned}Cov\left({}_4\widehat{\boldsymbol{\Psi}}_k, {}_4\widehat{\boldsymbol{\Psi}}_l\right) =& Cov_{S_A}\left\{E_{M_{N_k}}\left[E_{M_{\psi_k}}\left({}_4\widehat{\boldsymbol{\Psi}}_k\right)\right], E_{M_{N_l}}\left[E_{M_{\psi_l}}\left({}_4\widehat{\boldsymbol{\Psi}}_l\right)\right]\right\}\\ &+ E_{S_A}\left\{Cov_{M_N}\left[E_{M_{\psi_k}}\left({}_4\widehat{\boldsymbol{\Psi}}_k\right), E_{M_{\psi_l}}\left({}_4\widehat{\boldsymbol{\Psi}}_l\right)\right]\right\}\\ &+ E_{S_A}\left\{E_{M_N}\left[Cov_{M_\psi}\left({}_4\widehat{\boldsymbol{\Psi}}_k, {}_4\widehat{\boldsymbol{\Psi}}_l\right)\right]\right\}.\end{aligned}$$

The distributions for the model-based estimators of total number and proportions-at-age for the kth and lth hauls are independent, therefore there is no covariance between hauls with respect to those distributions and the second and third terms are zero. This leaves the covariance due only to the sampling distribution of aged otoliths,

$$\begin{aligned}&Cov\left({}_4\widehat{\boldsymbol{\Psi}}_k, {}_4\widehat{\boldsymbol{\Psi}}_l\right) =\\ &E_{M_{N_k}}\left(\widehat{N}_k\right)E_{M_{N_l}}\left(\widehat{N}_l\right)\frac{Cov_{S_A}\left\{E_{M_{\psi_k}}\left(\frac{N_{Om}}{n_{Am}}\sum_{i=1}^{n_{Ak}}\psi_i\right), E_{M_{\psi_l}}\left(\frac{N_{Om}}{n_{Am}}\sum_{j=1}^{n_{Al}}\psi_j\right)\right\}}{n_{Ok}n_{Ol}}\\ &= N_kN_l\frac{Cov_{S_A}\left(\frac{N_{Om}}{n_{Am}}n_{Ak}, \frac{N_{Om}}{n_{Am}}n_{Al}\right)}{n_{Ok}n_{Ol}}\mathbf{p}_k\mathbf{p}_l^T\\ &= \frac{Cov\left(\widehat{n}_{Ok}, \widehat{n}_{Ol}\right)}{n_{Ok}n_{Ol}}\boldsymbol{\Psi}_k\boldsymbol{\Psi}_l^T \qquad (5.114)\end{aligned}$$

where

$$Cov\left(\widehat{n}_{Ok}, \widehat{n}_{Ol}\right) = -N_{Om}\left(\frac{N_{Om}}{n_{Am}} - 1\right)\frac{N_{Om}P_{Ok}P_{Ol}}{N_{Om} - 1} \tag{5.115}$$

and $P_{Ok} = n_{Ok}/N_{Om}$. As mentioned before, the number of aged otoliths per haul is a hypergeometric random variable when a SRS of all otoliths are aged. Therefore, the vector of numbers aged in each haul is multivariate hypergeometric and the covariance term (Eq. 5.115) is given by an off-diagonal term in the covariance matrix of the estimator of the vector of numbers of otoliths in each haul (see Section A.2.2). In a different context, a derivation of the covariance between totals for two domains is given in Section A.7.2 of the appendix.

Using the the variance within hauls derived in Section 2.A.4 and the covariance of within-haul estimators (Eq. 5.114), the variance of Eq. 5.30 is

$$\begin{aligned}
V\left({}_1\widehat{\boldsymbol{\Psi}}_{amt}\right) =& V_1\left[\frac{G_t}{g_t}E_2\left(\frac{g_{at}}{g_{Ot}}\sum_{k=1}^{g_{Omt}}\boldsymbol{\Psi}_k|s_1\right)\right] + E_1\left[\left(\frac{G_t}{g_t}\right)^2 V_2\left(\frac{g_{at}}{g_{Ot}}\sum_{k=1}^{g_{Omt}}\boldsymbol{\Psi}_k|s_1\right)\right] \\
&+ E_1\left\{\left(\frac{G_t}{g_t}\right)^2 E_2\left[\left(\frac{g_{at}}{g_{Ot}}\right)^2\sum_{k=1}^{g_{Omt}}V\left({}_4\widehat{\boldsymbol{\Psi}}_k\right)|s_1\right]\right\} \\
&+ E_1\left\{\left(\frac{G_t}{g_t}\right)^2 E_2\left[\left(\frac{g_{at}}{g_{Ot}}\right)^2\sum_{k\neq l}^{g_{Omt}}\sum Cov\left({}_4\widehat{\boldsymbol{\Psi}}_k, {}_4\widehat{\boldsymbol{\Psi}}_l\right)|s_1\right]\right\} \\
=& \underbrace{V_1\left(\frac{G_t}{g_t}\sum_{k=1}^{g_{amt}}\boldsymbol{\Psi}_k\right)}_{V_1} + \underbrace{E_1\left[\left(\frac{G_t}{g_t}\right)^2 V_2\left(\frac{g_{at}}{g_{Ot}}\sum_{k=1}^{g_{Omt}}\boldsymbol{\Psi}_k|s_1\right)\right]}_{V_2} \\
&+ \underbrace{E_1\left[\left(\frac{G_t}{g_t}\right)^2\frac{g_{at}}{g_{Ot}}\sum_{k=1}^{g_{amt}}V\left({}_4\widehat{\boldsymbol{\Psi}}_k\right)\right]}_{V_3} \\
&+ \underbrace{E_1\left[\left(\frac{G_t}{g_t}\right)^2\frac{g_{at}(g_{Ot}-1)}{g_{Ot}(g_{at}-1)}\sum_{k\neq l}^{g_{amt}}\sum Cov\left({}_4\widehat{\boldsymbol{\Psi}}_k, {}_4\widehat{\boldsymbol{\Psi}}_l\right)\right]}_{V_4}
\end{aligned}$$

where

$$V_1\left(\frac{G_t}{g_t}\sum_{k=1}^{g_{amt}}\boldsymbol{\Psi}_k\right) = G_t\left(\frac{G_t}{g_t} - 1\right)\left[\frac{\sum_{k=1}^{G_{amt}}\boldsymbol{\Psi}_k\boldsymbol{\Psi}_k^T}{G_t - 1} - \frac{\left(\sum_{k=1}^{G_{amt}}\boldsymbol{\Psi}_k\right)\left(\sum_{k=1}^{G_{amt}}\boldsymbol{\Psi}_k\right)^T}{G_t(G_t-1)}\right]$$

and

$$V_2\left(\frac{g_{at}}{g_{Ot}}\sum_{k=1}^{g_{Omt}}\mathbf{\Psi}_k|s_1\right)=g_{at}\left(\frac{g_{at}}{g_{Ot}}-1\right)\left[\frac{\sum_{k=1}^{g_{amt}}\mathbf{\Psi}_k\mathbf{\Psi}_k^T}{g_{at}-1}-\frac{\left(\sum_{k=1}^{g_{amt}}\mathbf{\Psi}_k\right)\left(\sum_{k=1}^{g_{amt}}\mathbf{\Psi}_k\right)^T}{g_{at}(g_{at}-1)}\right].$$

The variance estimator (Eq. 5.31) is unbiased with respect to all expectations at higher sampling events than within hauls, but the within-haul variance estimator (Eq. 2.37) was shown approximately unbiased in Section 2.A.4 and this negligible bias is ignored here. I will show that estimators of each of the components $V_1,\ldots,V_4$ are unbiased (beyond within-haul complications). The sum of these estimators yields Eq. 5.31. The estimator for V_1 is

$$\begin{aligned}\widehat{V}_1 =& G_t\left(\frac{G_t}{g_t}-1\right)\frac{g_{at}}{g_t g_{Ot}}\left\{\sum_{k=1}^{g_{Omt}}\left[{}_4\widehat{\mathbf{\Psi}}_{k4}\widehat{\mathbf{\Psi}}_k^T-\widehat{V}\left({}_4\widehat{\mathbf{\Psi}}_k\right)\right]\right.\\&\left.-\frac{g_{at}-1}{(g_t-1)(g_{Ot}-1)}\sum\sum_{k\neq l}^{g_{Omt}}\left[{}_4\widehat{\mathbf{\Psi}}_{k4}\widehat{\mathbf{\Psi}}_l^T-\widehat{Cov}\left({}_4\widehat{\mathbf{\Psi}}_k,{}_4\widehat{\mathbf{\Psi}}_l\right)\right]\right\}\end{aligned}$$

which has expectation

$$\begin{aligned}E\left(\widehat{V}_1\right) =& G_t\left(\frac{G_t}{g_t}-1\right)E_1\left\{\frac{E_2\left[\frac{g_{at}}{g_{Ot}}\sum_{k=1}^{g_{Omt}}\mathbf{\Psi}_k\mathbf{\Psi}_k^T\right]}{g_t}\right.\\&\left.-\frac{E_2\left[\frac{g_{at}(g_{at}-1)}{(g_{Ot}(g_{Ot}-1)}\sum\sum_{k\neq l}^{g_{Omt}}\mathbf{\Psi}_k\mathbf{\Psi}_l^T\right]}{g_t(g_t-1)}\right\}\\=& G_t\left(\frac{G_t}{g_t}-1\right)E_1\left[\frac{\sum_{k=1}^{g_{amt}}\mathbf{\Psi}_k\mathbf{\Psi}_k^T}{g_t}-\frac{\sum\sum_{k\neq l}^{g_{amt}}\mathbf{\Psi}_k\mathbf{\Psi}_l^T}{g_t(g_t-1)}\right]\\=& G_t\left(\frac{G_t}{g_t}-1\right)\left[\frac{\sum_{k=1}^{G_{amt}}\mathbf{\Psi}_k\mathbf{\Psi}_k^T}{G_t}-\frac{\sum\sum_{k\neq l}^{G_{amt}}\mathbf{\Psi}_k\mathbf{\Psi}_l^T}{G_t(G_t-1)}\right].\end{aligned}$$

The estimator for V_2 is

$$\widehat{V}_2 = \left(\frac{G_t}{g_t}\right)^2 g_{at}\left(\frac{g_{at}}{g_{Ot}}-1\right)\left\{\frac{\sum_{k=1}^{g_{Omt}}\left[{}_4\widehat{\boldsymbol{\Psi}}_{k4}\widehat{\boldsymbol{\Psi}}_k^T - \widehat{V}\left({}_4\widehat{\boldsymbol{\Psi}}_k\right)\right]}{g_{Omt}}\right.$$
$$\left. - \frac{\sum\sum_{k\neq l}^{g_{Omt}}\left[{}_4\widehat{\boldsymbol{\Psi}}_{k4}\widehat{\boldsymbol{\Psi}}_l^T - \widehat{Cov}\left({}_4\widehat{\boldsymbol{\Psi}}_k, {}_4\widehat{\boldsymbol{\Psi}}_l\right)\right]}{g_{Ot}(g_{Ot}-1)}\right\}$$

which has expectation

$$\begin{aligned} E\left(\widehat{V}_2\right) =& E_1\left\{\left(\frac{G_t}{g_t}\right)^2 g_{at}\left(\frac{g_{at}}{g_{Ot}}-1\right)E_2\left[\frac{\sum_{k=1}^{g_{Omt}}\boldsymbol{\Psi}_k\boldsymbol{\Psi}_k^T}{g_{Omt}} - \frac{\sum\sum_{k\neq l}^{g_{Omt}}\boldsymbol{\Psi}_k\boldsymbol{\Psi}_l^T}{g_{Ot}(g_{Ot}-1)}\right]\right\} \\ =& E_1\left\{\left(\frac{G_t}{g_t}\right)^2 g_{at}\left(\frac{g_{at}}{g_{Ot}}-1\right)\frac{\sum_{k=1}^{g_{amt}}\boldsymbol{\Psi}_k\boldsymbol{\Psi}_k^T}{g_{amt}} - \frac{\sum\sum_{k\neq l}^{g_{amt}}\boldsymbol{\Psi}_k\boldsymbol{\Psi}_l^T}{g_{at}(g_{at}-1)}\right\}. \end{aligned}$$

The estimator for the third and fourth components are

$$\widehat{V}_3 = \left(\frac{G_t}{g_t}\frac{g_{at}}{g_{Ot}}\right)^2\sum_{k=1}^{g_{Omt}}\widehat{V}\left({}_4\widehat{\boldsymbol{\Psi}}_k\right)$$

and

$$\widehat{V}_4 = \left(\frac{G_t}{g_t}\frac{g_{at}}{g_{Ot}}\right)^2\sum\sum_{k\neq l}^{g_{Omt}}\widehat{Cov}\left({}_4\widehat{\boldsymbol{\Psi}}_k, {}_4\widehat{\boldsymbol{\Psi}}_l\right)$$

which have expectations

$$E\left(\widehat{V}_3\right) = E_1\left\{\left(\frac{G_t}{g_t}\right)^2 E_2\left[\left(\frac{g_{at}}{g_{Ot}}\right)^2\sum_{k=1}^{g_{Omt}}V\left({}_4\widehat{\boldsymbol{\Psi}}_k\right)\right]\right\} = E_1\left[\left(\frac{G_t}{g_t}\right)^2\frac{g_{at}}{g_{Ot}}\sum_{k=1}^{g_{amt}}V\left({}_4\widehat{\boldsymbol{\Psi}}_k\right)\right]$$

and

$$\begin{aligned} E\left(\widehat{V}_4\right) =& E_1\left\{\left(\frac{G_t}{g_t}\right)^2 E_2\left[\left(\frac{g_{at}}{g_{Ot}}\right)^2\sum\sum_{k\neq l}^{g_{Omt}}Cov\left({}_4\widehat{\boldsymbol{\Psi}}_k, {}_4\widehat{\boldsymbol{\Psi}}_l\right)\right]\right\} \\ =& E_1\left[\left(\frac{G_t}{g_t}\right)^2\frac{g_{at}(g_{Ot}-1)}{g_{Ot}(g_{at}-1)}\sum\sum_{k\neq l}^{g_{amt}}Cov\left({}_4\widehat{\boldsymbol{\Psi}}_k, {}_4\widehat{\boldsymbol{\Psi}}_l\right)\right]. \end{aligned}$$

5.A.3 Prediction Error Variance of $\widehat{N}^*_{amt}$ and $\widehat{\Theta}^*_{amt}$: Pots and Trawls

Let $Y_{mt}|X_{mt}$ denote the random variable for the number of the appropriate species found in s-prevalent catch or effort which is a proportion of a total catch or effort, X_{mt}. The covariate, X_{mt} denotes total weight of catch for trawls and total number of pots for pot vessels, hence the dual catch/effort terminology. Hereafter catch will be used. Also, let Z_{mt} be the random variable that indicates whether catch is s-prevelent. I assume the expectations:

$$E\left(Y_{mt}|Z_{mt}\right) = Z_{mt}\phi_{n,am}X_{mt},$$

$$V\left(Y_{mt}|Z_{mt}\right) = Z_{mt}\sigma^2_{n,am}X_{mt},$$

$E\left(Z_{mt}\right) = \alpha_m$ and $V\left(Z_{mt}\right) = \alpha_m\left(1-\alpha_m\right)\tau_{\alpha m}$. The prediction error variance for $\widehat{N}^*_{amt}$ is

$$V\left[X_{mt}\widehat{\alpha}_m\widehat{\phi}_{n,am} - Y_{mt}\right] = \underbrace{X^2_{mt}V\left(\widehat{\alpha}_m\widehat{\phi}_{n,am}\right)}_{V_1} + \underbrace{V\left(Y_{mt}\right)}_{V_2}.$$

The first component is

$$V_1 = X^2_{mt}\left[\alpha^2_m V\left(\widehat{\phi}_{n,am}\right) + \phi^2_{n,am}V\left(\widehat{\alpha}_m\right) + V\left(\widehat{\alpha}_m\right)V\left(\widehat{\phi}_{n,am}\right)\right]$$

which is unbiasedly estimated by

$$\widehat{V}_1 = X^2_{mt}\left[\widehat{\alpha}^2_m\widehat{V}\left(\widehat{\phi}_{n,am}\right) + \widehat{\phi}^2_{n,am}\widehat{V}\left(\widehat{\alpha}_m\right) - \widehat{V}\left(\widehat{\alpha}_m\right)\widehat{V}\left(\widehat{\phi}_{n,am}\right)\right].$$

The second component is

$$\begin{aligned} V_2 =& V\left[E\left(Y_{mt}|Z_{mt}\right)\right] + E\left[V\left(Y_{mt}|Z_{mt}\right)\right] \\ & \left(X_{mt}\phi_{n,am}\right)^2\alpha_m\left(1-\alpha_m\right)\tau_{\alpha m} + \alpha_m X_{mt}\sigma^2_{n,am} \end{aligned}$$

which is estimated by substituting parameter estimates.

Let $\mathbf{\Theta}_{mt}$ denote the random variable for numbers of fish in length or age classes in s-prevalent catch that is a proportion of the total catch, X_{mt} in the mth period/region for the undersampled trip. We assume the same information as we did for $\widehat{N}^*_{amt}$, but also

$$E\left(\boldsymbol{\Theta}_{mt}|Y_{mt}\right) = Y_{mt}\mathbf{p}_{\Theta,am},$$

$$V\left(\boldsymbol{\Theta}_{mt}|Y_{mt}\right) = Y_{mt}\left[\operatorname{diag}\left(\mathbf{p}_{\Theta,am}\right) - \mathbf{p}_{\Theta,am}\mathbf{p}_{\Theta,am}^T\right]\tau_{\Theta,am},$$

The prediction error variance for $\widehat{\boldsymbol{\Theta}}_{amt}^*$ is

$$V\left[X_{mt}\widehat{\alpha}_m\widehat{\phi}_{n,am}\widehat{\mathbf{p}}_{\Theta,am} - \boldsymbol{\Theta}_{mt}\right] = X_{mt}^2 V\left(\widehat{\alpha}_m\widehat{\phi}_{n,am}\widehat{\mathbf{p}}_{\Theta,am}\right) + V\left(\boldsymbol{\Theta}_{mt}\right).$$

Component-wise,

$$X_{mt}^2 V\left(\widehat{\alpha}_m\widehat{\phi}_{n,am}\widehat{\mathbf{p}}_{\Theta,am}\right) = V_1\mathbf{p}_{\Theta,am}\mathbf{p}_{\Theta,am}^T + E\left(\widehat{N}_{amt}^{*2}\right)V\left(\widehat{\mathbf{p}}_{\Theta,am}\right)$$

and

$$\begin{aligned} V\left(\boldsymbol{\Theta}_{mt}\right) =& V\left\{E\left[E\left(\boldsymbol{\Theta}_{mt}|Y,Z\right)\right]\right\} + E\left\{V\left[E\left(\boldsymbol{\Theta}_{mt}|Y,Z\right)\right]\right\} + E\left\{E\left[V\left(\boldsymbol{\Theta}_{mt}|Y,Z\right)\right]\right\} \\ =& N_{amt}\left[\operatorname{diag}\left(\mathbf{p}_{\Theta,am}\right) - \mathbf{p}_{\Theta,am}\mathbf{p}_{\Theta,am}^T\right]\tau_{\Theta,am} + V_2\mathbf{p}_{\Theta,am}\mathbf{p}_{\Theta,am}^T. \end{aligned}$$

The first component can be estimated by

$$X_{mt}^2\widehat{V}\left(\widehat{\alpha}_m\widehat{\phi}_{n,am}\widehat{\mathbf{p}}_{\Theta,am}\right) = \widehat{V}_1\widehat{\mathbf{p}}_{\Theta,am}\widehat{\mathbf{p}}_{\Theta,am}^T + \widehat{N}_{amt}^{*2}\widehat{V}\left(\widehat{\mathbf{p}}_{\Theta,am}\right) - \widehat{V}_1\widehat{V}\left(\widehat{\mathbf{p}}_{\Theta,am}\right)$$

and the second component can be estimated by substituting parameter estimtes.

5.A.4 Prediction Error Variance for $\widehat{\boldsymbol{\Lambda}}_{tam}^*$ and $\widehat{\boldsymbol{\Psi}}_{tam}^*$: Longliners

The predictor of the numbers-at-length or -age ($\boldsymbol{\Theta}$) in the mth region in prevalent hauls is

$$\widehat{\boldsymbol{\Theta}}_{tam} = H_{mt}\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Theta,am}.$$

The predictor can be thought of as the number of hooks set in the mth region multiplied by the estimated joint probability that the hook comes from a haul where the species is prevalent (α_m), a fish is caught on the hook in a haul where the species is prevalent (π_{am}) and the probability that the caught fish is a given length/age ($\mathbf{p}_{\Theta,am}$).

The prediction error covariance matrix is

$$V\left[H_{mt}\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Theta,am} - \sum_{i=1}^{H_{mt}} \mathbf{X}_i\right] = H_{mt}^2 V\left(\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Theta am}\right) + H_{mt} V\left(\mathbf{X}_i\right)$$

where $\mathbf{X}_i$ is a vector of indicator random variables for each of j age/length classes, such that $X_{ij} = 1$ when a hook is in an prevalent haul, the hook catches a fish and the fish is in the jth age/length class. Letting Y_i be an indicator such that $Y_i = 1$ when the hook is in an prevalent haul and the hook catches a fish and Z_i an indicator such that $Z_i = 1$ when a hook is in an prevalent haul, we have $E\left(\mathbf{X}_i|Y_i\right) = Y_i\mathbf{p}_{\Theta,am}$, $E\left(Y_i|Z_i\right) = Z_i\pi_{am}$ and $E\left(Z_i\right) = \alpha_m$. Furthermore, with the overdispersed models, $V\left(\mathbf{X}_i|Y_i\right) = Y_i\left[\text{diag}\{\mathbf{p}_{\Theta,am}\} - \mathbf{p}_{\Theta am}\mathbf{p}_{\Theta,am}^T\right]\tau_{\Theta_{am}}$, $V\left(Y_i|Z_i\right) = Z_i\pi_{am}(1-\pi_{am})\tau_{\pi_{am}}$ and $V\left(Z_i\right) = \alpha_m(1-\alpha_m)\tau_{\alpha_m}$. Given these expectations,

$$\begin{aligned}
V\left(\mathbf{X}_i\right) =& E\left\{E\left[V\left(\mathbf{X}_i|Y_i, Z_i\right)\right]\right\} + E\left\{V\left[E\left(\mathbf{X}_i|Y_i, Z_i\right)\right]\right\} + V\left\{E\left[E\left(\mathbf{X}_i|Y_i, Z_i\right)\right]\right\} \\
=& \alpha_m\pi_{am}\Big\{\left[\text{diag}\{\mathbf{p}_{\Theta,am}\} - \mathbf{p}_{\Theta,am}\mathbf{p}_{\Theta,am}^T\right]\tau_{\Theta_{am}} \\
&+ \left[(1-\pi_{am})\tau_{\pi_{am}} + (1-\alpha_m)\tau_{\alpha_m}\pi_{am}\right]\mathbf{p}_{\Theta,am}\mathbf{p}_{\Theta,am}^T\Big\}
\end{aligned}$$

and using an identical approach with independent parameter estimates from the observed catches,

$$\begin{aligned}
V\left(\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Theta,am}\right) = &\left[\alpha_m^2 + V\left(\widehat{\alpha}_m\right)\right]\left[\pi_{am}^2 + V\left(\widehat{\pi}_{am}\right)\right]\left[\mathbf{p}_{\Theta,am}\mathbf{p}_{\Theta,am}^T + V\left(\widehat{\mathbf{p}}_{\Theta,am}\right)\right] \\
&- \alpha_m^2\pi_{am}^2\mathbf{p}_{\Theta,am}\mathbf{p}_{\Theta,am}^T.
\end{aligned}$$

An asymptotically unbiased covariance matrix estimators for $\mathbf{X}_i$ is obtained by substituting the estimates of the various parameters from the observed data and an unbiased covariance matrix estimator for $\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Theta,am}$ is

$$\begin{aligned}
\widehat{V}\left(\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Theta,am}\right) =& \widehat{\alpha}_m^2\widehat{\pi}_{am}^2\widehat{\mathbf{p}}_{\Theta,am}\widehat{\mathbf{p}}_{\Theta,am}^T \\
&- \left[\widehat{\alpha}_m^2 - \widehat{V}\left(\widehat{\alpha}_m\right)\right]\left[\widehat{\pi}_{am}^2 - \widehat{V}\left(\widehat{\pi}_{am}\right)\right]\left[\widehat{\mathbf{p}}_{\Theta,am}\widehat{\mathbf{p}}_{\Theta,am}^T - \widehat{V}\left(\widehat{\mathbf{p}}_{\Theta,am}\right)\right]
\end{aligned} \tag{5.116}$$

For the prediction error covariance matrix of two undersampled trips we have

$$Cov\left(H_{mt}\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Theta,am} - \sum_{i=1}^{H_{mt}} \mathbf{X}_i, H_{mu}\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Theta,am} - \sum_{j=1}^{H_{mu}} \mathbf{X}_j \right)$$
$$= H_{mt}H_{mu}V\left(\widehat{\alpha}_m\widehat{\pi}_{am}\widehat{\mathbf{p}}_{\Theta,am}\right)$$

because of the independence of random variables being predicted (among and between trips) and the observed data from which the parameter estimates are derived. The covariance is estimated using Eq. 5.116.

5.A.5 Derivation of $\widehat{\mathbf{\Theta}}_v$

For the general vessel-specific estimator with the total number of trips known, we have uncertainty at the trip level and at the vessel level. I already assumed in Section 5.2 that the sampling distribution of trips within each vessel is given by SRS. Let p_v denote the sampling distribution of trips within the vessel, s_v denote a particular sample of trips for that vessel, and p_2 denote the distribution within and among trips. The distribution within trips may result from various model and/or sampling distributions. For example, when there is sufficient sampling in the tth trip, there can be models used within each haul because of problems repeatedly referred to in this dissertation and p_2 includes these within-haul conditional model-induced distributions and the sampling distribution for hauls in a hierarchical manner.

The expected value of the general vessel-specific estimator is

$$E\left(\widehat{\mathbf{\Theta}}_v\right) = E_{p_v}\left[\frac{C_v}{c_v}\sum_{t=1}^{c_v} E_{p_2}\left(\widehat{\mathbf{\Theta}}_t|s_v\right)\right] = E_{p_v}\left(\frac{C_v}{c_v}\sum_{t=1}^{c_v}\mathbf{\Theta}_t\right) = \sum_{t=1}^{C_v}\mathbf{\Theta}_t = \mathbf{\Theta}_v$$

and the variance is

$$
\begin{aligned}
V\left(\widehat{\Theta}_v\right) =& E_{p_v}\left[\left(\frac{C_v}{c_v}\right)^2 V_{p_2}\left(\sum_{t=1}^{c_v}\widehat{\Theta}_t|s_v\right)\right] + V_{p_v}\left[\frac{C_v}{c_v}\sum_{t=1}^{c_v} E_{p_2}\left(\widehat{\Theta}_t|s_v\right)\right] \\
=& E_{p_v}\left\{\left(\frac{C_v}{c_v}\right)^2\left[\sum_{t=1}^{c_v} V_{p_2}\left(\widehat{\Theta}_t\right) + \sum\sum_{t\neq u}^{c_v} Cov_{p_2}\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right)\right]\right\} \\
& + V_{p_v}\left(\frac{C_v}{c_v}\sum_{t=1}^{c_v}\Theta_t\right) \\
=& \underbrace{\left(\frac{C_v}{c_v}\right)\left[\sum_{t=1}^{C_v} V_{p_2}\left(\widehat{\Theta}_t\right) + \frac{c_v-1}{C_v-1}\sum\sum_{t\neq u}^{C_v} Cov_{p_2}\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right)\right]}_{V_1} \\
& \underbrace{+ C_v\left(\frac{C_v}{c_v}-1\right)\frac{\sum_{t=1}^{C_v}\left(\Theta_t - \overline{\Theta}_v\right)\left(\Theta_t - \overline{\Theta}_v\right)^T}{C_v-1}}_{V_2}.
\end{aligned}
$$

Now, unbiased estimators of each component, V_1 and V_2, are shown unbiased separately and the sum of the estimators will be shown to equal Eq. 5.72. The unbiased estimator of the first component is

$$
\widehat{V}_1 = \left(\frac{C_v}{c_v}\right)^2\left[\sum_{t=1}^{c_v}\widehat{V}_{p_2}\left(\widehat{\Theta}_t\right) + \sum\sum_{t\neq u}^{c_v}\widehat{Cov}_{p_2}\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right)\right]
$$

where $\widehat{V}_{p_2}\left(\widehat{\Theta}_t\right)$ and $\widehat{Cov}_{p_2}\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right)$ are unbiased estimators that depend on p_2. The expected value is

$$
\begin{aligned}
E\left(\widehat{V}_1\right) =& E_{p_v}\left\{\left(\frac{C_v}{c_v}\right)^2\left[\sum_{t=1}^{c_v} E_{p_2}\left[\widehat{V}_{p_2}\left(\widehat{\Theta}_t\right)|s_v\right]\right.\right. \\
& \left.\left. + \sum\sum_{t\neq u}^{c_v} E_{p_2}\left[\widehat{Cov}_{p_2}\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right)|s_v\right]\right]\right\} \\
=& \left(\frac{C_v}{c_v}\right)\left[\sum_{t=1}^{C_v} V_{p_2}\left(\widehat{\Theta}_t\right) + \frac{c_v-1}{C_v-1}\sum\sum_{t\neq u}^{C_v} Cov_{p_2}\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right)\right].
\end{aligned}
$$

The unbiased estimator of the second component is

$$\begin{aligned}
\widehat{V}_2 =& \frac{C_v}{c_v}\left(\frac{C_v}{c_v}-1\right)\left\{\sum_{t=1}^{c_v}\left[\widehat{\Theta}_t\widehat{\Theta}_t^T - \widehat{V}_{p2}\left(\widehat{\Theta}_t\right)\right] - \frac{\sum\sum_{t\neq u}^{c_v}\left[\widehat{\Theta}_t\widehat{\Theta}_u^T - \widehat{Cov}_{p2}\left(\widehat{\Theta}_t,\widehat{\Theta}_u\right)\right]}{c_v-1}\right\} \\
=& C_v\left(\frac{C_v}{c_v}-1\right)\frac{\sum_{t=1}^{c_v}\left(\widehat{\Theta}_t-\overline{\widehat{\Theta}}_v\right)\left(\widehat{\Theta}_t-\overline{\widehat{\Theta}}_v\right)^T}{c_v-1} \\
& -\frac{C_v}{c_v}\left(\frac{C_v}{c_v}-1\right)\left[\widehat{V}_{p2}\left(\widehat{\Theta}_t\right) - \frac{\sum\sum_{t\neq u}^{c_v}\widehat{Cov}_{p2}\left(\widehat{\Theta}_t,\widehat{\Theta}_u\right)}{c_v-1}\right]
\end{aligned}$$

To show that $\widehat{V}_2$ is unbiased, first realize that

$$E_{p2}\left[\widehat{\Theta}_t\widehat{\Theta}_t^T - \widehat{V}_{p2}\left(\widehat{\Theta}_t\right)\right] = \Theta_t\Theta_t^T$$

and

$$E_{p2}\left[\widehat{\Theta}_t\widehat{\Theta}_u^T - \widehat{Cov}_{p2}\left(\widehat{\Theta}_t,\widehat{\Theta}_u\right)\right] = \Theta_t\Theta_u^T,$$

then the expected value of $\widehat{V}_2$ is

$$\begin{aligned}
E\left(\widehat{V}_2\right) =& \frac{C_v}{c_v}\left(\frac{C_v}{c_v}-1\right)E_{p_v}\left\{\sum_{t=1}^{c_v}\Theta_t\Theta_t^T - \frac{\sum\sum_{t\neq u}^{c_v}\Theta_t\Theta_u^T}{c_v-1}\right\} \\
=& \left(\frac{C_v}{c_v}-1\right)\left[\sum_{t=1}^{C_v}\Theta_t\Theta_t^T - \frac{\sum\sum_{t\neq u}^{c_v}\Theta_t\Theta_u^T}{C_v-1}\right] \\
=& C_v\left(\frac{C_v}{c_v}-1\right)\frac{\sum_{t=1}^{C_v}\left(\Theta_t-\overline{\Theta}_v\right)\left(\Theta_t-\overline{\Theta}_v\right)^T}{C_v-1}
\end{aligned}$$

and the sum of the component estimators is

$$\begin{aligned}
\widehat{V}_1 + \widehat{V}_2 =& C_v\left(\frac{C_v}{c_v}-1\right)\frac{\sum_{t=1}^{c_v}\left(\widehat{\Theta}_t-\widehat{\overline{\Theta}}_v\right)\left(\widehat{\Theta}_t-\widehat{\overline{\Theta}}_v\right)^T}{c_v-1} \\
&+\left(\frac{C_v}{c_v}\right)\sum_{t=1}^{c_v}\widehat{V}_{p_2}\left(\widehat{\Theta}_t\right) \\
&+\left[\frac{C_v}{c_v(c_v-1)}\left(\frac{C_v}{c_v}-1\right)+\left(\frac{C_v}{c_v}\right)^2\right]\sum\sum_{t\neq u}^{c_v}\widehat{Cov}_{p_2}\left(\widehat{\Theta}_t,\widehat{\Theta}_u\right) \\
=& C_v\left(\frac{C_v}{c_v}-1\right)\frac{\sum_{t=1}^{c_v}\left(\widehat{\Theta}_t-\widehat{\overline{\Theta}}_v\right)\left(\widehat{\Theta}_t-\widehat{\overline{\Theta}}_v\right)^T}{c_v-1} \\
&+\left(\frac{C_v}{c_v}\right)\left[\sum_{t=1}^{c_v}\widehat{V}_{p_2}\left(\widehat{\Theta}_t\right)+\frac{C_v-1}{c_v-1}\sum\sum_{t\neq u}^{c_v}\widehat{Cov}_{p_2}\left(\widehat{\Theta}_t,\widehat{\Theta}_u\right)\right]
\end{aligned}$$

For vessels with 100% coverage, the estimator is just the sum of the within-trip estimators,

$$\widehat{\Theta}_v=\sum_{t=1}^{C_v}\widehat{\Theta}_t$$

because $C_v = c_v$ and the expectation is

$$E\left(\widehat{\Theta}_v=\sum_{t=1}^{C_v}E\left(\widehat{\Theta}_t\right)=\sum_{t=1}^{C_v}\Theta_t\right).$$

The variance is

$$V\left(\widehat{\Theta}_v\right)=\sum_{t=1}^{C_v}V\left(\widehat{\Theta}_t\right)+\sum\sum_{t\neq u}^{C_v}Cov\left(\widehat{\Theta}_t,\widehat{\Theta}_u\right)$$

and the variance is estimated using appropriate variance and covariance estimators.

When the total number of trips made for vessel with 30% coverage is unknown and the model assumed for the distribution of sampled trips in Section 5.2.1 holds, the expected value of Eq. 5.73 is

$$E\left(\widehat{\Theta}_v\right)=\frac{10}{3}E_m\left[E_s\left(\sum_{t=1}^{c_v}\widehat{\Theta}_t|c_v\right)\right]=\frac{10}{3}\frac{E_m\left(c_v\right)}{C_v}\sum_{t=1}^{C_v}\Theta_t=\sum_{t=1}^{C_v}\Theta_t=\Theta_v$$

where m denotes the distribution of number of sampled trips and s denotes the conditional sampling distribution of trips given the number of sampled trips as well as within-trip distributions. The variance is

$$
\begin{aligned}
V\left(\widehat{\Theta}_v\right) &= \left(\frac{10}{3}\right)^2 \left\{ E_m \left[V_s \left(\sum_{t=1}^{c_v} \widehat{\Theta}_t | c_v \right) \right] + V_m \left[E_s \left(\sum_{t=1}^{c_v} \widehat{\Theta}_t | c_v \right) \right] \right\} \\
&= \left(\frac{10}{3}\right)^2 \left\{ E_m \left[c_v \left(1 - \frac{c_v}{C_v} \right) \right] \mathbf{S}_v^2 + \frac{E_m\left(c_v\right)}{C_v} \sum_{t=1}^{C_v} V\left(\widehat{\Theta}_t\right) \right. \\
&\quad \left. + \frac{E_m\left[c_v(c_v-1)\right]}{C_v(C_v-1)} \sum_{t \neq u}^{C_v}\sum Cov\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right) + V_m\left(c_v\right) \overline{\Theta}_v \overline{\Theta}_v^T \right\} \\
&= \left[C_v \frac{7}{3} - \left(\frac{10}{3}\right)^2 \frac{\sigma_c^2}{C_v} \right] \mathbf{S}_v^2 + \left(\frac{10}{3}\right)^2 \frac{E_m\left(c_v\right)}{C_v} \sum_{t=1}^{C_v} V\left(\widehat{\Theta}_t\right) \\
&\quad + \left(\frac{10}{3}\right)^2 \left[\frac{E_m\left[c_v(c_v-1)\right]}{C_v(C_v-1)} \sum_{t \neq u}^{C_v}\sum Cov\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right) + \sigma_c^2 \overline{\Theta}_v \overline{\Theta}_v^T \right]
\end{aligned}
$$

where $\overline{\Theta}_v = \sum_{t=1}^{C_v} \Theta_t / C_v$ and

$$
\mathbf{S}_v^2 = \frac{\sum_{t=1}^{C_v} \left(\Theta_t - \overline{\Theta}_v\right)\left(\Theta_t - \overline{\Theta}_v\right)^T}{C_v - 1}.
$$

Consider the situation where the expected total number of trips (C_v) is known and an unbiased estimator of the variance in the number of trips sampled ($\widehat{\sigma}_c^2$) is available. An unbiased variance estimator is

$$
\begin{aligned}
\widehat{V}\left(\widehat{\Theta}_v\right) =& C_v \frac{7}{3} \widehat{\mathbf{S}}_v^2 + \left(\frac{10}{3}\right)^2 \left[\sum_{t=1}^{c_v} \widehat{V}\left(\widehat{\Theta}_t | c_v\right) + \sum_{t \neq u}^{c_v}\sum \widehat{Cov}\left(\widehat{\Theta}_t, \widehat{\Theta}_u | c_v\right) \right] \\
&+ \left(\frac{10}{3}\right)^2 \widehat{\sigma}_c^2 \left[\widehat{\overline{\Theta}}_v \widehat{\overline{\Theta}}_v^T - \widehat{V}\left(\widehat{\overline{\Theta}}_v | c_v\right) - \frac{\widehat{\mathbf{S}}_v^2}{C_v} \right]
\end{aligned}
$$

where an unbiased estimator of $\mathbf{S}_v^2$ conditional on c_v is

$$
\widehat{\mathbf{S}}_v^2 = \widetilde{\mathbf{S}}_v^2 - \frac{\sum_{t=1}^{c_v} \widehat{V}\left(\widehat{\Theta}_t | c_v\right)}{c_v} + \frac{\sum_{t \neq u}^{c_v}\sum \widehat{Cov}\left(\widehat{\Theta}_t, \widehat{\Theta}_u | c_v\right)}{c_v(c_v - 1)}
$$

and

$$\widetilde{\mathbf{S}}_v^2 = \frac{\sum_{t=1}^{c_v}\left(\widehat{\Theta}_t - \widehat{\overline{\Theta}}_v\right)\left(\widehat{\Theta}_t - \widehat{\overline{\Theta}}_v\right)^T}{c_v - 1}$$

$$\begin{aligned}\widehat{V}\left(\widehat{\overline{\Theta}}_v|c_v\right) = \frac{\widehat{V}\left(\widehat{\Theta}_v|c_v\right)}{C_v^2} = &\left(\frac{1}{c_v} - \frac{1}{C_v}\right)\widetilde{\mathbf{S}}_v^2 + \frac{1}{C_v c_v}\sum_{t=1}^{c_v}\widehat{V}\left(\widehat{\Theta}_t|c_v\right)\\ &+ \frac{C_v - 1}{C_v c_v(c_v-1)}\sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{\Theta}_t, \widehat{\Theta}_u|c_v\right)\end{aligned}$$

It turns out that when terms are consolidated, the variance estimator can be written as

$$\begin{aligned}\widehat{V}\left(\widehat{\Theta}_v\right) =&C_v\frac{7}{3}\widetilde{\mathbf{S}}_v^2 + \left(\frac{10}{3}\right)^2\widehat{\sigma}_c^2\left[\widehat{\overline{\Theta}}_v\widehat{\overline{\Theta}}_v^T - \frac{\widetilde{\mathbf{S}}_v^2}{c_v}\right] + \left[\left(\frac{10}{3}\right)^2 - \frac{7}{3}\frac{C_v}{c_v}\right]\sum_{t=1}^{c_v}\widehat{V}\left(\widehat{\Theta}_t|c_v\right)\\ &+\left[\left(\frac{10}{3}\right)^2 + \frac{7}{3}\frac{C_v}{c_v(c_v-1)} - \left(\frac{10}{3}\right)^2\frac{\widehat{\sigma}_c^2}{c_v(c_v-1)}\right]\sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{\Theta}_t, \widehat{\Theta}_u|c_v\right)\end{aligned}$$

Because there are no terms above involving the product of C_V and $\widehat{\sigma}_c^2$, substuting an unbiased estimator of the total number of trips for the true value will also give an unbiased estimators, assuming that the model for the distribution of the number of sampled trips is uncorrelated with the conditional sampling distribution of trips. When the unbiased estimator of the total number of trips, $\widehat{C}_v = 10c_v/3$, is substituted,

$$\begin{aligned}\widehat{V}\left(\widehat{\Theta}_v\right) =&c_v\frac{70}{9}\widetilde{\mathbf{S}}_v^2 + \left(\frac{10}{3}\right)^2\widehat{\sigma}_c^2\left[\widehat{\overline{\Theta}}_v\widehat{\overline{\Theta}}_v^T - \frac{\widetilde{\mathbf{S}}_v^2}{c_v}\right] + \frac{10}{3}\sum_{t=1}^{c_v}\widehat{V}\left(\widehat{\Theta}_t|c_v\right)\\ &+\frac{10}{3}\left[\frac{10}{3}\left(1 - \frac{\widehat{\sigma}_c^2}{c_v(c_v-1)}\right) + \frac{7}{3(c_v-1)}\right]\sum\sum_{t\neq u}^{c_v}\widehat{Cov}\left(\widehat{\Theta}_t, \widehat{\Theta}_u|c_v\right).\end{aligned}$$

Now assuming the estimator of the model variance for the number of sampled trips is unbiased, the expected value of the conservative variance estimator (Eq. 5.74) proposed in Section 5.2.1 is

$$E\left[\widehat{V}\left(\widehat{\Theta}_v\right)\right] = \left(\frac{10}{3}\right)^2 E_m\left\{\widehat{\sigma}_c^2\left[\overline{\Theta}_v\overline{\Theta}_v^T + V\left(\widehat{\overline{\Theta}}_v|c_v\right)\right]\right\} + \left(\frac{10}{3}\right)^2 \frac{E_m\left(c_v\right)}{C_v}\sum_{t=1}^{C_v} V\left(\widehat{\Theta}_t\right)$$
$$+ \frac{E_m\left[c_v(c_v-1)\right]}{C_v(C_v-1)}\sum\sum_{t\neq u}^{C_v} Cov\left(\widehat{\Theta}_t, \widehat{\Theta}_u\right) + \frac{7}{3}C_v\mathbf{S}_v^2 - \frac{10}{3}E_m\left(\frac{\widehat{\sigma}_c^2}{c_v}\right)\mathbf{S}_v^2$$

To a second order approximation,

$$E_m\left(\frac{\widehat{\sigma}_c^2}{c_v}\right) = \frac{10}{3}\frac{\sigma_c^2}{C_v}$$

and so

$$E\left[\widehat{V}\left(\widehat{\Theta}_v\right)\right] \doteq V\left(\widehat{\Theta}_v\right) + E_m\left[\widehat{\sigma}_c^2 V\left(\widehat{\overline{\Theta}}_v|c_v\right)\right].$$

Therefore, even when $\widehat{\sigma}_c^2$ is unbiased, the variance estimator (Eq. 5.74) is postively biased and hence conservative.

5.A.6 Derivation of $\widehat{V}\left(\widehat{\Theta}_U\right)$

The expected value of the pooled vessel variance estimator (Eq. 5.99) is

$$E\left[\widehat{V}\left(\widehat{\Theta}_U\right)\right] = \frac{V_U}{V_U-1}\left[\sum_{v=1}^{V_U} E\left(\widehat{\Theta}_v\widehat{\Theta}_v^T\right) - \frac{\sum_{v=1}^{V_U}\sum_{w=1}^{V_U} E\left(\widehat{\Theta}_v\widehat{\Theta}_w^T\right)}{V_U}\right]$$
$$+ \frac{V_U}{V_U-1}\sum\sum_{v\neq w}^{V_U} Cov\left(\widehat{\Theta}_v, \widehat{\Theta}_w\right)$$
$$= \frac{V_U}{V_U-1}\sum_{v=1}^{V_U}\left[\Theta_v\Theta_v^T + Cov\left(\widehat{\Theta}_v, \widehat{\Theta}_v\right)\right]$$
$$- \frac{\sum_{v=1}^{V_U}\sum_{w=1}^{V_U}\left[\Theta_v\Theta_w^T + Cov\left(\widehat{\Theta}_v, \widehat{\Theta}_w\right)\right]}{V_U-1}$$
$$+ \frac{V_U}{V_U-1}\sum\sum_{v\neq w}^{V_U} Cov\left(\widehat{\Theta}_v, \widehat{\Theta}_w\right)$$
$$= \frac{\sum_{v=1}^{V_U}\left(\Theta_v - \overline{\Theta}_U\right)\left(\Theta_v - \overline{\Theta}_U\right)^T}{V_U-1} + \sum_{v=1}^{V_U}\sum_{w=1}^{V_U} Cov\left(\widehat{\Theta}_v, \widehat{\Theta}_w\right)$$

Now,

$$\sum_{v=1}^{V_U}\sum_{w=1}^{V_U} Cov\left(\widehat{\Theta}_v, \widehat{\Theta}_w\right) = \sum_{v=1}^{V_U} V\left(\widehat{\Theta}_v\right) + \sum_{v \neq w}^{V_U}\sum Cov\left(\widehat{\Theta}_v, \widehat{\Theta}_w\right)$$

which is the true variance over all undersampled vessels and so, the variances for the elements of $\widehat{\Theta}_v$ along the diagonal of $V\left(\widehat{\Theta}_v\right)$ are overestimated by Eq. 5.99. However, the direction of bias (positive or negative) of the offdiagonal elements is unknown.

Chapter 6

ESTIMATORS IN APPLICATION

6.1 Introduction

The purely design-based estimators and corresponding variance estimators I presented in the previous chapters cannot be used with data that are currently collected by observers in the Alaskan groundfish fleets, but the estimators that integrate design- and model-based approaches (IDM estimators) were derived specifically for this purpose. The forms of the IDM estimators are necessarily general and it is informative to see how they may be applied in specific cases to obtain knowledge about the groundfish fisheries and the ecosystems with which they interact. As such, I have applied the IDM estimators to estimate catch parameters in time periods and/or regions that are of current interest.

I begin with regional estimation of black-footed albatross bycatch and bycatch rates aboard longline vessels in Section 6.2. I work through the levels of the sampling design with black-footed albatross and northern fulmar and yearly estimates for laysan albatross across all gear types are also given. Estimation of regional total catch weight of walleye pollock and Pacific cod is treated in Section 6.3. I give estimates of salmon bycatch in Section 6.4 and estimation of Steller sea lion and killer whale mortalities is the subject of Section 6.5. In Section 6.6, I present results for numbers and proportions-at-age and -length for a few species and I describe estimation separately for longline and trawl gears. Estimation for pot gear is omitted because the procedures are virtually identical to those for trawl gear.

There are a few important general conclusions from these examples. First, estimates of precision for all of these catch parameters are possible and not limited to the species used in the examples. Many of these types of catch attributes are currently calculated (using *ad hoc* methods) on a continual basis for management needs, but the resulting values have often been treated as known or without error. Also, these estimates are based solely on observer-collected data rather than the "known" values of catch for various "fisheries" that are calculated via the blend system by the Alaska Regional Office of NMFS (blend-based estimates). The blend-based estimates are produced through an *ad hoc* fusion of observer and fishing-industry supplied information and there is no way to know the quality of this information and the resulting estimates. Indeed, as demonstrated below there is substantial evidence that blend-based estimates and simplified assumptions may have consistent errors for some catch attributes.

6.2 Seabird bycatch

6.2.1 Black-footed Albatross in Longlines

To illustrate how the IDM estimators can be used to estimate seabird mortality in longline fishies, I present a detailed example for black-footed albatross. I begin with an observed trip aboard a longline vessel in the medium size class during the second quarter of 2001 (Table 6.1). The number of hooks set and sampled and the number of albatross observed during the tally periods for each sampled haul are the necessary information used in Eq. 2.3 to obtain the model-based haul estimates of total mortality. Notice that most of the sampled hauls have no albatross mortalities during the tally period. This is the case for most observed trips througout the North Pacific groundfish fleet. In fact, no black-footed albatross mortalities were observed for any sampled hauls during the fourth quarter of 2001. The same vessel made two observed trips during the second quarter but no albatross mortalities were incurred on the second observed trip (Table 6.2). Of the 9 hauls made during the first trip, 8 were sampled by the observer (Table 6.1). The estimates for each trip are obtained identically by using the haul estimates within the respective trips with the trip-level estimator (Eq. 5.14) and variance estimator (Eq. 5.15). The information presented for each trip is used with Eq. 5.73 and corresponding variance estimator (Eq. 5.74) to obtain $\widehat{N}_v = 9.75$ and $\widehat{SE}\left(\widehat{N}_v\right) = 12.7$.

Several trips made by vessels in either size class were undersampled and different methods are required to predict bycatch for these trips. In Table 6.3, the number of hooks deployed for all sampled hauls and the number of observed albatross mortalities for each NMFS statistical area during the second quarter of 2001 are listed. The estimated bycatch rates are necessary to make model-based predictions of mortalities

Table 6.1. Estimates of black-footed albatross bycatch ($\widehat{N}_k$) and standard errors and numbers of hooks deployed (H_k), numbers of hooks sampled (h_k) and number of seabird mortalities counted among sampled hooks (n_k) for hauls made during a trip by a vessel in the medium size class in the second quarter of 2001. The trip estimates are given in the bottom row and there is no sample information for one haul because only eight of nine hauls made during the trip were sampled.

Haul	H_k	h_k	n_k	$\widehat{N}_k$	$\widehat{SE}\left(\widehat{N}_k\right)$
1	4850	1552	0	0	0
2	4850	1552	0	0	0
3	4584	1530	0	0	0
4	5597	1738	0	0	0
5	4600	1472	0	0	0
6	6272	1766	0	0	0
7	4725	1511	2	6.254	3.646
8	3298	1164	0	0	0
9	4704				
Trip				7.036	4.523

Table 6.2. Estimates of black-footed albatross bycatch ($\widehat{N}_t$), standard errors and numbers of hauls made (G_t) and sampled (g_t) for two trips on a vessel in the medium size class in the second quarter of 2001. The first trip is that presented in Table 6.1 and the vessel estimate is given in the bottom row.

Trip	G_t	g_t	$\widehat{N}_t$	$\widehat{SE}\left(\widehat{N}_t\right)$
1	9	8	7.036	4.523
2	12	10	0	0
Vessel			23.456	28.679

for the undersampled trips in the second quarter of 2001. Albatross mortalities were only observed for hauls made in four management regions with one in the Bering Sea ($m = 517$) and the others in the Gulf of Alaska ($m = 630, 640, 650$). Using the number of hooks deployed, the management area where each set was made, the estimates in Table 6.3 and the seabird bycatch predictor (Eq. 5.48) and variance estimator (Eq. 5.49), we can predict bycatch and estimate corresponding variance for the undersampled trips. The resulting bycatch predictions and variance for one undersampled trip is presented in Table 6.4. Non-zero mortalities are predicted for this trip because the sets were made in one management area where mortalities were observed. Some sets have the same predicted mortalities because identical numbers of hooks are deployed.

Table 6.3. Number of hauls (g_m), observed hooks (h_m), black-footed albatross mortalities (n_m) and estimated bycatch rates ($\widehat{\pi}_m$) and dispersion parameters ($\widehat{\tau}_m$) for NMFS statistical area (m) during the second quarter of 2001.

m	n_m	h_m	g_m	$\widehat{\pi}_m$	$\widehat{\tau}_m$
509	0	687349	256	0	0
512	0	118109	42	0	0
513	0	144614	64	0	0
516	0	25716	5	0	0
517	1	244474	114	4.09×10^{-6}	1.255
518	0	101903	56	0	0
519	0	51087	34	0	0
521	0	3254122	873	0	0
523	0	253500	78	0	0
524	0	541852	152	0	0
541	0	300719	168	0	0
542	0	149671	95	0	0
543	0	234673	98	0	0
610	0	423698	241	0	0
620	0	189439	104	0	0
630	3	481279	290	6.233×10^{-6}	0.765
640	4	307590	201	1.3×10^{-5}	1.493
650	1	243849	158	4.101×10^{-6}	0.788
659	0	291	2	0	0

Table 6.4. The NMFS statistical area (m), total numbers of hooks (H_k), predicted black-footed albatross bycatches ($\widehat{N}_k^*$) and standard errors for four hauls made during an undersampled fishing trip on a medium size vessel in the second quarter of 2001.

Haul	m	H_k	$\widehat{N}_k^*$	$\widehat{SE}\left(\widehat{N}_k^*\right)$
1	640	3820	0.050	0.274
2	640	2865	0.037	0.237
3	640	3820	0.050	0.274
4	640	2865	0.037	0.237

At the vessel level, I use different estimators for different size classes. For the medium vessels that had more than one trip sampled I use the model-based estimator (Eq. 5.73) and variance estimator (Eq. 5.74) with the trip-specific estimates to make vessel-specific estimates. Table 6.5 presents the vessel-specific mortality estimates and corresponding standard error estimates for all medium vessels that had had more than one trip sampled during the second quarter of 2001. The bycatch estimates of the undersampled medium vessels (only one trip observed) are used with Eq. 5.99 to make a conservative variance estimate over all the undersampled medium vessels. The resulting standard error estimate is $\widehat{SE}\left(\widehat{N}_U\right) = 10.71$ and the estimated bycatch estimate is $\widehat{N}_U = \sum_{v=1}^{V_U} \widehat{N}_v = 10.72$. For vessels with 100% coverage, I use Eq. 5.71 and Eq. 5.72 because the total number of trips made by those vessels is known. Furthermore, the number of trips observed is equal to the number of trips made by the vessel and the bycatch estimator reduces to the sum of the trip estimates and the variance estimator reduces to the sum of the trip variance estimates and the sum of the estimates of the covariance between any undersampled trips.

As directed by Eq. 5.95 and Eq. 5.100, quarterly and yearly mortality estimates in Table 6.6 are made by simply summing estimates across vessels . However, the associated variance estimates made using Eq. 5.96 and Eq. 5.101 must account for any covariance of predictions for trips that are undersampled. When the models used for predictions are quarter-specific, there is covariance between trips within a quarter, but not between quarters. That is, the covariance of quarterly estimates, $Cov\left(\widehat{\Theta}_q, \widehat{\Theta}_r\right)$ is zero for all $q \neq r$.

Over the years 1993 to 2003, bycatch of black-footed albatross in the Gulf of Alaska

Table 6.5. Number of observed trips (c_v), estimates of black-footed albatross bycatch ($\widehat{N}_v$) and standard errors for each vessel in the medium size class with more than one trip sampled during the second quarter of 2001.

v	c_v	$\widehat{N}_v$	$\widehat{SE}\left(\widehat{N}_v\right)$
1	3	0	0
2	12	0	0
3	5	0.119	0.454
4	4	0	0
5	5	0	0
6	5	0	0
7	2	23.453	28.679
8	2	0	0
9	2	0	0
10	2	0	0
11	2	0	0
12	2	0.004	0.071
13	2	0	0
14	2	0	0
15	2	9.746	11.906
16	2	0	0
17	2	0	0
18	2	0	0
19	2	0	0
20	3	0	0
21	2	0	0
22	2	10.000	12.780
23	2	0	0
24	2	0	0
25	2	0	0
26	2	0	0
27	6	0	0
28	2	0	0

Table 6.6. Estimates of black-footed albatross bycatch ($\widehat{N}_q$) and standard errors for each quarter and the corresponding estimates for the entire year (2001) in the bottom row.

Quarter	$\widehat{N}_q$	$\widehat{SE}\left(\widehat{N}_q\right)$
1	18.09	18.61
2	68.03	35.74
3	16.95	11.87
4	0.00	0.00
Year	103.07	42.00

(GOA) are on average about eight times those in the Bering Sea/Aleutian Islands (BSAI), but the number of mortalities varies substantially over time and estimated precision is low (Table 6.7). For yearly estimates without regard to regions, there is some variability in standard errors for years where similar numbers of albatross mortalities are estimated, but there appears to be a relationship of precision to the numbers of albatross mortalities (Figure 6.1). The latter variability is in large part due to whether or not mortalities are observed aboard vessels requiring 100% coverage.

Yearly estimates of bycatch rate and corresponding variance are also calculated for the BSAI and GOA, separately, using Eq. 5.103 and Eq. 5.104. These estimators use the estimates of total bycatch (Table 6.7), estimates for number of deployed hooks (Table 6.8) and estimates of covariance between deployed hooks and mortality estimates which are obtained using Eq. 5.105. The estimated number of deployed hooks has increased dramatically over the last 10 years in the BSAI, but there has been no corresponding trend of increased bycatch in the BSAI over the same period (Table 6.7). This lack of increase in bycatch is reflected in the decreasing bycatch rate in the BSAI (Figure 6.2), but the most striking pattern is the drastically higher bycatch rates in the GOA. Indeed, the yearly bycatch rates are so much larger in the GOA that they correspond to the yearly total bycatch mortality (Figure 6.1) over the 11 year period despite the more extensive fishing effort in the BSAI.

Differences in yearly estimates of black-footed albatross by longliners reported

Table 6.7. Regional estimates of black-footed albatross bycatch ($\widehat{N}$) and standard errors in the longline fleet for years 1993 to 2003.

