Real Analysis – MATH2400 Full Lecture Notes

Contents

1 Introduction

1.1 The Real Numbers

1.1.1 Defining the Real Numbers

The *Real Numbers*, denoted $\mathbb{R}$, are numbers such that for all Cauchy sequences (which will be defined formally in Section 2.4), $(x_n)_n^\infty$, for $x_n \in \mathbb{Q}$ with $\lim_{n\to\infty} x_n = q^*$, where $q^* \in \overline{\mathbb{Q}}$, then $\mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$. That is, the Real Numbers are defined as the set of rational numbers and the limit-points of sequences of rational numbers which approach irrational numbers.

Remark. So we can think of the real numbers as the rational numbers, plus sequences of rational numbers which fill in the 'gaps' of the rationals that would be given by transcendental numbers like π or e or other numbers like $\sqrt{2}$, $\sqrt{17}$ and so on.

Example. Consider $\sqrt{2}$, which can be defined as the limit point of the following recursive sequence of rational numbers:

$$x_0 = 1, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \quad \Longrightarrow \quad \lim_{n\to\infty} x_{n+1} = \sqrt{2}$$

In general, we have the following order of sets,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

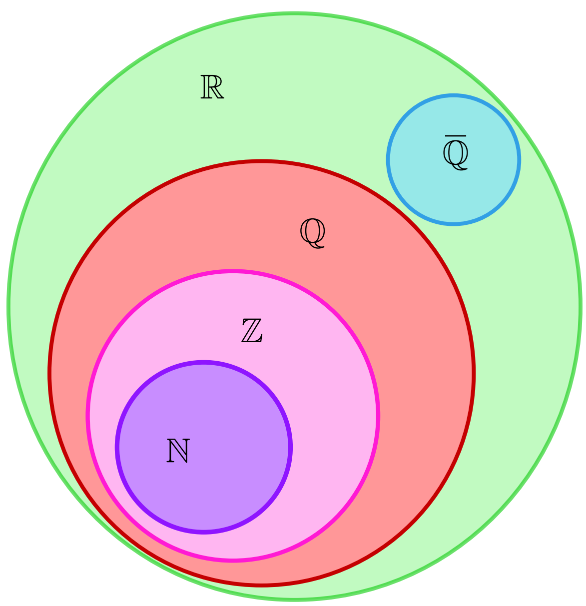

Figure 1: The Set Hierarchy

Remark. The real numbers consitute what we call a ***complete, ordered*** *field.*

1.1.2 The Properties of the Real Numbers

A **field**, $\mathcal{F}$ is a set of numbers A, equipped with the operations $+$ and $\cdot$ (which we know as addition and multiplication) which obey the following properties:

1. **Commutativity**: $\forall x,\ y \in \mathbb{R}$, $x + y = y + x$, and $x \cdot y = y \cdot x$.
2. **Associativity**: $\forall x,\ y,\ z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$, and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
3. **Distributivity** of multiplication over addition: $\forall x,\ y,\ y \in \mathbb{R}$, $(x + y) \cdot z = x \cdot z + y \cdot z$.
4. **Additive Identity**: There exists an element $0 \in \mathbb{R}$, called the *additive identity*, as it satisfies $x + 0 = x$, $\forall x \in \mathbb{R}$.
5. **Multiplicative Identity**: There exists an element $1 \in \mathbb{R}$, called the *multiplicative identity*, as it satisfies $x \cdot 1 = x$, $\forall x \in \mathbb{R}$.
6. **Additive Inverse**: Every real number x admits an *additive inverse*, $-x$, with $x+(-x) = 0$. Extension of this idea defines what we know as subtraction.
7. **Multiplicative Inverse**: Every non-zero real number x admits an *multiplicative inverse*, $\frac{1}{x}$, with $x \cdot \left(\frac{1}{x}\right) = 1$. Extension of this idea defines what we know as division.

A field $\mathcal{F}$ is said to be *totally ordered* if we can define the idea of $x \leq y$ and $x \geq y$. This ordering admits two important properties,

1. For any two real numbers x and y, exactly one of $x < y$, $x = y$ or $x > y$ is true, and
2. if $x \leq y$ and $y \leq z$, then $x \leq z$.

It is also true that this ordering (or what we will call an inequality from now on) allows multiplication and addition, following all of the previous additive and multiplicative rules of a field.