Year	BSAI		GOA	
	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$
1993	33.15	18.61	40.86	17.36
1994	36.75	13.14	12.05	9.60
1995	72.59	18.37	285.28	54.79
1996	19.77	9.67	755.31	138.37
1997	6.61	3.80	139.14	57.62
1998	7.87	5.16	191.62	85.79
1999	32.60	21.64	279.79	138.79
2000	34.95	23.37	239.27	142.78
2001	3.89	3.20	99.32	41.88
2002	0.00	0.00	54.40	44.84
2003	25.44	21.24	110.04	41.57

Table 6.8. Yearly estimates of numbers of deployed hooks ($\times 10^6$, $\widehat{H}$) and standard errors for the Bering Sea/Aleutian Islands (BSAI) and the Gulf of Alaska (GOA).

Year	BSAI		GOA	
	$\widehat{H}$	$\widehat{SE}\left(\widehat{H}\right)$	$\widehat{H}$	$\widehat{SE}\left(\widehat{H}\right)$
1993	146.548	7.336	38.719	3.963
1994	181.111	8.818	14.695	2.764
1995	180.851	7.814	40.173	3.899
1996	186.900	9.109	29.527	2.954
1997	207.982	6.787	24.469	2.688
1998	205.865	7.639	22.164	2.574
1999	189.993	7.556	21.180	2.206
2000	222.488	8.424	26.165	2.625
2001	254.662	8.969	30.486	3.777
2002	248.433	8.683	34.126	3.614
2003	310.098	10.218	31.163	3.104

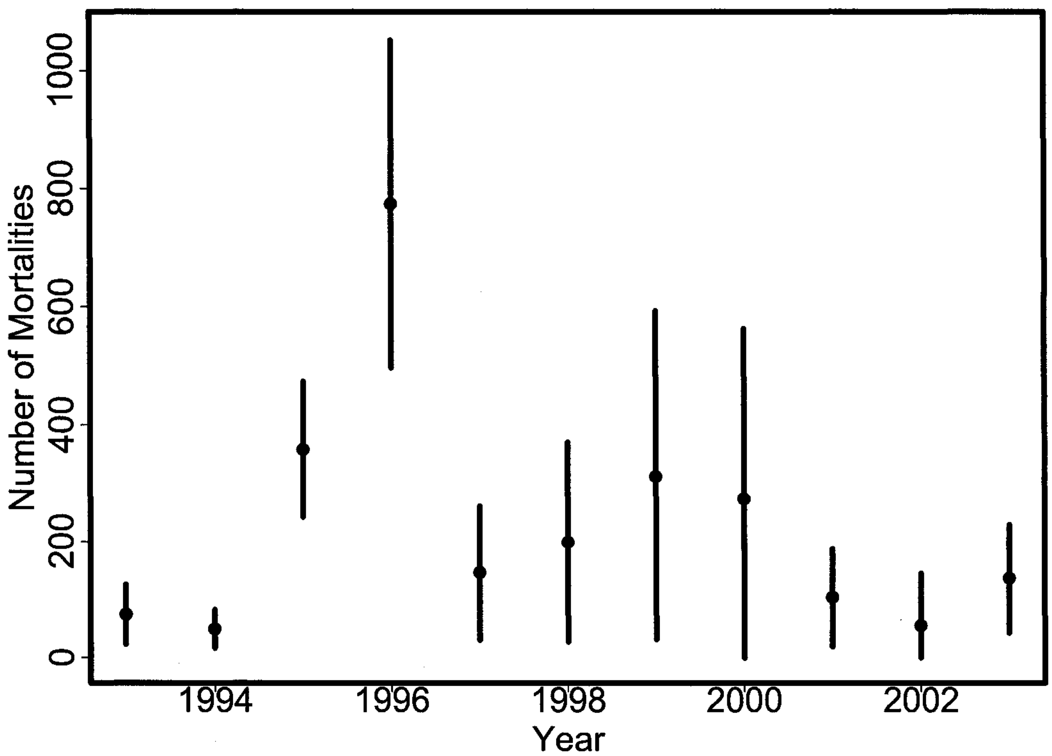

Figure 6.1. Yearly estimates of black-footed albatross bycatch due to fishing effort subject to NPGOP coverage. Vertical bars represent asymptotic 95% confidence intervals.

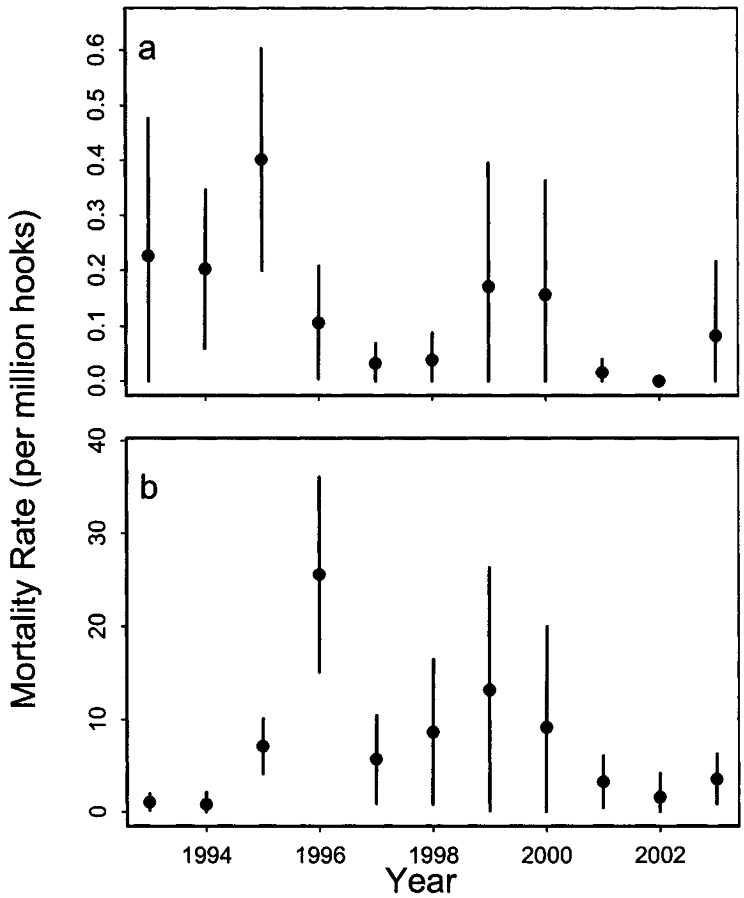

Figure 6.2. Estimates of black-footed albatross bycatch rates in the (a) Bering Sea and Aleutian Islands and (b) Gulf of Alaska due to fishing effort subject to NPGOP coverage from 1993 to 2003. Vertical bars represent asymptotic 95% confidence intervals.

by NPFMC (2003) and those based on my estimators are not detectable in either the BSAI or GOA regions because of poor precision of the estimates (Figure 6.3). However, point estimates that I calculated were generally greater than those reported in NPFMC (2003) for either region and my standard error estimates are on average $\sim$ 34% and $\sim$ 54% greater for the BSAI and GOA, respectively. The higher point estimates produced here could be obtained if there is a subtantial amount of black-footed albatross mortalities by vessels in the medium size class and coverage is on average greater than the assumed 30%. On the other hand, the estimates reported in NPFMC (2003) could be negatively biased if fishery catch estimates produced by the Alaska Regional Office are biased because this information is used by authors of NPFMC (2003) to produce their seabird bycatch estimates. In fact, there is no reason to presume that the estimation approach used in NPFMC (2003) would target the same values since information other than observer data (e.g., catch estimates reported by vessels) are employed. My estimates would be expected to have poorer precision because the uncertainty arising from various levels of sampling is accounted for whereas bycatch estimates reported in NPFMC (2003) treat total catches for various fisheries as known in a ratio estimation context.

6.2.2 Northern Fulmars in Trawls

To show how seabird bycatch is estimated for trawl gear I start with a set of hauls made during a trip in a fashion similar to the black-footed albatross example. Bycatch of black-footed albatross has not occurred on trawl or pot vessels, but other birds such as northern fulmars have. Therefore, I focus on northern fulmars for detailed description of estimation.

In the first quarter of 2001, all observed northern fulmar bycatch were in the BSAI aboard vessels with 100% observer coverage. During one trip, a large vessel made 22 hauls and fulmar mortalities were observed for three of those hauls (Table 6.9). When haul-specific estimates are desired, they can be accomplished with estimators provided in Section 3.6.2, but the ultimate goal here is to estimate yearly totals.

A little caution is needed in determining the appropriate estimator for the trip total. If we are interested in estimating the total regardless of region, then assuming model Eq. 3.33 is appropriate. However, here I focus on estimation for the two regions for which yearly black-footed albatross estimates I have presented above. Thus, I

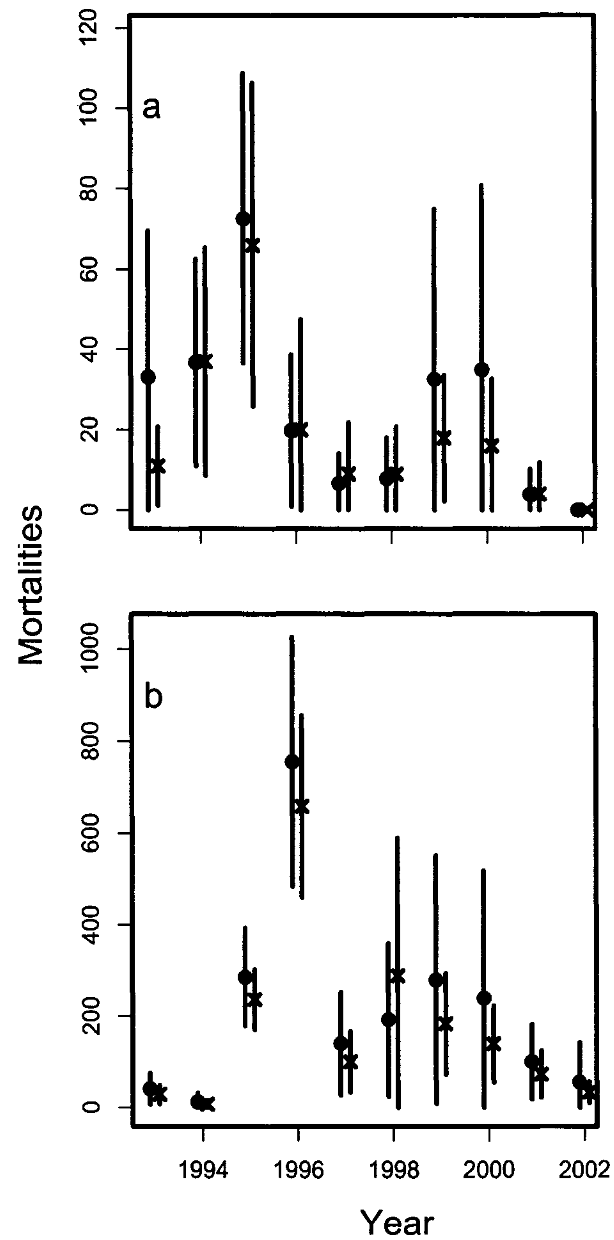

Figure 6.3. Comparison of estimated black-footed albatross bycatch due to longline effort (indicated by closed circle) in the (a) Bering Sea and Aleutian Islands and (b) Gulf of Alaska to those reported by NPFMC (2003) which are denoted by "x." Vertical bars represent asymptotic 95% confidence intervals.

Table 6.9. Total weight of each haul in kg (Υ_k), weight (kg) of sampled portion (υ_k) and number of observed northern fulmars (n_k) for a trip aboard a large size class vessel fishing trawl gear in the first quarter of 2001. All 22 hauls were sampled.

Haul	Υ_k	υ_k	n_k
1	88,530	39,999.00	0
2	5,290	5,290.00	0
3	5,940	5,940.00	1
4	65,110	23,825.00	0
5	24,570	10,354.00	0
6	11,440	5,449.00	0
7	3,360	3,360.00	1
8	38,470	15,133.00	0
9	72,250	32,142.00	1
10	177,330	37,276.00	0
11	106,450	37,090.00	0
12	9,090	639.83	0
13	76,990	31,913.00	0
14	83,700	42,080.00	0
15	100,070	46,326.00	0
16	80,400	26,041.00	0
17	81,690	44,947.00	0
18	30,320	14,676.00	0
19	107,790	48,226.00	0
20	73,390	30,445.00	0
21	64,350	34,947.00	0
22	85,810	41,936.00	0

resort to model Eq. 5.37 for domain-specific estimation. Note that there is only a difference in these two models in the rare event that hauls are made in more than one region during a given trip. The estimate of the rate of capture is $\widehat{\phi}_{n,t} = 6.67 \times 10^{-5}$ and the estimate of the trip total bycatch is $\widehat{N}_t = 92.88$ with corresponding variance estimate $\widehat{SE}\left(\widehat{N}_t\right) = 67.38$.

During the course of the first quarter of 2001, the vessel made 8 trips (defined by my criterion for trip distinction) with varying numbers of hauls and northern fulmar bycatch was observed for one of the other trips, but one trip was defined with only one haul and it required model-based methods for predicting the haul bycatch and variance because more than one haul is required for variance estimation (Table 6.10). The number of northern fulmars for the eighth trip is predicted using the estimators based on the assumed model in Eq. 5.65 which require the management region-specific total weights for the trip and the model parameter estimates in Table 6.11. Vessel estimates are obtained using the same estimators that were used for black-footed albatross on longline vessels as are estimates for quarters, years, etc.

Table 6.10. Number of hauls made (G_t) and observed (g_t) for each of eight trips made by a vessel in the first quarter of 2001 along with estimates of northern fulmar bycatch ($\widehat{N}_t$) and corresponding variance estimates.

Trip	G_t	g_t	$\widehat{N}_t$	$\widehat{SE}\left(\widehat{N}_t\right)$
1	46	46	4.20	2.16
2	22	22	7.23	3.36
3	13	13	0.00	0.00
4	4	4	0.00	0.00
5	22	22	0.00	0.00
6	7	7	0.00	0.00
7	20	20	0.00	0.00
8	1	1	0.01	0.10

Compared to longliners, yearly bycatch of northern fulmars by trawlers and pots is minimal and unlike black-footed albatross bycatch of fulmars is far higher in the BSAI than GOA (Table 6.12). However, the average yearly bycatch of fulmars is far higher than black-footed albatross ($\sim$ 8003 and $\sim$ 513 for the BSAI and GOA, respectively). Like black-footed albatross the yearly bycatch of fulmars is variable

Table 6.11. The number of northern fulmars mortalities in observer samples (n_m), total sampled weight in kg (v_m), sum of squared sampled weights ($\sum v_k^2$) and estimated parameters of the bycatch-rate models for each NMFS statistical area (m) in 2001.

m	n_m	v_m	$\sum v_k^2$	$\widehat{\phi}_{n,m}$	$\widehat{\sigma}^2_{n,m}$
509	11	53,675,098.06	4.35×10^{12}	2.05×10^{-7}	3.17×10^{-7}
513	7	58,381,072.26	4.33×10^{12}	1.2×10^{-7}	1.21×10^{-7}
516	0	30,822,739.58	2.61×10^{12}	0	0
517	3	54,353,706.30	6.39×10^{12}	5.52×10^{-8}	5.55×10^{-8}
518	0	1,636.22	2.66×10^{5}	0	0
519	2	9,053,564.42	4.18×10^{11}	2.21×10^{-7}	2.23×10^{-7}
521	8	41,028,593.12	3.65×10^{12}	1.95×10^{-7}	1.97×10^{-7}
523	0	146.20	2.14×10^{4}	0	0
524	0	536.27	1.57×10^{5}	0	0
541	4	2,040,752.75	5.04×10^{10}	1.96×10^{-6}	2.03×10^{-6}
542	0	630,075.06	1.76×10^{10}	0	0
543	0	171,232.42	1.51×10^{10}	0	0
610	0	715,462.46	3.63×10^{10}	0	0
620	0	3,587,739.18	1.45×10^{11}	0	0
630	0	2,811,266.23	9.54×10^{10}	0	0
649	0	0	0	0	0

from year to year in either region and precision is generally poor. Although bycatch of fulmars is relatively low among trawlers the estimates are noticably higher over the last five years compared to the previous five years.

6.2.3 Yearly Bycatch of Laysan Albatross in Longlines

Like northern fulmars and unlike black-footed albatross, laysan albatross exhibit higher bycatch in the BSAI than the GOA on average (Table 6.13). Like other seabird species discussed above, bycatch of laysan albatross is variable from year to year, but overall laysan albatross bycatch is higher than black-footed albatross on average ($\sim$ 739 laysan compared to $\sim$ 226 black-footed). There has been some bycatch of laysan albatross by trawl vessels in the BSAI in recent years, but none have been observed in the GOA.

6.3 Regional Total Catch Weight

In this section, I explain how estimates of total catch weight for a given species are obtained using Pacific cod as an example. In the Sections 6.3.1 and 6.3.2 I walk the reader through estimation for longline vessels and trawl/pot vessels, respectively. Trawl and pot vessels are treated together because estimation of catch weight is performed virtually identically. In Section 6.3.3, I give and discuss yearly estimates of catch weight for Pacific cod and walleye pollock.

6.3.1 Longline

To illustrate estimation of total catch weight for longline vessels, I use Pacific cod as an example. I begin with a trip made by a medium vessel in the GOA during the first quarter of 2001 (Table 6.14). For the 6 hauls that were sampled by the observer, the information in Table 6.14 is used to calculate haul-specific total weight estimates with Eq. 2.20 and corresponding variance estimates with Eq. 2.21. As with other catch parameters, the estimator to use for the trip total weight depends on whether or not we want a region/time period-specific estimate. When we are unconcerned with regions or time periods the appropriate estimator for the total weight is Eq. 5.16 and when we do want to estimate for a particular region/ time period Eq. 5.45 is apppropriate. Ultimately, regional estimates of total catch weight are of interest

Table 6.12. Yearly estimates of northern fulmar bycatch and standard errors from 1993-2003 in the Bering Sea/Aleutian Islands and Gulf of Alaska by gear type (longline and trawl).

Year	Bering Sea/Aleutian Is.		Gulf of Alaska	
	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$
longline				
1993	4,182.74	346.92	849.43	122.85
1994	5,561.18	409.95	238.43	46.50
1995	10,630.98	553.74	602.99	109.16
1996	6,067.09	419.80	668.29	133.99
1997	13,070.66	565.70	400.42	110.80
1998	15,597.79	565.00	943.57	268.38
1999	8,494.20	580.52	299.50	102.93
2000	11,863.74	847.29	278.71	76.89
2001	5,915.89	646.31	241.55	79.22
2002	793.67	85.20	218.36	102.10
2003	3,455.59	239.33	139.99	57.03
trawl				
1993	0	0	0	0
1994	50.52	37.67	0	0
1995	27.14	9.94	0	0
1996	30.01	10.32	0	0
1997	6.47	2.97	43.73	42.14
1998	112.80	74.78	111.29	110.60
1999	493.39	115.33	0	0
2000	428.76	171.45	8.55	10.47
2001	254.00	20.84	36.23	28.42
2002	135.65	41.71	336.76	191.51
2003	597.56	351.99	192.39	135.18
pot				
1993	0	0	0	0
1994	0	0	0	0
1995	19.12	14.19	0	0
1996	22.13	12.95	0	0
1997	24.41	16.77	0	0
1998	0	0	11.48	10.75
1999	106.46	24.24	9.33	6.88
2000	0	0	0	0
2001	12.15	12.00	15.46	15.46
2002	60.01	60.01	0	0
2003	68.11	35.54	0	0

Table 6.13. Yearly estimates of laysan albatross bycatch and standard errors from 1993-2003 in the Bering Sea/Aleutian Islands and Gulf of Alaska by gear type.

Year	**Bering Sea/Aleutian Is.**		**Gulf of Alaska**	
	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$
longline				
1993	607.41	81.45	188.49	58.82
1994	368.78	70.37	125.21	17.34
1995	656.05	138.46	119.08	72.28
1996	345.16	81.89	341.19	141.65
1997	389.21	51.84	24.40	14.03
1998	1,473.17	136.66	215.16	101.61
1999	800.03	158.54	402.70	120.00
2000	525.29	70.99	89.99	34.53
2001	613.54	174.19	84.83	45.81
2002	110.74	56.71	0	0
2003	157.60	41.11	41.84	37.71
trawl				
1993	0	0	0	0
1994	0	0	0	0
1995	0	0	0	0
1996	0	0	0	0
1997	6.35	5.87	0	0
1998	347.97	270.92	0	0
1999	9.20	2.09	0	0
2000	0	0	0	0
2001	10.04	4.97	0	0
2002	1.67	1.13	0	0
2003	80.17	79.20	0	0

so, I use Eq. 5.45 and the variance is estimated using Eq. 5.46. The resulting GOA-specific estimate of catch weight for this trip is $\widehat{W}_t = 25,601.61$ and the standard error is $\widehat{SE}\left(\widehat{W}_t\right) = 4623.89$.

Table 6.14. Total number of deployed hooks (H_k), number of observed hooks (h_k), number of Pacific cod counted during the tally period (n_k), extrapolated tally weight (w_k), average sample weight ($\overline{w}_k$) and the number of fish in the weight sample ($n_{w,k}$) for each sampled haul during an observed trip aboard a medium size-class, longline vessel. The trip occurred during the first quarter of 2001 and weight is measured in kilograms.

haul	H_k	h_k	n_k	w_k	$\overline{w}_k$	$n_{w,k}$
1	2,210	910	211	626.67	2.97	50
2	3,380	1170	89	266.11	2.99	50
3	2,340					
4	3,640					
5	2,080	650	95	351.69	3.70	50
6	2,210					
7	2,210					
8	2,600					
9	2,470					
10	2,340					
11	1,950					
12	2,080	650	293	1042.49	3.56	50
13	3,120	910	117	361.76	3.09	50
14	390					
15	1,950					
16	2,860					
17	2,340	780	122	390.64	3.20	50

The vessel made two trips during the first quarter and the second trip had a higher proportion of the hauls sampled which led to smaller variance estimate although the catch weight is much higher (Table 6.15). Analogous to seabird bycatch on longliners, estimation of the total catch made on all trips for a given vessel in the GOA whether observed or not is performed using Eq. 5.73 and the variance is estimated using Eq. 5.74. Estimation of quarterly and yearly total catch weights also follows that discussed for seabird bycatch on longliners.

Table 6.15. Number of hauls made (G_t) and observed (g_t) for the two observed trips on a medium-size class, longline vessel in the first quarter of 2001 with estimates of Pacific cod catch weight in kilograms ($\widehat{W}_t$) and corresponding standard errors.

trip	G_t	g_t	$\widehat{W}_t$	$\widehat{SE}\left(\widehat{W}_t\right)$
1	17	6	25,601.61	4,623.89
2	27	17	48,206.09	3,955.47

Only two trips during the first quarter were undersampled and to show how the model-based estimation is perfomed for undersampled trips we shift focus to one of those trips which, by my criterion for trip distinction, has only one haul. This haul was made in the GOA and sampled by the observer, but could just as well have not been sampled. In either case, variance estimation is not possible because an insufficient number of weight samples is available. In my description of seabird bycatch estimation, I use models that had parameters specific to each NMFS statistical area in the GOA and BSAI. However, here I assume models that are more general which are specific to the Bering Sea (BS), Aleutian Islands (AI) and the GOA. The total weight of the haul (and trip) is predicted using the total number of deployed hooks in the region (in this case $H_m = 6,870$ for the GOA) with estimates of the probability of capture for the region (Table 6.16) and the average weight of captured fish for the region (Table 6.17) in Eq. 5.54 which yields $\widehat{W}^*_{mt} = 3,969.45$ and the variance of the prediction is estimated using Eq. 5.55 which yields $\widehat{SE}\left(\widehat{W}^*_{mt}\right) = 1521.75$.

Table 6.16. Numbers of Pacific cod in sampled longline catches (n_m), numbers of observed hooks (h_m), numbers of observed trips (g_m) and estimated probabilites of capture ($\widehat{\pi}_m$) and dispersion parameter ($\widehat{\tau}_m$) for each management region (m): Aleutian Islands (AI), Bering Sea (BS) and Gulf of Alaska (GOA).

m	n_m	h_m	g_m	$\widehat{\pi}_m$	$\widehat{\tau}_m$
AI	450,946	3,485,809	1183	0.13	99.75
BS	1,833,898	10,363,902	2228	0.18	183.44
GOA	123,247	746,850	310	0.17	185.19

Table 6.17. Weight of Pacific cod in observer samples from longline catches (w_m), numbers of Pacific cod weighed ($n_{w,m}$) and estimated mean weight parameter ($\widehat{\mu}_{w,m}$) and variance parameter ($\widehat{\sigma}_{w,m}$) for each management region (m): Aleutian Islands (AI), Bering Sea (BS) and Gulf of Alaska (GOA).

m	w_m	$n_{w,m}$	$\widehat{\mu}_{w,m}$	$\widehat{\sigma}^2_{w,m}$
AI	286,714.93	56,915	5.04	169.44
BS	364,335.42	110,795	3.29	31.38
GOA	40,828.81	11,661	3.50	119.52

6.3.2 Trawls and Pots

For trawl and pot vessels, estimation of total catch weight is similar to estimation of total numbers and follow analogous methods. Therefore, the description here of estimating total catch weight for trawl and pot vessels follows closely that of estimating total seabird mortalities aboard trawl vessels in Section 6.2. Again, I use Pacific cod as an example and I begin with an observed trip made by a medium vessel in the GOA where there were 4 hauls made and all 4 were sampled by the observer (Table 6.18). Under the model-based approach, estimation of total catch for each haul is possible, but is not necessary for estimation of total catch for higher levels. As for seabird bycatch, I use the domain-specific estimation approach based on the model in Eq. 5.37 because of interest in regional estimates. The estimate of total catch in the GOA for the trip in Table 6.18 and associated variance estimate are $\widehat{W}_t = 1,835.62$ and $\widehat{SE}\left(\widehat{W}_t\right) = 1,198.76$, respectively.

Table 6.18. Total weight of each haul (Υ_k), weight of sampled portion (υ_k) and weight of sampled Pacific cod (w_k) for an observed trip aboard a medium size-class vessel fishing trawl gear in the first quarter of 2001. All 4 hauls were sampled.

haul	Υ_k	υ_k	w_k
1	38,680.00	524.56	5.22
2	36,850.00	280.17	0
3	143.16	143.16	13.22
4	22,250.00	280.67	4.59

The vessel made nine trips during the first quarter of 2001 and eight of them were sufficiently sampled (Table 6.19), but one was not. In the undersampled trip, four hauls were made and sampled, but only one occurred in the GOA (the others were in the BS). Thus, variance cannot be estimated for this trip under the model in Eq. 5.37. Instead, we use the information in Table 6.20 with the known weight of the GOA-specific catch ($\Upsilon_{mt} = 16,950$) in the predictor based on the closely related model in Eq. 5.65 to estimate the weight of Pacific cod caught in the trip. The resulting estimate and variance estimate for the undersampled trip are $\widehat{W}^*_{mt} = 6347.26$ and $\widehat{SE}\left(\widehat{W}^*_{mt}\right) = 1,019.22$.

Table 6.19. Number of hauls made (G_t) and observed (g_t) for eight sufficiently sampled observed trips of a medium-size class, trawl vessel in the first quarter of 2001 with estimates of Pacific cod catch weight in kilograms ($\widehat{W}_t$) and corresponding standard errors.

trip	G_t	g_t	$\widehat{W}_t$	$\widehat{SE}\left(\widehat{W}_t\right)$
1	4	4	1,835.62	1,198.76
2	3	3	0	0
3	5	3	0	0
4	3	3	0	0
5	3	3	0	0
6	3	3	0	0
7	4	4	0	0
8	2	2	0	0

Table 6.20. The sample weight of Pacific cod (w_m), total sampled weight (v_m), sum of squared sampled weights ($\sum v_k^2$) and estimated parameters of the catch-rate models for each region.

m	w_m	v_m	$\sum v_k^2$	$\widehat{\phi}_{w,m}$	$\widehat{\sigma}^2_{w,m}$
AI	155,832.75	311,378.0	1.21×10^8	0.5005	76.31
BS	484,784.73	80,670,100.4	2.66×10^{12}	0.0060	1.20
GOA	78,674.23	205,228.8	6.94×10^7	0.3833	58.06

Estimation of catches at the vessel level and higher is identical across gear types and proceeds in a fashion analogous to the description for seabirds bycatch. Furthermore, estimation of total catch for pot vessels follows the methods for trawl vessels, but the numbers of pots fished and sampled are the fundamental information (covariate) rather than the weight of the total catch and and the weight of the sampled portion.

6.3.3 Yearly Catch Estimates

6.3.3.1 Pacific Cod

Between 1993 and 2003, yearly estimates of total catch weight for Pacific cod are higher in the BS than the AI and GOA whether longline, trawl or pot gear are considered (approximately 66 – 86% of total catch on average in the BS depending on gear type; Table 6.21). In the GOA, trawlers catch substantially more Pacific cod by weight than longliners or pot vessels. In the BS, catch attributable to trawlers and longliners were similar historically, but recently catch by longliners has markedly exceeded that of trawlers. For longliners, the (scaled) precision of catch weight estimates in the GOA is much poorer on average ($\widehat{CV}\left(\widehat{W}\right) = 33.2\%$) than those of the BS and AI ($\widehat{CV}\left(\widehat{W}\right) = 4.2\%$ and $\widehat{CV}\left(\widehat{W}\right) = 10.2\%$, respectively). For trawlers, the precision of catch weight estimates in the GOA was also poorer ($\widehat{CV}\left(\widehat{W}\right) = 10.4\%$) than those in the BS and AI, but the difference was less dramatic ($\widehat{CV}\left(\widehat{W}\right) = 5.8\%$ and $\widehat{CV}\left(\widehat{W}\right) = 4.6\%$, respectively). Differences in precision regionally is most likely due to differences in regional proportions of effort for 30% and 100% coverage vessels, but targeting of many different fish species or occurrence of high species heterogeneity in catches will also lower precision.

Yearly total catch estimates for Pacific cod are also produced by the Alaska Regional Office of the National Marine Fisheries Service via the "blend-system" and used in stock assessments (Thompson and Dorn 2004; Thompson et al. 2004). Unlike my catch weight estimates, the blend-based estimates have no corresponding estimates of uncertainty. For the AI region, the reported point estimates are usually within corresponding constructed asymptotic confidence intervals based on my estimators, but the estimate for 1994 is lower and that for 2000 is higher (Figure 6.4). There is no consistent positive or negative difference between the point estimates and the

Table 6.21. Yearly estimates of Pacific cod total catch weight (metric tonnes) and standard errors by gear type (longline, trawl and pots) and region (Aleutian Islands, Bering Sea and Gulf of Alaska) from 1993 to 2003.

Year	Aleutian Is.		Bering Sea		Gulf of Alaska	
	$\widehat{W}$	$\widehat{SE}\left(\widehat{W}\right)$	$\widehat{W}$	$\widehat{SE}\left(\widehat{W}\right)$	$\widehat{W}$	$\widehat{SE}\left(\widehat{W}\right)$
longline						
1993	16,877.99	937.93	63,285.81	4,167.11	5,934.87	2,018.85
1994	8,101.64	898.61	102,562.11	5,471.01	4,408.40	1,878.00
1995	6,132.76	994.97	116,534.81	5,173.99	11,299.32	2,976.26
1996	8,765.02	1,886.20	108,727.58	5,551.54	4,557.63	1,631.04
1997	9,184.66	959.12	128,019.04	3,889.86	3,648.18	1,399.58
1998	15,977.07	1,029.77	93,196.97	3,343.72	1,919.21	1,232.78
1999	8,787.56	1,352.58	92,723.94	3,836.24	3,593.49	907.21
2000	13,583.27	135.30	91,146.46	3,415.70	5,397.94	1,551.95
2001	17,984.07	658.74	96,423.46	3,690.05	6,889.39	1,894.18
2002	3,342.78	553.79	109,461.92	3,931.75	11,320.52	2,359.25
2003	922.27	39.90	122,274.35	3,981.79	6,618.90	1,408.14
trawl						
1993	18,024.79	570.41	86,127.29	2,873.71	24,527.73	2,052.43
1994	16,593.41	303.39	144,696.32	6,268.14	15,815.86	2,573.90
1995	11,635.75	470.65	137,931.81	6,379.46	33,467.15	3,046.44
1996	22,108.88	616.58	121,457.61	4,342.08	30,581.69	2,537.76
1997	16,427.17	374.05	130,398.45	5,469.64	31,547.71	2,241.57
1998	20,648.31	248.82	78,075.76	4,269.85	30,954.21	2,631.59
1999	16,005.10	274.58	65,805.35	4,688.05	24,406.62	2,420.66
2000	19,305.61	1,124.62	66,962.62	4,112.45	14,719.76	1,898.80
2001	17,450.13	1,793.74	45,343.02	3,757.61	20,306.43	1,880.01
2002	28,023.50	2,451.19	64,586.47	4,822.15	16,955.49	2,084.71
2003	27,937.93	2,240.55	61,573.45	5,941.19	20,616.44	2,482.78
pot						
1993	0	0	2,161.56	518.90	3,843.39	477.64
1994	3.34	0.48	9,432.85	783.65	2,827.62	583.51
1995	1,602.58	479.25	15,872.29	1,397.38	6,389.25	825.24
1996	3,496.75	555.70	24,668.25	2,098.76	3,511.53	717.42
1997	515.25	142.63	20,609.30	2,475.95	1,987.35	546.21
1998	400.72	186.14	9,102.92	991.77	8,129.09	3,842.47
1999	3,342.32	655.49	8,715.01	871.39	5,409.83	1,079.18
2000	998.45	298.56	8,600.78	2,148.16	5,854.19	848.50
2001	1,158.46	757.67	13,562.15	1,935.38	2,979.11	1,017.59
2002	0.01	0.01	13,516.04	2,027.36	3,795.91	896.67
2003	7.00	5.51	19,511.65	2,725.53	4,114.94	874.51

reported values.

For the BS, many of the yearly blend-based catches reported in Thompson and Dorn (2004) are lower than the estimates derived from my estimators (Figure 6.5). The reported catch estimate for 1997 was higher than the corresponding estimate based on my estimator. The largest difference between yearly estimates was observed in 1994. The average yearly bias was approximately 10%.

In the GOA, blend-based catches reported by Thompson et al. (2004) are all greater than the corresponding estimates based on my estimators (Figure 6.6). The direction of bias for the GOA estimates is opposite that of the BS estimates, but the blend-based catch estimates in the GOA are on average nearly twice as high as mine (approximately 192%).

The discrepancy between estimates obtained from my estimators and the blend-system could arise from various causes. Some of the potential causes are similar to those I proposed in the seabird bycatch context. First, it is doubtful that my estimators and those used to produce the reported values would be estimating the same target. The values reported in Thompson and Dorn (2004) are obtained from a blend-system at the Alaska Regional Office which integrates data collected by observers on vessels and in shoreside processing plants with industry-provided data in a complex way. The quality of the industry-provided information is unknown, but because industry and observer information are integrated, the expected value of the blend-based estimates is unlikely to be the same as those made using my estimators (only observer data).

The consistent differences exhibited for the GOA and BS are of particular concern. The differences are opposite for the two regions and the discrepency may in part be due to allocation of information to different regions. If vessels incorrectly report catches to different regions when they are offloaded, then this could partly explain the differences seen here.

When observer coverage for the medium size class vessels is different from the assumed 30% on average, then bias would be introduced into my estimates. An incorrect observer coverage assumption as the sole reason for discrepancey implies that observer coverage in the BS would actually be higher than 30% and those for the GOA would actually be lower. However, an incorrect observer coverage assumption cannot be single factor because for some regions both positive and negative bias is

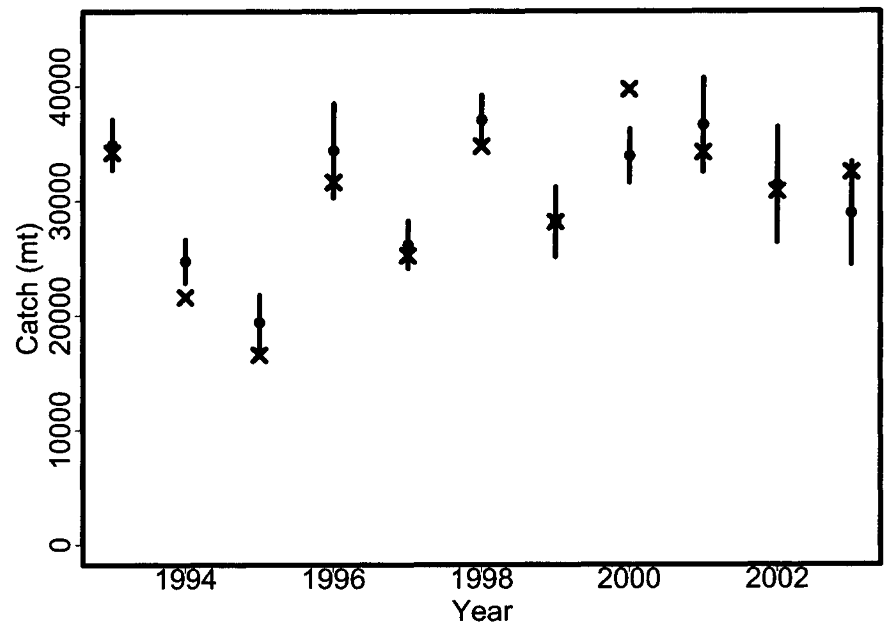

Figure 6.4. Yearly estimates of Pacific cod total catch weight from 1993 to 2003 in the Aleutian Islands (closed circle) compared with the corresponding values reported in Thompson and Dorn (2004) which are denoted by "x." The vertical lines represent asymptotic 95% confidence intervals.

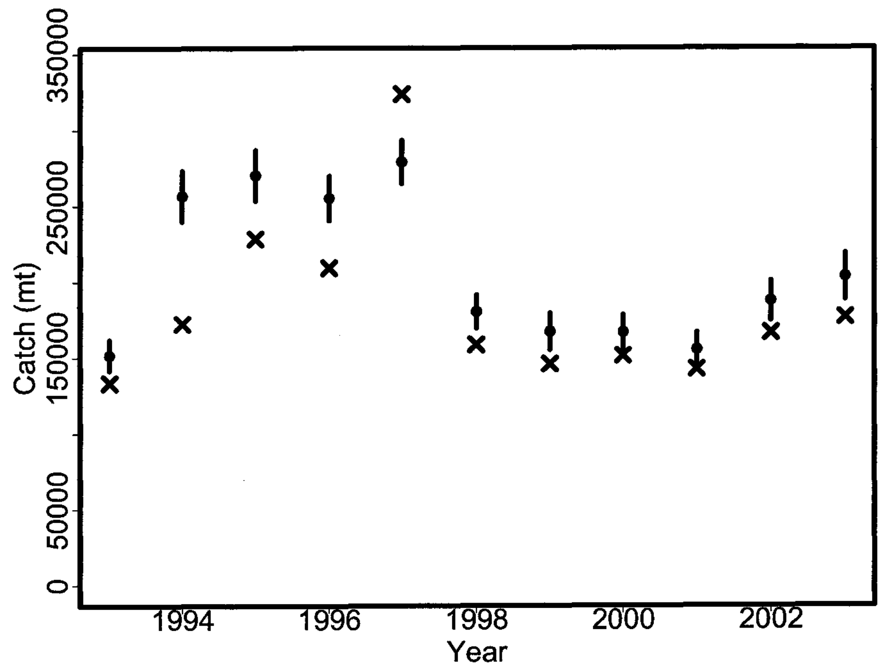

Figure 6.5. Yearly estimates of Pacific cod total catch weight from 1993 to 2003 in the Bering Sea (closed circle) compared with the corresponding values reported in Thompson and Dorn (2004) which are denoted by "x." The vertical lines represent asymptotic 95% confidence intervals.

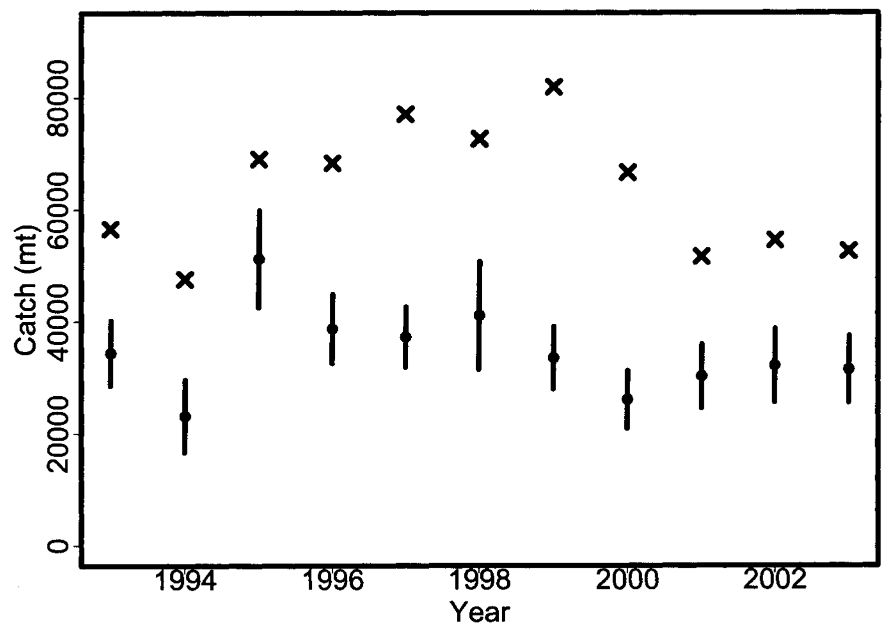

Figure 6.6. Yearly estimates of Pacific cod total catch weight from 1993 to 2003 in the Gulf of Alaska (closed circle) compared with the corresponding values reported in Thompson et al. (2004) which are denoted by "x." The vertical lines represent asymptotic 95% confidence intervals.

observed. Another source of discrepancy is the lack of coverage for those vessels less than 60 ft in length. My estimates do not account for this sector of smaller vessels whereas the currently used estimates may.

6.3.3.2 Walleye Pollock

The large majority of walleye pollock catches are made by trawlers regardless of region (Table 6.22). For any given region the average proportion of total walleye pollock catch attributable to longlines is less than 2% and estimates for pot vessels are not reported because they are far less than the longline estimates. The estimates for the eastern Bering Sea (BSE) are far larger than any other region (approximately 75% of total catch is in BSE on average), but the western Bering Sea (BSW) trawl catches have the highest average (scaled) precision ($\widehat{CV}\left(\widehat{W}\right) = 0.4\%$) and those for the GOA had the lowest average precision ($\widehat{CV}\left(\widehat{W}\right) = 9.1\%$). Average precision for the AI and BSE trawl catches are $\widehat{CV}\left(\widehat{W}\right) = 6.4\%$ and $\widehat{CV}\left(\widehat{W}\right) = 1.4\%$, respectively. The low precision of the GOA estimates is largely due to the higher proportion of vessels with 30% coverage that operate in the region, but could also be influenced by targeting of different fish species during fishing trips.

Between 1993 and 2003, yearly estimates of walleye pollock catch in the AI from all gear-type sectors derived from the blend-system and reported in Barbeaux et al. (2004) match my corresponding estimates well (Figure 6.7). Only in 1998 is the reported catch greater than the asymptotic confidence interval for my estimate. Walleye pollock catches were much higher between 1993 and 1995 than those in 1996 to 1998, and catches in 1999 to 2003 are substantially lower than the previous three years (near zero).

For the BSE, blend-system derived catches reported in Ianelli et al. (2004) are usually higher than my corresponding estimates, but the catch estimates for 2003 have the reverse relationship and those of 1993 and 1998 are well aligned (Figure 6.8). The catch estimates for the BSW behave more like those for the AI in that most of the estimates based on my estimators and the blend-system match well (Figure 6.9). However, my estimate for the BSW in 2003 is substantially higher than the corresponding blend-based estimate.

For the GOA, many of the yearly estimates of walleye pollock catches based on the blend-system and reported in Dorn et al. (2004) are similar to the corresponding

Table 6.22. Yearly estimates of walleye pollock total catch weight (metric tonnes) and standard errors by gear type (longline and trawl) and region (Aleutian Islands, eastern and western Bering Sea and Gulf of Alaska) from 1993 to 2003.

Year	Aleutian Is.		Eastern Bering Sea		Western Bering Sea		Gulf of Alaska	
	$\widehat{W}$	$\widehat{SE}\left(\widehat{W}\right)$	$\widehat{W}$	$\widehat{SE}\left(\widehat{W}\right)$	$\widehat{W}$	$\widehat{SE}\left(\widehat{W}\right)$	$\widehat{W}$	$\widehat{SE}\left(\widehat{W}\right)$
longline								
1993	33.61	3.20	348.49	26.78	2,255.18	191.14	16.86	7.52
1994	8.17	0.80	491.45	12.12	3,111.46	410.71	10.28	3.45
1995	31.68	9.23	1,266.55	24.20	2,679.92	261.71	32.19	12.41
1996	10.57	1.99	1,661.57	51.53	2,129.34	253.56	17.94	5.52
1997	15.61	1.51	2,318.40	54.69	2,687.45	185.66	8.58	3.12
1998	21.90	2.02	1,828.44	114.60	2,269.36	130.22	16.21	6.27
1999	14.66	2.80	2,979.47	57.14	1,806.29	115.38	22.98	8.14
2000	47.15	2.59	3,344.60	131.25	2,797.15	243.72	72.63	27.76
2001	82.03	6.29	3,198.03	136.81	4,007.98	249.58	76.16	31.76
2002	8.65	1.74	3,888.49	188.64	3,942.35	270.37	90.75	23.73
2003	10.37	2.07	2,439.97	171.03	5,847.39	199.75	19.39	5.88
trawl								
1993	57,196.72	978.39	1,076,958.14	10,345.52	225,175.70	824.66	106,595.53	6,577.96
1994	61,657.93	2,088.60	1,254,797.50	17,366.26	169,612.60	427.44	95,555.87	8,560.54
1995	67,828.44	1,013.41	1,275,675.03	15,507.75	89,527.18	140.68	90,993.58	5,861.31
1996	31,064.56	1,099.76	1,193,491.24	17,472.29	104,010.50	141.91	70,142.65	6,851.19
1997	25,043.83	1,065.93	874,865.61	11,981.65	302,284.19	255.36	84,980.16	6,368.29
1998	21,939.25	73.18	989,081.61	10,446.00	128,384.28	283.24	118,494.72	6,961.49
1999	594.95	64.03	855,835.99	14,022.11	204,196.09	544.67	84,781.86	6,531.59
2000	744.81	88.03	911,578.43	13,975.30	293,110.82	1,363.68	64,456.54	7,279.60
2001	748.81	103.34	1,038,286.50	16,276.53	425,674.55	4,528.55	43,499.92	5,475.45
2002	703.47	60.12	1,226,224.18	17,897.51	311,263.61	1,099.18	47,293.79	5,828.69
2003	1,627.64	171.99	1,014,412.98	17,860.57	563,234.86	4,901.60	49,075.49	5,527.86

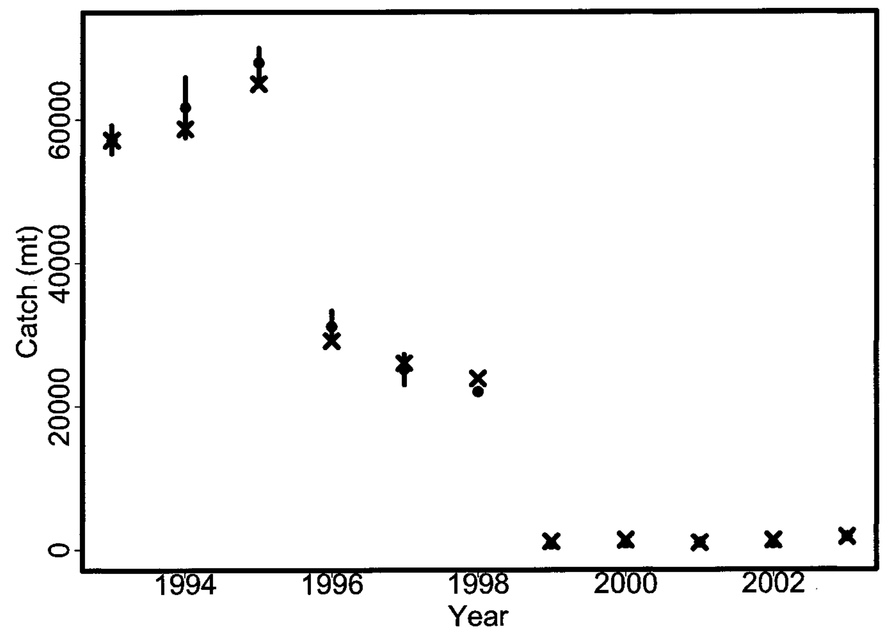

Figure 6.7. Yearly estimates of walleye pollock total catch weight from 1993 to 2003 in the Aleutian Islands (closed circle) compared with the corresponding values reported in Barbeaux et al. (2004) which are denoted by "x." The vertical lines represent asymptotic 95% confidence intervals.

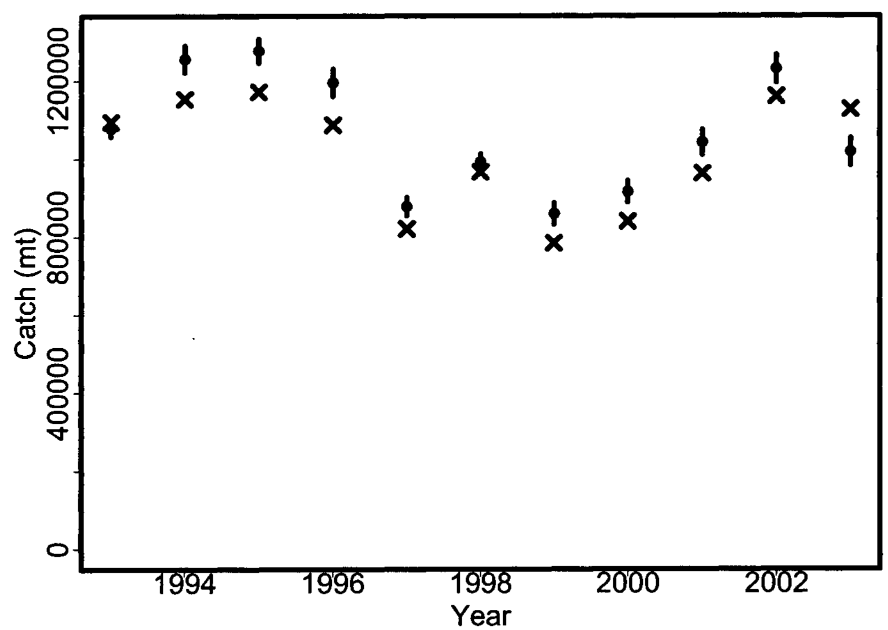

Figure 6.8. Yearly estimates of walleye pollock total catch weight from 1993 to 2003 in the eastern Bering Sea (closed circle) compared with the corresponding values reported in Ianelli et al. (2004) which are denoted by "x." The vertical lines represent asymptotic 95% confidence intervals.

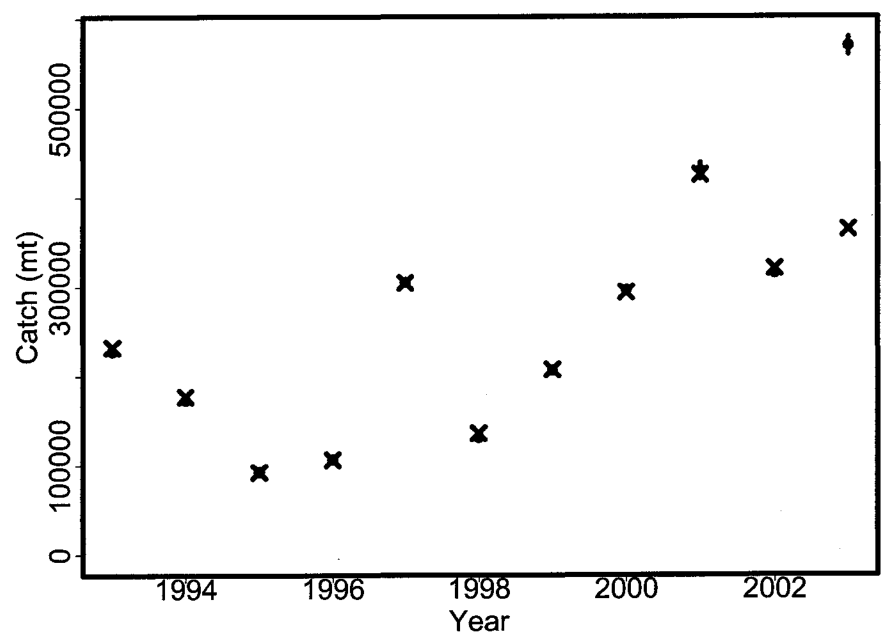

Figure 6.9. Yearly estimates of walleye pollock total catch weight from 1993 to 2003 in the western Bering Sea (closed circle) compared with the corresponding values reported in Ianelli et al. (2004) which are denoted by "x." The vertical lines represent asymptotic 95% confidence intervals.

estimates based on my estimators (Figure 6.10). However, blend-based estimates for 1995 and 1996 are lower than those based on my estimators and the blend-based estimate for 2001 is higher than that based on my estimator.

Because blend-based estimate and those based on my estimators match consistently for AI and BSW whereas those for the BSE show consistent differences and those for the GOA show some inconsistent differences, there is little evidence for a systematic bias in my estimators. In the GOA where there are inconsistent differences in yearly estimates, there is also relatively poor precision for my estimates. If there is also similar (and unmeasured) precision in the blend-based estimates, many of the differences would not be statistically significant by inspection of confidence intervals. In fact, while the point estimates for the BSE show systematic differences, the same lack of significance for those differences would be observed for most of the yearly estimates when precision of blend-based estimates is similar to precision of my estimates.

6.4 Bycatch of Chinook and Sockeye Salmon

Estimating bycatch of salmon is dependent on gear type, but within each gear type the methods are no different than estimation of bycatch of seabirds. Therefore, I do not redescribe the estimation methods.

Between 1993 and 2003, yearly bycatch of Chinook salmon is neglible for longliners (approximately 2.3, 17.6 and 14.0 on average for the AI, BS and GOA, respectively) compared to trawlers (approximately 2807.6, $51,764.7$ and $20,291.1$ on average for the AI, BS and GOA, respectively) regardless of region and most bycatch occurs in the BS (Table 6.23). There is also substantial Chinook salmon bycatch in the GOA, but far less in the AI. Precision of regional total Chinook salmon bycatch estimates is highest for the BS on average ($\widehat{CV}\left(\widehat{W}\right) = 4.7\%$) whereas average precision for the AI and GOA are similar ($\widehat{CV}\left(\widehat{W}\right) = 18.9\%$ and $\widehat{CV}\left(\widehat{W}\right) = 14.5\%$, respectively).

Compared to Chinook salmon, very few sockeye salmon are caught by any gear type (Table 6.24). Virtually no bycatch is observed in the AI (approximately 2.9 on average) and somewhat more bycatch in the BS and GOA (approximately 131.8 and 93.8 on average, respectively). However, all regional estimates have poor precision ($\widehat{CV}\left(\widehat{N}\right) = 47.4\%$, $\widehat{CV}\left(\widehat{N}\right) = 38.7\%$ and $\widehat{CV}\left(\widehat{N}\right) = 61.1\%$ on average for the AI,

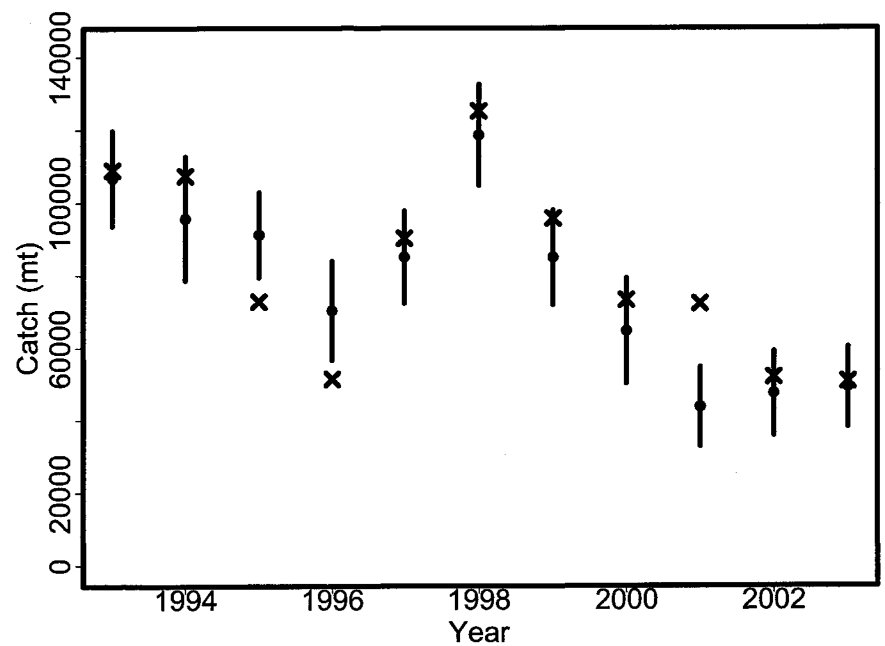

Figure 6.10. Yearly estimates of walleye pollock total catch weight from 1993 to 2003 in the Gulf of Alaska (closed circle) compared with the corresponding values reported in Dorn et al. (2004) which are denoted by "x." The vertical lines represent asymptotic 95% confidence intervals.

Table 6.23. Yearly estimates of chinook salmon bycatch numbers and standard errors by gear type (longline and trawl) and region (Aleutian Islands, Bering Sea and Gulf of Alaska) from 1993 to 2003.

Year	Aleutian Is.		Bering Sea		Gulf of Alaska	
	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$
longline						
1993	8.31	5.45	9.40	4.48	29.43	18.50
1994	0	0	57.37	24.66	6.81	6.45
1995	0	0	27.49	19.40	13.16	13.21
1996	0	0	7.63	4.20	17.57	11.99
1997	0	0	11.03	5.15	12.67	12.06
1998	0	0	4.41	3.86	16.18	16.19
1999	0	0	7.83	4.94	2.82	2.23
2000	11.22	11.06	3.10	2.46	47.16	33.19
2001	2.50	1.94	18.64	7.88	0	0
2002	0	0	39.08	21.34	8.34	10.07
2003	3.75	3.16	7.08	4.13	0	0
trawl						
1993	2,899.25	333.19	29,746.43	2,040.79	25,680.75	1,904.70
1994	3,677.43	634.95	42,948.58	2,738.69	12,891.04	2,365.93
1995	3,162.98	625.02	19,207.67	1,388.17	16,788.19	2,517.94
1996	1,686.73	486.20	64,016.33	2,171.93	13,346.42	1,932.32
1997	3,802.10	601.01	79,299.85	2,646.53	16,373.81	1,710.83
1998	3,130.84	673.85	76,122.62	2,720.04	17,874.21	1,749.40
1999	1,123.80	184.24	33,769.94	1,616.91	37,767.89	5,207.22
2000	2,489.67	411.83	26,445.62	1,266.11	37,508.16	7,408.06
2001	1,646.74	445.42	62,555.44	3,042.71	14,993.89	2,512.98
2002	4,302.74	757.24	56,133.88	1,387.94	12,741.99	2,127.92
2003	2,961.85	473.45	79,164.81	3,031.66	17,235.96	2,942.93
total						
1993	2,907.55	333.23	29,755.84	2,040.80	25,710.18	1,904.79
1994	3,677.43	634.95	43,005.96	2,738.80	12,897.85	2,365.94
1995	3,162.98	625.02	19,235.16	1,388.30	16,801.35	2,517.97
1996	1,686.73	486.20	64,023.96	2,171.93	13,363.99	1,932.36
1997	3,802.10	601.01	79,310.88	2,646.53	16,386.48	1,710.87
1998	3,130.84	673.85	76,127.02	2,720.04	17,890.39	1,749.48
1999	1,123.80	184.24	33,777.78	1,616.92	37,770.71	5,207.22
2000	2,500.88	411.98	26,448.72	1,266.11	37,555.32	7,408.13
2001	1,649.24	445.43	62,574.07	3,042.72	14,993.89	2,512.98
2002	4,302.74	757.24	56,172.97	1,388.11	12,750.33	2,127.94
2003	2,965.59	473.47	79,171.90	3,031.66	17,235.96	2,942.93

BS and GOA, respectively).

6.5 *Marine Mammal Mortalities*

Mortality of marine mammals caused directly by the actual *fishing* activities of the Alaskan groundfish fleet is very rare. Nevertheless, when mortalities occur the most frequent species is Steller sea lions. Some killer whales have also been incidentally captured as well. As mortalities are only observed for the trawl vessels, I will only describe the estimation of marine mammal bycatch for this gear type and I will use Steller sea lions as an example.

Definition of marine mammal mortality caused by fishing is necessary because there are various types of interactions that are counted. I define mortality as all three types of interactions recorded by observers that describe mortalities directly caused by fishing . In the first quarter of 2001, there were three Stellar sea lion mortalities observed in the trawl fleet. Two of these occurred in the BS and one in the AI and all three were aboard large vessels. Relevant information for one of the trips where a mortality was observed is provided in Table 6.25.

As I described in Section 3.5, there is no sampling uncertainty for marine mammal mortalities within hauls. Like other catch parameters I will focus on region-specific estimation and so use Eq. 5.42 to make an AI-specific estimate of Steller sea lion mortalities for the trip and estimate the variance with Eq. 5.43. The resulting estimate is shown in the second row of Table 6.26. The corresponding standard error estimate is greater than zero because only 10 of the 13 hauls made during the trip were sampled. Standard error estimates are zero for other trips where not all of the hauls are sampled because there is no variability in the number of mortalities (zero) for each haul in those trips. Estimation of mortalities for a given vessel, quarter or year is performed in the same way as that of seabird bycatch and any other parameters previously described.

Between 1993 and 2003, at least two Steller sea lion mortalities are estimated each year in the BS (Table 6.27). At least one yearly mortality is estimated in the AI except for 1993 and 2002 where no mortalities were observed. In the GOA, there are sporadic mortalities, but there is poor precision of the estimates because of occurence on vessels with only 30% observer coverage. Only in the BS are there any yearly mortality estimates for killer whales, but mortalities were observed in only 5 years between

Table 6.24. Yearly estimates of sockeye salmon bycatch numbers and standard errors by gear type (longline and trawl) and region (Aleutian Islands, Bering Sea and Gulf of Alaska) from 1993 to 2003.

Year	Aleutian Is.		Bering Sea		Gulf of Alaska	
	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$	$\widehat{N}$	$\widehat{SE}\left(\widehat{N}\right)$
longline						
1993	0	0	0	0	4.38	2.27
1994	0	0	0	0	2.59	2.02
1995	0	0	0	0	0	0
1996	0	0	11.52	11.33	0	0
1997	0	0	0	0	0	0
1998	0	0	0	0	26.85	26.69
1999	0	0	0	0	0	0
2000	0	0	0	0	0	0
2001	0	0	4.21	3.61	0	0
2002	0	0	0	0	0	0
2003	4.01	3.43	3.46	2.73	2.25	1.49
trawl						
1993	18.37	7.51	1.42	0.75	67.50	56.96
1994	8.99	1.41	2.82	1.12	112.16	55.63
1995	0	0	0	0	201.26	221.01
1996	0	0	1.25	0.55	10.30	5.31
1997	0	0	42.51	7.31	36.75	15.03
1998	0	0	205.55	99.56	238.33	35.96
1999	0	0	8.36	3.00	171.41	65.78
2000	0	0	192.03	47.11	0	0
2001	0	0	518.41	233.17	157.61	155.51
2002	0	0	24.17	3.65	0	0
2003	0	0	433.76	86.02	0	0
total						
1993	18.37	7.51	1.42	0.75	71.87	57.01
1994	8.99	1.41	2.82	1.12	114.75	55.66
1995	0	0	0	0	201.26	221.01
1996	0	0	12.78	11.35	10.30	5.31
1997	0	0	42.51	7.31	36.75	15.03
1998	0	0	205.55	99.56	265.18	44.79
1999	0	0	8.36	3.00	171.41	65.78
2000	0	0	192.03	47.11	0	0
2001	0	0	522.62	233.19	157.61	155.51
2002	0	0	24.17	3.65	0	0
2003	4.01	3.43	437.21	86.06	2.25	1.49

Table 6.25. Total weight of each haul (Υ_k) and number of Steller sea lion mortalities for each haul (Γ_k) for an observed trip aboard a large size-class vessel fishing trawl gear in the first quarter of 2001. Ten of the 13 hauls were sampled and blank spaces represent the unsampled hauls.

haul	Υ_k	Γ_k
1	74,200	0
2	110,400	0
3	96,970	0
4	83,000	
5	55,860	0
6	100,000	
7	129,000	0
8	55,000	
9	78,000	0
10	93,000	0
11	71,000	1
12	119,000	0
13	98,000	0

Table 6.26. Number of hauls made (G_t) and observed (g_t) for eight observed trips of a medium-size class, trawl vessel in the first quarter of 2001 with estimates of Steller sea lion mortalities ($\widehat{\Gamma}_t$) and corresponding standard errors.

trip	G_t	g_t	$\widehat{\Gamma}_t$	$\widehat{SE}\left(\widehat{\Gamma}_t\right)$
1	12	11	0.0	0.00
2	13	10	1.3	0.62
3	10	9	0.0	0.00
4	2	2	0.0	0.00
5	10	8	0.0	0.00
6	40	36	0.0	0.00
7	12	12	0.0	0.00
8	34	28	0.0	0.00

1993 and 2002 (Table 6.28). At most, approximately 3.5 mortalities are estimated in a given year. For either Stellar sea lions or killer whales, there are some non-zero estimates with standard error estimates of zero. The lack of uncertainty occurs because all hauls are monitored in a given trip where the mortality was observed and the trip was performed by a vessel with 100% observer coverage.

Table 6.27. Yearly estimates of Steller sea lion mortalities and standard errors by region (Aleutian Islands, Bering Sea and Gulf of Alaska) from 1993 to 2003.

Year	Aleutian Is.		Bering Sea		Gulf of Alaska	
	$\widehat{\Gamma}$	$\widehat{SE}(\widehat{\Gamma})$	$\widehat{\Gamma}$	$\widehat{SE}(\widehat{\Gamma})$	$\widehat{\Gamma}$	$\widehat{SE}(\widehat{\Gamma})$
1993	0	0	4.00	0	1.00	0
1994	1.00	0	6.00	0	3.33	3.33
1995	2.00	0	3.33	2.97	0	0
1996	1.00	0	2.00	0	0	0
1997	1.00	0	7.33	2.91	0	0
1998	3.00	0	2.00	0	1.00	0
1999	4.00	0	4.00	0	0	0
2000	1.00	0.05	7.83	2.96	0	0
2001	1.31	0.63	7.81	1.66	10.07	9.87
2002	0	0	5.80	1.83	0	0
2003	8.08	8.84	2.12	1.54	3.41	3.62

6.6 *Total Numbers in Length or Age Classes*

For both longline and trawl gear, Pacific cod is the species I use to describe estimation of length class numbers. I use sablefish and walleye pollock to describe estimation of age class numbers for longline and trawl gears, respectively. The length and age classes I use are the same as those used in current stock assessments (NPFMC 2004). For length class numbers of Pacific cod and age class numbers of walleye pollock, I also give some sex-specific estimates.

6.6.1 Total Numbers in Length Classes

6.6.1.1 Longline

My description of estimation of length class numbers begins within a haul in the BSAI during the first quarter of 2002. Table 6.29 shows the numbers of measured

Table 6.28. Yearly estimates of killer whale mortalities and standard errors by region (Aleutian Islands, Bering Sea and Gulf of Alaska) from 1993 to 2003.

Year	Aleutian Is.		Bering Sea		Gulf of Alaska	
	$\hat{\Gamma}$	$\widehat{SE}(\hat{\Gamma})$	$\hat{\Gamma}$	$\widehat{SE}(\hat{\Gamma})$	$\hat{\Gamma}$	$\widehat{SE}(\hat{\Gamma})$
1993	0	0	0	0	0	0
1994	0	0	0	0	0	0
1995	0	0	0	0	0	0
1996	0	0	0	0	0	0
1997	0	0	1.00	0	0	0
1998	0	0	1.00	0	0	0
1999	0	0	1.00	0	0	0
2000	0	0	0	0	0	0
2001	0	0	3.51	1.60	0	0
2002	0	0	1.05	0.22	0	0
2003	0	0	0	0	0	0

fish, proportions, and estimated total numbers of fish in each length class for the haul. The length class estimates are obtained using the available information with Eq. 2.31 where the total number of Pacific cod in the haul is estimated using Eq. 2.3. Likewise, the variance-covariance matrix (VCM) that corresponds to the vector of estimates of length class numbers is estimated using Eq. 2.32 where the variance of the total number estimate is estimated using Eq. 2.4. The resulting non-zero elements of the 27×27 VCM estimate are shown in Table 6.30.

The vessel made 36 hauls in the trip and all of them were made in the BSAI. Twenty-one of the hauls were sampled for species composition and all of the sampled hauls were s-prevalent in Pacific cod (see Section 5.1.2 for a definition). Recall from Section 5.1.3, that when we are interested in estimates for particular regions or time periods the estimators are different in form than when estimates without regard to regions or time periods are desired.

The same methods as described above for the first haul are used to estimate the length class numbers and the corresponding VCM estimates for the other 20 s-prevalent hauls. These results are used in Eq. 5.20 and Eq. 5.21 to estimate the length class numbers and the corresponding VCM for the entire trip. The resulting estimates are shown in Table 6.31. The non-zero elements of the estimated VCM for the trip make up a large 14×14 matrix and the elements of this matrix that

Table 6.29. The numbers of fish with length measurements by length class ($\boldsymbol{\lambda}_k$) with resulting estimated proportions ($\widehat{\mathbf{p}}_{\lambda,k}$) and total numbers ($\widehat{\boldsymbol{\Lambda}}_k$) in length classes for a haul made in the Bering Sea/Aleutian Islands region during the first quarter of 2002. The number of deployed and observed hooks were $H_k = 9304$ and $h_k = 3489$ with $n_k = 748$ Pacific cod counted on the observed hooks.

Length class	$\boldsymbol{\lambda}_k$	$\widehat{\mathbf{p}}_{\lambda,k}$	$\widehat{\boldsymbol{\Lambda}}_k$
<9	0	0	0
9-11	0	0	0
12-14	0	0	0
15-17	0	0	0
18-20	0	0	0
21-23	0	0	0
24-26	0	0	0
27-29	0	0	0
30-32	0	0	0
33-35	0	0	0
36-38	0	0	0
39-41	0	0	0
42-44	0	0	0
45-49	0	0	0
50-54	0	0	0
55-59	0	0	0
60-64	2	0.10	199.47
65-69	0	0	0
70-74	4	0.20	398.93
75-79	6	0.30	598.40
80-84	3	0.15	299.20
85-89	3	0.15	299.20
90-94	1	0.05	99.73
95-99	0	0	0
100-104	1	0.05	99.73
105-115	0	0	0
>115	0	0	0

Table 6.30. The non-zero variance and covariance estimates for the estimated total numbers in length classes $(\widehat{\mathbf{\Lambda}}_k)$ for the haul given in Table 6.29.

	60-64	70-74	75-79	80-84	85-89	90-94	100-104
60-64	18,860.20	−4,133.10	−6,199.65	−3,099.83	−3,099.83	−1,033.28	−1,033.28
70-74	−4,133.10	33,587.30	−12,399.30	−6,199.65	−6,199.65	−2,066.55	−2,066.55
75-79	−6,199.65	−12,399.30	44,181.30	−9,299.48	−9,299.48	−3,099.83	−3,099.83
80-84	−3,099.83	−6,199.65	−9,299.48	26,740.39	−4,649.74	−1,549.91	−1,549.91
85-89	−3,099.83	−6,199.65	−9,299.48	−4,649.74	26,740.39	−1,549.91	−1,549.91
90-94	−1,033.28	−2,066.55	−3,099.83	−1,549.91	−1,549.91	9,946.74	−516.64
100-104	−1,033.28	−2,066.55	−3,099.83	−1,549.91	−1,549.91	−516.64	9,946.74

correspond to the elements of the haul-specific VCM estimate (Table 6.30) are given in Table 6.32.

The form of estimators of the length class numbers from s-prevalent hauls and for a particular region (e.g., BSAI) and/or time period (e.g., April and May) at higher levels of aggregation is not gear-specific and is performed with estimators that are analogous to those given for univariate parameter estimation in earlier sections of this chapter. However, the estimated length class numbers for s-prevalent hauls can be "scaled-up" to all longline fishing using the approaches described in Section 5.6, but recall that this implicitly assumes that length frequencies in hauls where the species is prevalent are the same as those in hauls where the species is not prevalent.

Using Eq. 5.108 in particular, we must also calculate estimates for Pacific cod caught in the BSAI using Eq. 5.40 or Eq. 5.48 within trips where the estimator depends on whether trips are undersampled. Covariance of the domain estimates with the length classes must also be estimated and the appropriate estimators are described in Section 5.6.

In stock assessments for Pacific cod, proportions in length classes are estimated for three periods (January to May, June to August and September to December) within a given year and within two regions: BSAI and GOA (Thompson and Dorn 2004; Thompson et al. 2004). These proportions are multiplied by a total number caught for the same region/period to obtain estimates of numbers caught in each length class. However, the total number caught for the region/period is obtained by converting the total catch weight based on the blend-system discussed earlier in the context of catch weight estimation. Using my estimators, the total weight estimates are not required and total numbers in length classes can be estimated (along with measures of precision/correlation) directly from the data collected by observers on vessels.

For 2002, most Pacific cod were caught in the first period for either the BSAI or GOA (approximately 51.7% and 68.8%, respectively; Figure 6.11). No length measurements were obtained aboard vessels in the GOA between June and August and so no length class estimates are available, but very few Pacific cod were caught during that region/period ($\widehat{N} = 2880.2$, $\widehat{SE}\left(\widehat{N}\right) = 1149.5$). Many more Pacific cod are caught in the BSAI each period than the GOA ($89.0 - 99.9\%$ of total number caught each quarter are in the BSAI) and precision of estimates for the more common

Table 6.31. The estimated numbers-at-length for a longliner trip ($\widehat{\mathbf{\Lambda}}_{amt}$). The haul in Table 6.29 was one of the 21 hauls that were sampled and s-prevalent.

Length Class	$\widehat{\mathbf{\Lambda}}_{amt}$
<9	0
9-11	0
12-14	0
15-17	0
18-20	0
21-23	0
24-26	0
27-29	0
30-32	0
33-35	0
36-38	0
39-41	76.20
42-44	0
45-49	1,613.59
50-54	1,818.81
55-59	1,074.11
60-64	5,627.20
65-69	8,867.99
70-74	12,253.76
75-79	9,266.40
80-84	6,356.28
85-89	2,850.90
90-94	2,167.04
95-99	1,577.74
100-104	1,120.46
105-115	417.09
>115	0

Table 6.32. The elements of a trip-specific VCM estimate that correspond to the elements of the haul-specific VCM estimate that were non-zero in Table 6.30.

	60-64	70-74	75-79	80-84	85-89	90-94	100-104
60-64	936, 682.66	−57, 537.45	−235, 138.39	−164, 876.48	94, 791.44	−53, 087.32	−532.45
70-74	−57, 537.45	2, 304, 460.59	37, 107.54	−103, 406.85	2, 204.35	−181, 470.61	160, 147.75
75-79	−235, 138.39	37, 107.54	1, 607, 381.06	−103, 061.99	66, 155.21	192, 347.44	174, 047.12
80-84	−164, 876.48	−103, 406.85	−103, 061.99	800, 936.22	−57, 629.04	−378.29	−44, 125.02
85-89	94, 791.44	2, 204.35	66, 155.21	−57, 629.04	447, 007.13	−33, 119.89	96, 603.99
90-94	−53, 087.32	−181, 470.61	192, 347.44	−378.29	−33, 119.89	382, 850.90	−38, 681.91
100-104	−532.45	160, 147.75	174, 047.12	−44, 125.02	96, 603.99	−38, 681.91	307, 100.65

length classes in BSAI catches is better than corresponding estimates in the GOA because of differences in proportion of vessels with 30% observer coverage.

In estimating proportions-at-length in catches for Pacific cod stock assessments, scientists realize that the length measurements are not independently drawn from the entire catch for a given region/period and an overdispersed multinomial is assumed for the distribution of length classes of a sampled fish. The variance of the estimated proportion of fish in the ith length class under an overdispersed multinomial assumption is

$$V(\widehat{p}_i) = \frac{p_i (1 - p_i) \tau_i}{n}$$

where τ_i is a dispersion parameter and n is the number of length measurements (sample size). In the stock assessments, the dispersion parameter is essentially assumed to be $\tau_i = \sqrt{n}$ so that it is constant across all length classes (Thompson and Dorn 2004). To compare this assumption to the variance estimates provided by my approach, I calculated dispersion parameter estimates that make the overdispersed multinomial variance estimates equal to my variance estimates. Specifically, when my variance estimate for the ith length class is $\widehat{V}(\widehat{p}_i)$,

$$\widehat{V}(\widehat{p}_i) \equiv \frac{\widehat{p}_i (1 - \widehat{p}_i) \widehat{\tau}_i}{n} \Rightarrow \widehat{\tau}_i = \frac{\widehat{V}(\widehat{p}_i)}{\widehat{p}_i (1 - \widehat{p}_i) / n} \tag{6.1}$$

yields moment-based estimates of the dispersions of my variance estimates relative to those under multinomial sampling.

The proportions are estimated using Eq. 5.112 where the denominator is the sum of the estimated numbers in each length class (the same denominator as Eq. 5.108) and the corresponding VCM is estimated using Eq. 5.113. When these dispersions are estimated for data collected in 2002, we find that my estimates are more precise than the precision assumed in the stock assessments and the dispersions vary widely across length classes (Table 6.33). For all periods in the BSAI and the first period in the GOA, the variance due to sampling using my proportion estimators is at most 13% of the variance that would be assumed in the stock assessments and only in the GOA during the third period does the variance come near that assumed in the stock assessments where one or more length class variance estimates reach near 50% of the variance under the assesssment assumption. Estimates for catches in the BSAI during the second period (June-August) had lowest average dispersion (approximately

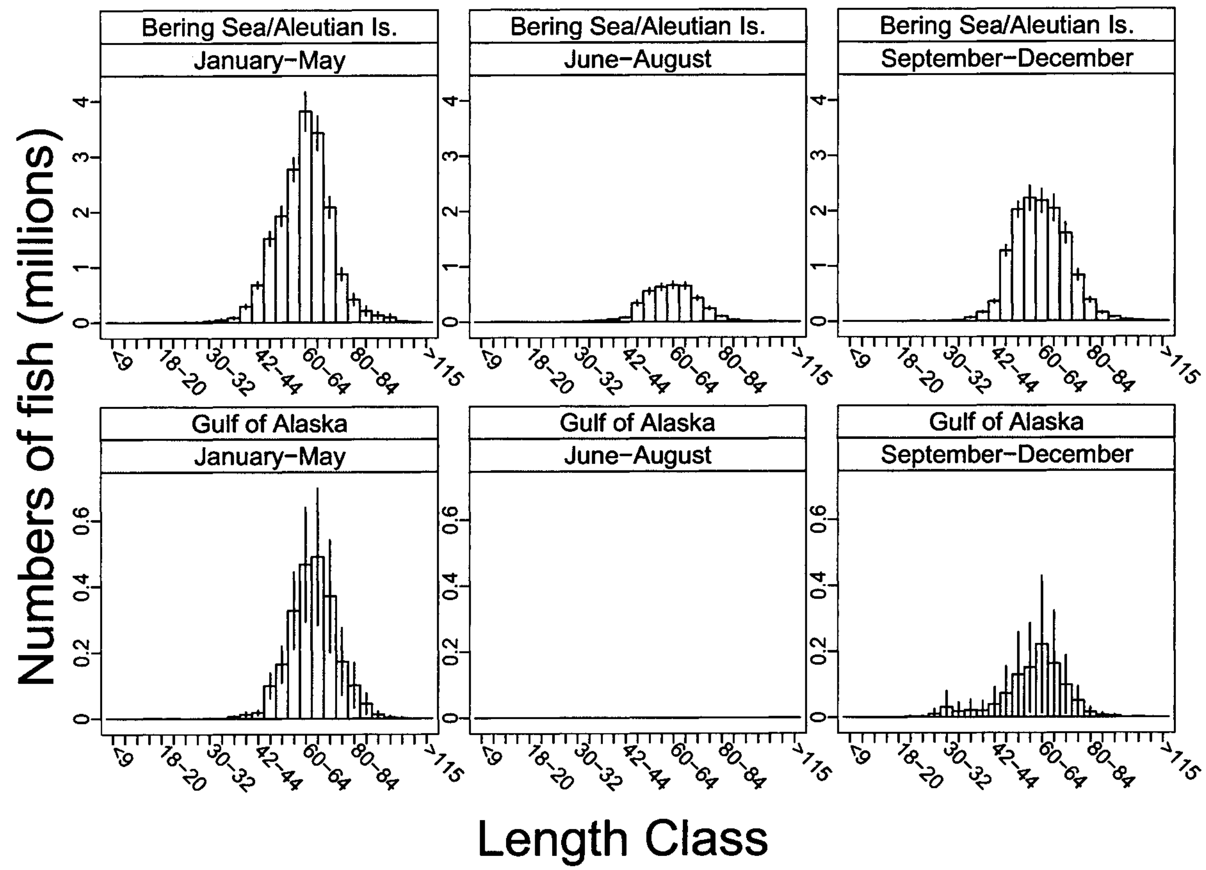

Figure 6.11. Estimated numbers in length classes for Pacific cod caught in the Bering Sea/Aleutian Islands and Gulf of Alaska during each of the three periods of 2002 (January-May, June-August, September-December). The vertical lines represent asymptotic 95% confidence intervals.

2.11), but those corresponding to the BSAI in September-December and the GOA in January-May were also low on average (approximately 3.21 and 2.73, respectively).

The dispersion estimates using Eq. 6.1 are useful for assessing whether the diagonal of the VCM is similar to the diagonal of a VCM under a multinomial assumption, but comparison of the off-diagonal elements, or correlations of length classes, is also important. Graphical comparisons of the correlations like those in Figure 6.12, which are for estimated proportions of the catch in different length classes in the BSAI, can be less confusing than tabular comparisons. For the first period in the BSAI, there is positive correlation of similar length classes at the low and high ends of the length class categories and there is small negative correlation of the middle length classes, but stronger negative correlation between the middle length classes and the larger length classes. Moreover, there is a positive correlation of the small length classes with the largest length class. There was mostly small negative correlation exhibited for the BSAI estimates in the second period, yet the pattern is still quite different than expected under multinomial assumptions. Like the first period, the BSAI estimates in the third period showed some positive correlation of a closely related set of length classes and strong negative correlation of those length classes with some larger length classes.

For the GOA estimates, correlations in the first period were positive for many of the closely related length classes and there is relatively strong negative correlation of two groups of middle length classes (Figure 6.13). In the third period, there is wide positive correlation among the smaller length classes, but strong negative correlation with the next larger set of length classes.

I have also plotted similar correlation plots for the Bering Sea (without the Aleutian Islands) and Gulf of Alaska when sex-specific length class proportions are estimated (Figures 6.14 and 6.15, respectively). These plots not only show whether patterns in correlation among length classes are different for males and females, but they allow one to discern whether patterns in correlation across sexes are different than within sexes.

The occurence of positive correlation of some proportion estimates can be due to positive covariance of length classes consistently across hauls, trips and vessel-quarter strata. The proportions within hauls are negatively correlated because of the multinomial assumption there, but covariance of length class estimates at higher

Table 6.33. Dispersion for variance estimates associated with estimated proportions of Pacific cod catch in length classes relative to corresponding variance estimates under multinomial sampling (Eq. 6.1) by region (Bering Sea/Aleutian Islands and Gulf of Alaska) and management period (Jan-May, Jun-Aug, Sep-Dec) in 2002. The dispersion assumed in Pacific cod stock assessments ($\sqrt{n}$) is given for comparison.

	Bering Sea/Aleutian Islands			**Gulf of Alaska**		
	Jan-May	Jun-Aug	Sep-Dec	Jan-May	Jun-Aug	Sep-Dec
$\sqrt{n}$	270.148	175.932	291.503	71.428	0	42.012
<9	2.355		0.707			
9-11	1.805		1.281			
12-14	4.016					
15-17	7.405					
18-20	8.135					
21-23	8.440	0.860		1.356		1.399
24-26	20.862	1.708	2.500			1.334
27-29	25.143	1.799	1.343			5.705
30-32	16.033	2.165	1.833	1.308		17.750
33-35	5.825	3.488	1.530	1.206		10.059
36-38	4.542	2.663	3.224	0.737		11.890
39-41	3.768	2.261	3.707	1.572		9.552
42-44	4.594	2.489	5.715	1.185		13.803
45-49	6.060	4.896	14.377	3.058		8.647
50-54	3.242	2.109	6.516	2.385		6.336
55-59	4.074	1.844	3.107	2.834		2.398
60-64	6.393	2.068	2.378	4.749		19.177
65-69	5.735	2.248	5.756	1.912		20.363
70-74	2.515	1.558	5.812	1.871		4.779
75-79	3.317	1.812	3.758	6.434		0.949
80-84	14.491	2.048	2.653	7.942		1.174
85-89	28.988	1.647	1.761	4.198		1.109
90-94	28.217	1.563	2.033	2.462		1.339
95-99	35.353	1.459	1.718	4.458		
100-104	4.469	1.108	0.936	2.124		0.367
105-115	2.137	2.567	1.008	0.122		
>115	3.071		0.122			

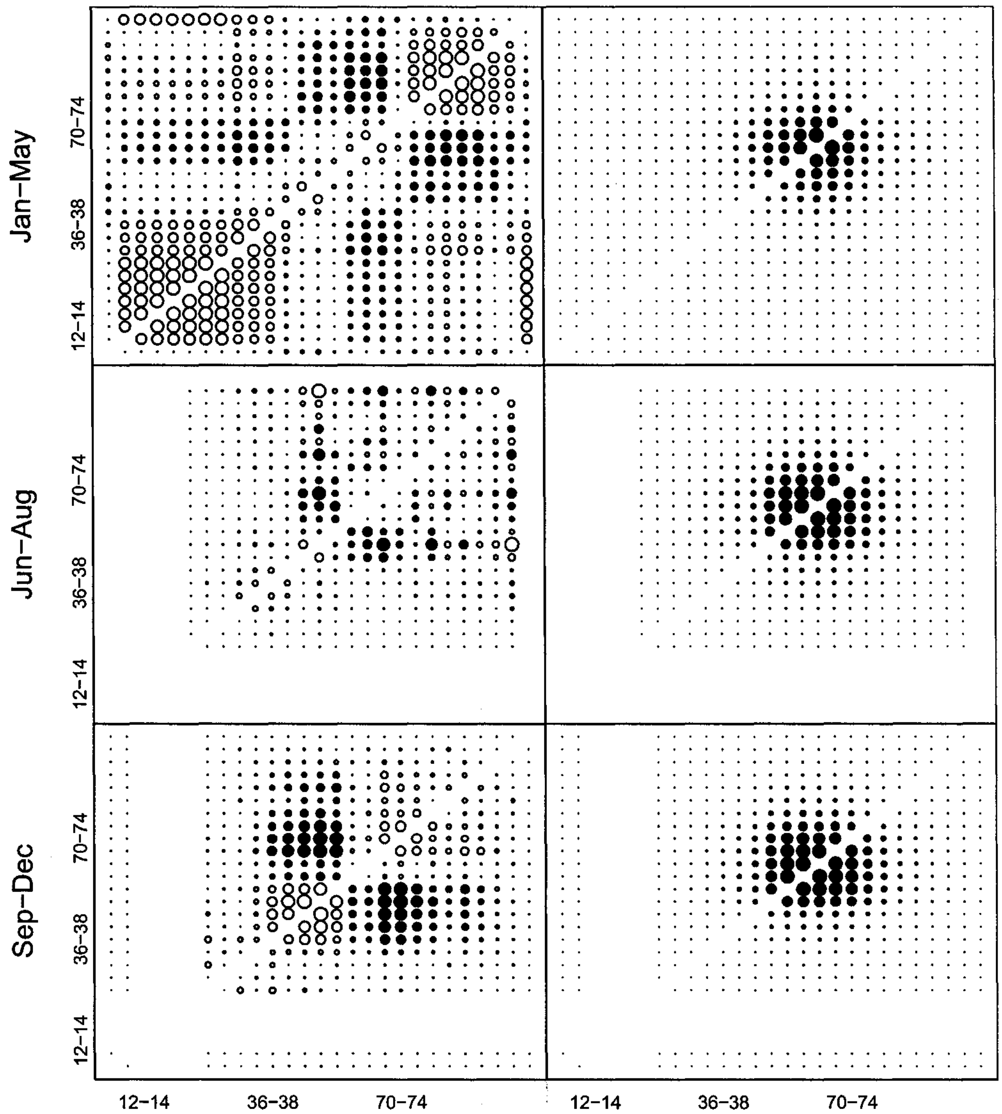

Figure 6.12. Correlation of estimated proportions of Pacific cod in length classes in the BSAI (left column) for each of three periods of 2002: January-May (top row), June-August (middle row), September-December (bottom row) and corresponding correlations under a multinomial assumption (right column). Open and closed circles represent positive and negative correlation, respectively, the size of the circle represents relative magnitude of the correlation and blank spaces indicate length classes not present in any observer samples.

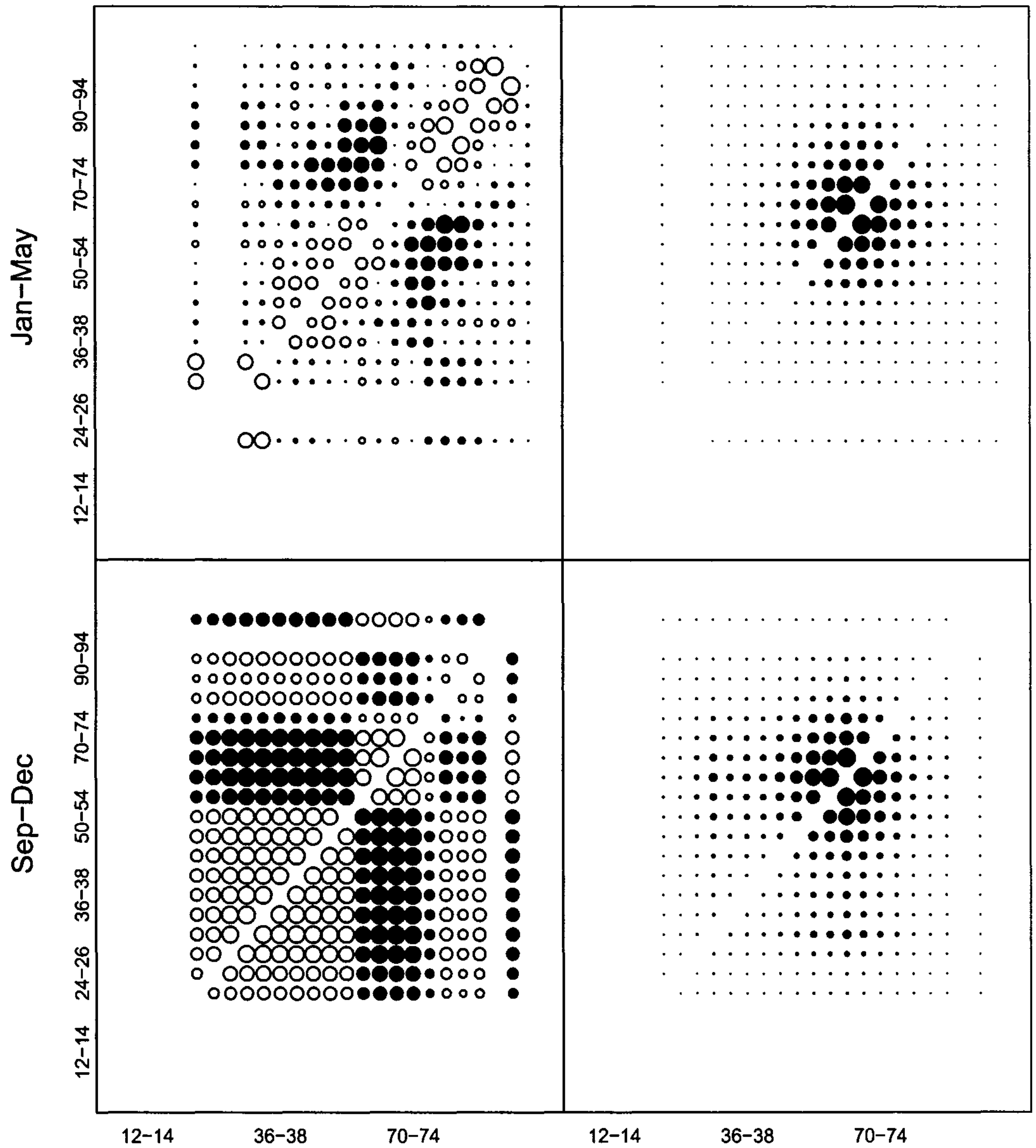

Figure 6.13. Correlation of estimated proportions of Pacific cod in length classes in the GOA (left column) for two periods of 2002: January-May (top row), September-December (bottom row) and corresponding correlations under a multinomial assumption (right column). Open and closed circles represent positive and negative correlation, respectively, the size of the circle represents relative magnitude of the correlation and blank spaces indicate length classes not present in any observer samples.

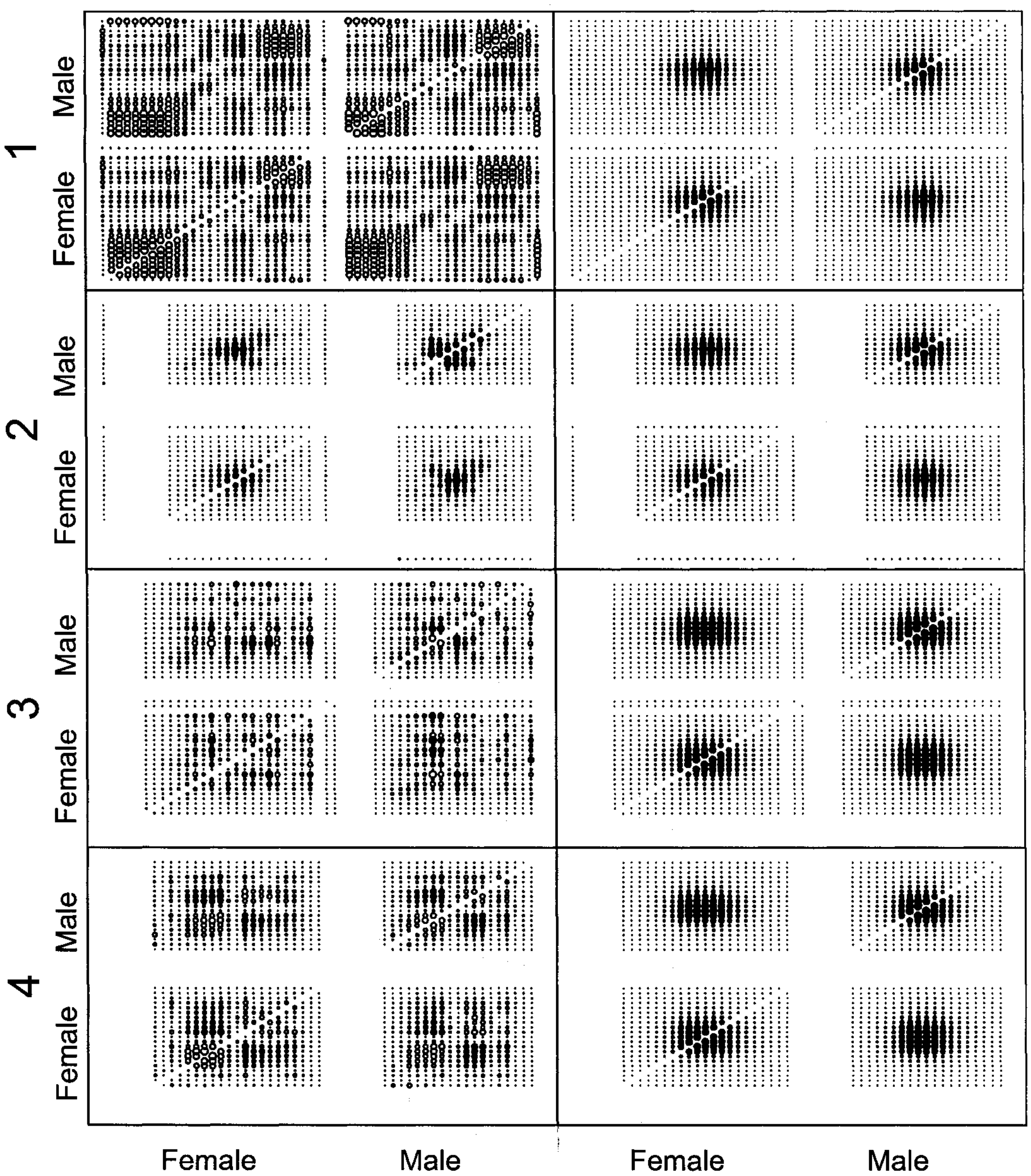

Figure 6.14. Correlation of estimated sex-specific proportions of Pacific cod in length classes for catches made in the Bering Sea (left column) during each of the four quarters (rows) of 2002 and corresponding correlations under a multinomial assumption (right column). Open and closed circles represent positive and negative correlation, respectively, the size of the circle represents relative magnitude of the correlation and blank spaces indicate length classes not present in any observer samples.

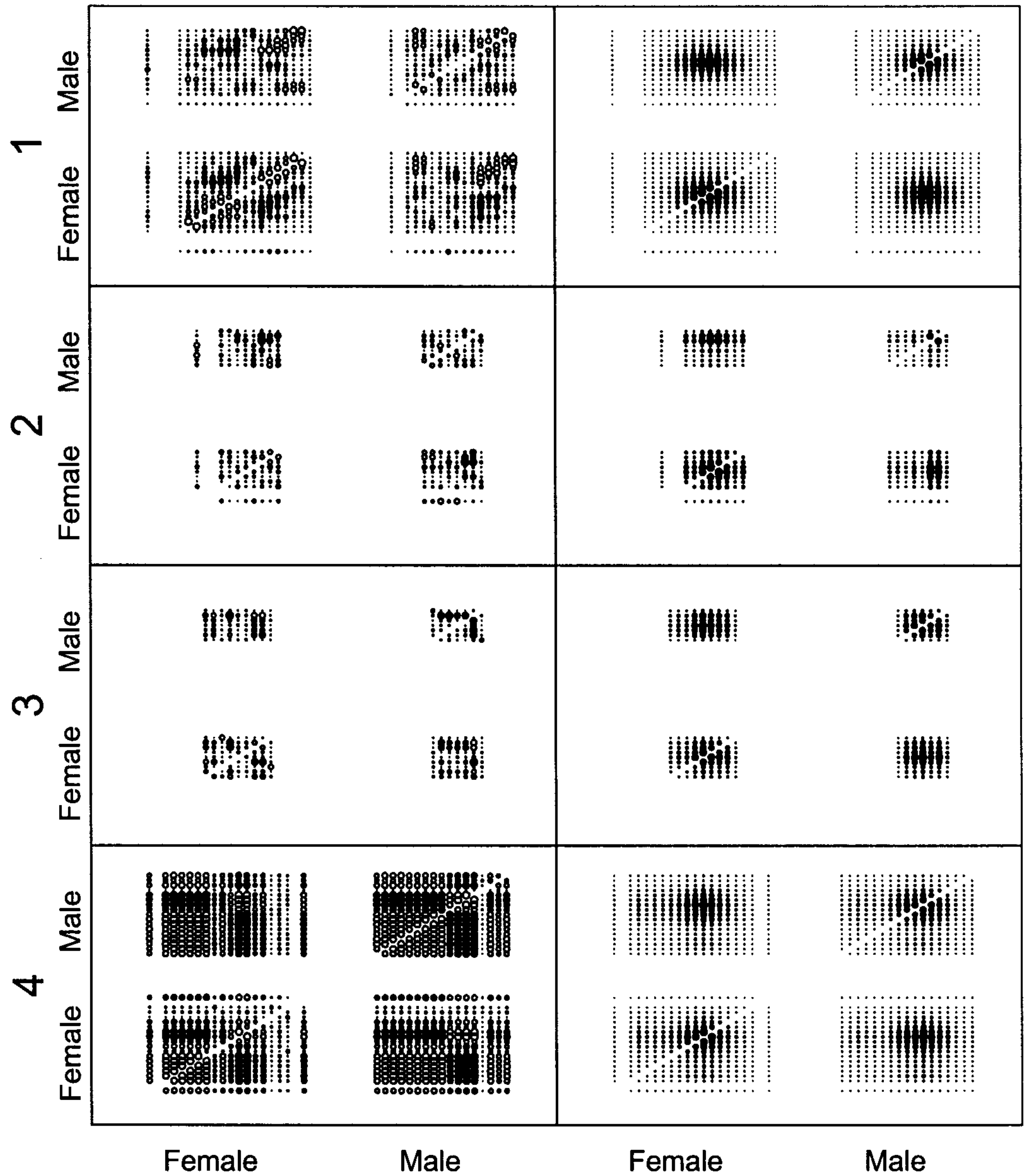

Figure 6.15. Correlation of estimated sex-specific proportions of Pacific cod in length classes in the Gulf of Alaska (left column) for each of the four quarters (rows) of 2002 and corresponding correlations under a multinomial assumption (right column). Open and closed circles represent positive and negative correlation, respectively, the size of the circle represents relative magnitude of the correlation and blank spaces indicate length classes not present in any observer samples.

levels of the design will dominate the within-haul covariances.

The very different precision and correlation of estimates for proportions in length classes obtained using my estimators from those used in stock assessments implies that there is a potential for substantial differences in stock assessment results using my estimation approach. The dispersion parameter assumed in the assessments ($\sqrt{n}$) accounts for sampling variance as well as model variance because the assumed dispersions are compared to those predicted by the assessment model. If so and my estimate of sampling variance is correct, then the component of overall variance due to the model would appear to be substantial. In any event, it would be both interesting and useful to run stock assessments using my proportion estimates and corresponding variance estimates to tease apart the two variance components. Perhaps estimates of the numbers caught in each length class could be used directly thus circumventing the need to multiply the proportion estimates by the blend-based estimate of total numbers caught.

6.6.1.2 Trawlers

Again, I begin with a haul made by a vessel during the first quarter of 2002, but here I consider sex-specific length classes (sex-length classes) and I treat the Bering Sea separately from the Aleutian Islands. Table 6.34 shows the numbers of the sampled fish in each sex-length class and the corresponding estimated proportions. The proportions are used with a haul-specific estimate of total overall numbers of fish caught to estimate total numbers in sex-length classes for the haul. The haul-specific estimates of total numbers for trawlers requires a trip-specific model and, as discussed in Section 5.1.2.5, I narrow the scope of the model to s-prevalent hauls. For these estimates, I used the estimators based on model in Eq. 5.25.

The vessel made 9 hauls during the trip, all of which occurred in the Gulf of Alaska. Eight of the hauls were sampled and 5 of the sampled hauls were s-prevalent for Pacific cod. The resulting estimated rate of capture among s-prevalent hauls is $\widehat{\phi}_{n,amt} = 0.1647$ and the estimated sex-length class numbers are given in Table 6.35. Because sex-length class numbers are being estimated, the VCM estimate is a 54×54 matrix and it is obtained using Eq. 5.23.

The same vessel made another trip where there were two hauls and neither were sampled by an observer. As such, the model-based approach for undersampled trips is

Table 6.34. The sex-length class numbers of Pacific cod with length measurements in length classes ($\boldsymbol{\lambda}_k$) and corresponding estimated proportions ($\mathbf{p}_{\lambda,k}$) for a trawler haul made in the Gulf of Alaska during the first quarter of 2002.

Length class	Female		Male	
	$\boldsymbol{\lambda}_k$	$\widehat{\mathbf{p}}_{\lambda,k}$	$\boldsymbol{\lambda}_k$	$\widehat{\mathbf{p}}_{\lambda,k}$
<9	0	0	0	0
9-11	0	0	0	0
12-14	0	0	0	0
15-17	0	0	0	0
18-20	0	0	0	0
21-23	0	0	0	0
24-26	0	0	0	0
27-29	0	0	0	0
30-32	0	0	0	0
33-35	0	0	0	0
36-38	0	0	0	0
39-41	0	0	0	0
42-44	0	0	0	0
45-49	0	0	1	0.06
50-54	0	0	0	0
55-59	5	0.28	2	0.11
60-64	0	0	1	0.06
65-69	0	0	3	0.17
70-74	1	0.06	1	0.06
75-79	0	0	0	0
80-84	2	0.11	0	0
85-89	1	0.06	0	0
90-94	1	0.06	0	0
95-99	0	0	0	0
100-104	0	0	0	0
105-115	0	0	0	0
>115	0	0	0	0

Table 6.35. Estimated sex-length class numbers for Pacific cod caught on a trawler trip in the Gulf of Alaska during the first quarter 2002.

Length Class	Female	Male
<9	0	0
9-11	0	0
12-14	0	0
15-17	0	0
18-20	0	0
21-23	0	0
24-26	0	0
27-29	0	0
30-32	0	0
33-35	0	0
36-38	0	0
39-41	0	0
42-44	0	110.82
45-49	24.58	7.93
50-54	0	19.95
55-59	311.21	104.42
60-64	462.30	230.66
65-69	258.28	341.36
70-74	229.56	414.86
75-79	76.05	11.29
80-84	22.50	0
85-89	7.93	0
90-94	7.93	0
95-99	0	0
100-104	0	0
105-115	0	0
>115	0	0

necessary and the estimator of sex-length class numbers and the corresponding VCM estimator have the form of Eq. 5.69 and Eq. 5.70, respectively. The known total weight of the two hauls made is used with the region-specific parameter estimates for the model of s-prevalence rate given in Table 6.36 and the region-specific parameter estimates for the the model of sex-length class proportions given in Table 6.37. Estimation for numbers in sex-length classes aboard trawlers at the vessel level and higher follows procedures analogous to those described for longline gear. Therefore, I will skip detailed description of those steps for trawl vessels.

Table 6.36. Numbers of sampled s-prevalent hauls (g_{am}), total number of sampled hauls (g_m), total number of observed trips (c_m), estimated probability of s-prevalence ($\widehat{\alpha}_m$) and estimated model dipsersion parameter for the probability of s-prevalence($\widehat{\tau}_{\alpha,m}$) for the trawl effort in the first quarter of 2002 by region (m): Bering Sea (BS), Gulf of Alaska (GOA) and Aluetian Islands (AI).

m	g_{am}	g_m	c_m	$\widehat{\alpha}_m$	$\widehat{\tau}_{\alpha,m}$
BS	706	8,520	958	0.083	7.693
GOA	303	647	151	0.468	3.699
AI	712	1,202	121	0.592	6.333

Estimates of the BS Pacific cod in various length classes caught by trawlers are similar across sexes in the first three quarters, but it appears fewer females were caught in the fourth quarter (Figure 6.16). The proportion of catches in the smaller length classes was higher in the second quarter than the other quarters. Estimates of the GOA Pacific cod in length classes caught by trawlers are less similar across sex classes and, except for the first quarter, the distribution of numbers in the sex-length classes lacks the smooth shape exhibited by corresponding distributions for the BS (Figure 6.17). Higher proportions of catches in some smaller length classes are also observed for GOA catches in the second quarter than other quarters.

Like the previous results for length classes for longline effort (Table 6.33), estimated proportions of the trawler-caught Pacific cod in sex-length classes are generally far more precise than would be assumed in stock assessments and the dispersions (Eq. 6.1) vary across sex-length classes (Tables 6.38 and 6.39). In fact, variance estimates are less than expected under the multinomial assumption for several proportions in

Table 6.37. Estimated probabilities of sex-length classes in s-prevalent hauls (table entries excluding the first row) and dispersion parameters ($\widehat{\tau}_{\lambda,m}$) for region-specific models of sex-length class frequencies in trawl gear during the first quarter of 2002. The regions are Bering Sea (BS), Gulf of Alaska (GOA) and Aluetian Islands (AI).

	AI		**BS**		**GOA**	
	Female	Male	Female	Male	Female	Male
$\widehat{\tau}_{\lambda,m}$	1.31		1.40		1.25	
<9	0	0	0	0	0	0
9-11	0	0	0	0	0	0
12-14	0	0	0	0	0	0
15-17	0	0	0	0	0	0
18-20	0	0	0	0	0.000	0.000
21-23	0.000	0	0	0	0.000	0.000
24-26	0.000	0	7.007×10^{-5}	0	0.001	0.001
27-29	0.000	0.000	2.803×10^{-4}	0.000	0.002	0.001
30-32	0.000	0.000	5.606×10^{-4}	0.001	0.006	0.006
33-35	0.003	0.002	4.905×10^{-4}	0.001	0.010	0.010
36-38	0.009	0.007	1.401×10^{-3}	0.001	0.009	0.010
39-41	0.016	0.016	2.803×10^{-3}	0.002	0.008	0.009
42-44	0.023	0.019	1.892×10^{-3}	0.003	0.011	0.011
45-49	0.038	0.033	3.293×10^{-3}	0.004	0.029	0.032
50-54	0.033	0.034	6.447×10^{-3}	0.006	0.036	0.038
55-59	0.056	0.064	6.306×10^{-3}	0.008	0.032	0.054
60-64	0.068	0.090	7.147×10^{-3}	0.016	0.061	0.095
65-69	0.078	0.108	1.766×10^{-2}	0.039	0.089	0.109
70-74	0.076	0.081	4.471×10^{-2}	0.086	0.092	0.075
75-79	0.050	0.035	7.358×10^{-2}	0.106	0.059	0.030
80-84	0.026	0.010	8.311×10^{-2}	0.099	0.026	0.015
85-89	0.009	0.004	7.813×10^{-2}	0.070	0.012	0.007
90-94	0.005	0.001	5.143×10^{-2}	0.043	0.005	0.004
95-99	0.001	0.000	3.37×10^{-2}	0.026	0.002	0.002
100-104	0.001	0.000	3.013×10^{-2}	0.018	0.001	0.000
105-115	0.000	0	2.172×10^{-2}	0.006	0.001	0.000
>115	0	0	2.102×10^{-4}	0	0	0

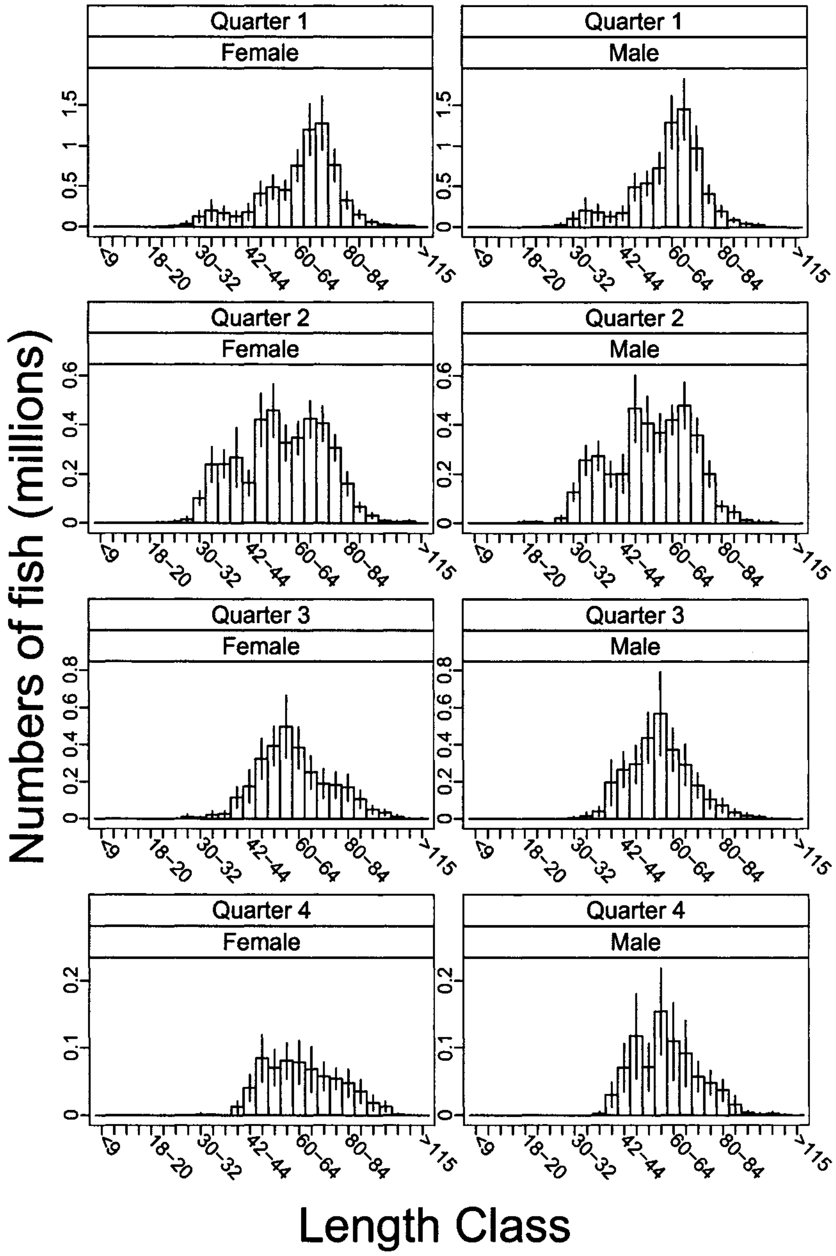

Figure 6.16. Estimated total numbers of Pacific cod caught by trawlers in length classes in the Bering Sea during 2002. Estimates are quarterly (rows) and sex-specific (columns). The vertical bars represent asymptotic 95% confidence intervals.

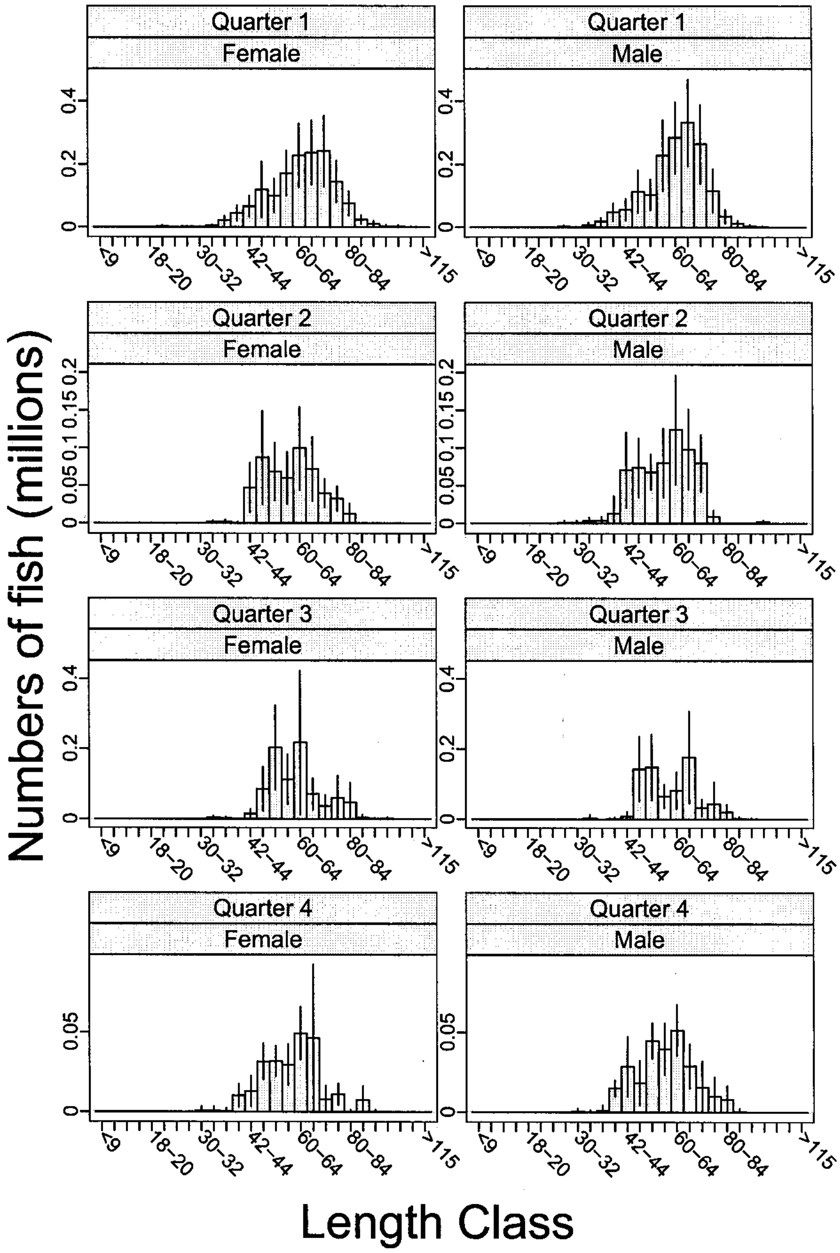

Figure 6.17. Estimated total numbers of Pacific cod caught by trawlers in length classes in the Gulf of Alaska during 2002. Estimates are quarterly (rows) and sex-specific(columns). The vertical bars represent asymptotic 95% confidence intervals.

either the BS or GOA. In the BS, the variance due to sampling using my proportion estimators is at most approximately 18% of the variance that would be assumed in the stock assessments for any given quarter. With the exception of one proportion in the third quarter the variance due to sampling using my proportion estimators is at most approximately 40% in the GOA. Proportion estimates for catches made during the fourth quarter in either the BS or GOA had lowest average dispersion (approximately 1.21), but those corresponding to the BS in the first quater and the GOA in in the third quarter had highest average dispersion (approximately 3.96 and 3.88, respectively).