From all of these rules we also derive some other simple properties for an ordered field, being that $0 \cdot x = 0$ for any $x \in \mathbb{R}$, $0 < 1$, and $x^2 \geq 0$.

Another important operation we have is that of the *absolute value*, which provides an idea of distance, defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

> **Remark.** This is also related to our idea of the *principal square root*, as $|x| = \sqrt{x^2}$, which is a property related to the *completeness* of $\mathbb{R}$. This idea of *completeness* is the distinguishing property of $\mathbb{R}$ compared to $\mathbb{Q}$.

A set X is *complete* if every finite subset of X, Y (so $Y \subseteq X$) which has an upper bound, admits a least upper bound as well. Any element of a subset of real numbers $y \in Y$ can be said to satisfy

$$y \leq u, \; \forall y \in Y \subseteq \mathbb{R}, \; u \in \mathbb{R}$$

where u is what we call an *upper bound*, which exists as a consequence of the *ordering* of Y.

However, we can say this is true of the rational numbers too, so what distinguishes the real numbers from the rationals? Well, we can extend this idea to say that in the set of all possible upper bounds U with, there exists an element of this set u_0, $u_0 \in U$, such that $\forall u \in U$,

$$u_0 \leq u$$

which we call the *least upper bound* of Y, or the supremum of Y, denoted

$$u_0 = \sup(Y).$$

A similar idea is that of the *greatest lower bound* of a set Y which is bounded below, which satisfies the property: For all possible lower bounds a, there exists an a_0, called the *greatest lower bound* or infimium of Y (denoted $\inf(Y)$) if $\forall a$,

$$a \leq a_0 = \inf(Y)$$

Diagram Here

If a set Y is bounded above and below, and $\sup(Y) \in Y$, then $\sup(Y) = \max(Y)$ and $\inf(Y) = \min(Y)$.

1.2 Intervals

We write a *closed* interval as

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\},$$

and an *open* intervals as,

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

So a closed interval contains its endpoints (and by extension it's minimal and maximal elements), while an open interval does not contain its endpoints.

Definition 1 (Least Upper Bound Property)**.** The *least upper bound property* of $\mathbb{R}$ (or completeness property):

For all non-empty, finite subsets of $\mathbb{R}$, $A \subseteq \mathbb{R}$, $\sup(A)$ exists, and $\sup(A) \in \mathbb{R}$.

Equivalently,

Definition 2 (Greatest Lower Bound Property)**.** The *greatest lower bound property* of $\mathbb{R}$:

For all non-empty, finite subsets of $\mathbb{R}$, $A \subseteq \mathbb{R}$, $\inf(A)$ exists, and $\inf(A) \in \mathbb{R}$.

1.3 The Absolute Value

The Absolute Value, which we defined earlier as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

satisfies the following properties:

- $|x| \geq 0$,
- $|x| = 0 \iff x = 0$,
- $|-x| = |x|$,
- $|xy| = |x|\,|y|$,
- $|x|^2 = (\sqrt{x^2})^2 = x^2$,
- $|x| \leq |y| \iff -y \leq x \leq y$,
- $-|x| \leq x \leq |x|$.

Proposition 1 (The Triangle Inequality)**.**

$$|x+y| \leq |x| + |y|$$

Proof. Let x, $y \in \mathbb{R}$. Then we know that

$$-|x| \leq x \leq |x|, \tag{1}$$

and,

$$-|y| \leq y \leq |y|. \tag{2}$$

Adding (1) and (2),

$$-(|x| + |y|) \quad \leq \quad x + y \quad \leq \quad |x| + |y|.$$

Then since $-a \leq b \leq a$ implies that $|b| \leq a$,

$$|x+y| \leq |x| + |y|.$$

□

Proposition 2 (The Reverse Triangle Inequality)**.**

$$||x| - |y|| \leq |x - y|.$$

Proof. **Just copy and paste from Assignment 1 later.** □

Corollary (The Generalised Triangle Inequality)**.**

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Proof. We define the proposition $P(n) : |x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$.

So by induction we first consider $n = 1$, where it is trivial to assess $P(1) : |x_1| \leq |x_1|$ as true, where it adopting the equality.