The correlations of the estimated proportion of trawler-caught Pacific cod in sex-length classes were both positive and negative like the corresponding length class proportions in longline catches (Figures 6.14 and 6.15) and unlike the correlation pattern implicit to a multinomial sampling assumption (Figures 6.18 and 6.19). Correlation patterns appear similar within and across sexes except there is less positive correlation of males in larger length classes than that of females in larger length classes during the fourth quarter for the BS.

The use of these sex-length class estimates and corresponding VCM estimates in stock assessments could yield differen results than assessments that use current proportion estimates. The dispersion or variance of the proportion estimates due to sampling is estimated directly from the observer data and there is no need to assume that the amount of overdispersion (*tau*) is constant across sex-length classes. Moreover, the estimated numbers in the sex-length classes could be used in the assessments with the associated estimated VCM.

6.6.2 Total Numbers in Age Classes

6.6.2.1 Longliners

I focus on sablefish for my example on estimating age class numbers in the longline sector of the Alaskan groundfish fishery and management regions used for sablefish stock assessments are the Aleutian Islands (AI), Bering Sea (BS), western and central Gulf of Alaska (WGOA and CGOA), western Yakutat (WY) and the Southeast (SE). I begin with a few hauls made by a longliner of the medium size class in the CGOA during the first quarter of 2002 (Table 6.40). The numbers-at-age in each haul are estimated with Eq. 2.35 and the VCM is estimated with Eq. 2.37. Notice that

Table 6.38. Dispersion for variance estimates corresponding to estimated proportions of Pacific cod trawl catch in sex-length classes relative to corresponding variance estimates under multinomial sampling (Eq. 6.1) for each quarter of 2002 in the Bering Sea. The dispersion assumed in Pacific cod stock assessments ($\sqrt{n}$) is given for each quarter for comparison.

	Quarter 1		Quarter 2		Quarter 3		Quarter 4	
	Female	Male	Female	Male	Female	Male	Female	Male
$\sqrt{n}$	116.288		106.141		63.781		33.317	
<9								
9-11								
12-14								
15-17								
18-20	0.052	0.052	1.075	0.976				
21-23	0.950	0.076	0.813	0.913				
24-26	3.699	1.124	2.038	0.483				
27-29	5.326	4.893	2.428	1.942	2.309			
30-32	6.761	11.027	1.327	1.019	1.844	1.844	0.652	
33-35	13.983	22.678	2.779	1.721	3.914	4.729	0.135	
36-38	5.460	9.066	1.384	1.496	1.501	2.017		0.654
39-41	4.242	5.023	10.227	3.168	3.964	11.555	1.446	1.084
42-44	8.576	6.222	2.263	5.085	5.940	2.702	0.770	0.923
45-49	4.625	4.957	5.177	3.743	3.060	2.957	0.591	1.948
50-54	1.713	1.827	3.966	4.802	0.604	1.041	0.102	2.338
55-59	1.310	2.853	1.591	0.833	2.251	8.256	0.870	1.005
60-64	1.929	0.945	1.613	2.368	1.453	1.566	0.323	2.648
65-69	5.946	2.944	2.469	4.060	2.091	3.281	0.989	2.548
70-74	4.128	3.645	2.867	4.402	4.077	2.552	2.308	0.509
75-79	3.247	2.336	2.562	2.542	3.701	3.530	1.757	2.008
80-84	3.509	3.980	2.388	1.958	4.440	2.756	2.350	1.656
85-89	2.103	1.407	1.314	3.510	3.011	1.386	2.399	1.796
90-94	2.100	1.175	0.965	0.626	1.311	2.370	1.553	0.635
95-99	1.241	2.019	0.352	2.246	1.209	1.870	1.032	0.481
100-104	1.410	1.320	1.144	1.660	1.455	0.516	0.275	0.419
105-115	1.443	0.729	2.016		1.021	0.451		0.411
>115				0.240	1.140			

Table 6.39. Dispersion for variance estimates corresponding to estimated proportions of Pacific cod trawl catch in sex-length classes relative to corresponding variance estimates under multinomial sampling (Eq. 6.1) for each quarter of 2002 in the Gulf of Alaska. The dispersion assumed in Pacific cod stock assessments ($\sqrt{n}$) is given for each quarter for comparison.

	Quarter 1		Quarter 2		Quarter 3		Quarter 4	
	Female	Male	Female	Male	Female	Male	Female	Male
$\sqrt{n}$	74.833		20.396		28.983		17.321	
<9								
9-11								
12-14								
15-17								
18-20								
21-23	3.256							
24-26	0.067							
27-29	2.143	3.337		0.410				
30-32	0.766	0.841		0.410		0.078	1.548	1.046
33-35	0.862	4.113	0.146	0.412	0.670	2.257	1.548	0.544
36-38	2.279	1.888	0.260	0.487	1.002		1.046	2.051
39-41	3.678	3.900	0.207	3.736	0.078	0.785	0.792	0.129
42-44	4.022	6.155	1.765	3.069	0.980	2.175	1.105	1.596
45-49	21.580	11.139	3.779	1.857	4.812	4.997	0.530	1.679
50-54	7.096	3.516	1.751	0.844	4.667	4.973	0.217	0.021
55-59	4.037	10.894	1.972	2.312	4.230	1.865	0.659	0.677
60-64	3.773	0.827	1.825	2.927	21.834	3.463	0.638	0.339
65-69	1.049	1.354	1.564	2.031	3.534	9.434	6.994	0.641
70-74	3.954	4.688	0.577	0.958	3.022	2.107	1.314	2.496
75-79	2.516	8.057	0.558	0.490	7.849	11.079	0.549	1.902
80-84	3.821	3.902	1.177	0.600	8.384	2.983	0.544	1.271
85-89	1.798	1.916			1.961	1.142	1.496	1.046
90-94	3.255	3.317			0.572	0.581	0.544	
95-99	1.650	1.620		0.182	0.868			
100-104	4.961	0.041						
105-115	1.208							
>115								

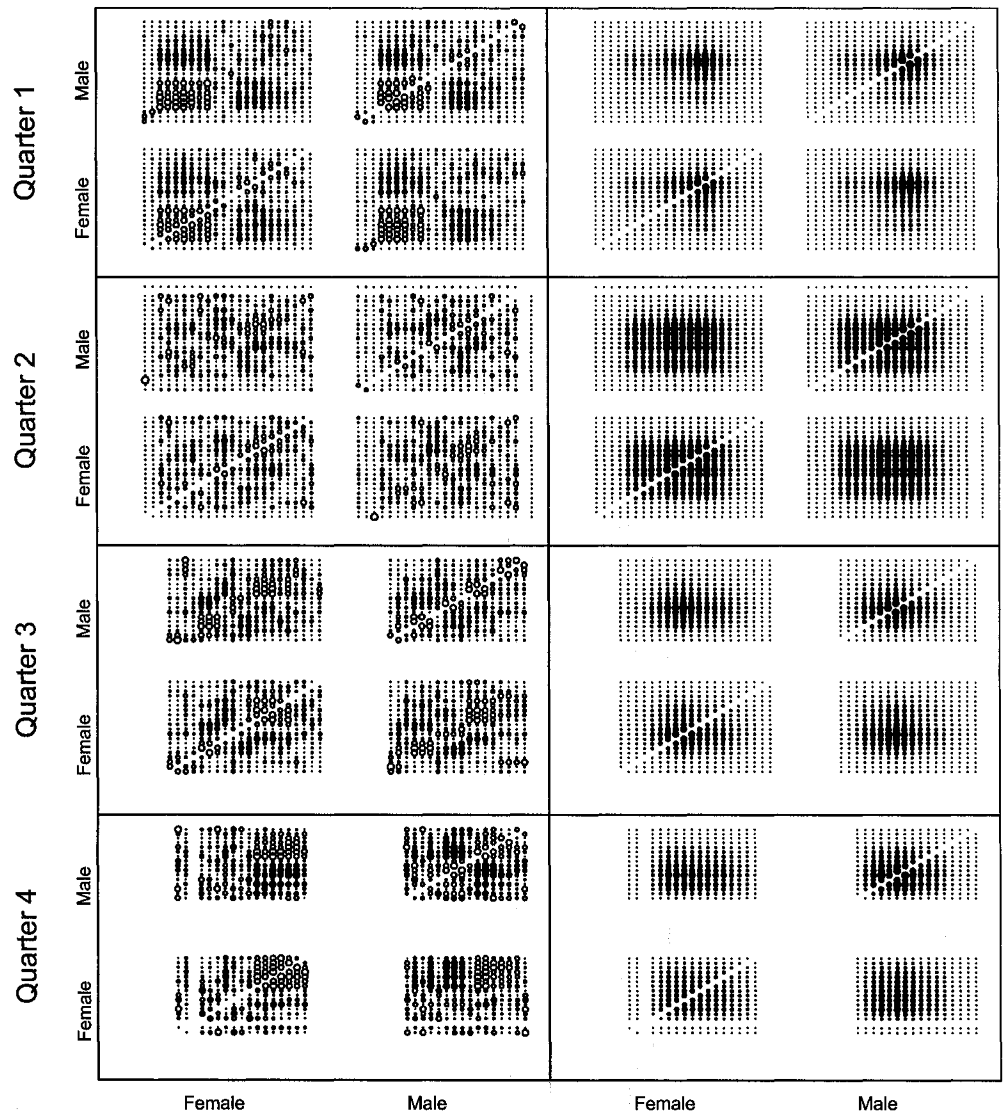

Figure 6.18. Correlation of estimated sex-length class proportions for Pacific cod in Bering Sea trawl catches during 2002. The estimates are quarterly (rows) and the correlations are based on my estimation procedure (left column) and under a multinomial assumption (right column). Open and closed circles represent positive and negative correlation, respectively, the size of the circle represents relative magnitude of the correlation and blank spaces indicate length classes not present in any observer samples.

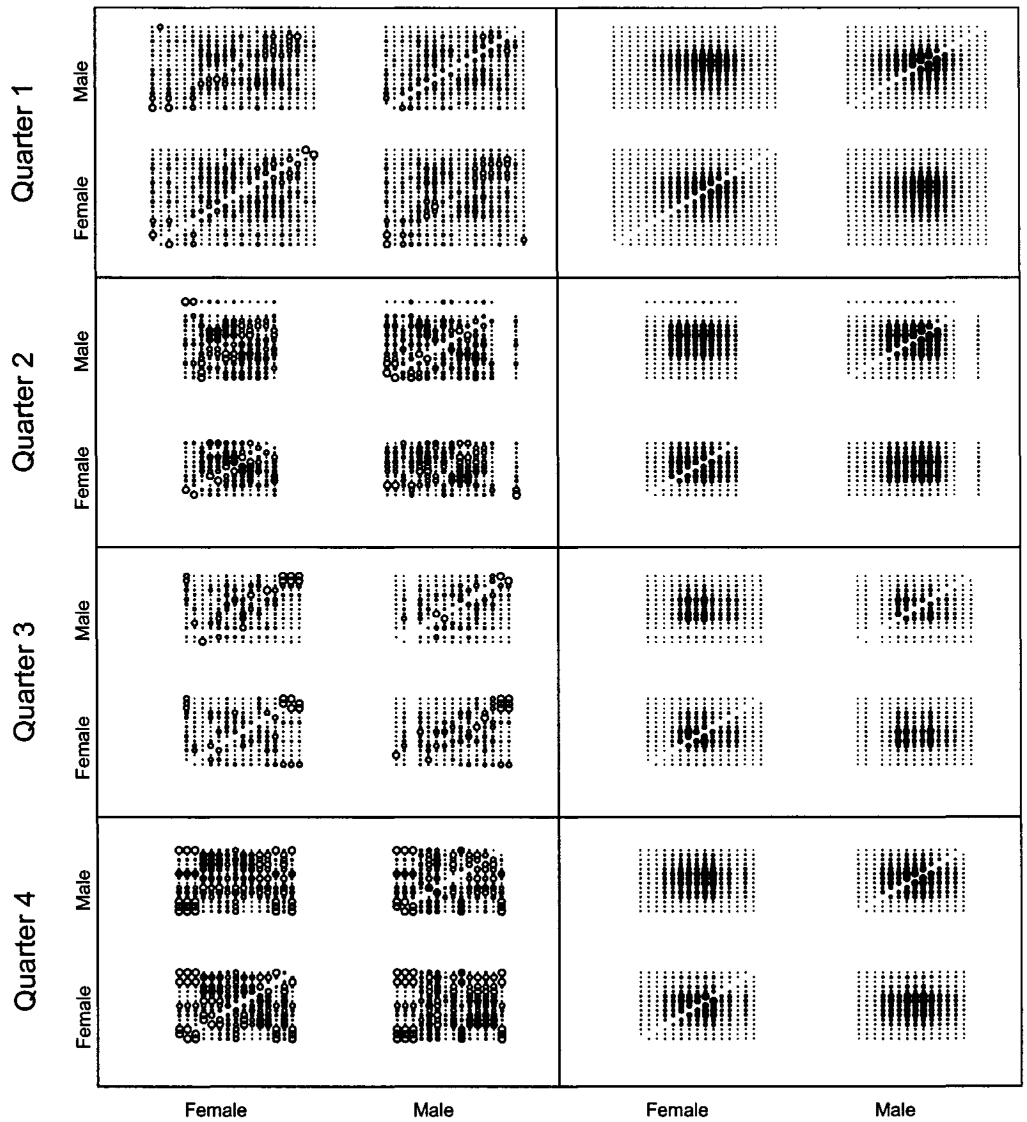

Figure 6.19. Correlation of estimated sex and length class proportions for Pacific cod in Gulf of Alaska trawl catches during 2002. The estimates are quarterly (rows) and the correlations are based on my estimation procedure (left column) and under a multinomial assumption (right column). Open and closed circles represent positive and negative correlation, respectively, the size of the circle represents relative magnitude of the correlation and blank spaces indicate length classes not present in any observer samples.

for one haul no fish were aged and estimates are zero although some otoliths were collected. However, for the second haul, more otoliths than expected were aged which results in estimated numbers-at-age that in sum are greater than the estimated total numbers for the haul. That is, $\sum_{i=1}^{A} \widehat{\Psi} ki > \widehat{N}_k = 420$. In expectation, the sum and the total number estimates are equal; the otolith subsampling for ageing just adds to the variance of the estimates. When making estimates at larger scales, the sum of the age class estimates and the total number estimates will converge.

The age class numbers in s-prevalent hauls made during the trip and the corresponding VCM are estimated using Eq. 5.30 and Eq. 5.31. Estimation of the age class numbers at the vessel level is performed analogously to the length class numbers, but as the estimator (Eq. 5.93) shows, estimation of the VCM must now account for covariance among trips that are sufficiently sampled in addition to covariance among trips that are undersampled. The covariance among sufficiently sampled trips which is due to the subsampling of otoliths for ageing, also occurs among vessels (see 5.98) and among quarter estimates within a year (see 5.102) when the age sample is chosen from all otoliths collected during a year.

The proportion of overall sablefish caught in hauls where they were prevalent was between ≈82–95% for all of the IFQ management regions except the Bering Sea where the proportion was only ≈10%. I used Eq. 5.108 to scale up the 2002 regional age class estimates in s-prevalent hauls (Figure 6.20). Relative to the number of Pacific cod measured for lengths, very few sablefish otoliths are aged and precision of estimated numbers-at-age is consequently much poorer.

The estimated dispersion of the proportion estimates relative to that of multinomially distributed ageing samples (using 6.1) are near one which indicates there is little overdispersion of the proportion estimates (Table 6.41). In fact, many of the regional proportion estimates have dispersions estimated to be less than that under a multinomial assumption. For most proportions, dispersion estimates imply that sampling variance is at most five times greater than expected when the samples are multinomially distributed.

The correlations of sablefish age class proportions are presented in Figure 6.21 for each region. In all regions there is a consistent pattern of positive correlation of the youngest individuals with one or more older age classes. However, which of the older age classes with which the youngest age classes positively correlate varies between

Table 6.40. Data collected and resulting estimates of numbers in age classes for 3 of 14 hauls made during a trip by a longline vessel of the medium size class in the central Gulf of Alaska in the first quarter of 2002. The number of hooks deployed (H_k) and sampled (h_k), number of counted sablefish (n_k), the number of otoliths sampled (n_{ok}) along with the numbers of aged otoliths in each age class and the total number of otoliths collected (N_{om} = 1038) and aged (n_{Am} = 206) in the region are used to estimate the total numbers in the age classes.

	1		2		3	
	data	$\widehat{\Psi}_{ki}$	data	$\widehat{\Psi}_{ki}$	data	$\widehat{\Psi}_{ki}$
H_k	4350		4350		3828	
h_k	1740		1740		1392	
n_k	111		168		229	
n_{ok}	3		2		2	
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	1	1,586.6
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	0	0	0	0	0	0
12	0	0	1	1,058.16	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	1	1,058.16	0	0
19	0	0	0	0	0	0
20	0	0	0	0	0	0
21	0	0	0	0	0	0
22	0	0	0	0	0	0
23	0	0	0	0	0	0
24	0	0	0	0	0	0
25	0	0	0	0	0	0
26	0	0	0	0	0	0
27	0	0	0	0	0	0
28	0	0	0	0	0	0
29	0	0	0	0	0	0
≥30	0	0	0	0	0	0

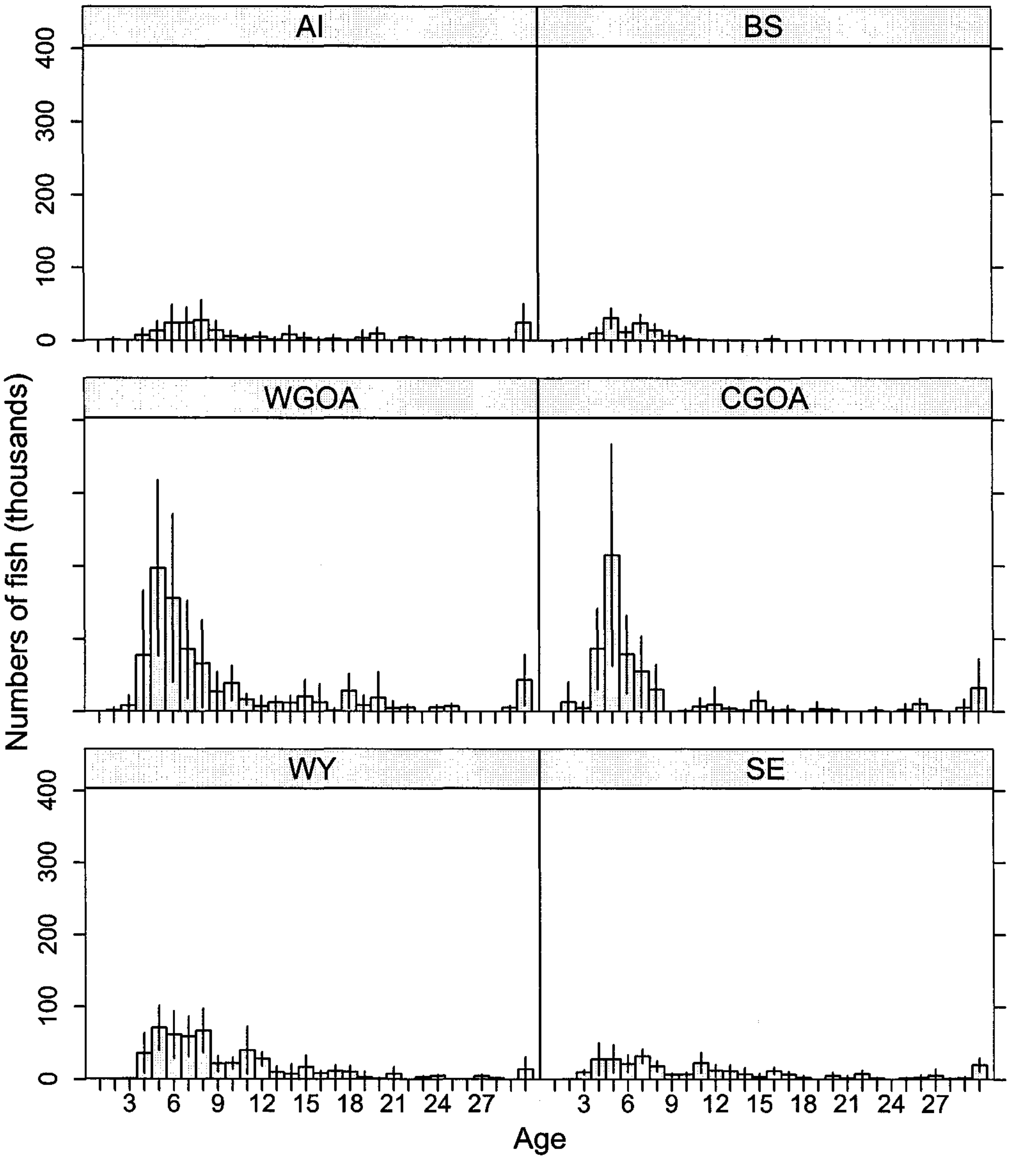

Figure 6.20. Estimated age class numbers of sablefish caught by longliners in each management region: Aleutian Islands (AI), Bering Sea (BS), western and central Gulf of Alaska (WGOA and CGOA), western Yakutat (WY) and the southeast (SE) during 2002. Vertical bars represent asymptotic 95% confidence intervals.

Table 6.41. Dispersion for variance estimates corresponding to estimated proportions of sablefish longline catch in age classes relative to corresponding variance estimates under multinomial sampling (Eq. 6.1) for 2002 by region: Aleutian Islands (AI), Bering Sea (BS), western and central Gulf of Alaska (WGOA and CGOA), western Yakutat (WY) and the Southeast (SE). Blank spaces occur where age classes are not present among aged otoliths.

Age Class	AI	BS	WGOA	CGOA	WY	SE
1						
2	1.817	0.778	4.240	0.689		
3		1.619	1.391	1.288	0.046	0.299
4	2.921	1.688	3.153	5.147	1.739	3.454
5	2.186	1.783	5.025	4.910	0.882	2.223
6	4.003	1.037	2.787	5.267	1.165	1.227
7	3.159	1.540	2.685	2.538	0.760	0.397
8	4.157	1.337	2.034	2.443	0.845	0.495
9	2.165	1.947		1.861	0.435	0.232
10	2.577	1.979	0.583	1.025	0.282	0.279
11	1.927	0.420	1.503	0.312	2.048	1.583
12	2.087	0.422	3.911	1.602	0.372	0.930
13	2.626	0.320	0.194	0.359	0.489	0.931
14	4.052	0.409	0.097	0.592	2.021	1.800
15	3.229		0.977	1.501	1.151	1.151
16	2.188	1.842	0.950	3.309	0.239	0.313
17	2.420		1.165	0.585	0.549	0.380
18	0.288			1.310	0.514	0.339
19	6.323	0.422	2.334	1.358	2.033	
20	2.120	0.201	0.125	3.982	0.065	0.628
21	0.411	0.369		1.376	1.147	0.966
22	0.510			0.174		0.676
23	0.962		1.237		0.068	0.157
24	0.890	0.418		0.141	0.083	
25	1.110	0.455	1.986	0.181		0.172
26	1.178	0.209	0.541			0.566
27	0.302		0.073		0.113	2.813
28					0.067	0.096
29	2.318	0.455	1.730	0.161		0.183
$\geq$30	7.336	0.350	2.669	1.720	1.451	0.646

regions.

6.6.2.2 Trawlers

For trawl effort, I turn to walleye pollock and I expand the set of categories to sex-specific age classes (sex-age classes) using data collected in 2001. Stock-assessment scientists make estimates for 20 age classes so with sex-age classes I estimate a 40×1 vector, in general. Table 6.42 shows relevant data for a haul made by a vessel that caught pollock in the GOA in the first quarter of 2001. These data from each s-prevalent haul made during the trip are used in Eq. 3.45 to estimate the sex-age class numbers for the trip. As with length and sex-length class estimators, the sex-age class numbers estimator (Eq. 3.45) incorporates an estimator of the total number of walleye pollock caught (Eq. 3.33) which relies on information collected throughout the trip.

When estimating the sex-age class numbers over the trip Eq. 5.30 is used and Eq. 5.31 is used to estimate the VCM. The VCM estimator accounts for the covariance among hauls which is a function of both the model-based methods for total numbers and the subsampling of otoliths for ageing and is estimated with Eq. 5.33. Aside from the different covariance estimator, estimation of numbers-at-age and -sex for longliners is similar to that for trawlers.

I scaled up the sex-age class numbers in s-prevalent hauls (Eq. 5.108) to obtain estimates pertaining to all walleye pollock catches by trawlers for each region: eastern Bering Sea (BSE), western Bering Sea (BSW) and Gulf of Alaska (GOA) in 2001 (Figure 6.22). Catches of males and females are not substantially different for any region, but the distribution of ages in catches does change across regions. Similar proportions of several younger age classes are estimated for the GOA, whereas there is a pronounced peak in ages of five or six years in the BSE and BSW. Oldest age classes are observed in the BSE. Estimates for the Aleutian Islands are not available because otoliths were not aged, but there was no directed fishery in this region.

As for other examples of length or age class estimation, I calculated the dispersion of sampling variance relative to that under multinomially distributed ageing samples (Eq. 6.1). Most sex-age class variance estimates are overdispersed to some degree with respect to multinomial sampling, but some sex-age classes are also underdispersed (Table 6.43). The dispersion estimates are similar in magnitude to those found for

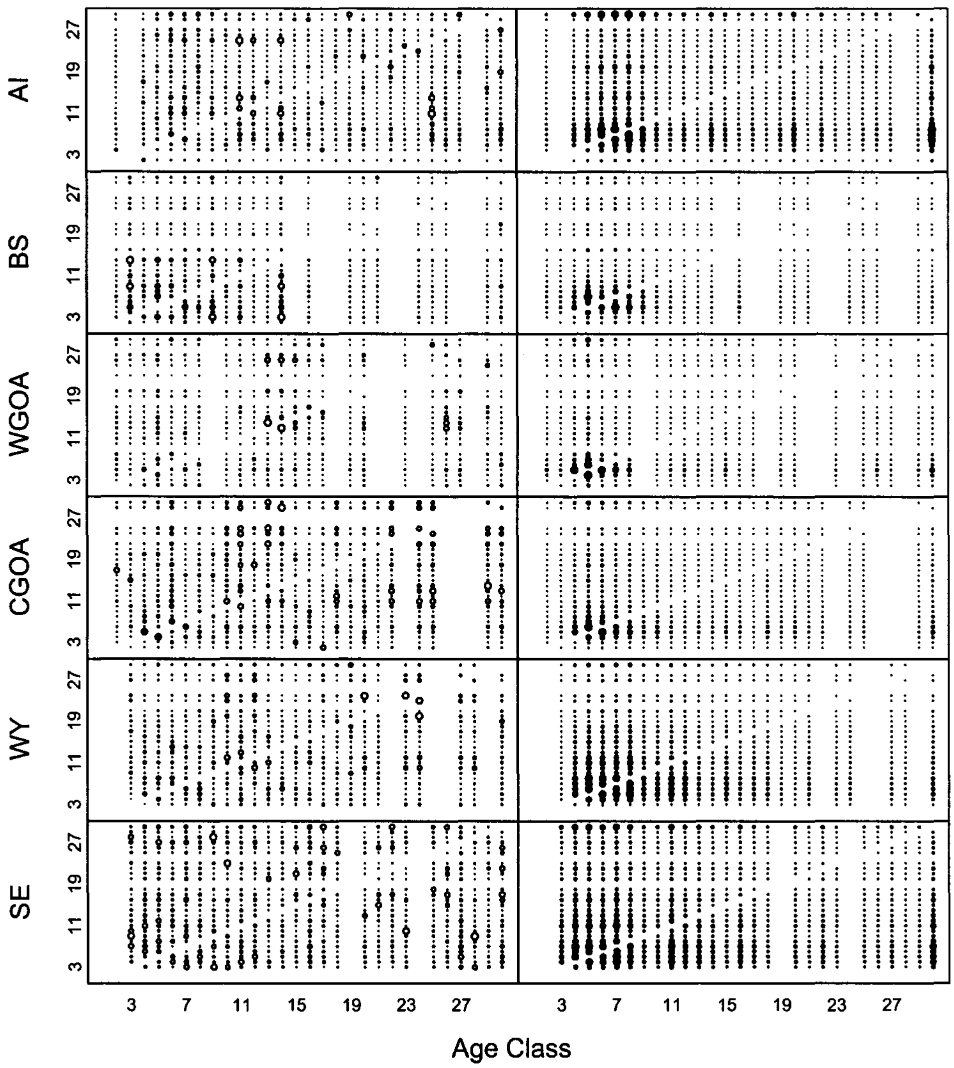

Figure 6.21. Correlation of estimated age class proportions of sablefish caught by longliners in 2002. The estimates are regional: Aleutian Islands (AI), Bering Sea (BS), western and central Gulf of Alaska (WGOA and CGOA), western Yakutat (WY) and the southeast (SE) and the left and right columns represent correlations based on my estimation procedure and corresponding correlations under a multinomial assumption, respectively. Open and closed circles represent positive and negative correlation, respectively, the size of the circle represents relative magnitude of the correlation and blank spaces indicate length classes not present in any observer samples.

Table 6.42. Data collected for a haul made during a trip by a trawler in the Gulf of Alaska in the first quarter of 2001. The total weight of the haul (Υ_k), sampled weight (v_k), number of sampled walleye pollock (n_k), the number of otoliths sampled (n_{ok}) and the numbers of aged otoliths in each sex-age class (number aged is less than n_{ok}). The total number of otoliths collected ($N_{om} = 1704$) and aged ($n_{Am} = 637$) in the region are also used to estimate the total numbers in the age classes.

	Female	Male
1	0	0
2	0	0
3	0	1
4	1	0
5	0	0
6	0	0
7	0	0
8	0	0
9	0	0
10	0	0
11	0	0
12	0	0
13	0	0
14	0	0
15	0	0
16	0	0
17	0	0
18	0	0
19	0	0
≥20	0	0
Υ_k	293.95	
v_k	293.95	
n_k	357.00	
n_{ok}	4	

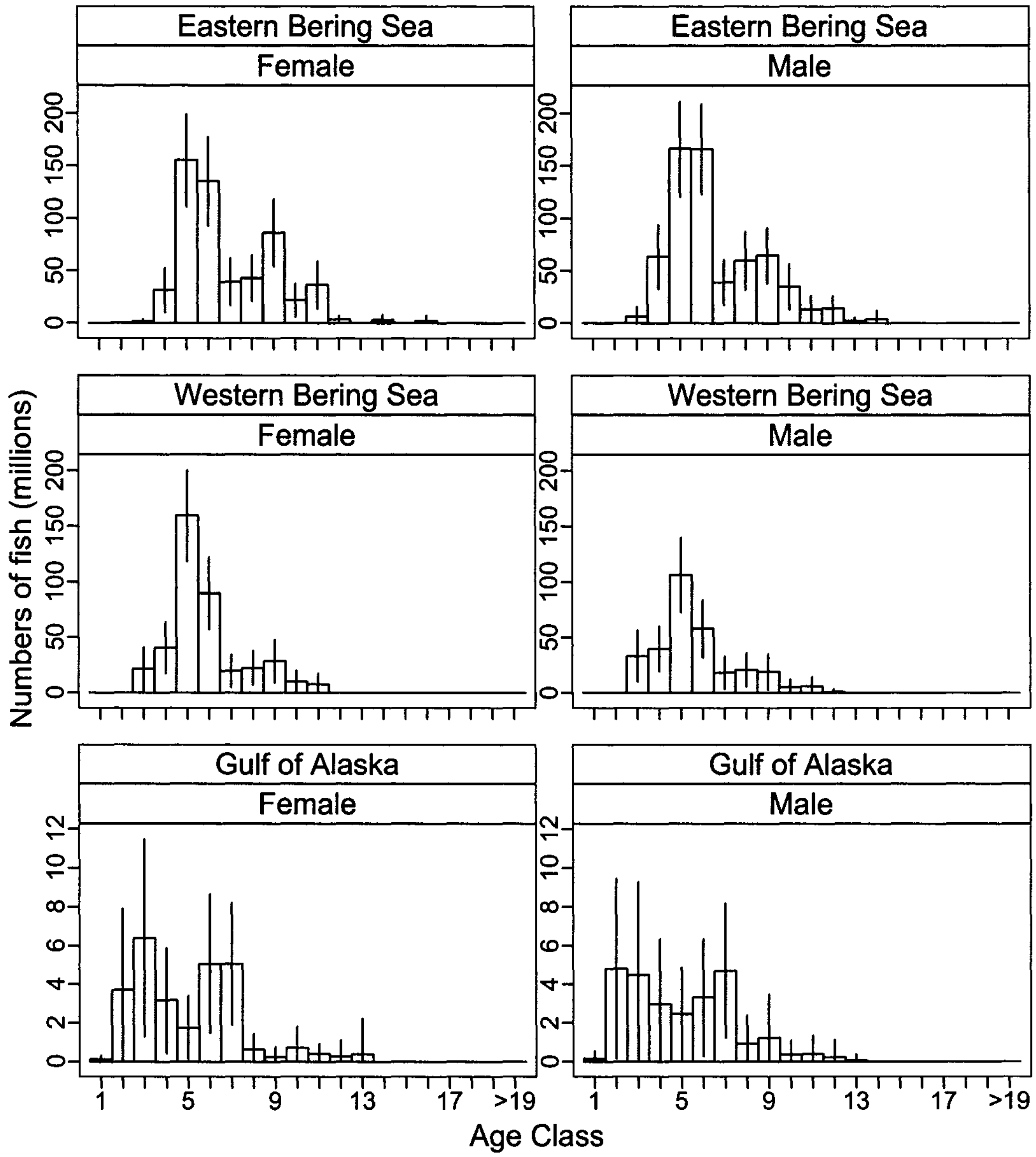

Figure 6.22. Estimated numbers in sex-age classes for walleye pollock caught by trawlers during 2001 by region: eastern Bering Sea (BSE), western Bering Sea (BSW) and Gulf of Alaska (GOA). Vertical bars represent asymptotic 95% confidence intervals.

sablefish age classes (Table 6.41). The correlations of most sex-age class proportion estimates are negative for all regions (Figure 6.23). In fact, correlations of proportion estimates for the BSW appear to be nearly identical to those under multinomial sampling.

Table 6.43. Dispersion for variance estimates associated with estimated proportions of walleye pollock trawl catch in sex-age classes relative to corresponding variance estimates under multinomial sampling (Eq. 6.1) for 2002 in the eastern Bering Sea (BSE), western Bering Sea (BSW) and Gulf of Alaska (GOA).

Age Class	**BSE**		**BSW**		**GOA**	
	Female	Male	Female	Male	Female	Male
1					0.905	2.648
2	0.011	0.137	0.012		13.797	13.038
3	1.040	3.406	3.056	2.590	14.964	17.355
4	3.114	3.210	2.240	1.625	6.765	11.743
5	2.328	2.381	0.802	1.672	4.663	7.091
6	2.722	1.941	1.719	1.740	6.133	7.357
7	2.868	2.620	2.005	2.111	4.832	6.892
8	2.439	2.690	1.788	1.910	2.681	6.095
9	2.422	2.295	2.351	2.482	3.380	11.459
10	2.496	2.952	1.828	1.767	4.645	4.034
11	3.144	2.876	2.387	2.121	1.901	7.031
12	0.833	2.330		1.528	7.667	9.956
13	0.009	1.046			24.291	4.740
14	2.049	4.005				
15	0.011					
16	3.015					
17						
18						
19						
$\geq$20						

Because the dispersions of the BSW sex-age class proportion estimates are similar across classes and the correlations of these estimates are very similar to that when the sampling distribution is multinomial (or overdispersed multinomial), this is the only case where stock assessments performed with these estimates would yield results that are similar to current practices. However, in many Alaskan groundfish stock assess-

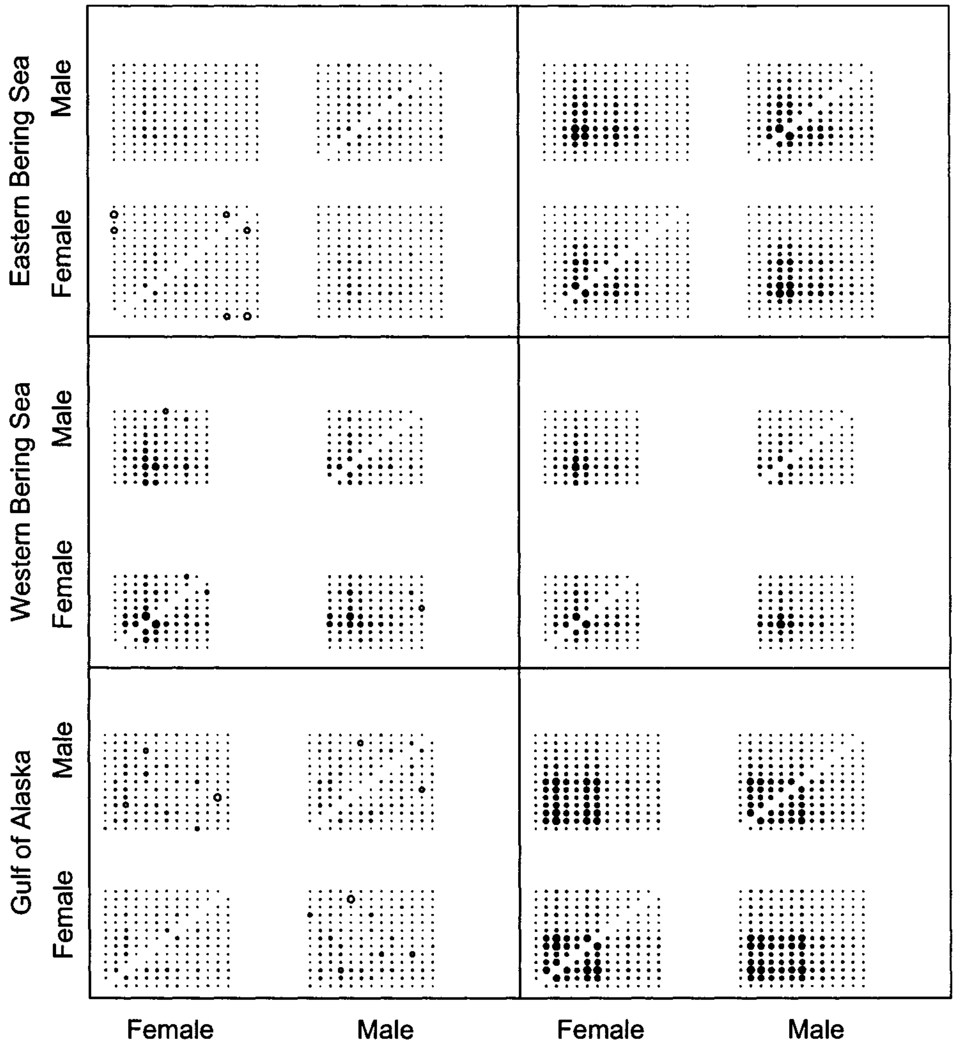

Figure 6.23. Correlation of estimated sex-age class proportions for walleye pollock in 2001 trawl catches by region: eastern Bering Sea (BSE), western Bering Sea (BSW) and Gulf of Alaska (GOA). Left and right columns represent correlations based on integrated model- and design-based methods and corresponding correlations under a multinomial assumption, respectively. Open and closed circles represent positive and negative correlation, respectively, the size of the circle represents relative magnitude of the correlation and blank spaces indicate length classes not present in any observer samples.

ments, sex and proportions-at-age are calculated differently (e.g., Kimura 1989) than as direct estimates of sex-age classes from observer data (James N. Ianelli, personal communication, 2005).

Chapter 7

OPTIMIZING OBSERVER COVERAGE

In this chapter, I examine the allocation of observer coverage that yields optimal (or maximized) precision of catch parameter estimates based on formulae I provide in Chapters 2 through 5. This optimization depends on a fixed amount of available coverage and the estimators I use for the optimization are the same integrated design- and model-based estimators I used in Chapter 6 to obtain yearly estimates for various catch parameters.

The optimal coverage I treat here is a special type of optimal sampling discussed in various sampling theory texts (e.g., Cochran 1977; Särndal et al. 1992). Optimal sampling is an important goal for any study that uses a well defined sampling design because there are costs associated with sampling a population and finances are rarely unlimited. When designs yield estimates with low precision, inferences based on the quantities of interest can be less certain. Therefore, ensuring that precision is as high as possible (or approximately so) with a given sample size is an important goal. Furthermore, in stratified designs, optimality results generally shed light on strata within the population that exhibit more or less variability than others because greater sample sizes will be optimal for strata with greater variablity when costs of sampling the strata are equal.

Current regulations require 100% observer coverage for trawl and longline vessels with length greater than 125 ft (large vessels) and 30% for trawl and longline vessels with length less than 125 ft and greater than 60 ft (medium vessels). All pot vessels greater than 60 ft in length require 30% observer coverage. There is no observer coverage required for any vessels less than 60 ft in length. For the stratified multi-phase design used by the North Pacific Groundfish Observer Program, determining optimal sampling fractions of trips for vessels by size class approximates optimization of observer coverage. On whatever catch parameter the optimization is based, maximum precision of its estimate is obtained. Current observer coverage levels have been in place for approximately 15 years and were originally based on financial impact to the vessels requiring coverage. In the current system, vessel owners pay for observer costs which increase with the number of observed fishing days. However, if the system used to pay for observer coverage was modified so that each vessel owner contributed a set amount or contributed as a function of their fishing effort, different allocation of observers to the size classes would be possible. An alternative observer coverage allocation could increase precision of catch parameter estimates. Since the NPGOP is currently considering such an alternative observer coverage scheme, a quantititative approach for determining coverage requirements is particularly relavent.

7.1 Analytical Approach

Some simplifications are necessary for determining optimal observer coverage. In my development of conservative variance estimation for vessels in the medium size class (Chapter 5), I resort to a model to describe the variability of the number of sampled trips. Here, I assume that steps will be taken by the NPGOP to make the total number of trips known for all vessels. Furthermore, as I have repeated throughout the previous chapters, within-haul variance estimation is not possible because of insufficient detail in recorded data, but I assume for future sampling, that this data will be recorded in appropriate detail. My model-based methods for undersampled trips cause covariance of predictions for those trips, but the optimal sampling results I present will ignore this covariance. Finally, future fishing will never be identical to past fishing in the number of trips made by each vessel. As such, it is more appropriate to optimize an expected rate of sampling for trips within vessels rather than actual numbers of elements to sample. This implies the assumption that any overall increase or decrease in fishing effort can be matched with a proportional increase in observer deployment and sampling.

I derive optimal observer coverage levels in arbitrary categories under a fixed cost constraint. When a single catch parameter is considered, my results regarding optimality are based on the minimization of variance for that catch parameter. When multiple catch parameters are considered, I minimize a linear combination (i.e., weighted sum) of relative variances (square of the coefficient of variation) for the respective catch parameters. In the multi-parameter context, other functions to minimize are concievable, but using the relative variance accounts for differences in scale of catch parameters and parameter-specific weights allows a management entity to determine the importance of each parameter relative to the others.

As I mentioned above there are two size classes of vessels that require observer coverage (medium and large) that are split by a length criterion of 125 ft (except for pot vessels). To reduce variance of catch parameter estimates with this length criterion, allocation of observed trips can be shifted between the size classes. Other classes may be defined for allocation of observed trips, but classes defined only by the length criterion is practical because current coverage is the same for some gear types (trawl and longline) of a given size class.

I further explore optimality under a varying length criterion such that it is constant across gear types or such that separate length criteria are allowed for each gear type. The fixed length criteria scenarios will have analytic solutions, but the variable length criteria extension will not. Rather, the optimal results for fixed length criteria are applied within a minimization algorithm.

7.2 Optimal Observer Coverage for Size Classes

7.2.1 Single Catch Parameter

In the context of the NPGOP sampling design, the strata within a given quarter are the vessels and there is a multi-phase design within each vessel. Many of the phases can be treated like stages in a multi-stage design including the first phase which is the sampling (observing) of trips within a vessel. As I have in previous chapters, I assume here that the set of observed trips for the vth vessel is obtained by SRS of all the trips made by that vessel in a given quarter and the variance of the estimator of the total for a generic catch parameter on the vth vessel, $\widehat{\Theta}_v$ is

$$V\left(\widehat{\Theta}_v\right) = C_v\left(\frac{C_v}{c_v} - 1\right) S^2_{\Theta,v} + \frac{C_v}{c_v}\sum_{t=1}^{C_v} V\left(\widehat{\Theta}_t\right). \tag{7.1}$$

where C_v is the total number of fishing trips made by the vth vessel, c_v is the number of observed trips,

$$S^2_{\Theta,v} = \frac{\sum_{t=1}^{C_v}\left(\Theta_t - \overline{\Theta}_v\right)^2}{C_v - 1}$$

is the variance among trips for the vth vessel and $V\left(\widehat{\Theta}_t\right)$ is the variance due to sampling within the tth trip. The variance (Eq. 7.1) is that of a two-stage design with SRS at the first stage and it is related to the variance presented in Eq. 5.72 except that covariance of trip estimators is ignored (see my simplifications stated in Section 7.1) or nonexistent. The variance can also be written as

$$V\left(\widehat{\Theta}_v\right) = \frac{C_v}{c_v}\left[C_v S^2_{\Theta,v} + \sum_{t=1}^{C_v} V\left(\widehat{\Theta}_t\right)\right] - C_v S^2_{\Theta,v} = \frac{A_v}{f_v} + B_v$$

where $A_v = C_v S^2_{\Theta,v} + \sum_{t=1}^{C_v} V\left(\widehat{\Theta}_t\right)$, $B_v = C_v S^2_{\Theta,v}$ and $f_v = c_v/C_v$.

The sampling fraction of trips, f_v, will not exactly equal the percent coverage that is required for a vessel when coverage is less than 100%, but f_v should approach the coverage requirement over time when there is no relationship between trip duration and presence of an observer. Ultimately the goal is to minimize the variance as a function of the expected sampling fraction of trips which I equate to percent observer coverage and which is analogous to the expected variable cost approach of Särndal et al. (1992, Section 12.8). When we do so, the expected sampling fraction is identical for all vessels in a size class so that the variance of the parameter estimate for all fishing effort in the qth quarter is

$$V\left(\widehat{\Theta}_q\right) = \sum_{s=1}^{S} \frac{1}{f_s} \sum_{v=1}^{V_s} A_v + \sum_{s=1}^{S} \sum_{v=1}^{V_s} B_v = \sum_{s=1}^{S} \frac{A_s}{f_s} + \sum_{s=1}^{S} B_s$$

where S is the number of size classes and V_s is the number of vessels in the sth size class and $B_s = \sum_{v=1}^{V_s} B_v$. For an aggregation of quarters, say for the yth year, the variance is just,

$$V\left(\widehat{\Theta}_y\right) = \sum_{q=1}^{4} V\left(\widehat{\Theta}_q\right) = \sum_{s=1}^{S} \frac{A_{sy}}{f_s} + \sum_{s=1}^{S} B_{sy}. \tag{7.2}$$

Notice that the variance (Eq. 7.2) derives from a stratified two-stage design.

When a single catch parameter is of interest, Eq. 7.2 is the function that is minimized given a cost constraint. Consider a general cost function for the yth year,

$$T_y = T_{0y} + \sum_{s=1}^{S} \sum_{v=1}^{V_{sy}} \sum_{t=1}^{c_v} T_{svt}$$

where T_{0y} is a cost component of the sampling project that is not affected by the amount of sampling (i.e., fixed cost) and T_{svt} is the cost of sampling the tth trip aboard the vth vessel in the sth size class. In truth, cost of sampling each trip increases with the duration of the trip because the vessel pays for each day the observer is on board, but cost is also complicated by the fact that observers are hired for set periods of time by contracting companies. A simpler cost function that ignores the differences in trip duration and assumes that a set amount of effort is available and that cost is equal for trips made by different vessels and different size classes is

$$T_y = T_0 + T_c \sum_{s=1}^{S} \sum_{v=1}^{V_{sy}} c_{sv} = T_0 + T_c \sum_{s=1}^{S} \sum_{v=1}^{V_{sy}} C_{sv} f_s = T_0 + T_c \sum_{s=1}^{S} f_s C_{sy}. \tag{7.3}$$

where T_c is the cost of sampling a trip and C_{sy} is the total number of trips made by vessels in the sth size class in the yth year. Ignoring the differences in trip duration should be appropriate for application to future sampling because the duration of future trips will be unknown. Moreover, there is no reason that the distribution of trip durations should be drastically different for future fishing. However, if there is consistent differences in trip duration between size classes, then trip duration should be considered.

With the objective function (Eq. 7.2) and the cost function (Eq. 7.3) the optimal sampling fractions or observer coverages for each size class can be obtained analytically using the Cauchy-Schwartz Inequality or Lagrange multipliers. However, notice that the second component of Eq. 7.2 and the first component of Eq. 7.3 are not functions of f_s and, therefore, the minimum of Eq. 7.2 with respect to f_s will not be affected by those components. By the Cauchy-Schwartz Inequality,

$$[T_y - T_{0y}]\left[V\left(\widehat{\Theta}_y\right) - \sum_{s=1}^{S} B_{sy}\right] = \left(\sum_{s=1}^{S} T_c f_s C_s\right)\left(\sum_{s=1}^{S} \frac{A_{sy}}{f_s}\right) \geq \left(\sum_{s=1}^{S} \sqrt{A_{sy} T_c C_{sy}}\right)^2$$

where equality is achieved when

$$\left(\frac{A_{sy}/f_s}{T_c f_s C_{sy}}\right)^{\frac{1}{2}}$$

are equal for all s. The resulting optimal sampling fraction for the sth size class in the yth year is

$$f_{sy}^{*} = \frac{T_y - T_0}{T_c} \frac{(A_{sy}/C_{sy})^{\frac{1}{2}}}{\sum_{s=1}^{S}(A_{sy}C_{sy})^{\frac{1}{2}}} = c_y \frac{(A_{sy}/C_{sy})^{\frac{1}{2}}}{\sum_{s=1}^{S}(A_{sy}C_{sy})^{\frac{1}{2}}} \tag{7.4}$$

where c_y is the number of trips sampled in the year (i.e., the total amount of sampling effort available. The result in Eq. 7.4 is identical in form to the analytical result for optimal sampling of a stratified design without multi-phase sampling within the strata (e.g., Cochran 1977, pg. 98). Specifically, $A_{sy} = N_{sy}S^2_{\Theta,sy}$ would give the more familiar result, but A_{sy} has more complicated structure due to the stratification by quarter and vessel and the multiphase sampling within those strata.

To use the optimal coverage result we have to use estimates of A_{sy} and also for C_{sy} when the true total number of trips for a size class is unknown. Recall that

$$A_{sy} = \sum_{q=1}^{4}\sum_{s=1}^{S}\sum_{v=1}^{V_{sq}} A_{sqv} = \sum_{q=1}^{4}\sum_{s=1}^{S}\sum_{v=1}^{V_{sq}} \left[C_v S^2_{\Theta,v} + \sum_{t=1}^{C_v} V\left(\widehat{\Theta}_t\right)\right]$$

so all we need are unbiased estimators for each vessel component which are

$$\widehat{A}_v = C_v \widetilde{S}^2_{\Theta,v}$$

where

$$\widetilde{S}^2_{\Theta,v} = \frac{\sum_{t=1}^{c_v}\left(\widehat{\Theta}_t - \widehat{\overline{\Theta}}_v\right)^2}{c_v - 1}.$$

The unknown number of trips made by vessels in the medium size class are predicted by $10c_v/3$.

Other inconveniences that must be dealt with are the vessels that are undersampled (see Section 5.3.2) and vessels that currently have 100% coverage (large size class), but only one trip sampled. We can use the average $\widetilde{S}^2_{\theta,v}$ among large vessels with more than one trip sampled as a predictor of $\widetilde{S}^2_{\theta,v}$ for large vessels with only one trip sampled. For the undercovered medium vessels, we can use $\widehat{V}\left(\widehat{\Theta}_U\right)/V_U$ as a predictor where $\widehat{V}\left(\widehat{\Theta}_U\right)$ is given in Eq. 5.99.

Although Eq. 7.4 provides an analytic result for optimal observer allocation, it is useful to see how the variance changes with different sampling effort. For some catch parameters there will be very little increase in variance in a large vicinity of the optimal allocation whereas for other catch parameters substantial changes in variance may be observed. Figure 7.1 shows example curves of coefficients of variation for total catch weight of Pacific cod in the Bering Sea, total bycatch in numbers of chinook salmon in the Gulf of Alaska, total bycatch in numbers of black-footed albatross in the Gulf of Alaska and total catch weight of walleye pollock in the eastern Bering Sea for the first quarter of 2003. To the right of the current coverage levels, the curve represents the coefficient of variation at constant amount of trip coverage, but reallocated to the different size classes. To the left of the current coverage levels, the curve still represents the coefficient of variation, but the amount of coverage is necessarily decreasing. For these results all pot vessels are in the medium stratum. Notice that for walleye pollock and Pacific cod, the estimated coefficients of variation are quite low for a wide range coverage levels. On the other hand, it appears estimates of chinook salmon and black-footed albatross bycatch would be much more efficient if coverage were shifted completely to the medium size class as there was no mortality for these species aboard vessels in the large size class. Of course, these results are for a single quarter and larger time scales are more relevant. However, these result display graphically the contradictory optimalities that result for different catch parameters and the different relationships of estimator efficiency to coverage levels.

7.2.2 Multiple Catch Parameters

The previous optimal observer coverage levels (Eq. 7.4) result from minimizing the variance of a single catch parameter given a fixed cost constraint. However, there are multiple catch parameters of interest to researchers that use the data collected by observers. Optimal levels of effort for many or all of the catch parameters that are deemed important by the NMFS should be the true focus. Texts on sampling theory discuss several related approaches to dealing with optimal allocation when multiple parameters or characteristics are of interest. A commonly mentioned analytic approach is to minimize the sum or average of variances (Cochran 1977; Särndal et al. 1992) and an obvious extension that allows managers to weight different parameters

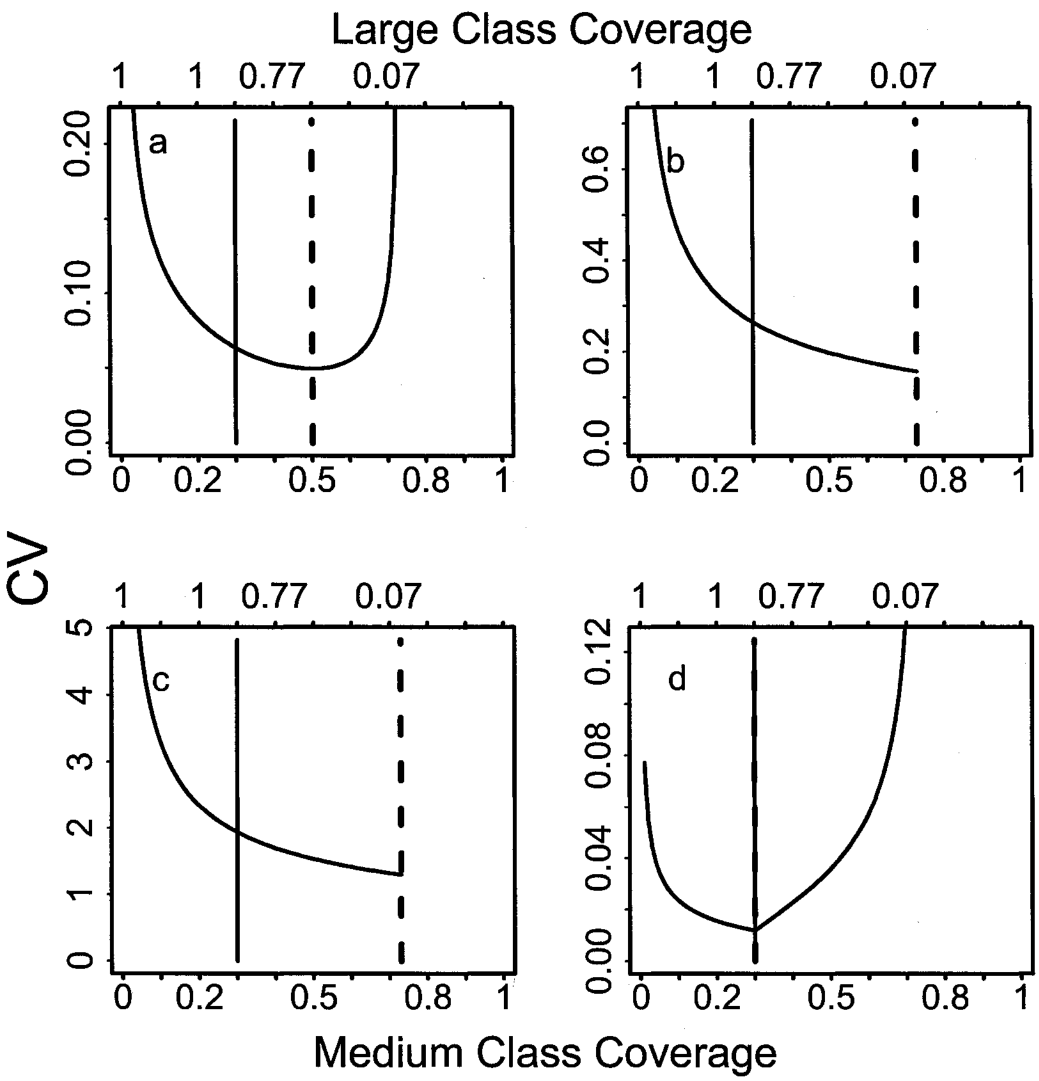

Figure 7.1. Predicted coefficients of variation at different allocations of observer coverage to the different size classes of vessels. The results pertain to the first quarter of 2003 and the catch parameters plotted are total catch weight of Pacific cod in the Bering Sea (a), total catch in numbers of chinook salmon in the Gulf of Alaska (b), total catch in numbers of black-footed albatross in the Gulf of Alaska (c) and total catch weight of walleye pollock (d). The bottom and top axes are medium and large size class coverage, respectively, the dashed line give the point of optimal coverage and the solid vertical line is at current coverage.

according to relative preference is

$$O_1 = \sum_{p=1}^{P} w_p V\left(\widehat{\Theta}_p\right) \tag{7.5}$$

where P is the total number of parameters to optimize over and w_p is an arbitrary weighting constant. This type of objective function is used by Kimura (1977) and Lai (1987, 1993). However, a problem intrinsic to Eq. 7.5 is that for some $p \neq p'$, $V\left(\widehat{\Theta}_p\right) \ll V\left(\widehat{\Theta}_{p'}\right)$ just by the nature of the distributions of the respective catch parameters in the population of interest. Sukhatme and Sukhatme (1970) suggest using the sum of relative variances (square of the coefficient of variation) which alleviates the problem. Again, an extension that allows managers to weight parameters is

$$O_2 = \sum_{p=1}^{P} w_p \frac{V\left(\widehat{\Theta}_p\right)}{\Theta_p^2} = \sum_{p=1}^{P} w_p RV\left(\widehat{\Theta}_p\right). \tag{7.6}$$

A weighted sum of the relative variances is not discussed in any of the sampling texts, but is an important candidate in the case of an observer program where managers determine relative importance of different catch parameters. An alternative objective function that is related to Eq. 7.6 and based on the coefficient of variation is

$$O_3 = \sum_{p=1}^{P} w_p \frac{\sqrt{V\left(\widehat{\Theta}_p\right)}}{\Theta_p} = \sum_{p=1}^{P} w_p CV\left(\widehat{\Theta}_p\right), \tag{7.7}$$

which apparently has only been applied recently. Manly et al. (2002) minimize a weighted function of the maximum, minimum and average coefficients of variation for the multiple parameters. Another objective function that incorporates the relative variances as well as the covariances of different catch parameters is

$$O_4 = \mathbf{w}^T \left[RV\left(\widehat{\Theta}\right)\right] \mathbf{w} \tag{7.8}$$

where $\mathbf{w}$ is a $P \times 1$ vector of weights and $RV\left(\widehat{\Theta}\right)$ is the $P \times P$ relative variance-covariance matrix corresponding to the P parameters of interest. Optimization of an objective function similar in flavor to Eq. 7.8 has been explored by previous authors when closely related parameters such as proportions-at-age are of interest (e.g., Smith and Sedransk 1982).

I focus on the second objective function (Eq. 7.6) which is relatively simple and has not been treated in sampling texts nor has it ever been applied in a fisheries context, but for which analytic solutions for optimal coverage of different size classes can be found. However, exploration of optimality based on Eq. 7.7 and Eq. 7.8 and comparison with results based on Eq. 7.6 would be worthwhile. Objectives Eq. 7.6 through Eq. 7.8 are more appropriate for multiple parameters because the scaled variances will not not favor optimization of parameters with naturally large variances, but research is still being published using the simpler type of objective function (Eq. 7.5) in statistics literature (e.g., Khan et al. 2003)

Using the multiparameter objective function (Eq. 7.6) and considering a given year y, we have

$$O_2 = \sum_{p=1}^{P} \frac{w_p}{\Theta_p^2} \left[\sum_{s=1}^{S} \frac{A_{psy}}{f_s} + \sum_{s=1}^{S} B_{psy} \right] = \sum_{s=1}^{S} \sum_{p=1}^{P} \frac{w_p A_{psy}}{\Theta_p^2 f_s} + \sum_{s=1}^{S} \sum_{p=1}^{P} \frac{w_p B_{psy}}{\Theta_p^2}$$
$$= \sum_{s=1}^{S} \frac{O_{sy}}{f_s} + \sum_{s=1}^{S} B_{sy}$$

where

$$O_{sy} = \sum_{p=1}^{P} \frac{w_p A_{psy}}{\Theta_p^2}.$$

Again, we can ignore the components with B present. Incorporating the same cost function (Eq. 7.3), we have the inequality,

$$\left(\sum_{s=1}^{S} \frac{O_{sy}}{f_s} \right) \left(\sum_{s=1}^{S} f_s C_{sy} \right) \geq \left(\sum_{s=1}^{S} \sqrt{O_{sy} C_{sy}} \right)^2$$

where at the minimized O_2 we obtain

$$f_s^{**} = c_y \frac{(O_{sy}/C_{sy})^{\frac{1}{2}}}{\sum_{s=1}^{S} (O_{sy} C_{sy})^{\frac{1}{2}}}. \tag{7.9}$$

The multivariate result (Eq. 7.9) can be used for any combination of catch parameters that are estimated from observer data.

To illustrate the utility of Eq. 7.9, I use it to obtained optimal coverage levels for all catch parameters listed in Table 7.1 with the corresponding weights. The parameters are chosen only to provide an example, but include representatives of all of the major types of catch parameters that I derived estimators for in Chapters 2 through 5 and include both targeted and prohibited fish species as well. The weights I use are such that similar parameters are treated equally and parameter "groups" are also treated equally. For example, catches of Pacific cod in the Gulf of Alaska and Bering Sea are treated equally as are the more general parameters, catches of Pacific cod and bycatches in numbers of black-footed albatross. This type of weighting is appropriate when we wish to treat species equally yet we also want to treat different stocks or regional populations of a given species equally. If Eq. 7.9 were used in a real setting by managers, most if not all of the various catch parameters that can be estimated from observer data would enter the objective function with weights determined by concensus of the managers.

I calculated the objective separately for each year from 2000 to 2003 and the corresponding plots are in Figure 7.2. Again, all pot vessels are in the medium stratum for these results. Notice that when optimizing over many catch parameters the optimal coverages for the medium size class is fairly consistent over time. However, because there are far more trips made by vessels in the medium size than the large size class and the objective function asymptotes at 0% coverage for the large vessels, there is actually a very limited range of coverages that can be optimal. The yearly (2000-2003) reduction in the objective function value by shifting to optimal coverage levels varies between 1 and 17%.

An important extension of the analytical result for optimal coverage is obtained by allowing the length criterion for splitting size classes to vary. Conditional on a given length criterion, we have the same objective function (Eq. 7.6), but the unconditional objective function does not have an analytic form. To obtain the objective function surface we evaluate the objective function for every non-trivial length criterion. In Figure 7.3, we can see the surface for data collected in 2000. The same parameters and weights in Table 7.1 are used here, but notice that when we allow a different length criterion we have a minimum located far from the current set of length criterion (125 ft) and coverage levels ($f = 0.3$ and $f = 1$ for medium and large size classes, respectively). Almost the same information obtained for optimal coverage levels with

Table 7.1. Catch parameters for which relative variance were used (along with corresponding weights) in the objective function to obtain multi-parameter optimality. Estimates are regional for all parameters where AI, BS and GOA denote the Aleutian Islands, Bering Sea and Gulf of Alaska, respectively whereas Eastern and Western Bering Sea are denoted by EBS and WBS. For numbers in length and age classes the number in parentheses denotes the number of classes.

Weights	Species	Parameter	Region
0.0227	walleye pollock	total catch weight	AI
0.0227	walleye pollock	total catch weight	EBS
0.0227	walleye pollock	total catch weight	WBS
0.0227	walleye pollock	total catch weight	GOA
0.0303	Pacific cod	total catch weight	AI
0.0303	Pacific cod	total catch weight	BS
0.0303	Pacific cod	total catch weight	GOA
0.0303	Steller sea lion	total mortalities	AI
0.0303	Steller sea lion	total mortalities	BS
0.0303	Steller sea lion	total mortalities	GOA
0.0303	Killer whale	total mortalities	AI
0.0303	Killer whale	total mortalities	BS
0.0303	Killer whale	total mortalities	GOA
0.0303	chinook salmon	total mortalities	AI
0.0303	chinook salmon	total mortalities	BS
0.0303	chinook salmon	total mortalities	GOA
0.0303	sockeye salmon	total mortalities	AI
0.0303	sockeye salmon	total mortalities	BS
0.0303	sockeye salmon	total mortalities	GOA
0.0455	norther fulmar	total mortalities	AI/BS
0.0455	norther fulmar	total mortalities	GOA
0.0455	black-footed albatross	total mortalities	AI/BS
0.0455	black-footed albatross	total mortalities	GOA
0.0455	laysan albatross	total mortalities	AI/BS
0.0455	laysan albatross	total mortalities	GOA
0.0011	walleye pollock	total number in age classes (20)	AI
0.0011	walleye pollock	total number in age classes (20)	EBS
0.0011	walleye pollock	total number in age classes (20)	WBS
0.0011	walleye pollock	total number in age classes (20)	GOA
0.0017	Pacific cod	total number in length classes (27)	AI/BS
0.0017	Pacific cod	total number in length classes (27)	GOA

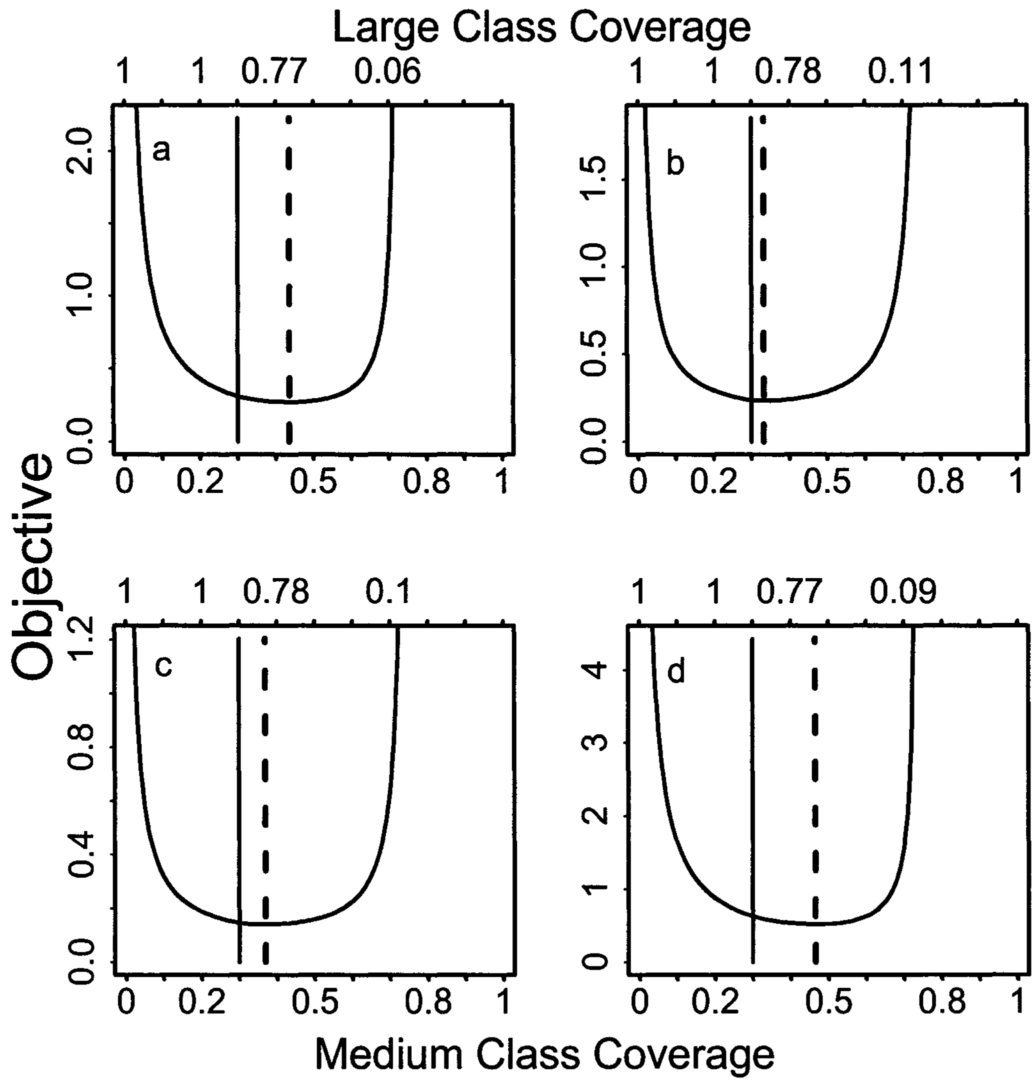

Figure 7.2. Predicted values for the objective function (Eq. 7.6) at different allocations of observer coverage to the different size classes of vessels. The weights and relative variances in the objective function correspond to all catch parameters given in Table 7.1 for data collected in years 2000 to 2003 (a to d). The bottom and top axes are medium and large size class coverage, respectively, the dashed line give the point of optimal coverage and the solid vertical line is current coverage.

the length criterion fixed at the current 125 ft is found in the horizontal slice through the plot in Figure 7.3. There are very small inconsistencies because a few medium size vessels delivering to a mothership have 100% coverage (large stratum) for those trips, but this component shifts to the medium stratum in the algorithms where length criterion is also optimized.

Figures 7.4 through 7.6 show analogous plots for years 2001 through 2003, respectively. With the ability to change both the length criterion and the coverage levels, the yearly (2000-2003) reduction in the objective function value by shifting to optimal coverage and length criterion is greater than when just coverage levels are optimized (9% to 25%). When the length criterion is allowed to vary, optimal coverages (and length criterion) are somewhat consistent from year to year, but the optimal length criterion varies (i.e., Figure 7.2). Optimal length criteria and coverage levels for the medium (including all pot vessels) and large size classes that are shown as points in Figures 7.3 to 7.7 are given in Table 7.2. When we optimize over all four years, we obtain results that average over the yearly results (Figure 7.7) and the reduction in the objective function reflects this (17%).

Table 7.2. Optimal length criteria for size classes in feet (S) and coverage levels (f) for the resulting Large (L) and Medium (M) size classes. The results are based on data from years 2000-2003 and the optimized objective function for a given length criterion is Eq. 7.6 where the relative variances for catch parameters given in Table 7.1 are used.

Year	Length	f_M	f_L
2000	228	0.489	1.000
2001	139	0.378	1.000
2002	124	0.266	0.861
2003	150	0.448	0.783
All Years	175	0.450	1.000
Current	125	0.300	1.000

A further extension of the multiparameter optimization is to allow coverage levels and length criteria to vary among gear types. In this scenario, the same objective

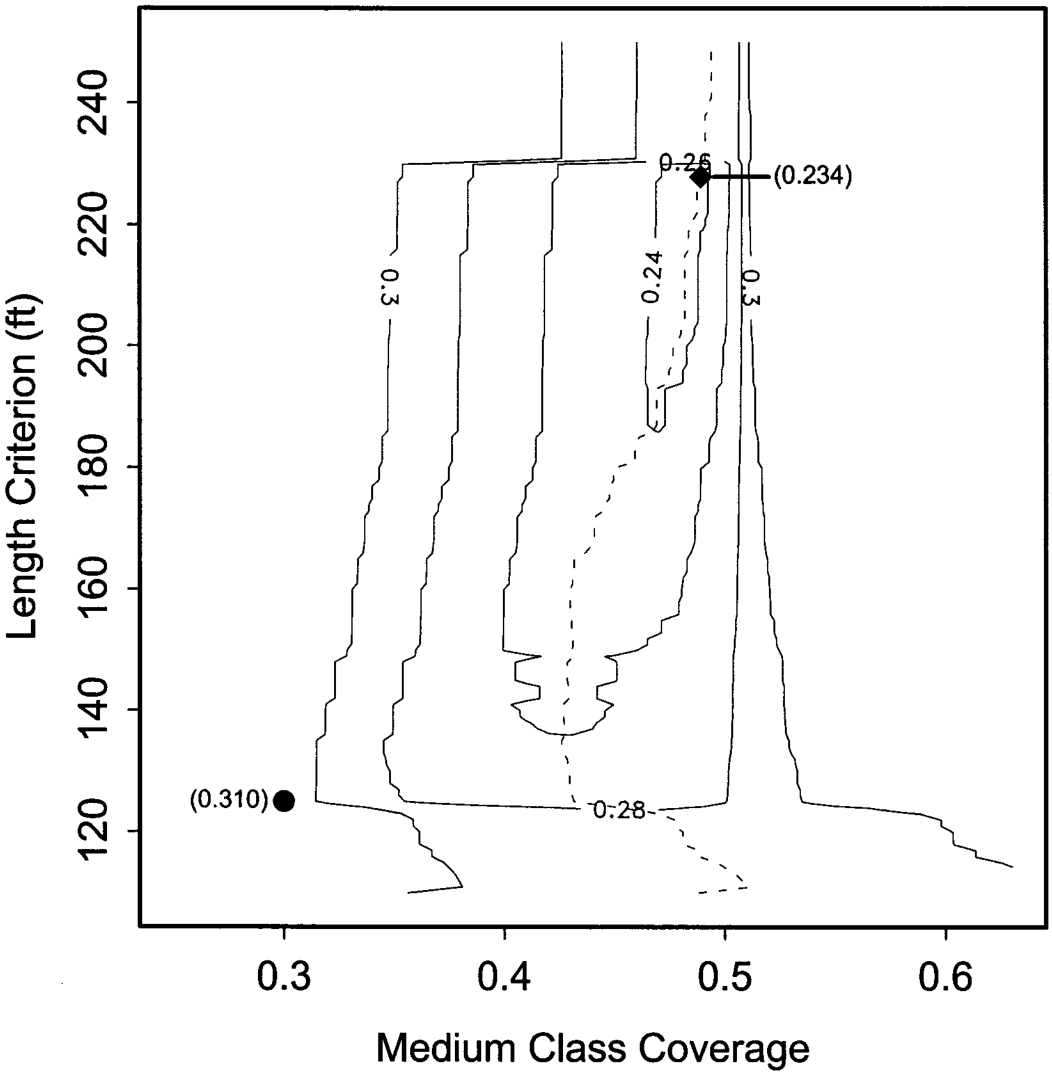

Figure 7.3. Surface of the objective function (Eq. 7.6) using catch parameters and weights given in Table 7.1 based on data collected in 2000. The length criterion for splitting size classes is constant across gear types and the closed circle is the current combination of the vessel length criterion for splitting size classes (125 ft) and observer coverage for the medium size class (0.3). The contours represent lines of constant objective function value, the closed diamond is the minimum across all size class splits where the corresponding coverage and size are given in Table 7.2 and the dashed line is the profile of the minimized objective function for a given size class split.

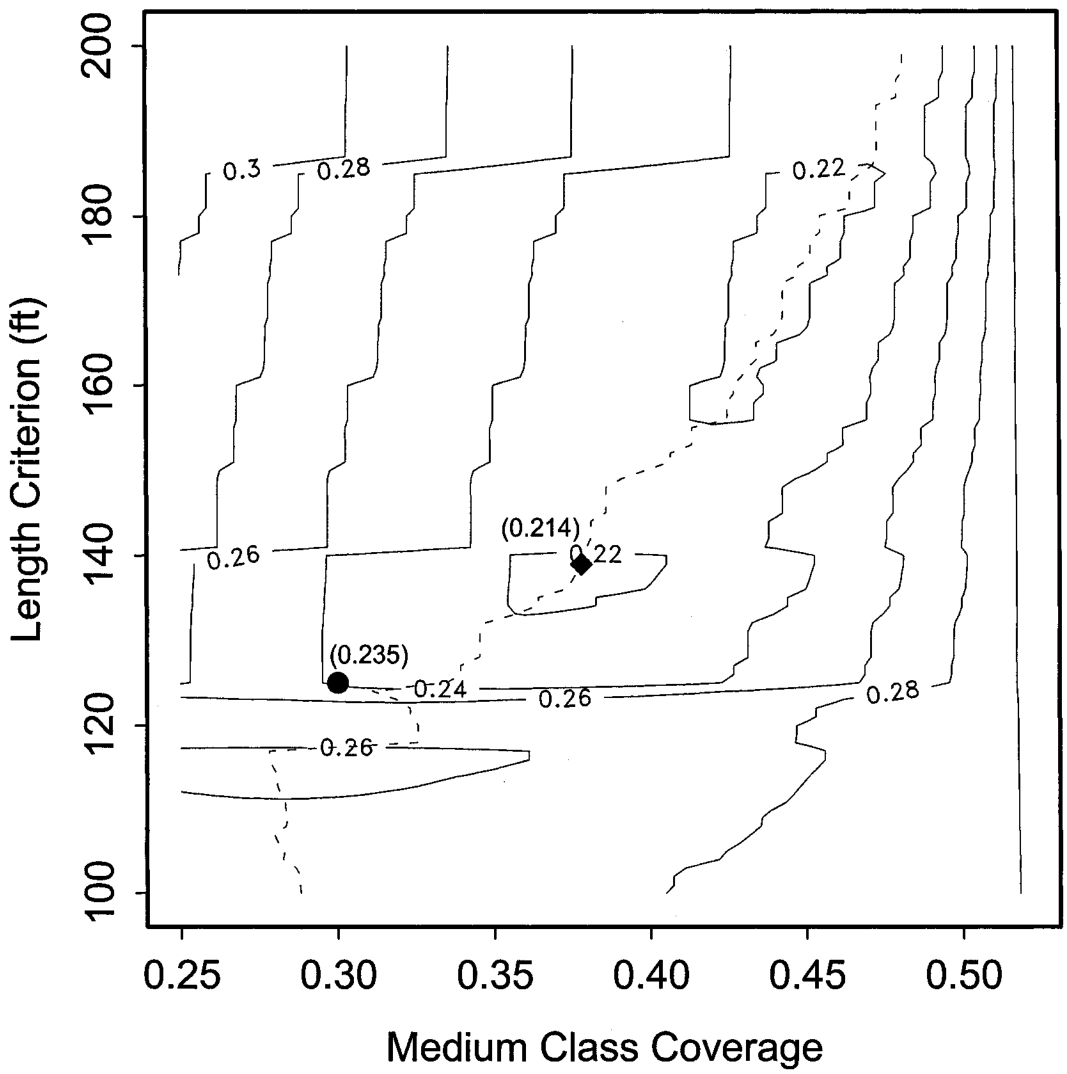

Figure 7.4. Surface of the objective function (Eq. 7.6) using catch parameters and weights given in Table 7.1 based on data collected in 2001. The length criterion for splitting size classes is constant across gear types and the closed circle is the current combination of the vessel length criterion for splitting size classes (125 ft) and observer coverage for the medium size class (0.3). The contours represent lines of constant objective function value, the closed diamond is the minimum across all size class splits where the corresponding coverage and size are given in Table 7.2 and the dashed line is the profile of the minimized objective function for a given size class split.

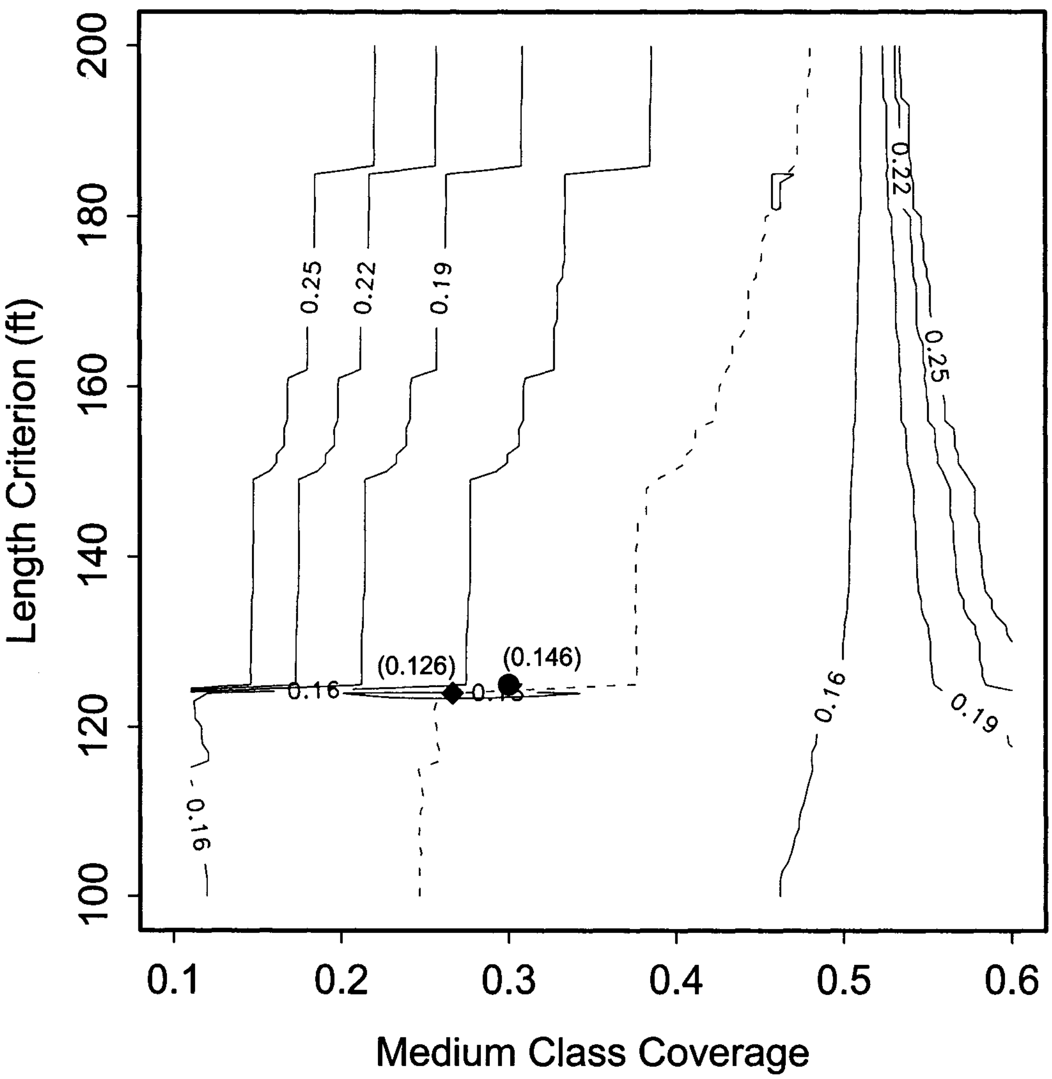

Figure 7.5. Surface of the objective function (Eq. 7.6) using catch parameters and weights given in Table 7.1 based on data collected in 2002. The length criterion for splitting size classes is constant across gear types and the closed circle is the current combination of the vessel length criterion for splitting size classes (125 ft) and observer coverage for the medium size class (0.3). The contours represent lines of constant objective function value, the closed diamond is the minimum across all size class splits where the corresponding coverage and size are given in Table 7.2 and the dashed line is the profile of the minimized objective function for a given size class split.

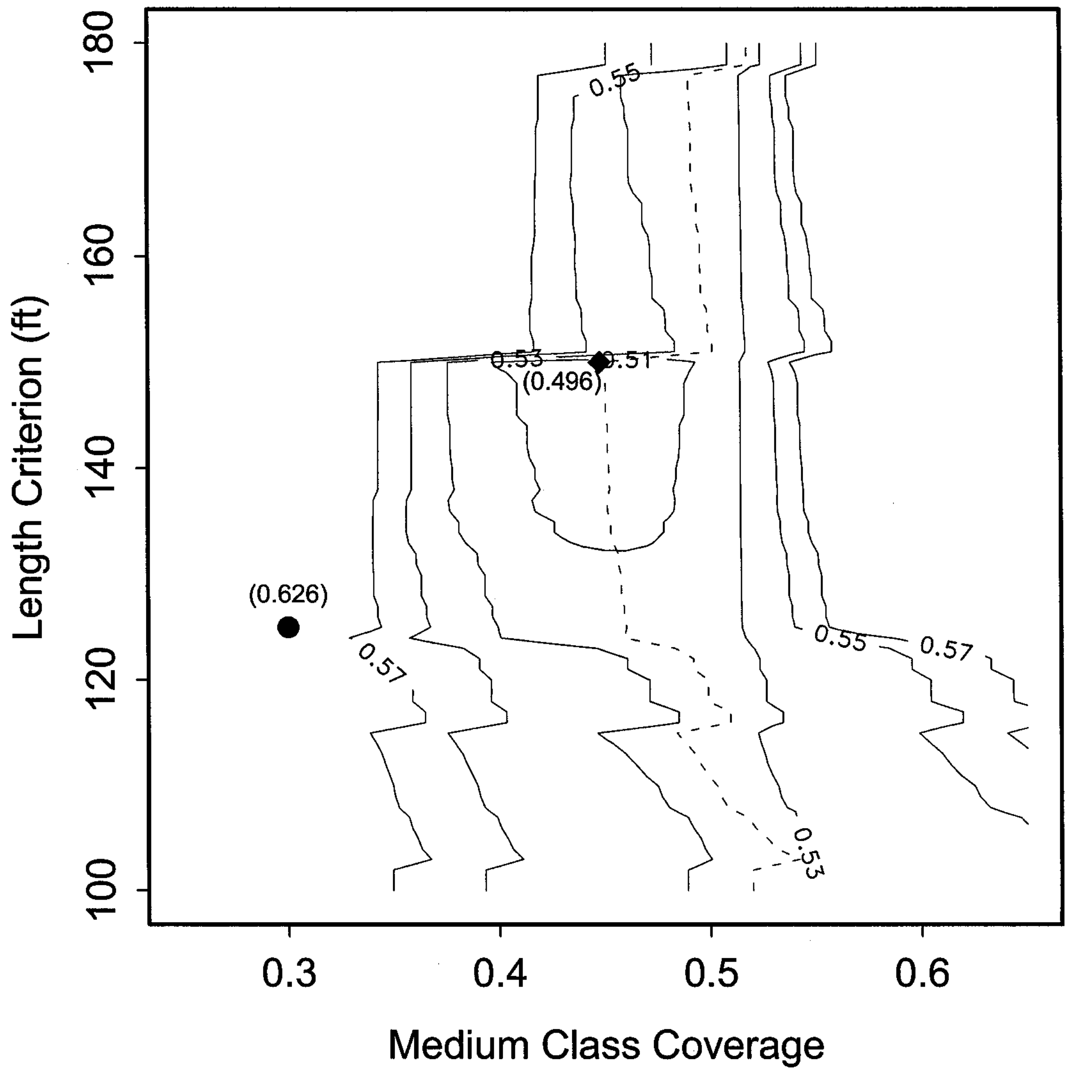

Figure 7.6. Surface of the objective function (Eq. 7.6) using catch parameters and weights given in Table 7.1 based on data collected in 2003. The length criterion for splitting size classes is constant across gear types and the closed circle is the current combination of the vessel length criterion for splitting size classes (125 ft) and observer coverage for the medium size class (0.3). The contours represent lines of constant objective function value, the closed diamond is the minimum across all size class splits where the corresponding coverage and size are given in Table 7.2 and the dashed line is the profile of the minimized objective function for a given size class split.

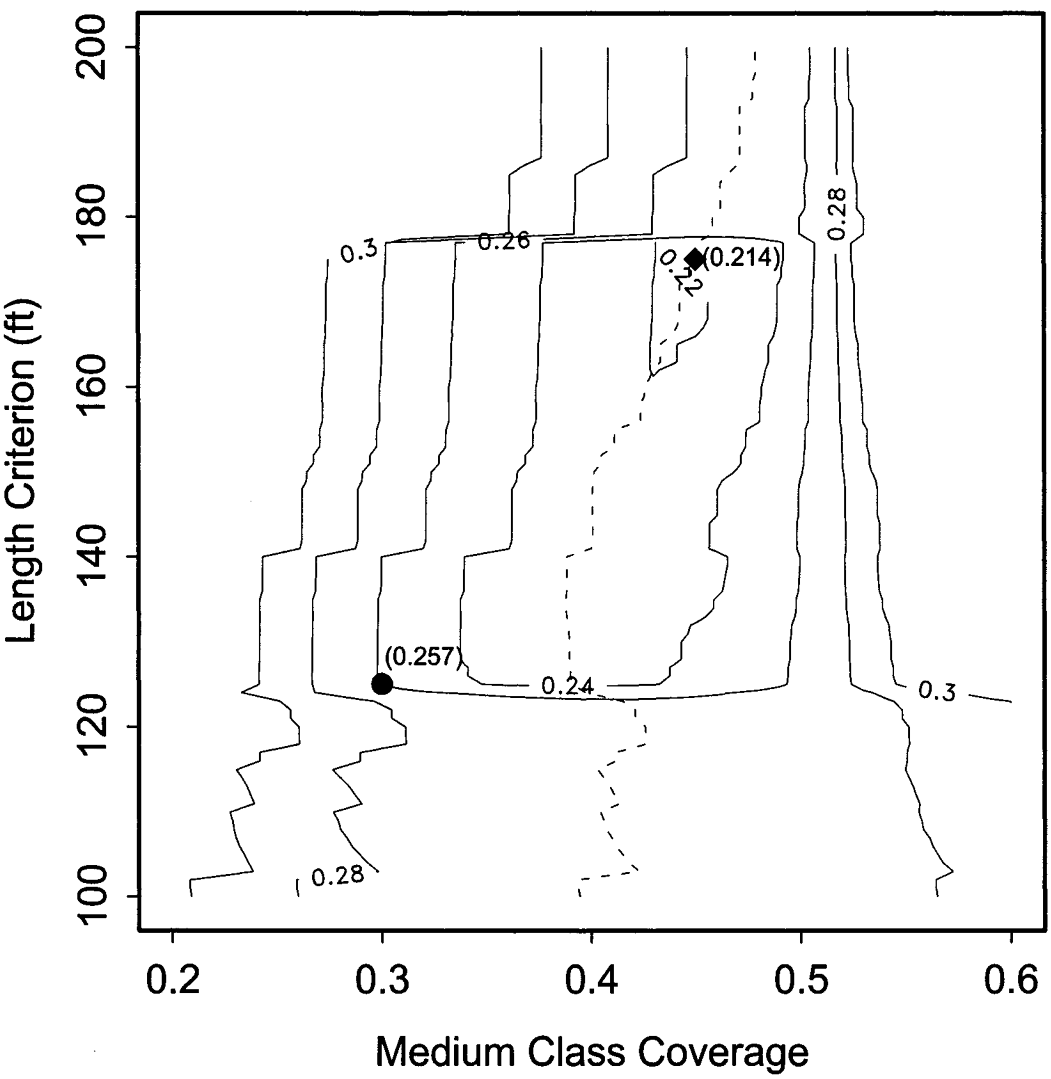

Figure 7.7. Surface of the objective function (Eq. 7.6) using catch parameters and weights given in Table 7.1 based on data collected in 2000-2003. The length criterion for splitting size classes is constant across gear types and the closed circle is the current combination of the vessel length criterion for splitting size classes (125 ft) and observer coverage for the medium size class (0.3). The contours represent lines of constant objective function value, the closed diamond is the minimum across all size class splits where the corresponding coverage and size are given in Table 7.2 and the dashed line is the profile of the minimized objective function for a given size class split.

function (Eq. 7.6) is used given the length criteria, but S denotes the number of size class-gear type combinations (six in the case of three gear types and 2 size classes) and s indexes each of those combinations. The minimization is efficiently performed using the conditional objective function within an algorithm that determines the optimal length criteria. Results using the parameters in Table 7.1 for years 2000 to 2003 and all years combined are in Table 7.3. There is a general trend in having very low coverage for vessels fishing pot gear, but the length criterion is variable. For the longline vessels, the yearly results show very high coverage is usually optimal and the optimal length criterion is lower than the current value for 2 of the 4 years. However, when all yearly data are combined, a large length criterion with low coverage in the resulting large size class is optimal for longliners. Except for 2000, both the optimal length criterion and coverage levels are relatively consistent and the current trawler coverage levels are near optimal, but the current length criterion is less than optimal. The disparatity between the yearly results and the multi-year result for longliners could be due to different influences on the objective function by different gear-types in alternate years. Finally, letting the length criterion and coverage levels vary by gear types allows the objective function to attain even lower values than when they are constant across gear types. The yearly reductions of the objective function by shifting to the optimal values from the current values are between 14% to 47% for 2000 to 2003. When all four years are considered together, optimized gear-specific coverage and length criteria yeild a 31% reduction in the objective function.

The optimal length criteria and coverage levels are useful in themselves, but we would somehow like to know how flat the multidimensional surface is with respect to the length criteria and coverage levels. More importantly, we would like to know how different the current relative variances for each catch parameter are from those at the optimal set of length criteria and coverage levels. We can approach this by observing the differences between the components of the relative variance for each catch parameter and the corresponding components at the minimized objective funtion. Letting

$$O_{py} = \frac{1}{\Theta_{py}^2} \sum_{s=1}^{S} \frac{A_{psy}}{f_s}$$

Table 7.3. Optimal length criteria and observer coverage levels (f) for resulting large (L) and medium (M) size classes by gear type (Longline, Trawl and Pot) and corresponding optimized objective function value (Objective). The results are based on data from years 2000-2003 and the optimized objective function for given length criteria is Eq. 7.6 where the relative variances for catch parameters given in Table 7.1 are used.

Year	**Longline**			**Trawl**			**Pot**			
	Length	f_M	f_L	Length	f_M	f_L	Length	f_M	f_L	Objective
2000	76	1	1.000	206	0.394	1.000	173	0.165	0.005	0.167
2001	126	1	0.395	140	0.299	1.000	127	0.445	0.548	0.202
2002	127	1	0.638	149	0.305	1.000	127	0.346	0.296	0.109
2003	118	1	1.000	150	0.378	0.798	151	0.121	0.133	0.330
All Years	180	1	0.258	139	0.308	0.958	107	0.111	0.130	0.177

The difference between parameter components is

$$G_p = O_{py} - \breve{O}_{py} = \frac{1}{\Theta_{py}^2} \sum_{s=1}^{S} \left(\frac{A_{psy}}{f_s} - \frac{\breve{A}_{psy}}{\breve{f}_s} \right) \tag{7.10}$$

where $\breve{A}_{psy}$ is the sum of the vessel components in the sth size class-gear type combination formed by the optimal length class criteria and $\breve{f}_s$ is the corresponding optimal coverage for the sth size class-gear type combination. It is worth reiterating that the set of vessels in the sth size class-gear type combination under the optimal length criterion is not, in general, the same set in the sth size class-gear type combination with the current length criterion. When G_p, is less than zero, estimation of the pth parameter is more efficient under the current set of length criterion and coverage levels than the optimal set. When the difference is greater than zero, estimation of the parameter gains efficiency by moving to the optimal set of length criteria and coverage levels.

For illustration, I have calculated G_p for all catch parameters (see Table 7.1) with the optimization carried out using all data collected in years 2000 to 2003 where observer coverage and length criteria are allowed to differ for each gear type. Table 7.4 gives the results for all non-length/age class estimation and Tables 7.5 and 7.6 give the results for numbers in length classes for Pacific cod and numbers in age classes for walleye pollock, respectively. The parameters that stand to gain most by shifting to optimal sampling are those for seabirds in any given region except for northern fulmars in the Bering Sea/Aleutian Islands. Relatively small changes would be obtained for most other catch attributes including length classes of Pacific cod age classes for walleye pollock. Precision of sockeye salmon caught in the Gulf of Alaska is the most adversely affected by shifting observer coverage and length criteria.

7.3 Comments

Assuming that the catch parameters and weightings I used in the example were those used to determine optimal gear-specific observer coverage and vessel size class length criteria, seabird bycatch estimation stands to gain the most by changing coverage levels and length criteria. Most other estimates would be minimally affected by the changes in observer coverage and length criteria. Seabird mortalities are almost en-

Table 7.4. Gains and losses of precision measured by G_p (Eq. 7.10) for various catch parameters by shifting to optimal observer coverage levels and length criteria when gear-specific length criterion and observer coverage levels are allowed. The results are based on data from years 2000-2003 and the optimized objective function for given length criteria is Eq. 7.6 where the relative variances for catch parameters given in Table 7.1 are used. Positive and negative values indicate gains and losses in precision, respectively.

Species	Parameter	Region	$G_p \times 10^3$
walleye pollock	catch weight	Aleutian Is.	−0.013
walleye pollock	catch weight	eastern Bering Sea	0.000
walleye pollock	catch weight	western Bering Sea	−0.001
walleye pollock	catch weight	Gulf of Alaska	0.004
Pacific cod	catch weight	Aleutian Is.	−0.028
Pacific cod	catch weight	Bering Sea	−0.014
Pacific cod	catch weight	Gulf of Alaska	−0.055
Steller sea lion	mortalities	Aleutian Is.	0.390
Steller sea lion	mortalities	Bering Sea	−0.047
Steller sea lion	mortalities	Gulf of Alaska	0.455
Killer whale	mortalities	Aleutian Is.	0.000
Killer whale	mortalities	Bering Sea	−0.460
Killer whale	mortalities	Gulf of Alaska	0.000
chinook salmon	mortalities	Aleutian Is.	−0.017
chinook salmon	mortalities	Bering Sea	−0.001
chinook salmon	mortalities	Gulf of Alaska	0.010
sockeye salmon	mortalities	Aleutian Is.	−0.077
sockeye salmon	mortalities	Bering Sea	−0.324
sockeye salmon	mortalities	Gulf of Alaska	−1.308
norther fulmar	mortalities	Bering Sea/Aleutian Is.	−0.270
norther fulmar	mortalities	Gulf of Alaska	0.795
black-footed albatross	mortalities	Bering Sea/Aleutian Is.	0.668
black-footed albatross	mortalities	Gulf of Alaska	3.678
laysan albatross	mortalities	Bering Sea/Aleutian Is.	49.607
laysan albatross	mortalities	Gulf of Alaska	27.984

Table 7.5. Gains and losses of precision measured by G_p (Eq. 7.10) for Pacific cod length classes by shifting to optimal observer coverage levels and length criteria when gear-specific length criterion and observer coverage levels are allowed. The results are based on data from years 2000-2003 and the optimized objective function for given length criteria is Eq. 7.6 where the relative variances for catch parameters given in Table 7.1 are used. Positive and negative values indicate gains and losses in precision, respectively.

Length Class	$G_p \times 10^3$	
	Bering Sea/Aleutian Is.	Gulf of Alaska
0-8	0.065	1.140
9-11	−0.914	0.000
12-14	0.288	0.024
15-17	0.247	0.024
18-20	0.035	0.004
21-23	0.018	0.128
24-26	0.056	0.086
27-29	0.007	0.053
30-32	0.001	0.115
33-35	0.001	0.018
36-38	0.002	0.006
39-41	0.001	0.003
42-44	0.001	0.003
45-49	0.001	0.003
50-54	−0.002	0.001
55-59	−0.003	0.000
60-64	−0.002	−0.008
65-69	−0.001	−0.018
70-74	−0.001	−0.019
75-79	−0.001	−0.010
80-84	0.000	−0.006
85-89	−0.001	0.004
90-94	0.000	0.020
95-99	−0.001	−0.007
100-104	−0.001	−0.161
105-115	−0.012	−0.202
≥ 116	0.011	0.000

Table 7.6. Gains and losses of precision measured by G_p (Eq. 7.10) for walleye pollock age classes by shifting to optimal observer coverage levels and length criteria when gear-specific length criterion and observer coverage levels are allowed. The results are based on data from years 2000-2003 and the optimized objective function for given length criteria is Eq. 7.6 where the relative variances for catch parameters given in Table 7.1 are used. Positive and negative values indicate gains and losses in precision, respectively.

Age class	$G_p \times 10^3$			
	Aleutian Is.	Bering Sea E.	Bering Sea W.	Gulf of Alaska
1	0.000	0.000	0	0.007
2	0.000	0.000	0	0.003
3	0.000	0.000	0	0.002
4	0.000	0.000	0	0.001
5	0.000	0.000	0	0.000
6	0.000	0.000	0	0.001
7	0.000	0.000	0	0.001
8	−0.003	0.000	0	0.001
9	−0.002	0.000	0	0.002
10	0.000	0.000	0	0.003
11	−0.001	−0.001	0	0.004
12	0.000	−0.001	0	0.006
13	0.000	−0.003	0	0.008
14	0.000	−0.011	0	0.008
15	0.000	−0.038	0	0.009
16	0.000	−0.015	0	0.000
17	0.000	0.000	0	0.000
18	0.000	0.000	0	0.000
19	0.000	0.002	0	0.000
$\geq$ 20	0.000	0.000	0	0.000

tirely encountered using longline gear and the gains in estimating these mortalities by switching to optimal coverage and length criteria are further appreciated by observing that optimal coverage for either resulting size class of longliners is high (except 2001 large size class, Table 7.3). With complete observer coverage for the longliners, the variance (and relative variance) of the seabird estimtates will be minimized with given rates of sampling of hauls within trips and sampling within hauls.

However, if the optimal solutions (Eq. 7.9) were used to change observer coverage levels, it is unlikely that the same catch parameters and weights that I use in the example would be chosen by the managers. In fact, any conclusions about gains and losses for particular catch parameters are dependent on the weights and parameters that enter the objective function. If the seabird bycatch estimates are weighted equally to or greater than other catch parameters and relative variances of other catch parameters are relatively small, then similar gains would still be found for the seabird bycatch estimates.

The optimal solutions provided by Eq. 7.9 can be a useful tool for fisheries managers to determine observer coverage levels that give efficient estimates for many catch parameters. The results are especially appealing because the coverage is determined methodically and with a high degree of transparency for the users of the data as well as the fishing industry which observer coverage impacts logistically and financially. Furthermore, under any of the different coverage scenarios (optimizing coverage with the current length criterion, optimizing coverage and length criterion or optimizing covererage and length criterion specific to gear-types) the gains and losses for any catch parameters (whether they are included in the objective function or not) by changing coverage from the current requirements can be assessed efficiently through Eq. 7.10.

The utility of the objective function that I optimized and the corresponding analytical solutions for optimal sampling fractions in various strata is not limited to the North Pacific Groundfish Observer Program. These ideas apply to any sampling design that is stratified with multi-phase or multi-stage sampling within each of the strata. These types of sampling designs are commonly used in studies of natural resources such as forest surveys or the periodic surveys carried out in marine systems for fishery-independent information. There is often many types of information collected in these surveys and the analytical results I provide in this chapter could be useful in determining appropriate allocation of sampling effort.

The purpose of my optimality results is to guide resource managers in effort allocation in the future, but optimality can only be calculated with data already collected. For this reason, the optimization is always an approximation, but this does not diminish its utility. Because fishing practices change over time and optimization using earlier data may not be as representative of fishing practices in the immediate future, I recommend optimization be based on the most current years of data archived by the North Pacific Groundfish Observer Program. However, the degree of variability of results for consecutive years argues for optimizing over as many years as possible. Optimization over multiple years may not give optimal allocations for the following year of fishing effort, but should provide near optimality for the next few years as long as fishing behavior does not change substantially. Furthermore, optimization over multiple years is likely to be more practical when observer coverage levels will not be changed frequently.

Chapter 8

CONCLUDING REMARKS

In this final chapter, I begin with a review of the contributions of the dissertation to Alaskan groundfish fishery science and fishery science, in general (Section 8.1). I also discuss recommendations for changes in data collection and reporting procedures that would allow improved estimation of catch parameters from data collected by observers in the Alaskan groundfish fleet (Section 8.2). I end this chapter by discussing some conclusions about employing my estimators in population or stock assessment models (Section 8.3), the nuances of design- and model-based inference approaches (Section 8.4) and how the ideas I have contributed in this dissertation may be extended (Section 8.5).

8.1 Contributions of this Dissertation to Fishery Science

This dissertation providesstatistically rigorous estimators for various catch parametersof interest to in-season managers, groundfish stock assessment scientists, the North Pacific Fisheries Management Council and those who study the impacts of fishing activities on other species within the Alaskan marine ecosystem (e.g., fish, marine mammals and seabirds). The general approach that I use to develop estimators and derive optimal observer coverage levels are also applicable to other observer programs or fishery resource surveys that employ well-defined sampling designs. The specific contributions of this dissertation to fishery science include the following:

Improved In-season Catch Estimation

The in-season managers at the Alaska Regional Office of NMFS can use the estimators I provide in this dissertation for catch in weight or numbers in particular management regions and/or time periods. These estimators account for the uncontrolled sample sizes within the region/period of interest and have corresponding variance estimators which allow managers to account for uncertainty in their estimates.

Currently, the Alaska Regional Office uses *ad hoc* estimation methods that integrate observer-collected data and catches reported by the fishing industry. These estimates have no measure of precision and the results in Chapter 6 show that (assuming my estimates are unbiased) there may be substantial and consistent bias for some targeted species (e.g., walleye polock in the Bering Sea). Consistent and incorrect catch estimates could cause the quotas to be mismanaged. Even if the point estimates were unbiased, there is no idea of how close the point estimates are to the true catches because the precision of the currently used estimates is not estimated. Therefore, the statistical rigor of in-season catch accounting will be improved by using my estimators. This improved rigor leads to sound inferences about the removals for species for which quotas are required. Furthermore, managers will also be able to consider uncertainty in their estimates and incorporate it into decisions about timing of fishery closures which have important biological and economic consequences.

Improved estimation of inputs to stock assessment models

Groundfish stock assessment scientists will also benefit from using my estimators for total catch (numbers and/or weight) and/or for total catch in length/age classes. Currently, stock assessments depend on the catch estimates made for in-season management which are provided by the Alaska Regional Office. As mentioned above these estimates may be biased and lack precision estimates. When these biased estimates are used in the stock assessment models the inferences about the status of current stocks could be incorrect. Incorrect assumptions about the precision of the catch estimates could also lead to incorrect inferences even if the catch estimates were unbiased. Asssessments typically use length measurements or aged ototliths (for the species or stocks being assessed) collected by observers to estimate proportions of the catches in length or age classes, but these estimates do not account for the sampling design. Estimation of numbers-at-age for many stock assessments follows methods given by Kimura (1989) which was designed for data collected prior to the NPGOP and does not account for the vessel-quarter stratification in the current sampling design. My estimators, which do account for the sampling design, can be used to estimate these parameters in age-length structure stock assessment models and corresponding sampling variance components. On the other hand, the estimates of total numbers caught in the length or age classes (and corresponding variance-covariance matrix) may be used directly in the assessment models. Even if the currently used catch estimates were unbiased and there was no apparent impact of incorrect precision assumptions, using estimators that reduce the number of assumptions implicit in assessment models will render them more realistic and defensible.

Ability to Determine Observer Coverage Based on Statistical Properties of Estimators

The multi-parameter optimization solutions I provide in Chapter 7 give resource managers, including the North Pacific Fisheries Management Council, tools needed to determine observer deployment based on statistical precision of the various catch parameters that are estimated from the data observers collect. The results are especially appealing because the observer coverage is determined methodically and with a high degree of transparency. With this transparency, users of the data as well as the fishing industry, which observer coverage impacts logistically and financially, can see clearly which catch parameters are used in the objective function and the relative weighting and importance they are given. This is important because all involved parties can see how observer coverage is determined and it informs the discussion of the weights for the various catch parameter estimators ultimately used in the objective function (Eq. 7.6). Furthermore, through Eq. 7.10, the involved parties can easily assess the gains and losses that result from changing coverage from the current requirements for any catch parameters.

There is also utility of the employed objective function beyond the North Pacific Groundfish Observer Program. My optimization approach and analytic solutions are applicable for any sampling design that is stratified with any sort of sampling within each of the strata. Sampling designs of this class may occur in studies of many natural resources including forest surveys or marine resource surveys. The analytical results I provide in Chapter 7 can be straightfowardly applied for these large scale surveys which often collect many types of information and make corresponding estimates. Even for other types of designs, the same objective function can be used to determine allocation of sampling effort.

A Reference for All Users of Data Archived by the North Pacific Groundfish Observer Program

This dissertation provides a detailed reference on estimation procedures for data collected by observers in the Alaskan groundfish fisheries. I have derived completely design-based estimators for various catch parameters that are appropriate under different sampling scenarios. Although these design-based estimators cannot be used with data that have been and are currently collected, they help identify the necessary changes in data collection procedures that can make the estimators usable in the future. Design-based estimators are desirable because no model assumptions are required for making inferences about the sampled catches.

I have also derived estimators for catch parameters that integrate design- and model-based methods to meet the problems in observer data collection and archiving that disallow completely design-based estimation. Integration of these two inference methods is an undiscussed idea in statistics literature and, therefore, this dissertation also makes contributions to statistical science as well as fishery science.

When researchers apply other estimators that do not reflect the randomization inherent in the sampling design, the estimates and corresponding variance estimates are likely to be biased and/or the implicit simplifying assumptions may go undeclared. With this dissertation, scientists have a suite of possible estimation procedures along with a clear explanation of the required assumptions. Furthermore, uniformity in inferences of catch parameters over time can be achieved and researchers can easily communicate the method of estimation they use. Using these estimators will produce more scientifically rigorous analyses and reflect the realities of the data collection procedures. Using other estimators that are biased leads to incorrect inferences and conclusions about the parameters that they estimate. Depending on the degree of bias, these incorrect conclusions could lead to management actions with undesirable effects on the Alaskan marine ecosystems and the associated fishing industry.

Another important benefit of using my estimation approach that is common to all uses of observer-collected data is the more realistic representation of precision in estimates. I have provided variance estimators for every catch parameter estimator that respect the observer sampling design. The examples in Chapter 6 shed some light on the level of uncertainty in estimates and possible disparities between estimates and the values reported by the Alaska Regional Office of NMFS.

The estimation approaches also facilitate ecosytem-based management. It would be straightforward to estimate the correlation of catches of any given group of species by methods analogous to those used for estimating the covariance of numbers in different length or age classes. The catch weights or numbers become the elements of the estimated vector rather than numbers in length or age classes and correlations between species catch estimates are obtained from the corresponding VCM estimate. These types of analyses can be very informative to our understanding of interactions between fishing and various targeted and untargeted species.

8.2 Recommended Changes for Improved Estimation

There are several changes in data collection and/or recording procedures that would require less reliance on model-based methods for estimation. Striving for design-based estimation should be a goal because models require assumptions that may not be realistic. Some of these changes have greater impacts than others. I suggest the following modifications in the data recording procedures of the North Pacific Groundfish Observer Program:

Distinguish fishing trips and track unobserved effort by vessels with less than 100% oberver coverage

When there is less than 100% observer coverage, the major source of uncertainty in estimates made using observer data is between-trip variation within vessel-quarter strata. This coupled with the uncertainty arising from the unknown numbers of trips made by vessels with less than 100% coverage argues for making fishing trips identifiable in the observer data base and tracking the numbers of unobserved trips made by each vessel with less than 100% coverage. Observed fishing trips must be distinguished in the NPGOP database so that we need not make assumptions to arbitrarily define fishing trips. This information could be provided by observers during their deployment. If managers also determined the numbers of unobserved fishing trips made by vessels with less than 100% coverage, then the models I assumed for numbers of observed trips made by these vessels would be unnecessary and better precision would be achieved. Either of these information components will increase the precision in and relax assumptions for estimates based on observer data. Furthermore, more extensive data on the unobserved trips such as the number of days, number of hauls, number hooks in deployed sets, etc., could provide further improvements to the precision of estimates.

Halt extrapolation by observers of information from sampled hauls to unsampled hauls during deployment

A practice that should be halted immediately is the extrapolation by observers of estimates for sampled hauls to those that were not sampled, but that they perceived as similar. At the very least these hauls should be more easily identified in the database where haul information is archived. This practice was begun mainly for the needs of the Alaska Regional Office to tally catches against quotas. However, the dependence of observers to determine "like" sets amounts to them performing model-based estimation because they are implicitly predicting catch attributes for the unsampled haul from the sampled haul. Because managers are more experienced with the groundfish fisheries, they should be better equipped to determine appropriate extrapolation of sampled hauls. Furthermore, other investigators that use the archived data are apt to treat the observer extrapolations as real data.

Record more detailed information for each sampled haul

If observers were able to record all data they collected within hauls rather than summary information, we would be able to relax model assumptions required for estimation at the trip and haul level under the current data collection scenario. Even if all model-assumptions were restricted to within-haul estimation, this would be an important improvement over the *status quo.*

Collect length measurements and otoliths from targeted species when they are not prevalent

Under current data collection procedures, length measurements and hard parts from commercially important species are generally only taken when the species is prevalent and scientists must assume that age and length distributions are the same whether or not the species is prevalent. The assumption that the length or age distribution of fish caught in hauls where they are prevalent is equal to the corresponding distribution of fish caught in hauls where they are not prevalent may be difficult to justify. For some species, very few individuals may be caught in hauls where they are not prevalent and the assumption would have little impact if it was invalid. However, for some species a substantial proportion of individuals are caught in hauls when they are not prevalent (e.g., sablefish in the Bering Sea or thornyhead rockfish). Even when inferences are limited to the domain of hauls where the species is prevalent, the determination of prevalence is by observer judgement which may be inconsistent between observers.

Modify haul sampling for pot vessels

Having observers determine the appropriate grouping of pots to call a string is unneccessary as long as the total number of deployed pots is tracked. Under this sampling scenario, observers would randomly sample pots throughout the trip and estimation of the trip total is based on the single-stage random sampling of pots rather than the two-stage random sampling of hauls (strings) and pots within hauls.

Also, the exclusion of lost pots from catch parameter estimates appears improper. If we wish to determine fishing-induced mortality for various species as with other gear types, it would seem that accounting for mortality due to unretrieved pots would also be important, however small that mortality is. The lost pots can be modeled as missing at random and prediction of catches made in these pots is possible from retrieved pots that are sampled.