Suppose that $P(n)$ is true for $n > 1$. Then we shall show that $P(n) \implies P(n+1)$. Then,

$$|(x_1 + x_2 + \cdots + x_n) + x_{n+1}| \leq |x_1 + x_2 + \cdots + x_n| + |x_{n+1}|$$

but by assumption that $P(n) : |x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$ holds,

$$\leq |x_1| + |x_2| + \ldots |x_n| + |x_{n+1}| \iff P(n+1).$$

Hence $P(n) \implies P(n+1)$, and by induction,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|,$$

holds for $n \geq 1$. □

2 Sequences

2.1 Definitions

Definition 3 (Sequences)**.** A *sequence* is a function,

$$x : \mathbb{N} \to \mathbb{R}, \quad x_n = f(n),$$

so that $(x_n)_{n=1}^{\infty} = \{x_1, x_2, \dots\}$.

Example. $\left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty}$, looks like $-1, \frac{1}{2}, -\frac{1}{3}, \dots$.

Definition 4 (Convergence of a Sequence)**.** A sequence $(x_n)_{n=1}^{\infty}$ is said to converge to a limit L, if $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $\forall n > N$,

$$|x_n - L| < \varepsilon,$$

and we write this as $\lim\limits_{n\to\infty} x_n = L$.

What this means, is that if a sequence converges to a value L, we can always find some 'stopping point' for the sequence (which is N), where we can guarantee that all of the terms in the sequence, x_n after this point ($n > N$) is some non-zero distance (ε) from L. If we cannot *always* find a 'stopping point' that shows that x_n is getting closer to L, then it does not converge to L and we say that x_n is divergent.

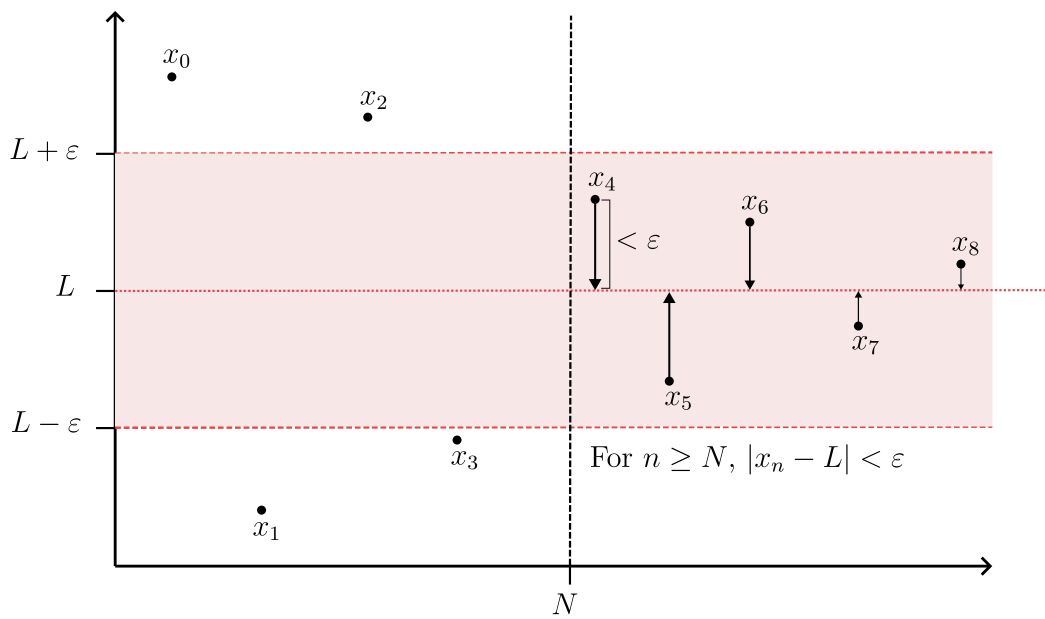

Figure 2: Limit of a sequence

Proposition 3.