Strive for some sampling of the fleet of vessels less than 60 ft in length

A minor problem with using observer data to estimate catch parameters is the lack of observer coverage of some sectors of the fleet. Specifically, vessels less than 60 ft in length require no coverage and if significant amounts of fishing effort are made by these vessels, then their catches must in some way be considered. There are practical difficulties of observing these small vessels, but observing effort in this fleet in some randomized fashion would allow more statistically rigorous estimation within this sector. The randomization for the small vessel sector could be completely different than the other size classes or other technologies (e.g., camera monitoring) could be used to allow for the realities of fishing activiities on these vessels. Even without randomized sampling in the small vessel class, an excellent alternative to current practices would be to limit any necessary assumptions about catch parameters to the unsampled sector and add the results to estimates obtained using my approaches for sectors with observer coverage.

Provide a chronological record of changes in observer sampling and deployment procedures

A readily available reference on changes in observer sampling protocols would be tremendously useful. Presently, people that wish to use observer data correctly, must ask employees of the NPGOP for details on sampling methodology and comb through each of the extensive yearly-updated observer manuals. Over time, fewer employees will be available to provide details on historical sampling protocols. A document that concisely denotes important changes in coverage regulations, the manner in which observers are deployed, and details on sampling protocols would be a more efficient means of relaying this important information to investigators.

8.3 Using Estimators in Population Modeling and Stock Assessment

There are some important consequences for population modeling and stock assessment that should result from my estimation approach. First, the correlation of various length or age classes provides insight into the structure of the population of caught fish. I found that strong co-ocurrence of some length classes can occur during certain times of the year, but patterns in correlations of numbers-at-age (over the entire year) were less pronounced. Either of these outcomes provides information about the nature of the catches and analysing the catches-at-length or -age over longer timespans will likely yield interesting patterns in the correlations for given time periods and/or regions in each year. Examples were provided in Chapter 6 to allow comparison with analogous estimates made for stock assessments, but different time periods and/or regions could be of interest for other investigations of age or length classes.

In addition, the correlation structure is obviously not what is assumed under the simpler overdispersed multinomial distribution (Fournier et al. 1990, 1998) or the multivariate logistic (Schnute and Richards 1995) often used in many stock assessment models based on catch-at-age or -length information. Although I do use multinomial assumptions within hauls, design-based and semiparametric methods (i.e., first and second moment assumptions) are used at higher levels in the hierarchy rather than specific distributions. So, negative correlation is implied for the conditional within-haul models and those used for prediction of undersampled trips, but positive correlation of different age or length classes among hauls within trips or among trips within vessels is allowed. Hrafnkelsson and Stefánson (2004) also found both positive and negative correlation of estimated proportions-at-length using completely model-based methods for Atlantic cod in groundfish surveys.

To deal with the lack of independent samples for catch-at-length and resulting overdispersion, Alaskan Pacific cod stock assessment scientists essentially fix a scalar multiple of the usual multinomial variance as the inverse of the square root of the overall sample size in each region/period. This results in an overdispersion assumed to be equal to the square root of the sample size which is constant across all length classes. My results show far less dispersion using the integrated design- and model-based (IDM) methods and the degree of overdispersion is not constant across length classes. Using the simpler modeling approach may have no effect on the consistency of the estimated proportions-at-length, but the estimated proportions-at-length may be far more efficient than currently thought. Moreover, when fitting stock assessment models for Pacific cod, the goodness-of-fit is evaluated in one respect by the deviations of the model-predicted sample sizes from the (assumed) square-root sample size from the observer data. If the dispersions estimated using my approach are correct, there could be significant effects on fitting the length-structured stock assessment models.

In general, stock assessment models that use various sources of data, can be viewed as a stochastic, multilevel, generalized nonlinear mixed effects model. The sampling distribution for length or age data is just one conditional distribution. Using any of the current methods requires some assumption on the degree of precision of proportion estimates and the goodness-of-fit of the stock assessment model is dependent on that assumption. I have provided an approach that allows estimation of error due to sampling, but the stock assessment model itself imposes an additional layer of uncertainty by linking proportions-at-length in catches over time. Because my proposed estimators are asymptotically normal and sample sizes are large, resulting estimates of numbers/proportions-at-age or -length can be incorporated into stock assessment models that require likelihood or Emperical Bayesian inferences. This would allow stock assessment scientists to tease apart the variance components due to the model and sampling and more properly assess the uncertainty of their results. The numbers/proportions-at-length can also be parameterized in a MULTIFAN-like fashion (Fournier et al. 1990, 1998) if length-at-age modeling is of interest. Alternatively, since length measurements are taken for all otoliths, estimation of proportions or numbers-at-length and -age could also be made using my aproach, but judging from estimates of numbers-at-age for sablefish, precision of numbers-at-length and -age would be very poor unless very large numbers of otoliths are aged.

Variance of total catches is also often an assumed value in stock assessment modeling, but observer-based estimates of total catch can be used to provide asymptotically normal inputs in stock assessment models. Moreover, the correlation of estimated total catch and numbers-at-length or -age can also be treated. This approach removes the necessity for cumbersome assumptions on the sampling precision (and the correlation structure) of catch data.

Finally, an area of research that may have an important role in ecosystem-based management of the Alaskan marine system is multi-species population modeling. It is necessary to consider fishing-induced mortality of all affected species in these models, including targeted and untargeted fish as well as seabirds and other animals. My estimation approach works well in this context because estimates of mortality for different species are not independent and the approach for one parameter presented here is easily generalized to vectors of parameters. Specific examples included in this dissertation are numbers in length or age classes for targeted fish species. Just as our single-species estimates can be incorporated in a single-species population model, vectors of mortality estimates and associated covariance or correlation matrices for various species can and should be made and incorporated in the multi-species modeling approach.

8.4 Should We Make Inferences Using Models or the Design?

I implied in the introductory chapter of this dissertation that whether one uses models or the sampling design to make inferences depends on the reluctance of the investigator to make assumptions about the collected data. However, I have discovered that whether or not assumptions need to be made is not always a decision that the investigator can make. For example, I have assumed that observers sample hauls via simple random sampling, this is clearly not true because there is no probability that observers will sample many hauls consecutively and the total number of hauls is unknown until the trip ends. Because the SRS assumption does not hold, I have effectively assumed independence of the sampling process for haul selection in the trip and the values the catch parameter of interest takes on for each haul in the trip.

Consider the total catch weight of walleye pollock by a trawler. Even if design-based estimation within hauls were possible, the total catch weight of walleye pollock for a trip is estimated under assumptions of a model with mean and variance parameters and exchangeability of hauls within the trip. I ultimately predict the catch weights of the unsampled hauls from the estimated catch weights of the sampled hauls because the randomization inherent to the actual sampling procedure used to select hauls is not the basis for inference. In the current situation with information limitations, model-based methods are required within hauls and we have assumptions of mean, variance, independence and exchangeablity that are fish-specific rather than haul-specific.

Under the philosophy of avoiding assumptions in our estimation approach we should base our inference on randomization mechanism used to select the hauls. However, the inclusion probabilities required to form the estimators are either intractactable or overly complex and we are left with model-based estimation as the only viable option. Thus, whether we should use models or the sampling design to make inferences depends on both whether we have a choice and our philosophy on inference.

Interestingly, this implicit model-based inference is not unique to my work here, but it is a rarely discussed issue in fisheries sampling contexts. The phenomenon is paralleled more generally in the application of SRS-based estimators when systematic sampling is performed which many sampling texts recommend under certain circumstances.

8.5 Further Research and Extensions of the Estimation Approach

The hierarchical models I use, especially for undersampled trips by longline vessels, allow overdispersion by not constraining the variance model to be the same across all hauls in the region and hence provide more conservative estimates of precision. However, my use of management region-specific models was arbitrary and other more fine-scale models may be appropriate. Whatever models are used should ultimately be justified. Objective ways of determining appropriate models include the usual likelihood ratio tests with generalized linear models or Wald tests with semiparametric models such as that in Eq. 5.65. The models I chose were merely for example, but they reflect a conservative variance estimation approach.

More sophisticated models could be used to estimate seabird bycatch aboard trawl vessels. As I noted in Section 3.6.1, when no seabirds are found in a sampled haul, the size of that sample is unknown. Observers may use different sample sizes for different catch attributes in a given haul and they are instructed to use the largest of those sample sizes for seabirds, but these instructions may or may not be followed. Theoretically, the probabilities of using different sample sizes can be estimated from hauls where observers do find seabirds and this process can be incorporated with my current model to estimate bycatch without assuming the maximum sample size.

Finally, an area of further work that would be beneficial is exploration of the sensitivity of the estimates and variance estimates for particular parameters to the assumptions defining trips. As the number of days of non-fishing between observed hauls increases, the number of defined trips will decrease and vice versa. Ideally, there will be little change in the estimates and variance estimates with different numbers of days of non-fishing. However, if there are substantial changes it is important to use that information and perhaps developing models for trip length would be useful.

BIBLIOGRAPHY

AFSC (Alaska Fisheries Science Center). 2003. North Pacific Groundfish Observer Manual. North Pacific Groundfish Observer Program, AFSC, 7600 Sand Point Way N.E., Seattle, Washington 98155.

AFSC (Alaska Fisheries Science Center). 2004. North Pacific Groundfish Observer Manual. North Pacific Groundfish Observer Program, AFSC, 7600 Sand Point Way N.E., Seattle, Washington 98155.

Andrews, N. L. and Chen, Y. 1997. Optimal sampling for estimating the size structure and mean size of abalone caught in a New South Wales fishery. Fishery Bulletin **95**: 403–417.

Barbeaux, S., Ianelli, J. N., and Brown, E. 2004. Stock assessment of Aleutian Islands Region pollock. *In* Stock assessment and fishery evaluation report for the groundfish resources of the Bering Sea/Aleutian Islands., North Pacific Fishery Management Council, Anchorage, Alaska.

Bolfarine, H. and Zacks, S. 1992. Prediction Theory for Finite Populations. Springer-Verlag, New York.

Cassel, C.-M., Särndal, C.-E., and Wretman, J. H. 1977. Foundations of Inference in Survey Sampling. Wiley & Sons, New York.

Chakravarti, I. M. 1955. On the problem of planning a multistage survey for multiple correlated characters. Sankhyā **14**: 211–216.

Chester, A. J. and Waters, J. R. 1985. Two-stage sampling for age distribution in the Atlantic menhaden fishery, with comments on optimal survey design. North American Journal of Fisheries Management **5**: 449–456.

Cochran, W. G. 1977. Sampling Techniques. John Wiley & Sons, New York, third edition.

Dalenius, T. 1952. Eine einfache geometrische veranschaulichung der theorie des geschichteten stichprobenverfahres. Mitteilungsblatt für Mathematische Statistik **4**: 121–128.

Dalenius, T. 1953. The multivariate sampling problem. Skandinavisk Aktuarietidskrift **56**: 92–102.

Dalenius, T. 1957. Sampling in Sweden. Almqvist and Wiksell, Stockholm.

Dorn, M. W., Barbeaux, S., Gaichas, S., Guttormsen, M., Megrey, B., Spalinger, K., and Wilkins, M. 2004. Assessment of walleye pollock in the Gulf of Alaska. *In* Stock assessment and fishery evaluation report for the groundfish resources of the Gulf of Alaska., North Pacific Fishery Management Council, Anchorage, Alaska.

Dorn, M. W., Gaichas, S. K., Fitzgerald, S. M., and Bibb, S. A. 1999. Measuring total catch at sea: use of a motion-compensated flow scale to evaluate observer volumetric methods. North American Journal of Fisheries Management **19**: 999–1016.

Draper, N. R. and Guttman, I. 1968. Bayesian stratified two-phase sampling results: k characteristics. Biometrika **55**: 587–589.

Ericson, W. A. 1965. Optimal allocation in stratified sampling using prior information. Journal of the American Statisical Association **60**: 750–771.

Fournier, D. A., Hampton, J., and Sibert, J. R. 1998. MULTIFAN-CL: a length-based, age-structured model for fisheries stock assessment, with application to South Pacific albacore. Canadian Journal of Fisheries and Aquatic Sciences **55**: 2063–2077.

Fournier, D. A., Sibert, J. R., Majkowski, J., and Hampton, J. 1990. MULTIFAN a likelihood-based method for estimating growth parameters and age composition from multiple length frequency data sets illustrated using data for southern bluefin tuna (*Thunnus maccoyii*). Canadian Journal of Fisheries and Aquatic Sciences **47**: 301–317.

Fredin, R. A. 1987. History of regulation of Alaska groundfish fisheries. Processed Report 87-02, Northwest Alaska Fisheries Center, National Marine Fisheries Service.

Godambe, V. P. 1955. A unified theory of sampling from finite populations. Journal of the Royal Statistical Society. Series B **17**: 269–278.

Godambe, V. P. 1960. An optimum property of regular maximum likelihood estimation. Annals of Mathematical Statistics **31**: 1208–1211.

Goodman, L. A. 1960. On the exact variance of products. Journal of the American Statisical Association **55**: 708–713.

Hansen, M. H., Madow, W. G., and Tepping, B. J. 1983. An evaluation of model-dependent and probability-sampling inference in sample surveys. Journal of the American Statisical Association **78**: 776–793.

Hanurav, T. V. 1968. Hyperadmissibility and optimum estimators for sampling finite populations. Annals of Mathematical Statistics **39**: 621–642.

Hartley, H. O. 1965. Multiple purpose optimum allocation in stratified sampling. Proceedings of the American Statistical Association, Social Statistics Section pp. 258–261.

Hartley, H. O. and Rao, J. N. K. 1968. A new estimation theory for sample surveys. Biometrika **55**: 547–557.

Horppila, J. and Peltonen, H. 1992. Optimizing sampling from trawl catches: contemporaneous multistage sampling for age and length structures. Canadian Journal of Fisheries and Aquatic Sciences **49**: 1555–1559.

Horvitz, D. G. and Thompson, D. J. 1952. A generalization of sampling without replacement from a finite universe. Journal of the American Statisical Association **47**: 663–685.

Hrafnkelsson, B. and Stefánson, G. 2004. A model for categorical length data from groundfish surveys. Canadian Journal of Fisheries and Aquatic Sciences **61**: 1135–1142.

Ianelli, J. N., Barbeaux, S., Walters, G., Honkalehto, T., and Williamson, N. 2004. Eastern Bering Sea walleye pollock assessment. *In* Stock assessment and fishery evaluation report for the groundfish resources of the Bering Sea/Aleutian Islands., North Pacific Fishery Management Council, Anchorage, Alaska.

Jessen, R. J. 1978. Statistical Survey Techniques. John Wiley & Sons, New York.

Jinn, J. H., Sedransk, J., and Smith, P. 1987. Optimal two-phase stratified sampling for estimation of the age composition of a fish population. Biometrics **43**: 343–353.

Kappenman, R. F. 1992. Statistical analyses of vessel incentive program data. Processed Report 92-02, Alaska Fisheries Science Center, National Marine Fisheries Service.

Karp, W. A. 1997. Observer coverage needs. Memorandum to: The North Pacific Fishery Management Council, Observer Advisory Committee.

Ketchen, K. S. 1949. Stratified subsampling for determining age distributions. Transactions of the American Fisheries Society **79**: 205–212.

Khan, M. G. M., Khan, E. A., and Ahsan, M. J. 2003. An optimal multivariate stratified sampling design using dynamic programming. Australian and New Zealand Journal of Statistics **45**: 107–113.

Kimura, D. K. 1977. Statistical assessment of the age-length key. Journal of the Fisheries Research Board of Canada **34**: 317–324.

Kimura, D. K. 1989. Variability in estimating catch-in-numbers-at-age and its impact on cohort analysis. *In* Effects of Ocean Variability on Recruitment and an evaluation of parameters used in stock assessment models, edited by R. J. Beamish and G. A. McFarlane, pp. 57–66, Canadian Special Publications of Fisheries and Aquatic Sciences 108.

Kokan, A. R. 1963. Optimum allocation in multivariate surveys. Journal of the Royal Statistical Society. Series A **126**: 557–565.

Kutkuhn, J. H. 1963. Estimating absolute age composition of California salmon landings. California Department of Fish and Game, Fish Bulletin No. 120.

Lai, H.-L. 1987. Optimum allocation for estimating age composition using age-length key. Fishery Bulletin **85**: 179–185.

Lai, H.-L. 1993. Optimal sample design for using the age-length key to estimate age composition of a fish population. Fishery Bulletin **91**: 382–388.

Manly, B. F. J., Akroyd, J.-A. M., and Walshe, K. A. R. 2002. Two-phase stratified random surveys on multiple populations at multiple locations. New Zealand Journal of Marine and Freshwater Research **36**: 581–591.

McCullagh, P. and Nelder, J. A. 1989. Generalized Linear Models. Chapman & Hall.

MRAG (Marine Resources Assessment Group Americas, Inc.). 2000. Independent review of the North Pacific Groundfish Observer Program. Report to: Alaska Fisheries Science Center, National Marine Fisheries Service.

MRAG (Marine Resources Assessment Group Americas, Inc.). 2003. Evaluation and analysis of current field sampling used in North Pacific groundfish fisheries. Report to: Alaska Fisheries Science Center, National Marine Fisheries Service.

Neyman, J. 1934. On the two different aspects of the representative method: the method of stratified sampling and the method of purposive selection. Journal of the Royal Statistical Society **97**: 558–625.

NPFMC (North Pacific Fishery Management Council). 2003. Stock assessment and fishery evaluation document for the BSAI and GOA. Appendix C: ecosystem considerations, Anchorage, Alaska.

NPFMC (North Pacific Fishery Management Council). 2004. Stock assessment and fishery evaluation document for the BSAI and GOA, Anchorage, Alaska.

Rao, C. R. 1971. Some aspects of statistical inference in problems of sampling from finite populations. *In* Proceedings of the Symposium on the Foundations of Statistical Inference, pp. 203–233, Holt, Rinehart & Winston, Toronto.

Robinson, J. 1978. An asymptotic expansion for samples from a finite population. The Annals of Statistics **6**: 1005–1011.

Royall, R. M. 1968. An old approach to finite population sampling. Journal of the American Statisical Association **63**: 1269–1279.

Royall, R. M. 1970. On finite population sampling theory under certain linear regression models. Biometrika **57**: 377–387.

Royall, R. M. 1976. The linear least-squares prediction approach to two-stage sampling. Journal of the American Statisical Association **71**: 657–664.

Royall, R. M. and Herson, J. 1973a. Robust estimation in finite populations i. Journal of the American Statistical Association **68**: 880–889.

Royall, R. M. and Herson, J. 1973b. Robust estimation in finite populations ii: stratification on a size variable. Journal of the American Statistical Association **68**: 890–893.

Särndal, C.-E. 1978. Design-based and model-based inference in survey sampling. Scandinavian Journal of Statistics **5**: 27–52.

Särndal, C.-E., Swensson, B., and Wretman, J. H. 1992. Model Assisted Survey Sampling. Springer, New York.

Schnute, J. and Richards, L. J. 1995. The influence of error on population estimates from catch-age models. Canadian Journal of Fisheries and Aquatic Sciences **52**: 2063–2077.

Schweigert, J. F. and Sibert, J. R. 1983. Optimizing survey design for determining age structure of fish stocks: an example from British Columbia Pacific herring (*Clupea harengus pallasi*). Canadian Journal of Fisheries and Aquatic Sciences **40**: 588–597.

Sen, A. R. 1986. Methodolgical problems in sampling commercial rockfish landings. Fishery Bulletin **84**: 409–421.

Smith, P. J. 1988. Survey design optimization for estimating the exploitable biomass of a fishery accounting for non-sampling errors. Applied Statistics **37**: 370–384.

Smith, P. J. 1989. Is two-phase sampling really better for estimating age composition? Journal of the American Statisical Association **84**: 916–921.

Smith, P. J. and Sedransk, J. 1982. Bayesian optimization of the estimation of the age composition of a fish population. Journal of the American Statisical Association **77**: 707–713.

Smith, T. M. F. 1976. The foundations of survey sampling: a review. Journal of the Royal Statistical Society. Series A **139**: 183–195.

Smith, T. M. F. 1994. Sample surveys 1975-1990; an age of reconciliation? International Statistical Review **62**: 5–19.

Smith, T. M. F. 2001. *Biometrika* centenary: sample surveys. Biometrika **88**: 167–194.

Solomon, H. and Zacks, S. 1970. Optimal design of sampling from finite populations: a critical review and indication of new research areas. Journal of the American Statistical Association **65**: 653–667.

Southward, G. M. 1976. Sampling landings of halibut for age composition. International Pacific Halibut Commision, Scientific Report No. 58.

Sugden, R. A. and Smith, T. M. F. 1997. Edgeworth approximations to the distribution of the sample mean under simple random sampling. Statistics & Probability Letters **34**: 293–299.

Sugden, R. A., Smith, T. M. F., and Jones, R. P. 2000. Cochran's rule for simple random sampling. Journal of the Royal Statistical Society. Series B **62**: 787–793.

Sukhatme, P. V. and Sukhatme, B. V. 1970. Sampling theory of surveys with applications. Iowa State University Press, Ames, Iowa.

Tanaka, S. 1953. Precision of age-determination of fish estimated by double sampling method using the length for stratification. Bulletin of the Japanese Society of Scientific Fisheries **19**: 657–670.

Thompson, G. G. and Dorn, M. W. 2004. Assessment of the Pacific cod stock in the Eastern Bering Sea and Aleutian Islands area. *In* Stock assessment and fishery evaluation report for the groundfish resources of the Bering Sea/Aleutian Islands., chapter 2, North Pacific Fishery Management Council, Anchorage, Alaska.

Thompson, G. G., Zenger, H. H., and Dorn, M. W. 2004. Assessment of Pacific cod in the Gulf of Alaska. *In* Stock assessment and fishery evaluation report for the groundfish resources of the Gulf of Alaska., chapter 2, pp. 131–232, North Pacific Fishery Management Council, Anchorage, Alaska.

Turnock, J. and Karp, W. A. 1997. Estimation of salmon bycatch in the 1995 pollock fishery in the Bering Sea/Aleutian Islands: a comparison of methods based on observer sampling and counts of salmon retained by fishing vessel and processing plant personnel, Alaska Fisheries Science Center, National Marine Fisheries Service.

Valliant, R., Dorfman, A. H., and Royall, R. M. 2000. Finite Population Sampling and Inference: a Prediction Approach. Wiley, New York.

Vølstad, J. H., Richkus, W., Gaurin, S., and Easton, R. 1997. Analytical and statistical review of procedures for collection and analysis of commercial fishery data used for management and assessment of groundfish stocks in the U.S. exclusive economic zone off Alaska. Report to: Alaska Fisheries Science Center, National Marine Fisheries Service.

Waters, J. R. and Chester, A. J. 1987. Optimal allocation in multivariate, two-stage sampling designs. American Statistician **41**: 46–50.

Witherell, D. and Pautzke, C. 1997. A brief history of bycatch management measures for Eastern Bering Sea groundfish fisheries. Marine Fisheries Review **59**: 15–22.

Yates, F. 1981. Sampling methods for censuses and surveys. MacMillan, New York, fourth edition.

Appendix A

SAMPLING THEORY

A.1 Preliminaries

The notation I use here is taken and sometimes modified from Särndal et al. (1992). Let $U = \{1, \ldots, i, \ldots, N\}$ be the population of all elements available to sample and y_i be the value of a variable for the ith element. Also, for a given design, let S be a set-valued random variable that takes on values that are all possible samples from U, such that $P(S = s) = p(s)$ is the probability of taking a particular sample. The indicator,

$$I_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}, \tag{A.1}$$

which Särndal et al. (1992) term the *sample membership indicator* is useful for calculating expectation because the probability of the ith element will be sampled is $P(i \in S) = P(I_i = 1)$. This is called the *first order inclusion probability,*

$$\pi_k = P(I_i = 1) = \sum_{s:i \in s} p(s).$$

The *second order inclusion probability,* $\pi_{ij} = P(I_i I_j = 1)$, is the probability that both the ith and jth elements are in the sample. It is also important to note that because the indicators are Bernoulli random variables π_i and π_{ij} are $E(I_i)$ and $E(I_i I_j)$, respectively. Furthermore,

$$Var(I_i) = \pi_i(1 - \pi_i) \equiv \Delta_{ii} \tag{A.2}$$

$$Cov(I_i, I_j) = E(I_i I_j) - E(I_i)E(I_j) = \pi_{ij} - \pi_i \pi_j \equiv \Delta_{ij} \tag{A.3}$$

Now, the Horvitz-Thompson estimator of the population total, $T = \sum_U y_i$, is

$$\widehat{T}^* = \sum_s \frac{y_i}{\pi_i}, \tag{A.4}$$

which can also be written as

$$\widehat{T}^* = \sum_U I_i \frac{y_i}{\pi_i}$$

(Horvitz and Thompson 1952). The expected value of this estimator is

$$E\left(\widehat{T}^*\right) = \sum_U \frac{y_i}{\pi_i} E(I_i) = \sum_U y_i = T.$$

The variance of Eq. A.4 is

$$\begin{aligned}
V\left(\widehat{T}^{*}\right) &= E\left(\widehat{T}^{*}-T\right)^{2} \\
&= E\left[\left(\sum\nolimits_{s}\frac{y_i}{\pi_i}\right)^{2}\right]-T^{2} \\
&= E\left[\sum\sum\nolimits_{s}\frac{y_i}{\pi_i}\frac{y_j}{\pi_j}\right]-\sum\sum\nolimits_{U}y_iy_j \\
&= E\left[\sum\sum\nolimits_{U}I_iI_j\frac{y_i}{\pi_i}\frac{y_j}{\pi_j}\right]-\sum\sum\nolimits_{U}\pi_i\pi_j\frac{y_i}{\pi_i}\frac{y_j}{\pi_j} \\
&= \sum\sum\nolimits_{U}(\Delta_{ij}+\pi_i\pi_j)\frac{y_i}{\pi_i}\frac{y_j}{\pi_j}-\sum\sum\nolimits_{U}\pi_i\pi_j\frac{y_i}{\pi_i}\frac{y_j}{\pi_j} \\
&= \sum\sum\nolimits_{U}\Delta_{ij}\frac{y_i}{\pi_i}\frac{y_j}{\pi_j}. \qquad \text{(A.5)}
\end{aligned}$$

The unbiased variance estimator is

$$\sum\sum\nolimits_{s}\frac{\Delta_{ij}}{\pi_{ij}}\frac{y_i}{\pi_i}\frac{y_j}{\pi_j}.$$

A couple of identities that are useful for deriving variances and estimators of variance under simple random sampling are

$$\begin{aligned}
\sum\nolimits_{s}y_i^2+a\underset{i\neq j}{\sum\sum\nolimits_{s}}y_iy_j &= (1-a)\sum\nolimits_{s}y_i^2+a\sum\sum\nolimits_{s}y_iy_j \\
&= (1-a)\left(\sum\nolimits_{s}y_i^2-\frac{1}{n}\sum\sum\nolimits_{s}y_iy_j\right) \\
&\quad +\left(a+\frac{1-a}{n}\right)\sum\sum\nolimits_{s}y_iy_j \\
&= (1-a)\sum\nolimits_{s}(y_i-\overline{y})^2+n\overline{y}^2(a(n-1)+1)
\end{aligned}$$

and

$$\begin{aligned}
\sum\nolimits_s (y_{ik} - \overline{y})^2 &= \sum\nolimits_{s_1} \sum\nolimits_{s_i} (y_{ik} - \overline{y})^2 \\
&= \sum\nolimits_{s_1} \sum\nolimits_{s_i} [(y_{ik} - \overline{y}_i) + (\overline{y}_i - \overline{y})]^2 \\
&= \sum\nolimits_{s_1} \sum\nolimits_{s_i} (y_{ik} - \overline{y}_i)^2 + \sum\nolimits_{s_1} n_i (\overline{y}_i - \overline{y})^2 \\
&= \sum\nolimits_{s_1} \sum\nolimits_{s_i} (y_{ik} - \overline{y}_i)^2 + \sum\nolimits_{s_1} n_i \overline{y}_i^2 - n\overline{y}^2 \\
&= \sum\nolimits_{s_1} [(n_i - 1)\widehat{S}_i^2 + n_i \overline{y}_i^2] - n\overline{y}^2 \qquad \text{(A.6)}
\end{aligned}$$

A.2 Simple Random Sampling

In simple random sampling each element has an equal chance of being in a sample of size, n. The probability that the ith element will be sampled is

$$\pi_i = P(I_i = 1) = \frac{\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{n}{N}$$

and the probability that the ith and jth elements are sampled is

$$\pi_{ij} = P(I_i I_j = 1) = \frac{\binom{N-2}{n-2}}{\binom{N}{n}} = \frac{n(n-1)}{N(N-1)}.$$

This yields

$$\Delta_{ii} = \frac{n}{N}\left(1 - \frac{n}{N}\right)$$

and

$$\Delta_{ij} = \frac{n(n-1)}{N(N-1)} - \left(\frac{n}{N}\right)^2 = -\frac{1}{N-1}\frac{n}{N}\left(1 - \frac{n}{N}\right).$$

The usual estimator of the total under simple random sampling is $N\overline{y}$ which is the Horvitz-Thompson estimator,

$$N\overline{y} = \frac{N}{n}\sum\nolimits_s y_i = \sum\nolimits_s \frac{y_i}{\pi_i}, \qquad \text{(A.7)}$$

and by Eq. A.5 the variance of this estimator is

$$\begin{aligned}
V\left(\widehat{T}^{*}\right) &= \sum\sum\nolimits_{U} \Delta_{ij} \frac{y_i}{\pi_i} \frac{y_j}{\pi_j} \\
&= \left(\frac{N}{n}\right)^2 \frac{n}{N}\left(1-\frac{n}{N}\right)\left(\sum\nolimits_U y_i^2 - \frac{1}{N-1}\sum_{i\neq j}\sum\nolimits_U y_i y_j\right) \\
&= \frac{N}{n}\left(1-\frac{n}{N}\right)\left[\frac{(N-1)\sum_U y_i^2 - \sum_{i\neq j}\sum_U y_i y_j}{N-1}\right] \\
&= N\left(\frac{N}{n}-1\right) S_U^2 \qquad \text{(A.8)}
\end{aligned}$$

where $S_U^2 = \frac{1}{N-1}\sum_U (y_i - \bar{y})^2$. Also, $S_s^2 = \frac{1}{n-1}\sum_s (y_i - \bar{y}_s)^2$ is an unbiased estimator of S_U^2. The estimator of the sampling variance is

$$\begin{aligned}
\widehat{V}\left(\widehat{T}^{*}\right) &= \sum\sum\nolimits_{s} \frac{\Delta_{ij}}{\pi_{ij}} \frac{y_i}{\pi_i} \frac{y_j}{\pi_j} \\
&= \left(\frac{N}{n}\right)^2 \left(1-\frac{n}{N}\right) \frac{\sum_s (n-1) y_i^2 - \sum_{i\neq j}\sum_s y_i y_j}{n-1} \\
&= N\left(\frac{N}{n}-1\right) \frac{\sum_s y_i^2 - n\bar{y}^2}{n-1} \\
&= N\left(\frac{N}{n}-1\right) S_s^2. \qquad \text{(A.9)}
\end{aligned}$$

Another way to show that the estimator is unbiased,

$$
\begin{aligned}
E\left(S_s^2\right) &= \frac{E\left(\sum_s y_i^2 - n\bar{y}_s^2\right)}{n-1} \\
&= \frac{1}{n-1}\left[\left(\frac{n-1}{N}\right)\sum_U y_i^2 - \frac{n-1}{N(N-1)}\sum\sum_{\substack{U \\ i\neq j}} y_i y_j\right] \\
&= \frac{1}{N}\sum_U y_i^2 - \frac{1}{N(N-1)}\sum\sum_{\substack{U \\ i\neq j}} y_i y_j \\
&= \frac{1}{N-1}\left(\frac{N-1}{N}\sum_U y_i^2 - \frac{1}{N}\sum\sum_{\substack{U \\ i\neq j}} y_i y_j\right) \\
&= \frac{1}{N-1}\left(\sum_U y_i^2 - \frac{1}{N}\sum\sum_U y_i y_j\right) \\
&= \frac{1}{N-1}\left(\sum_U y_i^2 - N\bar{y}^2\right) \\
&= \frac{1}{N-1}\sum_U \left(y_i - \bar{y}\right)^2. \qquad \text{(A.10)}
\end{aligned}
$$

So, $E\left[\widehat{V}\left(\widehat{T}^*\right)\right] = V\left(\widehat{T}^*\right)$.

A.2.1 Hypergeometric Distribution

When SRS is performed on elements in a universe and the characteristic of interest observed on those elements is binary in nature, the sampling distribution is hypergeometric. Thus, some results under this scenario are provided. If $x \sim Hyper(N, n, X)$, then

$$f(x) = \frac{\binom{X}{x}\binom{N-X}{n-x}}{\binom{N}{n}}.$$

It is straightforward to show that

$$E(x) = \frac{n}{N}X$$

and

$$V(x) = n\left(1 - \frac{n}{N}\right)\frac{NP(1-P)}{N-1}$$

where $P = X/N$. With large N,

$$V(x) \doteq n\left(1 - \frac{n}{N}\right) P(1-P).$$

An unbiased estimator of X is

$$\widehat{X} = \frac{N}{n} x \tag{A.11}$$

and the variance of Eq. A.11 is

$$V\left(\widehat{X}\right) = \left(\frac{N}{n}\right)^2 V(x) = N\left(\frac{N}{n} - 1\right) \frac{NP(1-P)}{N-1} \tag{A.12}$$

and with large N

$$V\left(\widehat{X}\right) \doteq N\left(\frac{N}{n} - 1\right) P(1-P)$$

An unbiased estimator of Eq. A.12 is

$$\widehat{V}\left(\widehat{X}\right) = \left(\frac{N}{n}\right)^2 \left(1 - \frac{n}{N}\right) \frac{x(n-x)}{n-1} = N\left(\frac{N}{n} - 1\right) \frac{np(1-p)}{n-1} \tag{A.13}$$

where $p = x/n$.

A proof of the unbiasedness of Eq. A.13.

$$\begin{aligned} E\left[\widehat{V}\left(\widehat{X}\right)\right] &= \frac{N}{n}\left(\frac{N}{n} - 1\right) \frac{nE(x) - E(x^2)}{n-1} \\ &= \frac{N(N-n)}{n^2(n-1)} \left[\frac{n^2 X}{N} - \frac{n}{N} X \frac{(N-X)(N-n)}{N(N-1)} - \left(\frac{n}{N} X\right)^2\right] \\ &= N\left(\frac{N}{n} - 1\right) \frac{NP\,(1-P)}{N-1} \end{aligned}$$

To see the hypergeometric distribution is connected to the sampling distribution under SRS, consider each of the n elements, $x_1, \ldots, x_i, \ldots, x_n$, that comprise the hypergeometric random variable. The random variable x_i takes on values, 1 or 0 and $\sum_{i=1}^{n} x_i = x$. Therefore, Eq. A.11 can be written as

$$\widehat{X} = \frac{N}{n} \sum_{i=1}^{n} x_i$$

which is the unbiased estimator under SRS given in Eq. A.7. Furthermore, because of the fact, $x_i^2 = x_i$, the variance under SRS is

$$
\begin{aligned}
N\left(\frac{N}{n}-1\right)\frac{\sum_{i-1}^{N}(x_i-\overline{x})^2}{N-1} =& N\left(\frac{N}{n}-1\right)\frac{\sum_{i=1}^{N}x_i^2-N\overline{x}^2}{N-1} \\
=& N\left(\frac{N}{n}-1\right)\frac{NP(1-P)}{N-1}
\end{aligned}
$$

which is the variance of the estimator of the total with the hypergeometric distribution. A similar identity can be shown for the variance estimators.

A.2.2 Multivariate Hypergeometric

The hypergeometric distribution is useful when the characteristic of interest takes on just one of two values, but in general, it may take on one of p values and the sampling distribution is then multivariate hypergeometric.

If $\mathbf{x} \sim mhyper(N, n, \mathbf{X})$ where $\mathbf{x}' = (x_1, \ldots, x_p)$ and $\mathbf{X}' = (X_1, \ldots, X_p)$, then

$$E(\mathbf{x}) = \frac{n}{N}\mathbf{X}$$

and

$$
\begin{aligned}
V(\mathbf{x}) =& n\left(1-\frac{n}{N}\right)\frac{\sum_{i=1}^{N}(\mathbf{x}_i-\overline{\mathbf{x}})(\mathbf{x}_i-\overline{\mathbf{x}})^T}{N-1} \\
=& n\left(1-\frac{n}{N}\right)\frac{N\left[\mathrm{diag}(\mathbf{P})-\mathbf{P}\mathbf{P}^T\right]}{N-1}
\end{aligned}
$$

where $\mathbf{P} = \mathbf{X}/N = \sum_{i=1}^{N}\mathbf{x}_i/N$ because $\mathbf{x}_i\mathbf{x}_i^T = \mathrm{diag}(\mathbf{x}_i)$.

Similar to the univariate hypergeometric distribution an unbiased estimator of $\mathbf{X}$ is

$$\widehat{\mathbf{X}} = \frac{N}{n}\mathbf{x}$$

or

$$\widehat{\mathbf{X}} = \frac{N}{n}\sum_{i=1}^{n}\mathbf{x}_i$$

and an unbiased variance-covariance matrix estimator is

$$\begin{aligned}\widehat{V}\left(\widehat{\mathbf{X}}\right) =& N\left(\frac{N}{n}-1\right)\frac{\sum_{i=1}^{n}\left(\mathbf{x}_i-\overline{\mathbf{x}}\right)\left(\mathbf{x}_i-\overline{\mathbf{x}}\right)^T}{n-1}\\ =& N\left(\frac{N}{n}-1\right)\frac{n\left[\operatorname{diag}\left(\mathbf{p}\right)-\mathbf{p}\mathbf{p}^T\right]}{n-1}\end{aligned}$$

where $\mathbf{p}=\sum_{i=1}^{n}\mathbf{x}_i/n$.

A.2.3 Simple Random Sample of a Simple Random Sample

Let n_1 be the size of a sample taken via SRS from N, then take a SRS subsample of size $n_2 \leq n_1$ of the first SRS. The inclusion probabilities are

$$\pi_i = \frac{n_1}{N}\frac{n_2}{n_1} = \frac{n_2}{N}$$

and

$$\pi_{ij} = \frac{n_1(n_1-1)}{N(N1-1)}\frac{n_2(n_2-1)}{n_1(n_1-1)} = \frac{n_2(n_2-1)}{N(N-1)}.$$

Using these inclusion probabilities the HTT, variance and estimator of variance are as if the first sample was never taken. That is, the marginal sampling distribution of the final sample is itself an SRS. Specifically,

$$\widehat{T}^* = \frac{N}{n_2}\sum\nolimits_s y_i,$$

$$V\left(\widehat{T}^*\right) = N\left(\frac{N}{n_2}-1\right)S^2$$

and

$$\widehat{V}\left(\widehat{T}^*\right) = \frac{N^2(1-f)}{n_2}\widehat{S}^2$$

where

$$\widehat{S}^2 = \frac{\sum_s(y_i-\overline{y})^2}{n_2-1}.$$

A.3 Two-Stage Sampling

In two-stage sampling the universe is split into N_1 groups of elements which will be indexed, U_i each with N_i elements where $i \in \{1, \ldots, N_1\}$. Call the restructured universe, $U_1 \equiv U$ and we can see that $\sum_{U_1} N_i = N$. The sample (s_1) of *primary sampling units* is selected via the first-stage design, $p(s_1)$, and the sample of elements (s_i) in each U_i (*secondary sampling units*) is selected via the second-stage designs, $p(s_i)$. Analogously, let n_1 and n_i be the number of primary and secondary units sampled, respectively. We also need π-notation for the first and second stages. Let π_i and π_{ij} be the first and second order inclusion probabilities for the first-stage design, respectively, and π_{ik} and π_{ikl} be the first and second order *conditional* inclusion probabilities for the second stage designs, respectively. The Δ-values are obtained similarly to those previously.

$$\Delta_{ii} = \pi_i(1 - \pi_i)$$

$$\Delta_{ij} = \pi_{ij} - \pi_i\pi_j$$

$$\Delta_{ikk} = \pi_{ik}(1 - \pi_{ik})$$

$$\Delta_{ikl} = \pi_{ikl} - \pi_{ik}\pi_{il}$$

If we let $\widehat{T}_i^* = \sum_{Ui} \frac{y_{ik}}{\pi_{ik}}$ we have the conditional Horvitz-Thompson estimator within the ith primary sampling and the sampling variance within the first stage elements is

$$V\left(\widehat{T}_i^*|s_1\right) = \sum\sum\nolimits_{U_i} \Delta_{ikl}\frac{y_{ik}y_{il}}{\pi_{ik}\pi_{il}} \equiv V_i.$$

To derive the variance of the estimator of the overall total, $\widehat{T}^*$, in a two-stage design it is useful to write the variance as

$$V\left(\widehat{T}^*\right) = V\left[E\left(\widehat{T}^*|s_1\right)\right] + E\left[V\left(\widehat{T}^*|s_1\right)\right]. \tag{A.14}$$

$E\left(\widehat{T}^*|s_1\right)$ and $V\left(\widehat{T}^*|s_1\right)$ are the expected value and variance, respectively, of the estimator given a particular first-stage sample. We will derive the equation for the variance under this design, but first let us show the result (Eq. A.14) above.

$$\begin{aligned}
V(\hat{\theta}) &= E(\hat{\theta}^2) - E(\hat{\theta})^2 \\
&= E[E(\hat{\theta}^2|\phi)] - E[E(\hat{\theta}|\phi)]^2 \\
&= E[V(\hat{\theta}|\phi) + E(\hat{\theta}|\phi)^2] - E[E(\hat{\theta}|\phi)]^2 \\
&= E[V(\hat{\theta}|\phi)] + \underbrace{E[E(\hat{\theta}|\phi)^2] - E[E(\hat{\theta}|\phi)]^2}_{V[E(\hat{\theta}|\phi)]}
\end{aligned}$$

We can also obtain a similar result for 3 variables.

$$\begin{aligned}
V(\hat{\theta}) =& E\{E[E(\hat{\theta}^2|\phi,\psi)]\} - E\{E[E(\hat{\theta}|\phi,\psi)]\}^2 \\
=& E\{E[V(\hat{\theta}|\phi,\psi) + E(\hat{\theta}|\phi,\psi)^2]\} - E\{E[E(\hat{\theta}|\phi,\psi)]\}^2 \\
=& E\{E[V(\hat{\theta}|\phi,\psi)]\} + E\{E[E(\hat{\theta}|\phi,\psi)^2]\} - E\{E[E(\hat{\theta}|\phi,\psi)]\}^2 \\
=& E\{E[V(\hat{\theta}|\phi,\psi)]\} + E\{V[E(\hat{\theta}|\phi,\psi)] + E[E(\hat{\theta}|\phi,\psi)]^2\} - E\{E[E(\theta|\phi,\psi)]\}^2 \\
=& E\{E[V(\hat{\theta}|\phi,\psi)]\} + E\{V[E(\hat{\theta}|\phi,\psi)]\} + E\{E[E(\hat{\theta}|\phi,\psi)]^2\} - E\{E[E(\hat{\theta}|\phi,\psi)]\}^2 \\
=& E\{E[V(\hat{\theta}|\phi,\psi)]\} + E\{V[E(\hat{\theta}|\phi,\psi)]\} + V\{E[E(\hat{\theta}|\phi,\psi)]\} \qquad \text{(A.15)}
\end{aligned}$$

The conditional expectation and variance of $\widehat{T}^*$ given the first stage sample, are

$$\begin{aligned}
E\left(\widehat{T}^*|s_1\right) &= E\left(\sum\nolimits_s \frac{y_k}{\pi_k}\bigg|s_1\right) \\
&= E\left(\sum\nolimits_{s_1}\sum\nolimits_{s_i} \frac{y_k}{\pi_i \pi_{k|i}}\bigg|s_1\right) \\
&= \sum\nolimits_{s_1} \frac{1}{\pi_i} E\left(\sum\nolimits_{s_i} \frac{y_k}{\pi_{k|i}}\bigg|s_1\right) \\
&= \sum\nolimits_{s_1} \frac{E\left(\widehat{T}_i^*|s_1\right)}{\pi_i} \\
&= \sum\nolimits_{s_1} \frac{T_i}{\pi_i}
\end{aligned}$$

and

$$
\begin{aligned}
Var\left(\widehat{T}^{*}|s_1\right) &= Var\left(\sum\nolimits_{s}\frac{y_k}{\pi_k}\middle|s_1\right) = \sum\nolimits_{s_1} Var\left(\sum\nolimits_{s_i}\frac{y_k}{\pi_i\pi_{k|i}}\middle|s_1\right)\\
&= \sum\nolimits_{s_1}\frac{Var\left(\widehat{T}_i^{*}\middle|s_1\right)}{\pi_i^2}\\
&= \sum\nolimits_{s_1}\frac{V_i}{\pi_i^2}
\end{aligned}
$$

where $T_i = \sum_{U_i} y_{ik} = y_i$. So, for the variance of a two-stage design we have

$$
\begin{aligned}
Var\left(\widehat{T}^{*}\right) &= E\left(\sum\nolimits_{s_1}\frac{V_i}{\pi_i^2}\right) + Var\left(\sum\nolimits_{s_1}\frac{T_i}{\pi_i}\right)\\
&= \sum\nolimits_{U_1} E\left(I_i\right)\frac{V_i}{\pi_i^2} + \sum\sum\nolimits_{U_1}\Delta_{ij}\frac{T_iT_j}{\pi_i\pi_j}\\
&= \sum\nolimits_{U_1}\frac{V_i}{\pi_i} + \sum\sum\nolimits_{U_1}\Delta_{ij}\frac{T_iT_j}{\pi_i\pi_j} \qquad \text{(A.16)}\\
&= V_{SSU} + V_{PSU}
\end{aligned}
$$

where V_{PSU} and V_{SSU} are the contributions to the sampling variance by the primary and secondary stages, respectively.

I will now show that

$$
\begin{aligned}
\widehat{V}\left(\widehat{T}^{*}\right) &= \underbrace{\sum\sum\nolimits_{s_1}\frac{\Delta_{ij}}{\pi_{ij}}\frac{\hat{y}_i\hat{y}_j}{\pi_i\pi_j} - \sum\nolimits_{s_1}\frac{1}{\pi_i}\left(\frac{1}{\pi_i}-1\right)\widehat{V}_i}_{\widehat{V}_{PSU}} + \underbrace{\sum\nolimits_{s_1}\frac{\widehat{V}_i}{\pi_i^2}}_{\widehat{V}_{SSU}}\\
&= \sum\sum\nolimits_{s_1}\frac{\Delta_{ij}}{\pi_{ij}}\frac{\hat{y}_i\hat{y}_j}{\pi_i\pi_j} + \sum\nolimits_{s_1}\frac{\widehat{V}_i}{\pi_i} \qquad \text{(A.17)}
\end{aligned}
$$

is an unbiased estimator of $V\left(\widehat{T}^{*}\right)$.

$$
\begin{aligned}
E\left(\widehat{V}_{PSU}\right) &= E_1\left[E_2\left(\widehat{V}_{PSU}\right)\right] \\
&= E_1\left[\sum\sum\nolimits_{s_1} \frac{\Delta_{ij}}{\pi_{ij}} \frac{E_2\left(\hat{y}_i\hat{y}_j\right)}{\pi_i\pi_j}\right] - E_1\left[\sum\nolimits_{s_1} \frac{1}{\pi_i}\left(\frac{1}{\pi_i}-1\right) E_2\left(\widehat{V}_i\right)\right] \\
&= E_1\left(\sum\nolimits_{s_1} \frac{\Delta_{ii}}{\pi_{ii}} \frac{V_i + y_i^2}{\pi_i\pi_i}\right) + E_1\left(\sum\sum\nolimits_{\substack{s_1 \\ i\neq j}} \frac{\Delta_{ij}}{\pi_{ij}} \frac{y_i y_j}{\pi_i\pi_j}\right) \\
&\quad -E_1\left[\sum\nolimits_{s_1} \frac{1}{\pi_i}\left(\frac{1}{\pi_i}-1\right) V_i\right] \\
&= \sum\nolimits_{U_1} \Delta_{ii}\frac{V_i + y_i^2}{\pi_i\pi_i} + \sum\sum\nolimits_{\substack{U_1 \\ i\neq j}} \Delta_{ij}\frac{y_i y_j}{\pi_i\pi_j} - \sum\nolimits_{U_1}\left(\frac{1}{\pi_i}-1\right) V_i \\
&= \sum\sum\nolimits_{U_1} \Delta_{ij}\frac{y_i y_j}{\pi_i\pi_j} + \sum\nolimits_{U_1} \frac{\Delta_{ii}}{\pi_i}\frac{V_i}{\pi_i} - (1-\pi_i)\frac{V_i}{\pi_i} \\
&= \sum\sum\nolimits_{U_1} \Delta_{ij}\frac{y_i y_j}{\pi_i\pi_j}.
\end{aligned}
$$

The summation on the right in the second to last step equals zero because $\Delta_{ii} = Var(I_i) = \pi_i(1-\pi_i)$. Also,

$$
\begin{aligned}
E\left(\widehat{V}_{SSU}\right) &= E\left(\sum\nolimits_{s_1} \frac{\widehat{V}_i}{\pi_i^2}\right) \\
&= E_1\left[\sum\nolimits_{s_1} \frac{E_2\left(\widehat{V}_i\right)}{\pi_i^2}\right] \\
&= E_1\left(\sum\nolimits_{s_1} \frac{V_i}{\pi_i^2}\right) \\
&= \sum\nolimits_{U_1} \frac{V_i}{\pi_i}
\end{aligned}
$$

and so, the variance estimator is unbiased.

A.3.1 Two-Stage Simple Random Sampling

In general, the Horvitz-Thompson estimator, $\widehat{T}^*$, in two-stage sampling with simple random sampling at both stages is

$$\widehat{T}^* = \sum_{s_1} \sum_{s_i} \frac{y_{ik}}{\pi_i \pi_{ik}} = \frac{N_1}{n_1} \sum_{s_1} \frac{N_i}{n_i} \sum_{s_i} y_{ik} \tag{A.18}$$

We know that

$$V\left(\frac{N_i}{n_i} \sum_{s_i} y_{ik} | s_1\right) = N_i \left(\frac{N_i}{n_i} - 1\right) S_i^2 \equiv V_i$$

where

$$S_i^2 = \frac{\sum_{Ui} (y_{ik} - \overline{y}_i)^2}{N_i - 1}$$

and $\overline{y}_i = \sum_{s_i} y_{ik}/N_i$. The variance of (Eq. A.18) is straighforwardly obtained using the result (Eq. A.16).

$$\begin{aligned} Var\left(\widehat{T}^*\right) &= E\left(\sum_{s_1} \frac{V_i}{\pi_i^2}\right) + Var\left(\sum_{s_1} \frac{T_i}{\pi_i}\right) \\ &= \left(\frac{N_1}{n_1}\right) \sum_{U_1} N_i \left(\frac{N_i}{n_i} - 1\right) S_i^2 + N_1 \left(\frac{N_1}{n_1} - 1\right) S_{T,1}^2 \end{aligned}$$

where

$$S_{T,1}^2 = \frac{\sum_{U_1} \left(T_i - \overline{T}_1\right)^2}{N_1 - 1},$$

and $\overline{T}_1 = \frac{1}{N_1} \sum_{U_1} T_i$. The variance estimator,

$$\widehat{Var}\left(\widehat{T}_{HT}\right) = \frac{N_1}{n_1} \sum_{s_1} N_i \left(\frac{N_i}{n_i} - 1\right) \widehat{S}_i^2 + N_1 \left(\frac{N_1}{n_1} - 1\right) \widehat{S}_{T,1}^2$$

is obtained using (Eq. A.17) where

$$\widehat{S}_i^2 = \frac{\sum_{s_i} (y_{ik} - \widehat{\overline{y}}_i)^2}{n_i - 1},$$

$\widehat{\overline{y}}_i = \sum_{s_i} y_{ik}/n_i$,

$$\widehat{S}_{T,1}^2 = \frac{\sum_{s_1} \left(\widehat{T}_i - \widehat{\overline{T}}_1\right)^2}{n_1 - 1}$$

and

$$\widehat{\widehat{T}}_1 = \frac{1}{n_1}\sum_{s_1}\widehat{T}_i = \frac{1}{n_1}\sum_{s_1}\frac{N_i}{n_i}\sum_{s_i} y_{ik}.$$

The variance of the unbiased estimator of the mean per element is

$$\begin{aligned}
Var\left(\overline{\overline{y}}\right) &= Var\left(\frac{\widehat{T}_{HT}}{\sum_{U_1} N_i}\right) = \frac{Var\left(\widehat{T}_{HT}\right)}{N_1^2\bar{N}_2^2} \\
&= \frac{1}{N_1 n_1}\sum_{U_1}\left(\frac{N_i}{\bar{N}_2}\right)^2 (1-f_i)\frac{S_i^2}{n_i} + \frac{1-f_1}{n_1}\frac{\sum_{U_1}\left(\frac{T_i}{\bar{N}_2}-\overline{\overline{y}}\right)^2}{N_1-1} \\
&= \frac{1}{N_1 n_1}\sum_{U_1}\left(\frac{N_i}{\bar{N}_2}\right)^2\left(\frac{1-f_i}{n_i}\right)S_i^2 + \left(\frac{1-f_1}{n_1}\right)\frac{\sum_{U_1}\left(\frac{T_i}{\bar{N}_2}-\overline{\overline{y}}\right)^2}{N_1-1}.
\end{aligned}$$

If we have the special case that all PSU's are of identical size, then

$$\begin{aligned}
Var\left(\overline{\overline{y}}\right) &= Var\left(\frac{\widehat{T}_{HT}}{N_1 N_2}\right) = \frac{Var\left(\widehat{T}_{HT}\right)}{N_1^2 N_2^2} \\
&= \frac{(1-f_2)}{n_1 n_2}\frac{\sum_{U_1}\sum_{U_i}(y_{ik}-\bar{y}_i)^2}{N_1(N_2-1)} + \left(\frac{1-f_1}{n_1}\right)\frac{\sum_{U_1}\left(\bar{y}_i-\overline{\overline{y}}\right)^2}{N_1-1}
\end{aligned}$$

A.4 Three-stage Sampling

We first need notation to describe elements, samples and universes at each stage. Let N_1, N_i and N_{ik} be the number of elements in the first stage, second stage of the ith first stage and third stage of the kth second stage of the ith first stage, respectively, such that total number of elements in the universe is $\sum_{U_1}\sum_{U_i} N_{ik} \equiv N$ where U_1 is the universe of first-stage elements, U_i is the universe of elements in the ith first-stage element and U_{ik} is the universe of elements in the kth second stage element of the ith first-stage element. The Horvitz-Thompson estimator in a three-stage design is

$$\widehat{T}^* = \sum_{s_1}\sum_{s_i}\sum_{s_{ik}}\frac{y_{ikm}}{\pi_i\pi_{ik}\pi_{ikm}}$$

To derive the variance of this estimator it is useful to refer to the result (A.15) derived earlier

$$V(\hat{\theta}) = E\{E[V(\hat{\theta}|\phi,\psi)]\} + E\{V[E(\hat{\theta}|\phi,\psi)]\} + V\{E[E(\hat{\theta}|\phi,\psi)]\}.$$

So, the variance of the HTT can be written as

$$\begin{aligned} V\left(\widehat{T}^*\right) = & E_1\left\{E_2\left[V_3\left(\widehat{T}^*|s_1,s_i\right)\right]\right\} + E_1\left\{V_2\left[E_3\left(\widehat{T}^*|s_1,s_i\right)\right]\right\} \\ & + V_1\left\{E_2\left[E_3\left(\widehat{T}^*|s_1,s_i\right)\right]\right\}. \end{aligned}$$

For the first term,

$$\begin{aligned} E_1\{E_2[V_3(\widehat{T}^*|s_1,s_i)]\} &= E_1\left\{E_2\left[V_3\left(\sum_{s_1}\sum_{s_i}\sum_{s_{ik}}\frac{y_{ikm}}{\pi_i\pi_{ik}\pi_{ikm}}\right)\right]\right\} \\ &= E_1\left\{\sum_{s_1}\frac{1}{\pi_i^2}E_2\left[\sum_{s_i}\frac{1}{\pi_{ik}^2}V_3\left(\sum_{s_{ik}}\frac{y_{ikm}}{\pi_{ikm}}\right)\right]\right\} \\ &= \sum_{U_1}\frac{1}{\pi_i}\sum_{U_i}\frac{V_{ik}}{\pi_{ik}} \end{aligned}$$

where

$$V_{ik} = \sum\sum_{U_{ik}} \Delta_{ikmn}\frac{y_{ikm}y_{ikn}}{\pi_{ikm}\pi_{ikn}}.$$

For the second term,

$$\begin{aligned} E_1\{V_2[E_3(\widehat{T}_{HT}|s_1,s_i)]\} &= E_1\left\{V_2\left[E_3\left(\sum_{s_1}\sum_{s_i}\sum_{s_{ik}}\frac{y_{ikm}}{\pi_i\pi_{ik}\pi_{ikm}}\right)\right]\right\} \\ &= E_1\left\{\sum_{s_1}\frac{1}{\pi_i^2}V_2\left[\sum_{s_i}\frac{1}{\pi_{ik}}E_3\left(\sum_{s_{ik}}\frac{y_{ikm}}{\pi_{ikm}}\right)\right]\right\} \\ &= E_1\left[\sum_{s_1}\frac{1}{\pi_i^2}V_2\left(\frac{y_{ik}}{\pi_{ik}}\right)\right] \\ &= \sum_{U_1}\frac{V_i}{\pi_i} \end{aligned}$$

where V_i is defined previously as

$$V_i = \sum\sum_{U_i}\Delta_{ikl}\frac{y_{ik}y_{il}}{\pi_{ik}\pi_{il}}.$$

Finally, for the third term,

$$
\begin{aligned}
V_1\{E_2[E_3(\widehat{T}_{HT}|s_1,s_i)]\} &= V_1\left\{E_2\left[E_3\left(\sum_{s_1}\sum_{s_i}\sum_{s_{ik}}\frac{y_{ikm}}{\pi_i\pi_{ik}\pi_{ikm}}\right)\right]\right\} \\
&= V_1\left\{\sum_{s_1}\frac{1}{\pi_i}E_2\left[\sum_{s_i}\frac{1}{\pi_{ik}}E_3\left(\sum_{s_{ik}}\frac{y_{ikm}}{\pi_{ikm}}\right)\right]\right\} \\
&= V_1\left[\sum_{s_1}\frac{1}{\pi_i^2}E_2\left(\frac{y_{ik}}{\pi_{ik}}\right)\right] \\
&= V_1\left(\sum_{U_1}\frac{y_i}{\pi_i}\right) \\
&= \sum\sum_{U_1}\Delta_{ij}\frac{y_iy_j}{\pi_i\pi_j}.
\end{aligned}
$$

So, all together,

$$
V\left(\widehat{T}^*\right) = \sum\sum_{U_1}\Delta_{ij}\frac{y_iy_j}{\pi_i\pi_j} + \sum_{U_1}\frac{V_i}{\pi_i} + \sum_{U_1}\frac{1}{\pi_i}\sum_{U_i}\frac{V_{ik}}{\pi_{ik}}.
$$

The estimator of variance is

$$
\widehat{V}\left(\widehat{T}^*\right) = \widehat{V}_{PSU} + \widehat{V}_{SSU} + \widehat{V}_{TSU}
$$

where V_{TSU} is the variance component due to the tertiary sampling units. The estimators for each component are

$$
\begin{aligned}
\widehat{V}_{PSU} &= \sum\sum_{s_1}\frac{\Delta_{ij}}{\pi_{ij}}\frac{\hat{y}_i\hat{y}_j}{\pi_i\pi_j} - \sum_{s_1}\left[\frac{1}{\pi_i}\left(\frac{1}{\pi_i}-1\right)\left(\widehat{V}_i + \sum_{s_i}\frac{\widehat{V}_{ik}}{\pi_{ik}}\right)\right] \\
\widehat{V}_{SSU} &= \sum_{s_1}\frac{1}{\pi_i^2}\left[\widehat{V}_i - \sum_{s_i}\frac{1}{\pi_{ik}}\left(\frac{1}{\pi_{ik}}-1\right)\widehat{V}_{ik}\right] \\
\widehat{V}_{TSU} &= \sum_{s_1}\frac{1}{\pi_i^2}\sum_{s_i}\frac{\widehat{V}_{ik}}{\pi_{ik}^2}
\end{aligned}
$$

where

$$
\widehat{V}_i = \sum\sum_{s_i}\frac{\Delta_{ikl}}{\pi_{ikl}}\frac{\hat{y}_{ik}\hat{y}_{il}}{\pi_{ik}\pi_{il}}
$$

and

$$
\widehat{V}_{ik} = \sum\sum_{s_{ik}}\frac{\Delta_{ikmn}}{\pi_{ikmn}}\frac{y_{ikm}y_{ikn}}{\pi_{ikm}\pi_{ikn}}
$$

I prove the unbiasedness of the variance estimator by treating each of the three components separately. For the first term,

$$
\begin{aligned}
E\left(\widehat{V}_{PSU}\right) =& E\left\{\sum\sum\nolimits_{s_1}\frac{\Delta_{ij}}{\pi_{ij}}\frac{\hat{y}_i\hat{y}_j}{\pi_i\pi_j}-\sum\nolimits_{s_1}\left[\frac{1}{\pi_i}\left(\frac{1}{\pi_i}-1\right)\left(\widehat{V}_i+\sum\nolimits_{s_i}\frac{\widehat{V}_{ik}}{\pi_{ik}}\right)\right]\right\} \\
=& E_1\left[\sum\nolimits_{s_1}\frac{\Delta_{ii}}{\pi_i}\frac{E_2E_3\left(\hat{y}_i^2\right)}{\pi_i^2}+\sum\sum\nolimits_{\substack{s_1\\ i\neq j}}\frac{\Delta_{ij}}{\pi_{ij}}\frac{y_iy_j}{\pi_i\pi_j}\right] \\
& -E_1\left\{\sum\nolimits_{s_1}\left[\frac{1}{\pi_i}\left(\frac{1}{\pi_i}-1\right)\left(E_2E_3\left(\widehat{V}_i\right)+E_2\left(\sum\nolimits_{s_i}\frac{V_{ik}}{\pi_{ik}}\right)\right)\right]\right\} \\
=& E_1\left[\sum\nolimits_{s_1}\frac{\Delta_{ii}}{\pi_i^3}E_2E_3\left(\sum\nolimits_{s_i}\frac{\hat{y}_{ik}}{\pi_{k|i}}\right)^2\right]+\sum\sum\nolimits_{\substack{U_1\\ i\neq j}}\Delta_{ij}\frac{y_iy_j}{\pi_i\pi_j} \\
& -E_1\left\{\sum\nolimits_{s_1}\left[\frac{1}{\pi_i}\left(\frac{1}{\pi_i}-1\right)\left(E_2E_3\left(\sum\sum\nolimits_{s_i}\frac{\Delta_{ikl}}{\pi_{ikl}}\frac{\hat{y}_{ik}\hat{y}_{il}}{\pi_{ik}\pi_{il}}\right)\right.\right.\right. \\
& \left.\left.\left.+\sum\nolimits_{U_i}V_{ik}\right)\right]\right\} \\
=& E_1\left[\sum\nolimits_{s_1}\frac{\Delta_{ii}}{\pi_i^3}E_2\left(\sum\nolimits_{s_i}\frac{V_{ik}+y_{ik}^2}{\pi_{ik}^2}+\sum\sum\nolimits_{\substack{s_i\\ k\neq l}}\frac{y_{ik}y_{il}}{\pi_{ik}\pi_{il}}\right)\right] \\
& +\sum\sum\nolimits_{\substack{U_1\\ i\neq j}}\Delta_{ij}\frac{y_iy_j}{\pi_i\pi_j}-E_1\left\{\sum\nolimits_{s_1}\left[\frac{1}{\pi_i}\left(\frac{1}{\pi_i}-1\right)E_2\left(\sum\sum\nolimits_{\substack{s_i\\ k\neq l}}\frac{\Delta_{ikl}}{\pi_{ikl}}\frac{y_{ik}y_{il}}{\pi_{ik}\pi_{il}}\right.\right.\right. \\
& \left.\left.\left.+\sum\nolimits_{s_i}\frac{\Delta_{ikk}}{\pi_{ik}^3}\left(V_{ik}+y_{ik}^2\right)\right)\right]\right\}-\sum\nolimits_{U_1}\left(\frac{1}{\pi_i}-1\right)\sum\nolimits_{U_i}V_{ik} \\
=& \sum\nolimits_{U_1}\frac{\Delta_{ii}}{\pi_i^2}\left(\sum\nolimits_{U_i}\frac{V_{ik}}{\pi_{ik}}+\sum\sum\nolimits_{U_i}\pi_{ikl}\frac{y_{ik}y_{il}}{\pi_{ik}\pi_{il}}\right)+\sum\sum\nolimits_{\substack{U_1\\ i\neq j}}\Delta_{ij}\frac{y_iy_j}{\pi_i\pi_j} \\
& -\sum\nolimits_{U_1}\left[\left(\frac{1}{\pi_i}-1\right)\left(V_i+\sum\nolimits_{U_i}\left(\frac{1}{\pi_{ik}}-1\right)V_{ik}\right)\right] \\
& -\sum\nolimits_{U_1}\left(\frac{1}{\pi_i}-1\right)\sum\nolimits_{U_i}V_{ik} \\
=& \sum\nolimits_{U_1}\left[\left(\frac{1}{\pi_i}-1\right)\left(V_i+\sum\nolimits_{U_i}\frac{V_{ik}}{\pi_{ik}}\right)\right]+\sum\sum\nolimits_{U_1}\Delta_{ij}\frac{y_iy_j}{\pi_i\pi_j} \\
& -\sum\nolimits_{U_1}\left[\left(\frac{1}{\pi_i}-1\right)\left(V_i+\sum\nolimits_{U_i}\frac{V_{ik}}{\pi_{ik}}-\sum\nolimits_{U_i}V_{ik}\right)\right] \\
& -\sum\nolimits_{U_1}\left(\frac{1}{\pi_i}-1\right)\sum\nolimits_{U_i}V_{ik} \\
=& \sum\sum\nolimits_{U_1}\Delta_{ij}\frac{y_iy_j}{\pi_i\pi_j}.
\end{aligned}
$$

For the second term,

$$
\begin{aligned}
E\left(\widehat{V}_{SSU}\right) =& E\left\{\sum\nolimits_{s_1}\frac{1}{\pi_i^2}\left[\widehat{V}_i-\sum\nolimits_{s_i}\frac{1}{\pi_{ik}}\left(\frac{1}{\pi_{ik}}-1\right)\widehat{V}_{ik}\right]\right\}\\
=& E_1\left\{\sum\nolimits_{s_1}\frac{1}{\pi_i^2}\left[E_2E_3\left(\widehat{V}_i\right)-\sum\nolimits_{U_i}\left(\frac{1}{\pi_{ik}}-1\right)V_{ik}\right]\right\}\\
=& E_1\left\{\sum\nolimits_{s_1}\frac{1}{\pi_i^2}\left[E_2E_3\left(\sum\sum\nolimits_{s_i}\frac{\Delta_{ikl}}{\pi_{ikl}}\frac{\hat{y}_{ik}\hat{y}_{il}}{\pi_{ik}\pi_{il}}\right)-\sum\nolimits_{U_i}\left(\frac{1}{\pi_{ik}}-1\right)V_{ik}\right]\right\}\\
=& E_1\left\{\sum\nolimits_{s_1}\frac{1}{\pi_i^2}\left[E_2\left(\sum\sum\nolimits_{\substack{s_i\\ k\neq l}}\frac{\Delta_{ikl}}{\pi_{ikl}}\frac{y_{ik}y_{il}}{\pi_{ik}\pi_{il}}+\sum\nolimits_{s_i}\frac{\Delta_{ikk}}{\pi_{ik}^3}\left(V_{ik}+y_{ik}^2\right)\right)\right]\right\}\\
&-\sum\nolimits_{U_1}\frac{1}{\pi_i}\sum\nolimits_{U_i}\left(\frac{1}{\pi_{ik}}-1\right)V_{ik}\\
=&\sum\nolimits_{U_1}\frac{1}{\pi_i}\left[\left(V_i+\sum\nolimits_{U_i}\left(\frac{1}{\pi_{ik}}-1\right)V_{ik}\right)\right]-\sum\nolimits_{U_1}\frac{1}{\pi_i}\sum\nolimits_{U_i}\left(\frac{1}{\pi_{ik}}-1\right)V_{ik}\\
=&\sum\nolimits_{U_1}\frac{V_i}{\pi_i}.
\end{aligned}
$$

Finally, for the third part,

$$
\begin{aligned}
E\left(\widehat{V}_{TSU}\right) =& E_1\left\{\sum\nolimits_{s_1}\frac{1}{\pi_i^2}E_2\left[\sum\nolimits_{s_i}\frac{E_3\left(\widehat{V}_{ik}\right)}{\pi_{ik}^2}\right]\right\}\\
=& E_1\left[\sum\nolimits_{s_1}\frac{1}{\pi_i^2}E_2\left(\sum\nolimits_{s_i}\frac{V_{ik}}{\pi_{ik}^2}\right)\right]\\
=&\sum\nolimits_{U_1}\frac{1}{\pi_i}\sum\nolimits_{U_i}\frac{V_{ik}}{\pi_{ik}}.
\end{aligned}
$$

A.5 Covariance of Estimators

We often estimate more than one parameter total of a population in a particular sampling design, but to properly make inferences on these parameters we need to account for the covariance of the estimators. Suppose under the same design we have the estimates $\widehat{y}$ and $\widehat{z}$. In a single-stage sample design, the covariance of the estimators is

$$\begin{aligned}
Cov\left(\widehat{y},\widehat{z}\right) &= E\left(\widehat{y}\widehat{z}\right)-E\left(\widehat{y}\right)E\left(\widehat{z}\right)\\
&= E\left(\sum\nolimits_{s}\frac{y_i}{\pi_i}\sum\nolimits_{s}\frac{z_i}{\pi_i}\right)-\sum\nolimits_{U}y_i\sum\nolimits_{U}z_i\\
&= E\left(\sum\sum\nolimits_{s}\frac{y_iz_j}{\pi_i\pi_j}\right)-\sum\sum\nolimits_{U}y_iz_j\\
&= \sum\sum\nolimits_{U}\pi_{ij}\frac{y_iz_j}{\pi_i\pi_j}-\sum\sum\nolimits_{U}\pi_i\pi_j\frac{y_iz_j}{\pi_i\pi_j}\\
&= \sum\sum\nolimits_{U}\Delta_{ij}\frac{y_iz_j}{\pi_i\pi_j}.
\end{aligned}$$

We can see that the covariance of the two estimators takes a form analogous to that of the variance of a single estimator of the total (see A.1). The estimator of covariance for a single-stage design is

$$\widehat{Cov}\left(\widehat{y},\widehat{z}\right)=\sum\sum\nolimits_{U}\frac{\Delta_{ij}}{\pi_{ij}}\frac{y_iz_j}{\pi_i\pi_j}.$$

In a two-stage design, we have

$$\begin{aligned}
Cov\left(\widehat{y},\widehat{z}\right) &= E\left(\widehat{y}\widehat{z}\right)-\sum\nolimits_{U_1}y_i\sum\nolimits_{U_1}z_i\\
&= E\left(\sum\nolimits_{s_1}\frac{\widehat{y}_i}{\pi_i}\sum\nolimits_{s_1}\frac{\widehat{z}_i}{\pi_i}\right)-\sum\nolimits_{U_1}y_i\sum\nolimits_{U_1}z_i\\
&= E\left[\sum\sum\nolimits_{s_1}\frac{E(\widehat{y}_i\widehat{z}_j|s_1)}{\pi_i\pi_j}\right]-\sum\sum\nolimits_{U_1}y_iz_j\\
&= E\left[\sum\nolimits_{s_1}\frac{Cov\left(\widehat{y}_i,\widehat{z}_i\right)+y_iz_i}{\pi_i^2}+\sum\sum_{i\neq j}\nolimits_{s_1}\frac{y_iz_j}{\pi_i\pi_j}\right]-\sum\sum\nolimits_{U_1}\pi_i\pi_j\frac{y_iz_j}{\pi_i\pi_j}\\
&= \sum\sum\nolimits_{U_1}\Delta_{ij}\frac{y_iz_j}{\pi_i\pi_j}+\sum\nolimits_{U_1}\frac{Cov\left(\widehat{y}_i,\widehat{z}_i\right)}{\pi_i}
\end{aligned}$$

where

$$Cov\left(\widehat{y}_i,\widehat{z}_i\right)=\sum\sum\nolimits_{U_i}\Delta_{ikl}\frac{y_{ik}z_{il}}{\pi_{ik}\pi_{il}}.$$

When there is two-stage estimation for one character, y, and only one-stage estimation for the other, z, then the covariance of the estimators is

$$
\begin{aligned}
Cov\,(\widehat{y},\widehat{z}) &= E(\widehat{y}\widehat{z}) - \sum\nolimits_{U_1} y_i \sum\nolimits_{U_1} z_i \\
&= E\left(\sum\nolimits_{s_1} \frac{\widehat{y}_i}{\pi_i} \sum\nolimits_{s_1} \frac{z_i}{\pi_i}\right) - \sum\nolimits_{U_1} y_i \sum\nolimits_{U_1} z_i \\
&= E\left[\sum\sum\nolimits_{s_1} \frac{E(\widehat{y}_i|s_1) z_j}{\pi_i \pi_j}\right] - \sum\sum\nolimits_{U_1} y_i z_j \\
&= \sum\sum\nolimits_{U_1} \pi_{ij} \frac{y_i z_j}{\pi_i \pi_j} - \sum\sum\nolimits_{U_1} \pi_i \pi_j \frac{y_i z_j}{\pi_i \pi_j} \\
&= \sum\sum\nolimits_{U_1} \Delta_{ij} \frac{y_i z_j}{\pi_i \pi_j}
\end{aligned}
$$

which is just the first-stage covariance.

Now, suppose we take a SRS for estimation of one character,y, and use the rest of the universe to estimate another character,z. The Horvitz-Thompson estimators are

$$\widehat{y} = \frac{N}{n} \sum\nolimits_s y_i$$

and

$$\widehat{z} = \frac{N}{N-n} \sum\nolimits_{U-s} z_j$$

The covariance of the estimators is

$$
\begin{aligned}
Cov\,(\widehat{y},\widehat{z}) =& Cov\left[\frac{N}{n} \sum\nolimits_s y_i, \frac{N}{N-n}\left(z - \sum\nolimits_s z_i\right)\right] \\
=& -\frac{N}{n}\frac{N}{N-n}\left(\frac{n}{N}\right)^2 Cov\left(\frac{N}{n} \sum\nolimits_s y_i, \frac{N}{n} \sum\nolimits_s z_i\right) \\
=& -N\frac{\sum_U (y_i - \overline{y})(z_i - \overline{z})}{N-1}.
\end{aligned}
$$

A.6 Estimation of Domain Totals

We often perform sampling of a universe to estimate one or more parameters of this universe, but sometimes we would also like to estimate parameters for parts of the universe for which sampling effort was not controled in the sampling design. The parts of the universe of secondary interest have been termed domains or subpopulations in sampling theory texts (Cochran 1977; Yates 1981; Särndal et al. 1992). Let the universe, U, of elements be partitioned into $U_1, \ldots, U_d, \ldots, U_D$ domains. Also, let the elements be divided into clusters $U_1, \ldots, U_i, \ldots, U_C$ so that $U_d \cap U_i = U_{id}$ is the set of elements in the ith cluster that are also in the dth domain. With this, now define

$$y_{id} = \sum\nolimits_{U_{id}} y_{ik}$$

as the total of the ith cluster that is also in the dth domain. If there is random sampling within the ith cluster, then the Horvitz-Thompson estimator is

$$\widehat{y}_{id} = \sum\nolimits_{s_{id}} \frac{y_{ik}}{\pi_{ik}}$$

where s_{id} is the sampled elements that also occur in the dth domain. Notice that the first-order inclusion probabilities are not a function of the domain. In fact, this is also the case for the second-order inclusion probabilities. The variance of the estimator is

$$V\left(\widehat{y}_{id}\right) = \sum\sum\nolimits_{U_{id}} \Delta_{ikl} \frac{y_{ik} y_{il}}{\pi_{ik} \pi_{il}}.$$

The unbiased variance estimator is

$$\widehat{V}\left(\widehat{y}_{id}\right) = \sum\sum\nolimits_{s_{id}} \frac{\Delta_{ikl}}{\pi_{ikl}} \frac{y_{ik} y_{il}}{\pi_{ik} \pi_{il}}.$$

An interesting aspect of these estimators is the fact that no elements in the domain of interest need be sampled for unbiased estimation. If we have a SRS of elements within the ith cluster, then the estimator of the total in that cluster is

$$\widehat{y}_{id} = \frac{N_i}{n_i} \sum\nolimits_{s_{id}} y_{ik}$$

and the variance is

$$
\begin{aligned}
V\left(\widehat{y}_{id}\right) =& \frac{1-f_i}{f_i}\sum_{U_{id}} y_{ik}^2 - \frac{1-f_i}{f_i(N_i-1)}\sum_{k\neq l}\sum_{U_{id}} y_{ik}y_{il} \\
=& \frac{N_i^2(1-f_i)}{n_i(N_i-1)}\left(\sum_{U_{id}} y_{ik}^2 - \frac{\sum\sum_{U_{id}} y_{ik}y_{il}}{N_i}\right) \\
=& \frac{N_i^2(1-f_i)}{n_i(N_i-1)}\left(\sum_{U_{id}} y_{ik}^2 - N_{id}\overline{y}_{id}^2 + N_{id}\overline{y}_{id}^2 - \frac{N_{id}^2}{N_i}\overline{y}_{id}^2\right) \\
=& N_i\left(\frac{N_i}{n_i}-1\right)\frac{(N_{id}-1)S_{id}^2 + N_{id}(1-P_{id})\overline{y}_{id}^2}{N_i-1}
\end{aligned}
$$

where $f_i = n_i/N_i$, $P_{id} = N_{id}/N_i$, $\overline{y}_{id} = \sum_{U_{id}} y_{ik}/N_{id}$ and

$$
S_{id}^2 = \frac{\sum_{U_{id}}(y_{ik}-\overline{y}_{id})^2}{N_{id}-1}.
$$

Another form for the variance is

$$
V\left(\widehat{y}_{id}\right) = N_i\left(\frac{N_i}{n_i}-1\right)\left[\frac{\sum_{U_{id}} y_{ik}^2}{N_i-1} - \frac{\left(\sum_{U_{id}} y_{ik}\right)^2}{N_i(N_i-1)}\right]. \tag{A.19}
$$

The unbiased variance estimator can be shown in a similar manner to be

$$
\widehat{V}\left(\widehat{y}_{id}\right) = N_i\left(\frac{N_i}{n_i}-1\right)\frac{(n_{id}-1)\widetilde{S}_{id}^2 + n_{id}(1-p_{id})\widetilde{\overline{y}}_{id}^2}{n_i-1}
$$

where $p_{id} = n_{id}/n_i$, $\widetilde{\overline{y}}_{id} = \sum_{s_{id}} y_{ik}/n_{id}$ and

$$
\widetilde{S}_{id}^2 = \frac{\sum_{s_{id}}(y_{ik}-\widetilde{\overline{y}}_{id})^2}{n_{id}-1}.
$$

Another form for the variance estimator that parallels Eq. A.19 is

$$
\widehat{V}\left(\widehat{y}_{id}\right) = N_i\left(\frac{N_i}{n_i}-1\right)\left[\frac{\sum_{s_{id}} y_{ik}^2}{n_i-1} - \frac{\left(\sum_{s_{id}} y_{ik}\right)^2}{n_i(n_i-1)}\right]. \tag{A.20}
$$

The covariance of estimators for two different domains, d and e, is

$$\begin{aligned}Cov\left(\widehat{y}_{id}, \widehat{y}_{ie}\right) =& E\left(\widehat{y}_{id}\widehat{y}_{ie}\right) - y_{id}y_{ie}\\ =& \left(\frac{N_i}{n_i}\right)^2 E\left[\left(\sum\nolimits_{s_{id}} y_{ik}\right)\left(\sum\nolimits_{s_{ie}} y_{il}\right)\right] - y_{id}y_{ie}\\ =& \left(\frac{N_i}{n_i}\right)^2 \frac{n_i(n_i-1)}{N_i(N_i-1)}\left(\sum\nolimits_{U_{id}} y_{ik}\sum\nolimits_{U_{ie}} y_{il}\right) - y_{id}y_{ie}\\ =& -\left(\frac{N_i}{n_i}-1\right)\frac{y_{id}y_{ie}}{N_i-1} = -N_i\left(\frac{N_i}{n_i}-1\right)\frac{N_i\overline{y}_{id}\overline{y}_{ie}}{N_i-1}\end{aligned}$$

and an unbiased estimator is

$$\widehat{Cov}\left(\widehat{y}_{id}, \widehat{y}_{ie}\right) = -\left(1-\frac{n_i}{N_i}\right)\frac{\widehat{y}_{id}, \widehat{y}_{ie}}{n_i-1} = -N_i\left(\frac{N_i}{n_i}-1\right)\frac{n_i\widehat{\overline{y}}_{id}\widehat{\overline{y}}_{ie}}{n_i-1}. \tag{A.21}$$

The covariance estimator is unbiased because

$$\begin{aligned}E\left[\widehat{Cov}\left(\widehat{y}_{id}, \widehat{y}_{ie}\right)\right] =& -\left(1-\frac{n_i}{N_i}\right)\left(\frac{N_i}{n_i}\right)^2\frac{E\left[\left(\sum_{s_{id}} y_{ik}\right)\left(\sum_{s_{ie}} y_{il}\right)\right]}{n_i-1}\\ =& -\left(\frac{N_i}{n_i}-1\right)\frac{\left(\sum_{U_{id}} y_{ik}\right)\left(\sum_{U_{ie}} y_{il}\right)}{N_i-1}\end{aligned}$$

Now, instead assume we subsample within each of the elements, y_{ik}, in the ith cluster. We have already implied that each element, y_{ik}, is entirely inside or outside the domain, d by estimating the total of these elements in that domain above. Therefore, we have no complications of sampling inside and outside the domain at this stage which gives the conditional estimator,

$$\widehat{y}_{ik} = \sum\nolimits_{s_{ik}} \frac{y_{ikm}}{\pi_{ikm}}$$

and the estimator for the domain total is

$$\widehat{y}_{id} = \sum\nolimits_{s_{id}} \frac{1}{\pi_{ik}} \sum\nolimits_{s_{ik}} \frac{y_{ikm}}{\pi_{ikm}}.$$

The variance and variance estimators are

$$V\left(\widehat{y}_{id}\right) = \sum\sum\nolimits_{U_{id}} \Delta_{ikl}\frac{y_{ik}y_{il}}{\pi_{ik}\pi_{il}} + \sum\nolimits_{U_{id}} \frac{V_{ik}}{\pi_{ik}},$$

and

$$\widehat{V}(\widehat{y}_{id}) = \sum\sum\nolimits_{s_{id}} \frac{\Delta_{ikl}}{\pi_{ikl}} \frac{\widehat{y}_{ik}\widehat{y}_{il}}{\pi_{ik}\pi_{il}} + \sum\nolimits_{s_{id}} \frac{\widehat{V}_{ik}}{\pi_{ik}}$$

which are identical to those derived for two-stage designs, Eq. A.16 and Eq. A.17, except that our interest here is in the dth domain.

If SRS is performed within the clusters, then the variance and variance estimator have the forms

$$V(\widehat{y}_{id}) = N_i\left(\frac{N_i}{n_i} - 1\right)\frac{(N_{id}-1)S_{id}^2 + N_{id}(1-P_{id})\overline{y}_{id}^2}{N_i - 1} + \frac{N_i}{n_i}\sum\nolimits_{U_{id}} V(\widehat{y}_{ik}) \qquad \text{(A.22)}$$

and

$$\widehat{V}(\widehat{y}_{id}) = N_i\left(\frac{N_i}{n_i} - 1\right)\frac{(n_{id}-1)\widetilde{S}_{id}^2 + n_{id}(1-p_{id})\widetilde{\overline{y}}_{id}^2}{n_i - 1} + \frac{N_i}{n_i}\sum\nolimits_{s_{id}} \widehat{V}(\widehat{y}_{ik}) \qquad \text{(A.23)}$$

where $\widetilde{\overline{y}}_{id} = \sum_{s_{id}} \widehat{y}_{ik}/n_{id}$ and

$$\widetilde{S}_{id}^2 = \frac{\sum_{s_{id}}\left(\widehat{y}_{ik} - \widetilde{\overline{y}}_{id}\right)^2}{n_{id} - 1}.$$

Now that we have the appropriate domain estimators for the ith cluster, we can derive the estimators for the total in the dth domain over all clusters when there is sampling of clusters. Let,

$$y_d = \sum\nolimits_{U_C} y_{id}$$

be the total in domain d over all clusters. The Horvitz-Thompson estimator is

$$\widehat{y}_d = \sum\nolimits_{s_C} \frac{\widehat{y}_{id}}{\pi_i}.$$

This is a three-stage estimator and the appropriate variance and variance estimator are derived in A.4. If there is simple random sampling at each stage the variance is

$$V(\widehat{y}_d) = N_C\left(\frac{N_C}{n_C} - 1\right)S_{Cd}^2 + \frac{N_C}{n_C}\sum\nolimits_{U_C} V_{id} + \frac{N_C}{n_C}\sum\nolimits_{U_C}\frac{N_i}{n_i}\sum\nolimits_{U_{id}} V_{ik}$$

where N_C and n_C are the total number of clusters and number of clusters sampled.

Also,

$$S_{Cd}^2 = \frac{\sum_{U_C} \left(y_{id} - \overline{y}_{Cd}\right)^2}{N_C - 1},$$

V_{id} is given by Eq. A.22 and

$$V_{ik} = N_{ik} \left(\frac{N_{ik}}{n_{ik}} - 1 \right) S_{ik}^2.$$

The variance estimator is

$$\widehat{V}\left(\widehat{\overline{y}}_d\right) = N_C \left(\frac{N_C}{n_C} - 1 \right) \tilde{S}_{Cd}^2 + \frac{N_C}{n_C} \sum\nolimits_{s_C} \widehat{V}_{id} + \frac{N_C}{n_C} \sum\nolimits_{s_C} \frac{N_i}{n_i} \sum\nolimits_{s_{id}} \widehat{V}_{ik}$$

where $\widehat{V}_{id}$ is that presented in Eq. A.23 and

$$\tilde{S}_{Cd}^2 = \frac{\sum_{s_C} \left(\widehat{y}_{id} - \widehat{\overline{y}}_{Cd} \right)^2}{n_C - 1}.$$

A.6.1 Estimation of Total Number and Proportion of Elements in a Domain

Consider a simple, single-stage sample design. The total number of elements in a domain can be written as

$$N_d = \sum\nolimits_{U_d} 1.$$

This estimator is a generalization of the hypergeometric random variable in that the sample need not be collected by simple random sampling. The Horvitz-Thompson estimator for the number in a domain is simply

$$\widehat{N}_d = \sum\nolimits_{s_d} \frac{1}{\pi_i} \tag{A.24}$$

and the variance and variance estimator are

$$V\left(\widehat{N}_d\right) = \sum\sum\nolimits_{U_d} \Delta_{ij} \frac{1}{\pi_i \pi_j} = \sum\sum\nolimits_{U_d} \frac{\pi_{ij}}{\pi_i \pi_j} - N_d^2 \tag{A.25}$$

and

$$\widehat{V}\left(\widehat{N}_d\right) = \sum\sum\nolimits_{s_d} \frac{\Delta_{ij}}{\pi_{ij}} \frac{1}{\pi_i \pi_j} = \sum\sum\nolimits_{s_d} \frac{1}{\pi_i \pi_j} - \frac{1}{\pi_{ij}}, \tag{A.26}$$

respectively. The proportion of elements in a domain is

$$P_d = \frac{N_d}{N} = \frac{1}{N}\sum\nolimits_{U_d} 1$$

and the unbiased estimator as a function of the HTT is

$$\widehat{P}_d = \frac{1}{N}\sum\nolimits_{s_d} \frac{1}{\pi_i}. \tag{A.27}$$

The variance and design-unbiased estimator of variance are also merely functions of Eq. A.25 and Eq. A.26, respectively. If the total number of elements is estimated rather than known, then an asymptotically unbiased estimator of the proportion in the dth domain is

$$\widehat{P}_d^* = \frac{\widehat{N}_d}{\widehat{N}}$$

where $\widehat{N}_d$ is given in Eq. A.24 and $\widehat{N}$ is consistent, but likely a function of a different sample design than that of the domain total. Using a first-order Taylor Series approximation, the asymptotic variance is

$$AV\left(\widehat{P}_d^*\right) = \frac{1}{N^2}\left[V\left(\widehat{N}_d\right) + P_d^2 V\left(\widehat{N}\right) - 2P_d Cov\left(\widehat{N}_d, \widehat{N}\right)\right].$$

A.6.2 Known Total Number of Elements in the Domain

If we know the total number of elements in the ith cluster that are in the dth domain an assymptotically unbiased and more precise estimator of the total is

$$\widehat{y}_{id}^* = N_{id}\frac{\widehat{y}_{id}}{\widehat{N}_{id}} = N_{id}\frac{\sum_{s_{id}} \frac{y_{ik}}{\pi_{ik}}}{\sum_{s_{id}} \frac{1}{\pi_{ik}}}$$

This is a "ratio" estimator and its asymptotic variance can be found using the delta-method,

$$AV\left(\widehat{y}_{id}^*\right) = V\left(\widehat{y}_{id}\right) + \left(\frac{y_{id}}{N_{id}}\right)^2 V\left(\widehat{N}_{id}\right) - 2\frac{y_{id}}{N_{id}} Cov\left(\widehat{y}_{id}, \widehat{N}_{id}\right)$$

Now, the variance of $\widehat{N}_{id}$ with simple random sampling is obtained by realizing the hypergeometric nature of the sample size in the domain and, thus

$$V\left(\widehat{N}_{id}\right) = N_i\left(\frac{N_i}{n_i} - 1\right)\frac{N_i P_{id}(1 - P_{id})}{N_i - 1}.$$

We know that

$$Cov(\widehat{y}_{id}, \widehat{N}_{id}) = \sum\sum\nolimits_{U_{id}} \Delta_{ikl}\frac{y_{ik}}{\pi_i \pi_j}$$

and when we have simple random sampling

$$\begin{aligned} Cov(\hat{y}_{id}, \hat{N}_{id}) &= \frac{N_i^2(1 - f_i)}{n_i(N_i - 1)}\left(\sum\nolimits_{U_{id}} y_{ik} - \frac{\sum\sum_{U_{id}} y_{ik}}{N_i}\right) \\ &= N_i\left(\frac{N_i}{n_i} - 1\right)\frac{N_i \overline{y}_{id}(1 - P_{id})}{N_i - 1} \end{aligned}$$

So, with these components the asymptotic variance is

$$\begin{aligned} AV\left(\widehat{y}_{id}^{*}\right) &= \frac{N_i^2(1 - f_i)}{n_i(N_i - 1)}\left[(N_{id} - 1)S_{id}^2 + N_{id}(1 - P_{id})\overline{y}_{id}^2 + \left(\frac{y_{id}}{N_{id}}\right)^2 N_{id}(1 - P_{id})\right. \\ &\quad \left. -2\left(\frac{y_{id}}{N_{id}}\right) y_{id}(1 - P_{id})\right] \\ &= N_i\left(\frac{N_i}{n_i} - 1\right)\frac{(N_{id} - 1)S_{id}^2}{N_i - 1} \end{aligned}$$

If there is simple random sampling within U_i for y, then in A.5 we also have found that the covariance between $\widehat{N}_{id}$ and $\widehat{y}_{id}$ is still just a function of first-stage elements and the variance is

$$AV\left(\widehat{y}_{id}^{*}\right) = N_i\left(\frac{N_i}{n_i} - 1\right)\frac{(N_{id} - 1)S_{id}^2}{N_i - 1} + \frac{N_i}{n_i}\sum\nolimits_{U_{id}} N_{ik}\left(\frac{N_{ik}}{n_{ik}} - 1\right) S_{ik}^2.$$

Finally, using the ratio estimator to estimate the total when there is simple random sampling of clusters we have

$$\begin{aligned} AV\left(\widehat{y}_{d}^{*}\right) &= N_C\left(\frac{N_C}{n_C} - 1\right) S_{Cd}^2 + \frac{N_C}{n_C}\sum\nolimits_{U_C} N_i\left(\frac{N_i}{n_i} - 1\right)\frac{(N_{id} - 1)S_{id}^2}{N_i - 1} \\ &\quad + \frac{N_C}{n_C}\sum\nolimits_{U_C}\frac{N_i}{n_i}\sum\nolimits_{U_{id}} N_{ik}\left(\frac{N_{ik}}{n_{ik}} - 1\right) S_{ik}^2. \end{aligned}$$

A.7 Two-phase Sampling

Two-phase sampling is a generalization of two-stage sampling in that inclusion probabilities at the second stage may now vary across first stage samples. The following is an adapted derivation of a design-unbiased estimator of the total and estimator of variance given by Särndal et al. (1992). Let π_{ak} be the inclusion probability for the kth element in the first phase and $p_a(s_a)$ be the probability distribution of first-phase samples. Furthermore, define $\pi_{k|s_a}$ to be the the second-phase inclusion probability given a particular first-phase sample and $p_{b|a}(s_{b|a})$ the conditional probability distribution of second-phase samples. I will write s where $p(s) = p_a p_{b|a}$ for the ultimate sample achieved using the two-phase design. Under this design the sample inclusion probabilities are

$$\pi_k^* = \pi_{ak}\pi_{k|s_a}.$$

An unbiased estimator of the total is

$$\widehat{y}_{\pi^*} = \sum\nolimits_s \frac{y_k}{\pi_{ak}\pi_{k|s_a}}$$

because

$$\begin{aligned} E\left(\widehat{y}_{\pi^*}\right) &= E_{p_a}\left[E\left(\sum\nolimits_s \frac{y_k}{\pi_{ak}\pi_{k|s_a}}\middle| s_a\right)\right] \\ &= E_{p_a}\left(\sum\nolimits_{s_a} \frac{y_k}{\pi_{ak}}\right) = \sum\nolimits_U y_k \end{aligned}$$

The variance of the estimator is

$$\begin{aligned} V\left(\widehat{y}_{\pi^*}\right) &= Var_{p_a}\left[E\left(\widehat{y}_{\pi^*}\middle| s_a\right)\right] + E_{p_a}\left[V\left(\widehat{y}_{\pi^*}\middle| s_a\right)\right] \\ &= \sum\sum\nolimits_U \Delta_{akl}\frac{y_k}{\pi_{ak}}\frac{y_l}{\pi_{al}} + E_{p_a}\left(\sum\sum\nolimits_{s_a} \Delta_{kl|s_a}\frac{y_k}{\pi_{ak}\pi_{k|s_a}}\frac{y_l}{\pi_{al}\pi_{l|s_a}}\right). \end{aligned}$$

The unbiased estimator of variance is

$$\widehat{V}\left(\widehat{y}_{\pi^*}\right) = \sum\sum\nolimits_s \frac{\Delta_{akl}}{\pi_{akl}\pi_{kl|s_a}}\frac{y_k y_l}{\pi_{ak}\pi_{al}} + \sum\sum\nolimits_s \frac{\Delta_{kl|s_a}}{\pi_{kl|s_a}}\frac{y_k y_l}{\pi_{ak}\pi_{k|s_a}\pi_{al}\pi_{l|s_a}}. \tag{A.28}$$

Similarly, a design-unbiased estimator of the total in the dth domain is

$$\widehat{y}_{\pi^* d} = \sum\nolimits_{s_d} \frac{y_k}{\pi_{ak}\pi_{k|s_a}}$$

and the estimator of the total number of elements in the domain is

$$\widehat{N}_{\pi^* d} = \sum\nolimits_{s_d} \frac{1}{\pi_{ak}\pi_{k|s_a}}.$$

The respective variances are

$$V\left(\widehat{y}_{\pi^* d}\right) = \sum\sum\nolimits_{U_d} \Delta_{akl} \frac{y_k}{\pi_{ak}} \frac{y_l}{\pi_{al}} + E_{p_a}\left(\sum\sum\nolimits_{s_{ad}} \Delta_{kl|s_a} \frac{y_k}{\pi_{ak}\pi_{k|s_a}} \frac{y_l}{\pi_{al}\pi_{l|s_a}}\right) \quad \text{(A.29)}$$

and

$$V\left(\widehat{N}_{\pi^* d}\right) = \sum\sum\nolimits_{U_d} \Delta_{akl} \frac{1}{\pi_{ak}} \frac{1}{\pi_{al}} + E_{p_a}\left(\sum\sum\nolimits_{s_{ad}} \Delta_{kl|s_a} \frac{1}{\pi_{ak}\pi_{k|s_a}} \frac{1}{\pi_{al}\pi_{l|s_a}}\right). \quad \text{(A.30)}$$

The respective variance estimators are

$$\widehat{V}\left(\widehat{y}_{\pi^* d}\right) = \sum\sum\nolimits_{s_d} \frac{\Delta_{akl}}{\pi_{akl}\pi_{kl|s_a}} \frac{y_k y_l}{\pi_{ak}\pi_{al}} + \sum\sum\nolimits_{s_d} \frac{\Delta_{kl|s_a}}{\pi_{kl|s_a}} \frac{y_k y_l}{\pi_{ak}\pi_{k|s_a}\pi_{al}\pi_{l|s_a}} \quad \text{(A.31)}$$

and

$$\widehat{V}\left(\widehat{N}_{\pi^* d}\right) = \sum\sum\nolimits_{s_d} \frac{\Delta_{akl}}{\pi_{akl}\pi_{kl|s_a}} \frac{1}{\pi_{ak}\pi_{al}} + \sum\sum\nolimits_{s_d} \frac{\Delta_{kl|s_a}}{\pi_{kl|s_a}} \frac{1}{\pi_{ak}\pi_{k|s_a}\pi_{al}\pi_{l|s_a}}. \quad \text{(A.32)}$$

A.7.1 Bernoulli Sample of a Simple Random Sample

In bernoulli sampling (BS) each element is chosen with a probability, p, so that the sample size (n) is not fixed. Instead, $n \sim Bin(N, p)$ so that

$$E(n) = Np$$

and

$$V(n) = Np(1-p).$$

The first and second order inclusion probabilities are $\pi_i = p$ and $\pi_{ij} = p^2$ for $i \neq j$ and the Horvitz-Thompson estimator is

$$\widehat{y} = \frac{\sum_s y_i}{p}.$$

The variance and variance estimators are

$$V(\widehat{y}) = \left(\frac{1-p}{p}\right)\sum\nolimits_U y_i^2$$

and

$$\widehat{V}(\widehat{y}) = \frac{1}{p}\left(\frac{1-p}{p}\right)\sum\nolimits_s y_i^2.$$

When a SRS of size n is taken and a BS with probability p is taken from the SRS the inclusion probabilities are

$$\pi_i = \frac{n}{N}p$$

and

$$\pi_{ij} = \begin{cases} \frac{n}{N}p & \text{for } i = j \\ \frac{n(n-1)}{N(N-1)}p^2 & \text{for } i \neq j \end{cases}$$

The Horvitz-Thompson estimator is

$$\widehat{y} = \frac{N}{np}\sum\nolimits_s y_k$$

and the variance and variance estimator are

$$\begin{aligned} V(\widehat{y}) &= \frac{fp-(fp)^2}{(fp)^2}\sum\nolimits_U y_k^2 + \frac{fp^2\frac{n-1}{N-1}-(fp)^2}{(fp)^2}\sum_{k\neq l}\sum\nolimits_U y_k y_l \\ &= \left(\frac{1}{fp}-\frac{1}{f}\right)\sum\nolimits_U y_k^2 + \left(\frac{1}{f}-1\right)\sum\nolimits_U y_k^2 + \left(\frac{n-1}{f(N-1)}-1\right)\sum_{k\neq l}\sum\nolimits_U y_k y_l \\ &= \frac{N}{n}\frac{1-p}{p}\sum\nolimits_U y_k^2 + \frac{N^2(1-f)}{n}S^2 \end{aligned}$$

and

$$
\begin{aligned}
\widehat{V}\left(\widehat{y}\right) =& \frac{fp-(fp)^2}{(fp)^3}\sum\nolimits_s y_k^2 + \frac{f\frac{n-1}{N-1}p^2-(fp)^2}{f^3\frac{n-1}{N-1}p^4}\sum_{k\neq l}\sum\nolimits_s y_k y_l \\
=& \frac{1}{fp}\left[\frac{1}{fp}\sum\sum\nolimits_s y_k y_l - \sum\nolimits_s y_k^2 - \frac{N-1}{p(n-1)}\sum_{k\neq l}\sum\nolimits_s y_k y_l\right] \\
=& \left[\left(\frac{1}{fp}\right)^2 - \frac{N-1}{fp^2(n-1)}\right]\sum\sum\nolimits_s y_k y_l + \frac{1}{fp}\left(\frac{N-1}{p(n-1)}-1\right)\sum\nolimits_s y_k^2 \quad \text{(A.33)} \\
=& \frac{1}{p^2 n}\left\{\frac{N\left[N-1-p(n-1)\right]\sum_s y_k^2 - N(N-n)n\overline{y}^{*2}}{n-1}\right\} \\
=& \frac{N}{n}\left(\frac{1-p}{p^2}\right)\sum\nolimits_s y_k^2 + \left(\frac{N}{p}\right)^2\frac{(1-f)}{n}\frac{\sum_s y_k^2 - n\overline{y}^{*2}}{n-1} \quad \text{(A.34)}
\end{aligned}
$$

where $\overline{y}^* = \sum_s y_k/n$ which is, in general, different from the usual sample mean because the number of elements in the sample is on average np.

A.7.2 SRS of Clusters and SRS of Elements in All Clusters Sampled at the First Phase

When there is a simple random sample of clusters at the first phase, the first-phase inclusion probabilities are

$$
\begin{aligned}
\pi_{ak} &= \frac{n_1}{N_1} \\
&\text{and} \\
\pi_{akl} &= \begin{cases} \frac{n_1(n_1-1)}{N_1(N_1)} & \text{if } k\in U_i, l\in U_j, i\neq j \\ \frac{n_1}{N_1} & \text{if } k,l\in U_i \end{cases} .
\end{aligned}
$$

When there is a simple random sample of elements from all clusters sampled at the first stage the second-phase inclusion probabilities are

$$
\begin{aligned}
\pi_{k|s_a} &= \frac{n_2}{N_2} \\
&\text{and} \\
\pi_{kl|s_a} &= \begin{cases} \frac{n_2(n_2-1)}{N_2(N_2)} & \text{if } k\neq l \\ \frac{n_2}{N_2} & \text{if } k=l \end{cases}
\end{aligned}
$$

where $N_2 = \sum_{s_a} N_i$ is the total number of elements in all clusters sampled at the first phase and $n_2 = \sum_{s_a} n_{2i}$ is the number of elements sampled from N_2 at the second phase. With these inclusion probabilities we can derive an unbiased estimators of the total and an associated unbiased variance estimator specific to this sample design using Eq. A.28. The estimator of the total is

$$\widehat{y} = \frac{N_1 N_2}{n_1 n_2} \sum_{s} y_k = \frac{N_1 N_2}{n_1 n_2} \sum_{s_a} \sum_{s_i} y_{ik}.$$

The variance is

$$\begin{aligned} V(\widehat{y}) =& V\left[E(\widehat{y}|s_a)\right] + E\left[V(\widehat{y}|s_a)\right] \\ =& N_1 \left(\frac{N_1}{n_1} - 1\right) \frac{\sum_{s_a} (y_i - \overline{y})^2}{N_1 - 1} + \left(\frac{N_1}{n_1}\right)^2 E\left[N_2 \left(\frac{N_2}{n_2} - 1\right) \frac{\sum_{s_b} (y_k - \overline{y}_b)^2}{N_2 - 1}\right] \end{aligned}$$

but the second component can also be written as

$$E\left[V(\widehat{y}|s_a)\right] = \left(\frac{N_1}{n_1}\right)^2 E\left[V\left(\sum_{s_a} \widehat{y}_i | s_a\right)\right]$$

where the conditional within-cluster estimator is $\widehat{y}_i = N_2 \sum_{s_i} y_{ik}/n_2$. The conditional variance of the sum of within cluster estimators is

$$V\left(\sum_{s_a} \widehat{y}_i | s_a\right) = \sum_{s_a} V(\widehat{y}_i|s_a) + \sum_{i \neq j} \sum_{s_a} Cov(\widehat{y}_i, \widehat{y}_j | s_a)$$

where $V(\widehat{y}_i|s_a)$ and $Cov(\widehat{y}_i, \widehat{y}_j|s_a)$ are the variance and covariance of domain estimators given by Eq. A.20 and Eq. A.6 where the conditional variances within each cluster are those of domains under SRS because the sample sizes within clusters are not fixed (see Section A.6).

When we are interested in the total number of elements in the dth domain the estimator becomes

$$\widehat{N}_d = \frac{N_1 N_2}{n_1 n_2} \sum_{s_d} 1 = \frac{N_1 N_2 n_{2d}}{n_1 n_2} \equiv \frac{N_1 N_2}{n_1} p_{2d}.$$

and the variance estimator becomes

$$\begin{aligned}\widehat{V}\left(\widehat{N}_d\right) =& N_1\left(\frac{N_1}{n_1}-1\right)\frac{\sum_{s_a}\left(\widehat{N}_{id}-\widehat{\overline{N}}_d\right)^2}{n_1-1}+\frac{N_1}{n_1}\sum_{s_a}\widehat{V}\left(\widehat{N}_{id}|s_a\right)\\ &+\frac{N_1}{n_1}\left(\frac{N_1}{n_1}-1\right)\frac{\sum\sum_{s_a, i\neq j}\widehat{Cov}\left(\widehat{N}_{id},\widehat{N}_{jd}|s_a\right)}{n_1-1}\end{aligned} \tag{A.35}$$

where

$$\widehat{V}\left(\widehat{N}_{id}|s_a\right)=N_2\left(\frac{N_2}{n_2}-1\right)\frac{n_2p_{id}\left(1-p_{id}\right)}{n_2-1},$$

$p_{id}=n_{id}/n_2$ and

$$\widehat{Cov}\left(\widehat{N}_{id},\widehat{N}_{jd}|s_a\right)=-N_2\left(\frac{N_2}{n_2}-1\right)\frac{n_2p_{id}p_{jd}}{n_2-1}.$$

A.7.3 SRS of Clusters and All Elements in Selected Clusters, then Post-stratification of Elements by Cluster

Here we select clusters of elements by SRS, then we select an SRS of all elements in the clusters selected by SRS at the first phase. Finally, we post-stratify the elements selected at the second-phase by the cluster where the elements occur.

Let Ω_1 be the event that $n_{2i}\geq 1$, $\forall\ i\in s_a$ and $\boldsymbol{n}_2=(n_{21},\ldots,n_{2n_1})'$ where n_{2i} is the number of second-phase elements in the ith cluster, s_a is the first-phase sample of clusters and n_1 is the number of clusters sampled at the first phase. Furthermore, if we only consider the subset of samples where $\boldsymbol{n}_2$ is fixed, then this subset comprises the distribution of samples under stratified simple random sampling and

$$E_{p_{st}}\left(\sum_{s_a}\frac{N_i}{n_{2i}}\sum_{s_{ai}}y_{ik}\Big|s_a,\Omega_1,\boldsymbol{n}_2\right)=\sum_{s_a}\sum_{U_i}y_{ik}=\sum_{s_a}y_i$$

where p_{st} is the probability distribution of all samples under stratified simple random sampling and N_i is the total number of elements in the ith cluster. Notice that the above result does not depend on the sample sizes $\boldsymbol{n}_2$. Therefore, the expectation over all samples sizes is,

$$E_{p(\boldsymbol{n}_2)}\left[E_{p_{st}}\left(\sum\nolimits_{s_a}\frac{N_i}{n_{2i}}\sum\nolimits_{s_{ai}}y_{ik}\middle|s_a,\Omega_1,\boldsymbol{n}_2\right)\right]=E_{p(\boldsymbol{n}_2)}\left(\sum\nolimits_{s_a}y_i\middle|s_a,\Omega_1\right)=\sum\nolimits_{s_a}y_i$$

and the estimator of the total

$$\widehat{y}=\frac{N_1}{n_1}\sum\nolimits_{s_a}\frac{N_i}{n_{2i}}\sum\nolimits_{s_{ai}}y_{ik} \tag{A.36}$$

is unbiased conditional on Ω_1 because

$$E_{p_a}\left(\frac{N_1}{n_1}\sum\nolimits_{s_a}\frac{N_i}{n_{2i}}\sum\nolimits_{s_{ai}}y_{ik}\middle|\Omega_1\right)=E_{p_a}\left(\frac{N_1}{n_1}\sum\nolimits_{s_a}y_i\middle|\Omega_1\right)=\sum\nolimits_{U_a}y_i=y.$$

The variance is

$$\begin{aligned}V\left(\widehat{y}\middle|\Omega_1\right)=&\underbrace{E_{p_a}\left\{E_{p(\boldsymbol{n}_2)}\left[V_{p_{st}}\left(\widehat{y}\middle|\Omega_1,s_a,\boldsymbol{n}_2\right)\right]\right\}}_{V_I}+\underbrace{E_{p_a}\left\{V_{p(\boldsymbol{n}_2)}\left[E_{p_{st}}\left(\widehat{y}\middle|\Omega_1,s_a,\boldsymbol{n}_2\right)\right]\right\}}_{V_{II}}\\&+\underbrace{V_{p_a}\left\{E_{p(\boldsymbol{n}_2)}\left[E_{p_{st}}\left(\widehat{y}\middle|\Omega_1,s_a,\boldsymbol{n}_2\right)\right]\right\}}_{V_{III}}\end{aligned}$$

where the second term on the right, V_{II}, is zero because there is no variation in the conditional expectation, $E_{p_{st}}\left(\widehat{y}\middle|\Omega_1,s_a,\boldsymbol{n}_2\right)$, over the different second-phase sample sizes and the first term is

$$\begin{aligned}V_I=&E_{p_a}\left\{E_{p(\boldsymbol{n}_2)}\left[\sum\nolimits_{s_a}N_i\left(\frac{N_i}{n_{2i}}-1\right)S_i^2\middle|\Omega_1,s_a\right]\right\}\\=&E_{p_a}\left\{\sum\nolimits_{s_a}N_i\left[N_iE_{p(\boldsymbol{n}_2)}\left(\frac{1}{n_{2i}}\right)-1\right]S_i^2\middle|\Omega_1\right\}\\\approx&E_{p_a}\left\{\sum\nolimits_{s_a}N_i\left[N_i\left(\frac{1}{E(n_{2i})}+\frac{V(n_{2i})}{E(n_{2i})^3}\right)-1\right]S_i^2\middle|\Omega_1\right\}\end{aligned} \tag{A.37}$$

where $E(1/n_{2i})$ is approximated by a second-order Taylor series expansion. Because the vector of within-cluster sample sizes is multivariate hypergeometric, $E(n_{2i})=n_2\frac{N_i}{N_2}$ and

$$V(n_{2i})=\frac{N_2}{N_2-1}\left(1-\frac{n_2}{N_2}\right)\left(1-\frac{N_i}{N_2}\right)\frac{N_i}{N_2}n_2\approx\left(1-\frac{n_2}{N_2}\right)\left(1-\frac{N_i}{N_2}\right)E(n_{2i}).$$

After some algebra, Eq. A.37 becomes

$$
\begin{aligned}
V_I =& E_{p_a}\left\{\frac{1}{n_2^2}(N_2-n_2)\left[(n_2-1)\sum\nolimits_{s_a} N_i S_i^2+\sum\nolimits_{s_a} S_i^2\right]\Bigg|\,\Omega_1\right\} \\
=& \frac{n_2-1}{n_2}\left[\frac{E\left(N_2\sum_{s_a} N_i S_i^2\right)}{n_2}-E\left(\sum\nolimits_{s_a} N_i S_i^2\right)\right] \\
&+\frac{1}{n_2}\left[\frac{E\left(N_2\sum_{s_a} S_i^2\right)}{n_2}E\left(\sum\nolimits_{s_a} S_i^2\right)\right] \\
=& \frac{n_2-1}{n_2}\left\{\frac{Cov\left(N_2,\sum_{s_a} N_i S_i^2\right)}{n_2}+E\left(\sum\nolimits_{s_a} N_i S_i^2\right)\left[\frac{E(N_2)}{n_2}-1\right]\right\} \\
&+\frac{1}{n_2}\left\{\frac{Cov\left(N_2,\sum_{s_a} S_i^2\right)}{n_2}+E\left(\sum\nolimits_{s_a} S_i^2\right)\left[\frac{E(N_2)}{n_2}-1\right]\right\}
\end{aligned}
$$

where $N_2=\sum_{s_a} N_i$,

$$
S_i^2=\frac{\sum_{U_i}(y_{ik}-\bar{y}_i)^2}{N_i-1},\quad E\left(N_2\right)=\frac{n_1}{N_1}\sum\nolimits_{U_a} N_i=\frac{n_1}{N_1}N,
$$

$$
E\left(\sum\nolimits_{s_a} S_i^2\right)=\frac{n_1}{N_1}\sum\nolimits_{U_a} S_i^2 \;\; E\left(\sum\nolimits_{s_a} N_i S_i^2\right)=\frac{n_1}{N_1}\sum\nolimits_{U_a} N_i S_i^2,
$$

$$
\begin{aligned}
Cov\left(N_2,\sum\nolimits_{s_a} N_i S_i^2\right) =& n_1\left(1-\frac{n_1}{N_i}\right)\frac{\sum_{U_a}(N_i-\overline{N})(N_i S_i^2-\overline{NS^2})}{N_1-1} \\
\equiv& n_1\left(1-\frac{n_1}{N_i}\right)S_{NNS}^2,
\end{aligned}
$$

$$
\begin{aligned}
Cov\left(N_2,\sum\nolimits_{s_a} S_i^2\right) =& n_1\left(1-\frac{n_1}{N_i}\right)\frac{\sum_{U_a}(N_i-\overline{N})(S_i^2-\overline{S})}{N_1-1} \\
\equiv& n_1\left(1-\frac{n_1}{N_i}\right)S_{NS}^2,
\end{aligned}
$$

$\overline{NS^2} = \sum_{U_a} N_i S_i^2 / N_1$ and $\overline{S} = \sum_{U_a} S_i^2 / N_1$. The third variance component is

$$
\begin{aligned}
V_{III} =& V_{p_a}\left(\frac{N_1}{n_1}\sum_{s_a} y_i\right) \\
=& N_1\left(\frac{N_1}{n_1}-1\right)S_1^2
\end{aligned}
$$

where N_1 is the total number of clusters in the first-phase universe (U_a) and

$$
S_1^2 = \frac{\sum_{U_a}(y_i - \overline{y}_1)^2}{N_1 - 1}.
$$

With these pieces the approximate variance of Eq. A.36 is

$$
\begin{aligned}
AV(\widehat{y}) =& N_1\left(\frac{N_1}{n_1}-1\right)S_1^2 + \frac{n_1}{N_1 n_2}\left\{(n_2-1)\left[\left(1-\frac{n_1}{N_1}\right)\frac{S_{NNS}^2}{n_2} + \eta\left(\frac{n_1 N}{N_1 n_2}-1\right)\right]\right. \\
&\left. + \left(1-\frac{n_1}{N_1}\right)\frac{S_{NS}^2}{n_2} + S\left(\frac{n_1 N}{N_1 n_2}-1\right)\right\} \qquad \text{(A.38)}
\end{aligned}
$$

where $\eta = \sum_{U_a} N_i S_i^2$ and $S = \sum_{U_a} S_i^2$. The corresponding variance estimator is obtained by substituting sample estimates for the unknown values,

$$
\widehat{S}_1^2 = \frac{\sum_{s_a}(\widehat{y}_i - \overline{\widehat{y}})^2}{n_1 - 1} - \frac{1}{n_1}\sum_{s_a} N_i\left(\frac{N_i}{n_{2i}}-1\right)\widehat{S}_i^2,
$$

$$
\widehat{S}_{NNS}^2 = \frac{\sum_{s_a}(N_i - \widehat{\overline{N}})(N_i\widehat{S}_i^2 - \widehat{\overline{NS}})}{n_1 - 1}, \quad \widehat{S}_{NS}^2 = \frac{\sum_{s_a}(N_i - \widehat{\overline{N}})(\widehat{S}_i^2 - \widehat{\overline{S}})}{n_1 - 1}
$$

$$
\widehat{S}_i^2 = \frac{\sum_{s_{ai}}(y_{ik} - \overline{y}_i)^2}{n_{2i} - 1}, \quad \widehat{N} = \frac{N_1}{n_1}\sum_{s_a} N_i, \quad \widehat{\eta} = \frac{N_1}{n_1}\sum_{s_a} N_i S_i^2, \quad \widehat{S} = \frac{N_1}{n_1}\sum_{s_a} S_i^2
$$

and $\widehat{\overline{NS}} = \sum_{s_a} N_i S_i^2 / n_1$. Notice that the estimator of variance requires the condition, $\Omega_2 = \{s : n_{2i} \geq 2, \forall\, i \in s_a\}$.

A.7.4 SRS of Clusters, Then SRS of Elements within Clusters, Then BS of Elements

Here we first select clusters of elements by SRS, then within each cluster we sample the elements by SRS and finally take a Bernoulli subsample of the elements with probability ϕ. Here the SRS of clusters can be considered the first stage of a two-stage sampling design. The sampling at the second stage is independent across clusters, but within each cluster is a SRS-BS two-phase design. We know from Section A.2 the estimator for the total and corresponding variance and using the Horvitz-Thompson theorem for two-stage sampling (Eq. A.17) the estimator of the total and variance over both stages are

$$\widehat{y} = \frac{N_1}{n_1}\sum\nolimits_{s_1} \frac{N_i}{n_i\phi}\sum\nolimits_{s_i} y_{ik}$$

and

$$\begin{aligned} V(\widehat{y}) =& \sum\nolimits_{U_1} \Delta_{ij}\frac{y_i}{\pi_i}\frac{y_j}{\pi_j} + \sum\nolimits_{U_1} \frac{V(\widehat{y}_i)}{\pi_i} \\ =& N_1^2\left(1-\frac{n_1}{N_1}\right)\frac{S_1^2}{n_1} + \frac{N_1}{n_1}\sum\nolimits_{U_1}\left[\frac{N_i}{n_i}\frac{1-\phi}{\phi}\sum\nolimits_{U_i} y_{ik}^2 + N_i^2\left(1-\frac{n_i}{N_i}\right)\frac{S_i^2}{n_i}\right] \end{aligned}$$

where $\overline{y}_{Ui}^* = \sum_{U_i} y_{ik}/N_i$. The varianc estimator is

$$\begin{aligned} \widehat{V}(\widehat{y}) =& \sum\nolimits_{s_1} \frac{\Delta_{ij}}{\pi_{ij}}\frac{\widehat{y}_i}{\pi_i}\frac{\widehat{y}_j}{\pi_j} + \sum\nolimits_{s_1} \frac{\widehat{V}(\widehat{y}_i)}{\pi_i} \\ =& N_1^2\left(1-\frac{n_1}{N_1}\right)\frac{\widetilde{S}_1^2}{n_1} \\ &+ \frac{N_1}{n_1}\sum\nolimits_{s_1}\left[\frac{N_i}{n_i}\frac{1-\phi}{\phi^2}\sum\nolimits_{s_i} y_{ik}^2 + \left(\frac{N_i}{\phi}\right)^2\frac{1-\frac{n_i}{N_i}}{n_i}\frac{\sum_{s_i} y_{ik}^2 - n\overline{y}_{si}^{*2}}{n_i-1}\right] \end{aligned}$$

where $\overline{y}_{si}^* = \sum_{s_i} y_{ik}/n_i$. Notice that when $\phi = 1$ we have the usual Horvitz-Thompson estimator under a two-stage SRS design.

A.7.5 SRS of Clusters, Then SRS of All Elements in Selected Clusters, Then BS of Elements

Here we first select clusters of elements by SRS, then we select elements by SRS from all elements in the clusters selected at the first phase. Finally, we take a Bernoulli subsample of the elements with probability, ϕ. Once the clusters are chosen at the first phase, the sampling of elements is again a SRS-BS design and the unbiased estimator is

$$\widehat{y} = \frac{N_1}{n_1}\sum\nolimits_{s_1} \frac{N_2}{n_2\phi}\sum\nolimits_{s_2} y_k$$

where N_2 is the total number of elements in the clusters sampled at the first phase. The variance is

$$\begin{aligned} V(\widehat{y}) =& V\left[E(\widehat{y}|s_1)\right] + E\left[V(\widehat{y}|s_1)\right] \\ =& N_1^2\left(1-\frac{n_1}{N_1}\right)\frac{S_1^2}{n_1} + \left(\frac{N_1}{n_1}\right)^2 E\left[N_2^2\left(1-\frac{n_2}{N_2}\right)\frac{S_2^2}{n_2}\right] \\ &+ \left(\frac{N_1}{n_1}\right)^2 E\left[\frac{N_2}{n_2}\frac{1-\phi}{\phi}\sum_{s_1}\sum_{U_i} y_{ik}^2\right]. \end{aligned}$$

From the results for a Bernoulli sample of a simple random sample (Eq. A.34) we know an unbiased estimator of the second summand on the right hand side, but to obtain an unbiased estimate of the first summand recognize that

$$S_1^2 = \frac{1}{N_1 - 1}\left[\sum_{U_1} y_i^2 - \frac{\sum\sum_{U_1} y_i y_j}{N_1}\right]$$

and under SRS of first-phase elements only an unbiased estimator is

$$\widehat{S}_1^2 = \frac{1}{n_1 - 1}\left[\sum_{s_1} y_i^2 - \frac{\sum\sum_{s_1} y_i y_j}{n_1}\right]$$

When there are unbiased estimates and of variance estimates within each sampled cluster given the first-phase sample of clusters an unbiased estimate of S_1^2 is

$$\begin{aligned} \widehat{S}_1^2 =& \frac{1}{n_1-1}\left\{\sum_{s_1}\left[\widehat{y}_i^2 - \widehat{V}(\widehat{y}_i|s_1)\right] - \frac{\sum\sum_{s_1}\left[\widehat{y}_i\widehat{y}_j - \widehat{Cov}(\widehat{y}_i,\widehat{y}_j|s_1)\right]}{n_1}\right\} \\ =& \frac{\sum_{s_1}(\widehat{y}_i - \overline{y}_{s_1})^2}{n_1-1} - \frac{1}{n_1}\left[\sum_{s_1}\widehat{V}(\widehat{y}_i|s_1) - \frac{1}{n_1-1}\sum\sum_{\substack{s_1 \\ i\neq j}}\widehat{Cov}(\widehat{y}_i,\widehat{y}_j|s_1)\right]. \end{aligned} \tag{A.39}$$

The within cluster estimators are examples of domain estimators because sampling is with respect to all elements in all clusters sampled at the first phase. General results for these estimators are presented in Section A.6, but in this case with a SRS-BS design at the second phase $\widehat{y}_i = N_2 \sum_{s_{2i}} y_{ik}/(n_2\phi)$,

$$V\left(\widehat{y}_i|s_1\right) = N_2\left(\frac{N_2}{n_2}-1\right)\frac{(N_{2i}-1)S_{2i}^2 + N_{2i}\left(1-\frac{N_{2i}}{N_2}\right)\overline{y}_{2i}^2}{N_2-1} + \frac{N_2}{n_2}\frac{1-\phi}{\phi}\sum_{s_i} y_{ik}^2$$

and

$$\widehat{V}\left(\widehat{y}_i|s_1\right) = \left(\frac{N_2}{\phi^2}\right)\left(\frac{N_2}{n_2}-1\right)\frac{\sum_{s_i} y_{ik}^2 - n_{2i}\overline{y}_{2i}^{*2} + n_{2i}\left(1-\frac{n_{2i}}{n_2}\right)\overline{y}_{s2i}^{*2}}{n_2-1} + \frac{N_2}{n_2}\frac{1-\phi}{\phi^2}\sum_{s_i} y_{ik}^2$$

where $\overline{y}_{s2i}^* = \sum_{s_i} y_{ik}/n_{2i}$. The covariance and covariance estimator of two domain estimators for $i \neq j$ is

$$\begin{aligned} Cov\left(\widehat{y}_i, \widehat{y}_j|s_1\right) =& E\left(\widehat{y}_i\widehat{y}_j|s_1\right) - y_iy_j = \left(\frac{N_2}{n_2\phi}\right)^2 E\left(\sum_{s_i} y_{ik}\sum_{s_j} y_{jl}\right) - y_iy_j \\ =& \left(\frac{N_2}{n_2\phi}\right)^2 \frac{n_2(n_2-1)\phi^2}{N_2(N_2-1)}\left(\sum_{U_i} y_{ik}\sum_{U_j} y_{jl}\right) - y_iy_j \\ =& -N_2\left(\frac{1}{n_2}-\frac{1}{N_2}\right)\frac{y_iy_j}{N_2-1} \end{aligned}$$

and

$$\widehat{Cov}\left(\widehat{y}_i, \widehat{y}_j|s_1\right) = -\left(1-\frac{n_2}{N_2}\right)\frac{\widehat{y}_i\widehat{y}_j}{n_2-1}$$

The unconditional variance estimator is then

$$\begin{aligned} \widehat{V}\left(\widehat{y}\right) =& N_1\left(\frac{N_1}{n_1}-1\right)\widehat{S}_1^2 \\ &+ \left(\frac{N_1}{n_1}\right)^2\left[\frac{N_2}{\phi^2}\left(\frac{N_2}{n_2}-1\right)\frac{\sum_s y_k^2 - n_2\overline{y}_s^{*2}}{n_2-1} + \frac{N_2}{n_2}\frac{1-\phi}{\phi^2}\sum_s y_k^2\right] \end{aligned} \tag{A.40}$$

where $\widehat{S}_1^2$ is given in Eq. A.39 with the derived conditional variance and covariance estimators for the within-cluster totals.

If we are interested in a estimating the total number of elements in a domain the estimator is

$$\widehat{N}_d = \frac{N_1N_2}{n_1n_2\phi}n_{sd}$$

where n_{sd} is the number of sampled elements that occur in the dth domain. The variance estimator becomes

$$\widehat{V}\left(\widehat{N}_d\right) = N_1\left(\frac{N_1}{n_1}-1\right)\widehat{S}_1^2 + \left(\frac{N_1}{n_1}\right)^2\left[\frac{N_2}{\phi^2}\left(\frac{N_2}{n_2}-1\right)\frac{n_2 p_{sd}^*(1-p_{sd}^*)}{n_2-1} + \frac{N_2}{n_2}\frac{1-\phi}{\phi^2}n_{sd}\right]$$

where $p_{sd}^* = n_{sd}/n_2$ and $\widehat{S}_1^2$ is given by Eq. A.39, but

$$\widehat{V}\left(\widehat{N}_{id}|s_1\right) = \frac{N_2}{\phi^2}\left(\frac{N_2}{n_2}-1\right)\frac{n_2 p_{sid}^*(1-p_{sid}^*)}{n_2-1} + \frac{N_2}{n_2}\frac{1-\phi}{\phi^2}n_{sid}$$

and

$$\widehat{Cov}\left(\widehat{N}_{id}, \widehat{N}_{jd}|s_1\right) = -\left(1-\frac{n_2}{N_2}\right)\frac{\widehat{N}_{id}\widehat{N}_{jd}}{n_2-1}.$$

where $p_{sid}^* = n_{sid}/n_2$ and $\widehat{N}_{id} = N_2 n_{sid}/(n_2\phi)$.

A.7.6 SRS of Elements, Then SRS of a Domain

If there is a SRS of n elements from universe of N elements and the dth domain is indicated on the elements sampled the number of elements in the domain is $n_d = \sum_{s_{ad}} 1$. Suppose a SRS of n_2 elements is taken from those elements in the domain so that

$$\pi_{k|s_a} = \frac{n_2}{n_d}$$

and

$$\pi_{kl|s_a} = \frac{n_2(n_2-1)}{n_d(n_d-1)}$$

for $k \neq l$. The design-unbiased estimator of the domain total is

$$\widehat{y}_d = \frac{\sum_s y_k}{\pi_{ak}\pi_{k|s_a}} = \frac{N}{n}\frac{n_d}{n_2}\sum_s y_k$$

because

$$\begin{aligned} E\left(\widehat{y}_d\right) &= E\left[E\left(\widehat{y}_d|s_a\right)\right] = E\left[E\left(\frac{N}{n}\frac{n_d}{n_2}\sum_s y_k\middle| s_a\right)\right] \\ &= E\left(\frac{N}{n}\sum_{s_{ad}} y_k\right) = \sum_{U_d} y_k \end{aligned}$$

The variance of the estimator is

$$\begin{aligned} V\left(\widehat{y}_d\right) =& V\left[E\left(\widehat{y}_d|s_a\right)\right] + E\left[V\left(\widehat{y}_d|s_a\right)\right] \\ =& N\left(\frac{N}{n}-1\right)\frac{(N_d-1)S_d^2 + N_d\left(1-\frac{N_d}{N}\right)\overline{y}_d^2}{N-1} + \left(\frac{N}{n}\right)^2 E\left[n_d\left(\frac{n_d}{n_2}-1\right)S_{d_a}^2\right] \end{aligned}$$

where $\overline{y}_d = \sum_{U_d} y_k/N_d$, $\overline{y}_{d_a} = \sum_{s_a d} y_k/n_d$,

$$S_d^2 = \frac{\sum_{U_d}\left(y_k - \overline{y}_d\right)^2}{N_d - 1}$$

and

$$S_{d_a}^2 = \frac{\sum_{s_a d}(y_k - \overline{y}_{d_a})^2}{n_d - 1}.$$

The unbiased variance estimator is then,

$$\begin{aligned} \widehat{V}\left(\widehat{y}_d\right) =& N\left(\frac{N}{n}-1\right)\frac{(n_d-1)S_2^2 + n_d\left(1-\frac{n_d}{n}\right)\left[\overline{y}_2^2 - \left(\frac{1}{n_2}-\frac{1}{N_2}\right)S_2^2\right]}{n-1} \\ &+ \left(\frac{N}{n}\right)^2 n_d\left(\frac{n_d}{n_2}-1\right)S_2^2 \end{aligned}$$

where

$$S_2^2 = \frac{\sum_s(y_k - \overline{y}_s)^2}{n_2 - 1},$$

$\overline{y}_s = \sum_s y_k/n_2$ and s is the SRS sample of size n_2 from the dth domain.

A.7.7 SRS of Elements, Then a Two-stage Sample in a Domain

Consider the same scenario as described in Section A.7.6 but with the extension of a Bernoulli sample of elements in the domain-specific SRS and an unspecified sample design is carried out within each of the chosen elements. Let s_1 define the first SRS sample, $s_{d|1}$ define the SRS of the domain available from the first sample and $s_{b|d,1}$ define the Bernoulli sample (BS) of the elements available from $s_{d|1}$. Assuming that n elements are obtained from N in s_1 and $n_{d|1}$ is the number of elements from $N_{d|1}$

within the domain for $s_{d|1}$, the unbiased estimator is

$$\widehat{y}_d = \frac{N}{n}\frac{N_{d|1}}{n_{d|1}}\frac{1}{p}\sum\nolimits_{s}\widehat{y}_k$$

where p is the probability associated with the Bernoulli sampling and $\widehat{y}_k$ is the unbiased estimator of the total in the kth element. The variance of this estimator is

$$\begin{aligned}
V\left(\widehat{y}_d\right) =& V\left[E\left\{E\left[E\left(\widehat{y}_d|s_{b|d,1}, s_{d|1}, s_1\right)\right]\right\}\right] + E\left[V\left\{E\left[E\left(\widehat{y}_d|s_{b|d,1}, s_{d|1}, s_1\right)\right]\right\}\right] \\
&+ E\left[E\left\{V\left[E\left(\widehat{y}_d|s_{b|d,1}, s_{d|1}, s_1\right)\right]\right\}\right] + E\left[E\left\{E\left[V\left(\widehat{y}_d|s_{b|d,1}, s_{d|1}, s_1\right)\right]\right\}\right] \\
=& N\left(\frac{N}{n}-1\right)\frac{(N_d-1)S_d^2 + N_d(1-\frac{N_d}{N})\overline{y}_d^2}{N-1} \\
&+ \left(\frac{N}{n}\right)^2 E\left[N_{d|1}\left(\frac{N_{d|1}}{n_{d|1}}-1\right)S_{d|1}^2\right] + \left(\frac{N}{n}\right)^2\frac{1-p}{p}E\left[\frac{N_{d|1}}{n_{d|1}}\sum\nolimits_{s_d} y_k^2\right] \\
&+ \left(\frac{N}{n}\right)^2\frac{1}{p}E\left[\frac{N_{d|1}}{n_{d|1}}\sum\nolimits_{s_d} V\left(\widehat{y}_k\right)\right] \qquad \text{(A.41)}
\end{aligned}$$

where s_d is the set of elements in the domain obtained through the first SRS and

$$S_{d|1}^2 = \frac{\sum_{s_{d|1}}\left(y_k - \overline{y}_d\right)^2}{n_{d|1}-1}.$$

The unbiased variance estimator is

$$\begin{aligned}
\widehat{V}\left(\widehat{y}_d\right) =& \frac{N^2(1-f)}{n}\frac{N_{d|1}}{nn_{d|1}p}\left\{\sum\nolimits_{s_{b|d,1}}\left[\widehat{y}_k^2 - \widehat{V}\left(\widehat{y}_k\right)\right] - \frac{(N_{d|1}-1)\sum\sum_{\substack{s_{b|d,1}\\ k\neq l}}\widehat{y}_k\widehat{y}_l}{(n-1)(n_{d|1}-1)p}\right\} \\
&+ \left(\frac{N}{n}\right)^2\left[\frac{N_{d|1}(1-p)}{n_{d|1}p^2}\sum\nolimits_{s_{b|d,1}}\widehat{y}_k^2 + \frac{N_{d|1}^2}{p^2 n_{d|1}}\left(1-\frac{n_{d|1}}{N_{d|1}}\right)\frac{\sum_{s_{b|d,1}}\widehat{y}_k^2 - n_{d|1}\overline{\widehat{y}}^{*2}}{n_{d|1}-1}\right] \\
&+ \left(\frac{N}{n}\right)^2\frac{N_{d|1}}{n_{d|1}p}\sum\nolimits_{s_{b|d,1}}\widehat{V}\left(\widehat{y}_k\right) \qquad \text{(A.42)}
\end{aligned}$$

where $\overline{\widehat{y}}_k^* = \sum_{s_{b|d,1}}\widehat{y}_k/n_{d|1}$.

The unbiasedness of Eq. A.42 follows by recognizing the unbiasedness of the components. The first summand in Eq. A.41 is the variance of a domain estimator under SRS (Section A.6) and

$$\frac{(N_d-1)S_d^2+N_d(1-\frac{N_d}{N})\overline{y}_d^2}{N-1}=\frac{1}{N}\sum_{U_d}y_k^2-\frac{1}{(N-1)N}\sum\sum_{\substack{U_d\\k\neq l}}y_ky_l$$

which can be shown to be unbiasedly estimated by

$$\frac{N_{d|1}}{nn_{d|1}p}\sum_{s_{b|d,1}}\left[\widehat{y}_k^2-\widehat{V}\left(\widehat{y}_k\right)\right]-\frac{N_{d|1}\left(N_{d|1}-1\right)}{n(n-1)n_{d|1}(n_{d|1}-1)p^2}\sum\sum_{\substack{s_{b|d,1}\\k\neq l}}\widehat{y}_k\widehat{y}_l.$$

Conditional on the domain of the first SRS sample, the second, third and fourth summands in Eq. A.41 are variance components due to a particular two-stage sample design where the first-stage elements are chosen by SRS-BS two-phase sampling and the sampling at the second stage is unspecified. The variance estimator for a SRS-BS two-phase design is given in Eq. A.34 and from Eq. A.17 we see that unbiased variance estimators due to SRS-BS sampling at the first stage and sampling at the second stage are

$$\left(\frac{N}{n}\right)^2\left[\frac{N_{d|1}(1-p)}{n_{d|1}p^2}\sum_{s_{b|d,1}}\widehat{y}_k^2+\frac{N_{d|1}}{p^2}\left(\frac{N_{d|1}}{n_{d|1}}-1\right)\frac{\sum_{s_{b|d,1}}\widehat{y}_k^2-n_{d|1}\widehat{\overline{y}}^{*2}}{n_{d|1}-1}\right]$$
$$-\left(\frac{N}{n}\right)^2\frac{N_{d|1}}{n_{d|1}p}\left(\frac{N_{d|1}}{n_{d|1}p}-1\right)\sum_{s_{b|d,1}}\widehat{V}\left(\widehat{y}_k\right)$$

and

$$\left(\frac{N}{n}\frac{N_{d|1}}{n_{d|1}p}\right)^2\sum_{s_{b|d,1}}\widehat{V}\left(\widehat{y}_k\right),$$

respectively. Summing the expected values of the components yields Eq. A.41. For similar results without the final phase of Bernoulli sampling, but still having unspecified sampling within first-stage elements the estimator, variance and variance estimator are

$$\widehat{y}_d=\frac{N}{n}\frac{N_{d|1}}{n_{d|1}}\sum_s\widehat{y}_k,$$

$$
\begin{aligned}
V\left(\widehat{y}_d\right) =& N\left(\frac{N}{n}-1\right)\frac{(N_d-1)S_d^2+N_d(1-\frac{N_d}{N})\overline{y}_d^2}{N-1} \\
&+\left(\frac{N}{n}\right)^2 E\left[N_{d|1}\left(\frac{N_{d|1}}{n_{d|1}}-1\right)\frac{S_{d|1}^2}{n_{d|1}}+\frac{N_{d|1}}{n_{d|1}}\sum\nolimits_{s_d} V\left(\widehat{y}_k\right)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{V}\left(\widehat{y}_d\right) =& N\left(\frac{N}{n}-1\right)\frac{(N_{d|1}-1)\widehat{S^2}_d+N_{d|1}(1-\frac{N_{d|1}}{n})\widehat{\overline{y}^2}_d}{n-1} \\
&+\left(\frac{N}{n}\right)^2 N_{d|1}\left(\frac{N_{d|1}}{n_{d|1}}-1\right)\frac{\widetilde{S}_{d|1}^2}{n_{d|1}}+\frac{N_{d|1}}{n_{d|1}}\sum\nolimits_{s}\widehat{V}\left(\widehat{y}_k\right)
\end{aligned}
$$

where $\widehat{S^2}_d=\widetilde{S}_{d|1}^2-\sum_s\widehat{V}\left(\widehat{y}_k\right)/n_{d|1}$,

$$
\widetilde{S}_{d|1}^2=\frac{\sum_s\left(\widehat{y}_k-\overline{y}_s\right)^2}{n_{d|1}-1}
$$

and

$$
\widehat{\overline{y}^2}_d=\overline{y}_s^2-\frac{\widehat{V}\left(\widehat{y}_{d|1}|s_1\right)}{N_{d|1}^2}=\overline{y}_s^2-\left(1-\frac{n_{d|1}}{N_{d|1}}\right)\frac{\widetilde{S}_{d|1}^2}{n_{d|1}}-\frac{1}{N_{d|1}n_{d|1}}\sum\nolimits_s\widehat{V}\left(\widehat{y}_k\right).
$$

The unbiasedness of the variance estimator follows because

$$
E\left(\widetilde{S}_{d|1}^2|s_1\right)=S_{d|1}^2+\sum_{s_d}V\left(\widehat{y}_k\right)/N_{d|1},
$$

$E\left[\widehat{V}\left(\widehat{y}_d|s_1\right)\right]=E\left(\widehat{y}_{d|1}^2|s_1\right)-y_{d|1}^2$. With these results and the form of the unbiased variance estimator of domain totals in Section A.6 the first summand on the right hand side is obtained. The unbiasedness of the second and third summands is shown by recognizing this as the form of an unbiased variance estimator for a total under a two-stage sample design with SRS at the first stage.

A.8 Model-based Inference

A.8.1 Prediction Approach with a Single Super-population

In model-based inference a super-population model is assumed to generate the finite population of interest. This super-population model is a multivariate distribution with an N-dimensional parameter space. Here we will consider the simple distribution of $\mathbf{Y}$ $(N \times 1)$ where $E_M(\mathbf{Y}) = \mu\mathbf{1}$ and $V_M(\mathbf{Y}) = \sigma^2\mathbf{I}$ and expectations are with respect to the model. In the prediction approach used by Bolfarine and Zacks (1992) and Valliant et al. (2000) and initially proposed by Royall (1970) the values of the sampled elements are used to predict the values of the unobserved elements in the finite population. With the present model the best linear unbiased predictor (BLUP) of the total for the population is

$$\widehat{Y} = \sum\nolimits_s Y_k + \sum\nolimits_r \widehat{Y}_k = n\widehat{\mu} + (N-n)\widehat{\mu} = N\widehat{\mu}$$

where $\widehat{Y}_k = N\widehat{\mu} = \sum_s Y_k/n$ and $r = \{U : k \notin s\}$ is the unsampled portion of the population. The results of interest with this approach all rely on properties of the prediction error which is the difference between the parameter associated with the unsampled portion of the population r and its predictor,

$$\delta = \widehat{Y}_r - \sum\nolimits_r Y_k = \frac{N-n}{n}\sum\nolimits_s Y_k - \sum\nolimits_r Y_k = \left(\frac{N}{n} - 1\right)\sum\nolimits_s Y_k - \sum\nolimits_r Y_k.$$

Notice that δ is the difference of two random variables. The expected value of the estimator is $E_M(\widehat{\delta}) = 0$ and the variance is

$$V_M(\delta) = \left(\frac{N-n}{n}\right)^2 \sum\nolimits_s V(Y_k) + \sum\nolimits_{U-s} V(Y_k) = N\left(\frac{N}{n} - 1\right)\sigma^2 \equiv V_M\left(\widehat{Y}\right).$$

See Bolfarine and Zacks (1992) or Valliant et al. (2000) for proof of BLUP properties and more sophisticated results that consider the more general case of linear models of auxiliary variables. An unbiased estimator of the prediction variance uses an unbiased estimator of the super-population dispersion parameter, σ^2,

$$\widehat{\sigma}^2 = \frac{\sum_s (y_k - \widehat{\mu})^2}{n-1}$$

so that,

$$\widehat{V}_M\left(\widehat{Y}\right) = N\left(\frac{N}{n} - 1\right)\widehat{\sigma}^2.$$

Now, consider a super-population with $M = \sum_{U_1} M_i$ elements and $i = 1, \ldots, N$ where $E_M(Y_k) = \mu$ and $V_M(Y_k) = \sigma^2$, but the only information we have is $\overline{Y}_{s_1}, \ldots, \overline{Y}_{s_n}$ and $m_1, \ldots, m_n$ where $\overline{Y}_{s_i} = \sum_{s_i} Y_{ik}/m_i$. Letting $m = \sum_{s_1} m_i$, the BLUP of the population total is,

$$\widehat{Y} = M\widehat{\mu} = M\frac{\sum_{s_1} m_i \overline{Y}_i}{m}$$

and the variance is

$$V_M\left(\widehat{Y}\right) = M\left(\frac{M}{m} - 1\right)\sigma^2. \tag{A.43}$$

The BLUP for the total of the subpopulation U_i is

$$\widehat{Y}_i = m_i\overline{Y}_{si} + \sum\nolimits_{r_i} \widehat{Y}_{ik} = Y_{s_i} + (M_i - m_i)\widehat{\mu} \tag{A.44}$$

where $r_i = \{U_i : k \notin s_i\}$. The prediction error variance is

$$\begin{aligned} V_M\left[(M_i - m_i)\widehat{\mu} - \sum\nolimits_{r_i} Y_k\right] =& V_M\left[\frac{M_i - m_i}{m}\sum\nolimits_{s} Y_k - \sum\nolimits_{r_i} Y_k\right] \\ =& \left(\frac{M_i - m_i}{m} + 1\right)(M_i - m_i)\,\sigma^2 \end{aligned} \tag{A.45}$$

If we use the squared differences of the subpopulation means we obtain,

$$\begin{aligned} E_M\left[\frac{\sum_{s_1}\left(\overline{Y}_i - \overline{\overline{Y}}\right)^2}{n-1}\right] =& \frac{1}{n-1}\left[\sum\nolimits_{s_1} E_M\left(\overline{Y}_i^2\right) - \frac{E_M\left(\sum\sum_{s_1}\overline{Y}_i\overline{Y}_j\right)}{n}\right] \\ =& \frac{1}{n-1}\left[\sum\nolimits_{s_1}\left(\frac{\sigma^2}{m_i} + \mu^2\right) - \frac{\sum_{s_1}\left(\frac{\sigma^2}{m_i} + \mu^2\right) + \sum\sum_{s_1}\limits_{i\neq j}\mu^2}{n}\right] \\ =& \frac{\sigma^2}{n}\sum\nolimits_{s_1}\frac{1}{m_i} \end{aligned}$$

and thus an unbiased estimator of the dispersion parameter is

$$\widehat{\sigma}^2 = \frac{n}{\sum_{s_1} \frac{1}{m_i}} \frac{\sum_{s_1} \left(\overline{Y}_i - \overline{\overline{Y}} \right)^2}{n-1}$$

and an unbiased estimator of the prediction variance for the population is

$$\widehat{V}_M \left(\widehat{Y} \right) = M \left(\frac{M}{m} - 1 \right) \widehat{\sigma}^2.$$

Likewise, if prediction is required for any number of subpopulations that are members of a domain d, then the unbiased predictor is

$$\widehat{Y}_d = \sum\nolimits_{s_d} m_i \overline{Y}_{si} + \sum\nolimits_{s_d} (M_i - m_i) \widehat{\mu}.$$

The prediction error variance of the estimator is

$$V \left(\widehat{Y}_d \right) = (M_d - m_d) \left(M_d + \sum\nolimits_{s_{d'}} m_i \right) \frac{\sigma^2}{\sum_{s_1} m_i}$$

where $s_{d'}$ is the sample elements that do not occur in the domain, $M_d = \sum_{s_d} M_i$ and $m_d = \sum_{s_d} m_i$. The model-unbiased variance estimator is

$$\widehat{V}_M \left(\widehat{Y}_d \right) = (M_d - m_d) \left(M_d + \sum\nolimits_{s_{d'}} m_i \right) \frac{\widehat{\sigma}^2}{\sum_{s_1} m_i},$$

Now consider the possibility that the total number of elements within each subpopulation and/or the population is estimated rather than known. The estimator of the population total,

$$\widehat{Y}_b = \widehat{M} \widehat{\mu}, \tag{A.46}$$

is unbiased if $\widehat{M}$ and $\widehat{\mu}$ are independent random variables and the prediction error variance in this case is

$$\begin{aligned}V\left(\widehat{Y}_b\right) =&V\left[\left(\widehat{M}-m\right)\widehat{\mu}-\sum\nolimits_{U-s}Y_k\right]=V\left(\widehat{M}\widehat{\mu}-m\widehat{\mu}-\sum\nolimits_{U-s}Y_k\right)\\ =&V_M\left(M\widehat{\mu}-m\widehat{\mu}-\sum\nolimits_{U-s}Y_k\right)+E_M\left[\widehat{\mu}^2V\left(\widehat{M}\right)\right]\\ =&M(M-m)\frac{\sigma^2}{m}+V\left(\widehat{M}\right)\left(\mu^2+\frac{\sigma^2}{m}\right)=V_M\left(\widehat{Y}\right)+V\left(\widehat{M}\right)\left(\mu^2+\frac{\sigma^2}{m}\right)\end{aligned}$$

where $m=\sum_{s_1}m_i$ and $\widehat{Y}$ is defined in Eq. A.43. A consistent estimator of the variance of the total weight is

$$\begin{aligned}\widehat{V}_1\left(\widehat{Y}_b\right) =&\frac{\widehat{M}(\widehat{M}-m)\widehat{\sigma}^2}{m}+\widehat{V}\left(\widehat{M}\right)\widehat{\mu}^2\\ =&\frac{\widehat{M}(\widehat{M}-m)}{m}\frac{n}{\sum_{s_1}\frac{1}{m_i}}\frac{\sum_{s_1}\left(\overline{Y}_i-\overline{\overline{Y}}\right)^2}{n-1}+\widehat{V}\left(\widehat{M}\right)\left(\frac{\sum_{s_1}\sum_{s_i}Y_{ik}}{m}\right)^2\end{aligned}\tag{A.47}$$

and an unbiased estimator of the variance is

$$\widehat{V}_2\left(\widehat{Y}_b\right)=\left[\frac{\widehat{M}^2-\widehat{V}\left(\widehat{M}\right)}{m}-\widehat{M}\right]\widehat{\sigma}^2+\widehat{V}\left(\widehat{M}\right)\widehat{\mu}^2\tag{A.48}$$

because

$$E\left[\widehat{M}^2-\widehat{V}\left(\widehat{M}\right)\right]=M^2.$$

An estimator of the total for the ith subpopulation is $\widehat{Y}_{ib}=\widehat{M}_i\widehat{\mu}$. In the case of independence of the estimators $\widehat{M}_i$ and $\widehat{\mu}$, the variance for the ith subpopulation is

$$\begin{aligned}V\left(\widehat{Y}_{bi}\right) =&V\left[\left(\widehat{M}_i-m_i\right)\widehat{\mu}-\sum\nolimits_{U_i-s_i}Y_{ik}\right]=V\left(\widehat{M}_i\widehat{\mu}-m_i\widehat{\mu}-\sum\nolimits_{U_i-s_i}Y_{ik}\right)\\ =&V_M\left(M_i\widehat{\mu}-m_i\widehat{\mu}-\sum\nolimits_{U_i-s_i}Y_{ik}\right)+E_M\left[\widehat{\mu}^2V\left(\widehat{M}_i\right)\right]\\ =&V_M\left(\widehat{Y}_i\right)+V\left(\widehat{M}_i\right)\left(\mu^2+\frac{\sigma^2}{m}\right)\end{aligned}$$

where $V\left(\widehat{Y}_i\right)$ is defined in Eq. A.45. A consistent variance estimator obtained by substitution is

$$\widehat{V}_1\left(\widehat{Y}_{bi}\right) = \frac{\widehat{M}_i^2 + \widehat{M}_i\left(\sum_{s_1'} m_j - m_i\right) - m_i \sum_{s_1'} m_j}{m}\widehat{\sigma}^2 + \widehat{V}\left(\widehat{M}_i\right)\widehat{\mu}^2$$

and again an unbiased variance estimator is

$$\widehat{V}_2\left(\widehat{Y}_{bi}\right) = \frac{\widehat{M}_i^2 - \widehat{V}\left(\widehat{M}_i\right) + \widehat{M}_i\left(\sum_{s_1'} m_j - m_i\right) - m_i \sum_{s_1'} m_j}{m}\widehat{\sigma}^2 + \widehat{V}\left(\widehat{M}_i\right)\widehat{\mu}^2.$$

Similarly, if estimation is required for a domain including any number of subpopulations the unbiased estimator is $\widehat{Y}_{bd} = \widehat{M}_d\widehat{\mu}$ and the variance is

$$V\left(\widehat{Y}_{bd}\right) = V_M\left(\widehat{Y}_d\right) + V\left(\widehat{M}_d\right)\left(\mu^2 + \frac{\sigma^2}{m}\right)$$

As for the entire population or subpopulation there are two different possible variance estimator for the domain. The consistent and approximately unbiased estimator is

$$\begin{aligned}\widehat{V}_1\left(\widehat{Y}_{bd}\right) = & \frac{\widehat{M}_d^2 + \widehat{M}_d\left(\sum_{s_d'} m_j - \sum_{s_d} m_i\right) - \left(\sum_{s_d} m_i\right)\left(\sum_{s_d'} m_j\right)}{m}\widehat{\sigma}^2 \\ & + \widehat{V}\left(\widehat{M}_d\right)\widehat{\mu}^2\end{aligned}$$

and the unbiased variance estimator is

$$\begin{aligned}\widehat{V}_2\left(\widehat{Y}_{bd}\right) = & \frac{\widehat{M}_d^2 - \widehat{V}\left(\widehat{M}_d\right) + \widehat{M}_d\left(\sum_{s_d'} m_j - \sum_{s_d} m_i\right) - \left(\sum_{s_d} m_i\right)\left(\sum_{s_d'} m_j\right)}{m}\widehat{\sigma}^2 \\ & + \widehat{V}\left(\widehat{M}_d\right)\widehat{\mu}^2.\end{aligned}$$

A.8.2 Simple Random Sampling of Super-populations

Consider a finite number, N, of super-populations each with M_c elements and distinct mean vectors $\mu_c \mathbf{1}_{M_c}$ and covariance matrices $\sigma_c^2 I_{M_c \times M_c}$ where $c = 1, \ldots, N$ so that each super-population model is identical to that proposed above in Section A.8.1. If we have take a simple random sample of size n super-populations and we subsequently take a sample s_c of size m_c from the cth sampled super-populations we can form an estimator of the total over all N super-populations and corresponding variance that has both design- and model-based properties. The unbiased estimator of the total is

$$\widehat{Y} = \frac{N}{n}\sum\nolimits_s \widehat{Y}_c = \frac{N}{n}\sum\nolimits_s \frac{M_c}{m_c}\sum\nolimits_{sc} Y_{ck}$$

because

$$\begin{aligned} E\left(\widehat{Y}\right) &= E_p\left[E_M\left(\widehat{Y}\middle|s\right)\right] = E_p\left[\frac{N}{n}\sum\nolimits_s E_M\left(\widehat{Y}_c\middle|s\right)\right] \\ &= E_p\left(\frac{N}{n}\sum\nolimits_s Y_c\right) = \sum\nolimits_U Y_c = Y \end{aligned}$$

where E_p denotes expectation with respect to the distribution of samples. The variance of the estimator is

$$\begin{aligned} V\left(\widehat{Y}\right) &= V_p\left[E_M\left(\widehat{Y}\middle|s\right)\right] + E_p\left[V_M\left(\widehat{Y}\middle|s\right)\right] \\ &= V_p\left(\frac{N}{n}\sum\nolimits_s Y_c\right) + \left(\frac{N}{n}\right)^2 E_p\left[\sum\nolimits_s V_M\left(Y_i\right)\right] \\ &= N\left(\frac{N}{n}-1\right)\frac{\sum_U\left(Y_c-\overline{Y}_U\right)^2}{N-1} + \frac{N}{n}\sum\nolimits_U V_M\left(\widehat{Y}_c\right) \end{aligned}$$

where $V_M\left(\widehat{Y}_i\right)$ is the prediction error variance of the model-based estimator for the cth super-population and $\overline{Y}_U = \sum_U Y_c/N$. An unbiased variance estimator is

$$\widehat{V}\left(\widehat{Y}\right) = N\left(\frac{N}{n}-1\right)\frac{\sum_s\left(\widehat{Y}_c-\overline{\widehat{Y}}_s\right)^2}{n-1} + \frac{N}{n}\sum\nolimits_s \widehat{V}_M\left(\widehat{Y}_c\right)$$

where $\widehat{V}_M\left(\widehat{Y}_c\right)$ is the model-based estimator of the prediction error for the cth super-population and $\overline{\widehat{Y}}_s = \sum_s \widehat{Y}_c/n$. The variance estimator is unbiased because

$$
\begin{aligned}
E\left[\widehat{V}\left(\widehat{Y}\right)\right] =& \frac{N}{n-1}\left(\frac{N}{n}-1\right)E_p\left[\sum\nolimits_s E_M\left(\widehat{Y}_c^2\middle|s\right)+\frac{\sum_s E\left(\widehat{Y}_c^2\middle|s\right)+\sum\sum_{s,\,c\neq d} Y_cY_d}{n}\right]\\
&+\frac{N}{n}E_p\left\{\sum\nolimits_s E_M\left[\widehat{V}_M\left(\widehat{Y}_c\right)|s\right]\right\}\\
=&\frac{N}{n-1}\left(\frac{N}{n}-1\right)E_p\left[\frac{n-1}{n}\sum\nolimits_s V_M\left(\widehat{Y}_c\right)+\sum\nolimits_s Y_c^2 - n\overline{Y}_s\right]\\
&+\frac{N}{n}E_p\left[\sum\nolimits_s V_M\left(\widehat{Y}_c\right)\right]\\
=&N\left(\frac{N}{n}-1\right)\frac{\sum_U\left(Y_c-\overline{Y}_U\right)^2}{N-1}+\frac{N}{n}\sum\nolimits_U V_M\left(\widehat{Y}_c\right).
\end{aligned}
$$

Note that the prediction error variance and its estimator for each super-population could be those derived in Section A.8.1 above.

In the previous section we also considered the case when the total number of elements within each superpopulation is estimatated rather than known, but that the estimators, $\widehat{M_i}$ and $\widehat{\mu}_c$ were independent. If this is the case, we can form an unbiased estimator, $\widehat{Y}_b = \frac{N}{n}\sum_s \widehat{Y}_{bc}$, where $\widehat{Y}_{bc}$ is the estimator of the total within the cth super-population (Eq. A.46). The variance and variance estimator in this case are

$$
V_M\left(\widehat{Y}_{bc}\right) = N\left(\frac{N}{n}-1\right)\frac{\sum_U\left(Y_c-\overline{Y}_U\right)^2}{N-1}+\frac{N}{n}\sum\nolimits_U V_M\left(\widehat{Y}_{bc}\right)
$$

and

$$
\widehat{V}\left(\widehat{Y}\right) = N\left(\frac{N}{n}-1\right)\frac{\sum_s\left(\widehat{Y}_{bc}-\overline{\widehat{Y}}_{bs}\right)^2}{n-1}+\frac{N}{n}\sum\nolimits_s \widehat{V}_M\left(\widehat{Y}_{bc}\right)
$$

where $\overline{\widehat{Y}}_{bs} = \sum_s \widehat{Y}_{bc}/n$ and $\widehat{V}_M\left(\widehat{Y}_{bc}\right)$ may be the consistent or unbiased estimators of variance presented in Eq. A.47 and Eq. A.48,respectively.

A.8.3 Mixing Design-based and Model-based Inference

Sometimes estimators are products or other functions of multiple estimators and some of the component estimators may be design-based and others model-based. Suppose we have L strata of three-stage universes U_k. There are N_r primary sampling units in the rth stratum, N_{ri} elements in the ith primary sampling unit and N_{rik} elements in the kth secondary sampling unit. Furthermore, there is simple random sampling at the first and second stage, but model-based methods are used in the tertiary stage. There are n_r, n_{ri} and n_{rik} elements sampled at each of the respective stages. Finally, there is a SRS of m_d elements from the $M_d = \sum^L \sum^{n_r} \sum^{n_{dri}} n_{rik}$ elements ultimately sampled within the dth domain. All tertiary elements in the ith secondary sampling unit are either in or out of the domain. In this scenario the subsample size within each secondary sampling unit (m_{drik}) is random. The models within each secondary sampling unit are independent and all tertiary elements are uncorrelated, $E_m(y_{rika}) = \mu_{rik}$ and $V_m(y_{rika}) = \sigma^2_{rik}$. Also, suppose that independent design-based estimators of the total number of tertiary elements is available. The expected value of the tertiary sum is

$$E\left(\sum\nolimits_{s_{rik}} y_{rika}\right) = E_d\left[E_m\left(\sum\nolimits_{s_{rik2}} y_{rikm}|m_{drik}\right)\right] = E_d\left(m_{drik}\mu_i\right) = \frac{m_d}{M_d} n_{rik}\mu_i$$

where s_{rik2} denotes the subsample of tertiary sampling units available. An unbiased estimator of the cluster total using the independent estimator of the total number of elements in the cluster is

$$\widehat{y}_{rik} = \frac{M_d}{m_d}\frac{\widehat{N}_{rik}}{n_{rik}} \sum\nolimits_{s_{rik2}} y_{rika}.$$

The variance of the estimator is a function of the variance of three random components: the cluster total estimator, the subsample size in the cluster and the random variable associated with each subsampled element. The prediction variance of the model-based estimator of the total conditional on the within-cluster subsample size is

$$V_{m_{rik}}\left(\sum\nolimits_{s_{rik2}} y_{rik}|m_{drik}\right) = m_{drik}\left(1 - \frac{m_{drik}}{N_{rik}}\right)\sigma^2_{rik} \doteq m_{drik}\sigma^2_{rik}$$

where the approximation is appropriate when the cluster size is much larger than the subsample size. Using this approximation the unconditional variance is

$$\begin{aligned} V\left(\widehat{y}_{rik}\right) =& V_{\widehat{N}}\left\{E_d\left[E_m\left(\widehat{y}_{rik}|m_{drik},\widehat{N}_{rik}\right)\right]\right\} + E_{\widehat{N}}\left\{V_d\left[E_m\left(\widehat{y}_{rik}|m_{drik},\widehat{N}_{rik}\right)\right]\right\} \\ &+ E_{\widehat{N}}\left\{E_d\left[V_m\left(\widehat{y}_{rik}|m_{drik},\widehat{N}_{rik}\right)\right]\right\} \\ \doteq& \mu_{rik}^2 V\left(\widehat{N}_{rik}\right) + \left[N_{rik}^2 + V\left(\widehat{N}_{rik}\right)\right]\left\{\left(\frac{\mu_{rik}}{n_{rik}}\right)^2 V_d\left(\widehat{n}_{rik}\right) + \frac{\sigma_{rik}^2}{n_{rik}}\frac{M_d}{m_d}\right\} \end{aligned} \tag{A.49}$$

where

$$V\left(\widehat{n}_{rik}\right) = M_d\left(\frac{M_d}{m_d} - 1\right)\frac{M_d P_{rik}(1-P_{rik})}{M_d - 1}$$

and $P_{rik} = n_{rik}/M_d$. The estimator for Eq. A.49 is

$$\begin{aligned} \widehat{V}\left(\widehat{y}_{rik}\right) =& \left[\widehat{\mu}_{rik}^2 - \widehat{V}_m\left(\widehat{\mu}_{rik}|m_{drik}\right)\right]\widehat{V}\left(\widehat{N}_{rik}\right) \\ &+ \widehat{N}_{rik}^2\left\{\frac{\widehat{\mu}_{rik}^2 - \widehat{V}_m\left(\widehat{\mu}_{rik}|m_{drik}\right)}{n_{rik}^2} V_d\left(\widehat{n}_{rik}\right) + \frac{\widehat{\sigma}_{rik}^2 M_d}{n_{rik} m_d}\right\} \end{aligned}$$

where $\widehat{\mu}_{rik} = \sum_{s_{rik}} y_{rika}/m_{drik}$,

$$\widehat{\sigma}_{rik}^2 = \frac{\sum_{s_{rik}}\left(y_{rika} - \overline{y}_{rik}\right)^2}{m_{drik} - 1},$$

$\widehat{V}_m\left(\widehat{\mu}_{rik}|m_{drik}\right) = \widehat{\sigma}_{rik}^2/m_{drik}$ and

$$\widehat{V}_d\left(\widehat{n}_{rik}\right) = m_d\left(1 - \frac{m_d}{M_d}\right)\frac{m_d p_i(1-p_i)}{m_d - 1}.$$

From the information given, we do not need to estimate the variance of the subsample size in the ith cluster and we can use $V_d\left(\widehat{n}_{rik}\right)$ instead of the variance estimator.

Now, suppose that the total over all N_d clusters is to be estimated. The unbiased estimator is

$$\widehat{y}_{ri} = \frac{N_{ri}}{n_{ri}}\frac{M_d}{m_d}\sum_{s_{ri}}\frac{\widehat{N}_{rik}}{n_{rik}}\sum_{s_{rik}} y_{rika}$$

For this estimator there is now a fourth source of variability: sampling of clusters.

Furthermore, there is covariance of the subsample sizes within each cluster. The variance of the estimator is

$$\begin{aligned}
V(\widehat{y}_{ri}) =& V_s\left\{E_d\left[E_m\left[E_{\widehat{N}}(\widehat{y}_{ri}|m_{drik},s)\right]\right]\right\} + E_s\left\{V_d\left[E_m\left[E_{\widehat{N}}(\widehat{y}_{ri}|m_{drik},s)\right]\right]\right\}\\
&+ E_s\left\{E_d\left[V_m\left[E_{\widehat{N}}(\widehat{y}_{ri}|m_{drik},s)\right]\right]\right\} + E_s\left\{E_d\left[E_m\left[V_{\widehat{N}}(\widehat{y}_{ri}|m_{drik},s)\right]\right]\right\}\\
=& N_{ri}\left(\frac{N_{ri}}{n_{ri}}-1\right)\frac{\sum_{U_{ri}} y_{rik}^2 - \sum\sum_{U_{ri}} y_{rik}y_{ril}/N_{ri}}{N_{ri}-1}\\
&+\left(\frac{N_{ri}}{n_{ri}}\right)^2\left\{E_s\left[\sum_{s_{ri}}(N_{rik}\mu_{rik})^2\frac{V_d(\widehat{n}_{rik})}{n_{rik}^2}\right.\right.\\
&\left.+\sum_{k\neq l}\sum_{s_{ri}} N_{rik}\mu_{rik}N_{ril}\mu_{ril}\frac{Cov_d(\widehat{n}_{rik},\widehat{n}_{ril})}{n_{rik}n_{ril}}\right]\\
&\left.+\sum_{s_{ri}}\left[1+\frac{V_d(\widehat{n}_{rik})}{n_{rik}^2}\right]\mu_{rik}^2 V\left(\widehat{N}_{rik}\right)+\sum_{s_{ri}}\frac{\left[N_{rik}^2+V\left(\widehat{N}_{rik}\right)\right]M_d\sigma_{rik}^2}{n_{rik}m_d}\right\}
\end{aligned}$$

where

$$Cov_d(\widehat{n}_{rik},\widehat{n}_{ril}) = -M_d\left(\frac{M_d}{m_d}-1\right)\frac{M_dP_{rik}P_{ril}}{M_d-1}.$$

The first summand on the left hand side is unbiasedly estimated by

$$\begin{aligned}
&N_{ri}\left(\frac{N_{ri}}{n_{ri}}-1\right)\frac{\sum_{s_{ri}}\widehat{y}_{rik}^2-\sum\sum_{s_d}\widehat{y}_{rik}\widehat{y}_{ril}/n_{ri}}{n_{ri}(n_{ri}-1)}\\
&\quad+\frac{N_{ri}}{n_{ri}}\left(\frac{N_{ri}}{n_{ri}}-1\right)\left[\frac{1}{n_{ri}-1}\sum_{i\neq j}\sum_{s_d}\widehat{N}_{rik}\widehat{\mu}_{rik}\widehat{N}_{ril}\widehat{\mu}_{ril}\frac{Cov_d(\widehat{n}_{rik},\widehat{n}_{ril})}{n_{rik}n_{ril}}\right.\\
&\quad\left.-\sum_{s_{ri}}\widehat{V}(\widehat{y}_{rik})\right]
\end{aligned}$$

because

$$E\left(\sum_{s_{ri}}\widehat{y}_{rik}^2\Big|s\right)=\sum_{s_d}\left[y_{rik}^2+V(\widehat{y}_{rik})\right]$$

and

$$E\left(\sum_{k\neq l}\sum_{s_{ri}}\widehat{y}_{rik}\widehat{y}_{ril}\Bigg|s\right)=\sum_{i\neq j}\sum_{s_d}\left[y_{rik}y_{ril}+Cov(\widehat{y}_{rik},\widehat{y}_{ril})\right]$$

where

$$Cov\left(\widehat{y}_{rik},\widehat{y}_{ril}\right)=N_{rik}\mu_{rik}M_{ril}\mu_{ril}\frac{Cov_d\left(\widehat{n}_{rik},\widehat{n}_{ril}\right)}{n_{rik}n_{ril}}.$$

Conditional of the sample s, The second summand is simply

$$\left(\frac{M_d}{m_d}\right)^2\sum\nolimits_{s_{ri}}V\left(\widehat{y}_{rik}\right)+\sum_{k\neq l}\sum\nolimits_{s_{ri}}N_{rik}\mu_{rik}N_{ril}\mu_{ril}\frac{Cov_d\left(\widehat{n}_{rik},\widehat{n}_{ril}\right)}{n_{rik}n_{ril}}$$

and so is estimated by

$$\left(\frac{M_d}{m_d}\right)^2\sum\nolimits_{s_{ri}}\widehat{V}\left(\widehat{y}_{rik}\right)+\sum_{k\neq l}\sum\nolimits_{s_{ri}}\widehat{N}_{rik}\widehat{\mu}_{rik}\widehat{N}_{ril}\widehat{\mu}_{ril}\frac{Cov_d\left(\widehat{n}_{rik},\widehat{n}_{ril}\right)}{n_{rik}n_{ril}}$$

and all together the variance estimator is

$$\begin{aligned}\widehat{V}\left(\widehat{y}_{ri}\right)=&N_{ri}\left(\frac{N_{ri}}{n_{ri}}-1\right)\frac{\sum_{s_{ri}}\widehat{y}^2_{rik}-\sum\sum_{s_{ri}}\widehat{y}_{rik}\widehat{y}_{ril}/n_{ri}}{n_{ri}-1}+\frac{N_{ri}}{n_{ri}}\sum\nolimits_{s_{ri}}\widehat{V}\left(\widehat{y}_{rik}\right)\\&+\frac{N_{ri}\left(N_{ri}-1\right)}{n_{ri}\left(n_{ri}-1\right)}\sum_{k\neq l}\sum\nolimits_{s_{ri}}\widehat{N}_{rik}\widehat{\mu}_{rik}\widehat{N}_{ril}\widehat{\mu}_{ril}\frac{Cov_d\left(\widehat{n}_{rik},\widehat{n}_{ril}\right)}{n_{rik}n_{ril}}.\end{aligned}$$

It was mentioned earlier that there are multiple strata from which M_d originate. For estimation of a primary sampling unit total

$$\widehat{y}_r=\frac{N_r}{n_r}\sum\nolimits_{s_r}\widehat{y}_{ri}.$$

A.9 Asymptotic Covariance: Modified Delta Method

The delta method is useful in obtaining an approximate variance of a non-linear function of p random variables, $T_1,\ldots,T_p$, with expected values $\theta_1,\ldots,\theta_p$. However, I have not seen a presentation of the approximate covariance of two non-linear functions of the same random variables. Let the two functions be $g_1\left(\boldsymbol{T}\right)$ and $g_2\left(\boldsymbol{T}\right)$.

The first-order Taylor series expansions of each function about $\boldsymbol{\theta}$ are

$$g_1\left(\boldsymbol{t}\right)=g_1\left(\boldsymbol{\theta}\right)+\sum^p\frac{\partial g_1}{\partial t_i}\bigg|_{\boldsymbol{t}=\boldsymbol{\theta}}\left(t_i-\theta_i\right)+R$$

and

$$g_2(\boldsymbol{t}) = g_2(\boldsymbol{\theta}) + \sum^{p} \left.\frac{\partial g_2}{\partial t_i}\right|_{\boldsymbol{t}=\boldsymbol{\theta}} (t_i - \theta_i) + R$$

We approximate the variance of each function by dropping the remainder and writing

$$V(g_1) = E\left\{[g_1(\boldsymbol{T}) - g_1(\boldsymbol{\theta})]^2\right\} \approx E\left\{\left[\sum^{p} \left.\frac{\partial g_1}{\partial t_i}\right|_{\boldsymbol{t}=\boldsymbol{\theta}} (t_i - \theta_i)\right]^2\right\},$$

but we have a similar form for the approximate covariance as

$$\begin{aligned}
C(g_1, g_2) &= E\left\{[g_1(\boldsymbol{T}) - g_1(\boldsymbol{\theta})]\,[g_2(\boldsymbol{T}) - g_2(\boldsymbol{\theta})]\right\} \\
&\approx E\left\{\left[\sum \left.\frac{\partial g_1}{\partial t_i}\right|_{\boldsymbol{t}=\boldsymbol{\theta}} (t_i - \theta_i)\right]\left[\sum \left.\frac{\partial g_2}{\partial t_i}\right|_{\boldsymbol{t}=\boldsymbol{\theta}} (t_i - \theta_i)\right]\right\} \\
&= \sum \left.\frac{\partial g_1}{\partial t_i}\frac{\partial g_2}{\partial t_i}\right|_{\boldsymbol{t}=\boldsymbol{\theta}} V(T_i) + \sum_{i \neq j}\sum \left.\frac{\partial g_1}{\partial t_i}\frac{\partial g_2}{\partial t_j}\right|_{\boldsymbol{t}=\boldsymbol{\theta}} C(T_i, T_j) \\
&\equiv AC(g_1, g_2)
\end{aligned}$$

Notice that we can have functions with random variables that are not present in both functions. The terms associated with the random variables in the approximate covariance vanish.

When $g_1 = \mathbf{g}_1$ and $g_2 = \mathbf{g}_2$ are functions of parameter vectors, $\mathbf{T} = \mathbf{T}_1, \ldots, \mathbf{T}_p$, the asymptotic covariance matrix is

$$\begin{aligned}
Cov(\mathbf{g}_1, \mathbf{g}_2) =& E\left\{[\mathbf{g}_1(\mathbf{T}) - \mathbf{g}_1(\boldsymbol{\Theta})]\,[\mathbf{g}_2(\mathbf{T}) - \mathbf{g}_2(\boldsymbol{\Theta})]^T\right\} \\
\approx& E\left\{\left[\sum_{i=1}^{p} \left.\frac{\partial \mathbf{g}_1}{\partial \mathbf{t}_i}\right|_{\mathbf{t}=\boldsymbol{\Theta}} (\mathbf{t}_i - \boldsymbol{\Theta}_i)\right]\left[\sum_{i=1}^{p} \left.\frac{\partial \mathbf{g}_2}{\partial \mathbf{t}_i}\right|_{\mathbf{t}=\boldsymbol{\Theta}} (\mathbf{t}_i - \boldsymbol{\Theta}_i)\right]^T\right\} \\
=& \left.\left[\sum_{i=1}^{p}\sum_{j=1}^{p} \left(\frac{\partial \mathbf{g}_1}{\partial \mathbf{t}_i}\right) C(\mathbf{T}_i, \mathbf{T}_j) \left(\frac{\partial \mathbf{g}_2}{\partial \mathbf{t}_j}\right)^T\right]\right|_{\mathbf{t}=\boldsymbol{\Theta}} \\
\equiv& AC(\mathbf{g}_1, \mathbf{g}_2)\,.
\end{aligned}$$

Covariance estimates are obtained by substituting parameters (and parameter variances and covariances) with corresponding estimates.

VITA

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