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

Proof. Set $\varepsilon > 0$. Then if we want to show that $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$, then we want

$$\begin{aligned} |x_n - L| &< \varepsilon \\ \left|\frac{1}{n}\right| &= \frac{1}{n} \\ &< \varepsilon \end{aligned}$$

Take $N = \lceil \frac{1}{\varepsilon} \rceil$, then $\forall n > N$,

$$\begin{aligned} \frac{1}{n} \le \frac{1}{N} &< \varepsilon \\ \frac{1}{\varepsilon} &< N \\ &= \left\lceil \frac{1}{\varepsilon} \right\rceil \end{aligned}$$

$\square$

Proposition 4. $\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

Proof. Suppose by contradiction that there exists a limit L, so that $\lim_{n\to\infty}(-1)^n = L$. Then for all $\varepsilon > 0$ there would exist $N \in |N$ such that $\forall n > N$,

$$|(-1)^n - L| < \varepsilon.$$

Choose $\varepsilon = \frac{1}{2}$ and assume that there exists some N such that $|(-1)^n - L| < \varepsilon$ holds $\forall n > N$.

Consider n is even, then $(-1)^n = 1$, and for $n > N_1$

$$|(-1)^n - L| = |1 - L| < \frac{1}{2}. \tag{3}$$

Consider n is odd, then $(-1)^n = -1$, and for $n > N_1$

$$|(-1)^n - L| = |-1 - L| < \frac{1}{2}. \tag{4}$$

Now, note that we can write 2 as, $2 = |(1 - L) - (-1 - L)|$. Then by the triangle inequality,

$$\begin{aligned} 2 &= |(1 - L) - (-1 - L)| \\ &< |1 - L| + |-1 - L| \end{aligned}$$

take $N = \max(N_1, N_2)$ and $n > N$, then from (3) and (4),

$$
\begin{aligned}
&< \frac{1}{2} + \frac{1}{2} \\
2 &< 1.
\end{aligned}
$$

This is a contradiction, and hence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent. □

2.2 Properties of Sequences

Definition 5 (Boundedness)**.** A sequence $(x_n)_{n=1}^{\infty}$ is *bounded*, if there exists $M \in \mathbb{R}$, such that $\forall n \in \mathbb{N}$

$$
|x_n| \leq M. \tag{5}
$$

Proposition 5 (Convergence $\implies$ Boundedness)**.** A convergent sequence is bounded.

Proof. Suppose $\lim_{n\to\infty} x_n = L$.

Choose $\varepsilon = 1$, then there exists $N \in \mathbb{N}$, such that for $n > N$, $|x_n - L| < 1$. Suppose that $n > N$,

$$
\begin{aligned}
|x_n| &= |(x_n - L) + L|\,, \\
&\leq |x_n - L| + |L|\,, \text{ (by the triangle inequality)}, \\
&< 1 + |L|\,. \qquad (6)
\end{aligned}
$$

We know that $|x_n| < 1 + |L|$ for any $n > N$, but we don't know about the terms for $1 \leq n \leq N$. So, let

$$
M = \max\{\underbrace{|x_1|\,, |x_2|\,, \ldots, |x_N|}_{\text{bound terms } 1 \leq n \leq N}, \underbrace{1 + |L|}_{\text{from (6)}}\},
$$

then for all $n \geq 1$,

$$
|x_n| \leq M.
$$

□

Note. While Convergence does imply Boundedness, Boundedness **does not** imply Convergence. Consider $\{(-1)^n\}_{n=1}^{\infty}$, which we showed is divergent in Proposition 4. However, $|(-1)^n| \leq 1$.

Proposition 6. A convergent sequence admits a unique limit.

Proof. Suppose that $\{x_n\}_{n=1}^{\infty}$ has two seperate limits L_1 and L_2. Then for $\varepsilon > 0$, we have that $\lim\limits_{n\to\infty} x_n = L_1$, which implies that there exists $N_1 \in \mathbb{N}$, such that for all $n \geq N_1$,

$$
|x_n - L_1| < \frac{\varepsilon}{2}.
$$

Similarly, $\lim_{n\to\infty} x_n = L_2$ implies that there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$|x_n - L_2| < \frac{\varepsilon}{2}.$$

Take $N = \max\{N_1, N_2\}$, then for all $n \geq N$,

$$\begin{aligned} |L_2 - L_1| &= |L_2 - x_n + x_n - L_1| \\ &\leq |x_n - L_1| + |L_2 - x_n| \\ &= |x_n - L_1| + |x_n - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ |L_2 - L_1| &< \varepsilon. \end{aligned}$$

Remember that if $0 \leq |L_2 - L_1| < \varepsilon$ holds for **every** $\varepsilon > 0$, then for $|L_2 - L_1| < \varepsilon$ and $0 \leq |L_2 - L_1|$ to hold, $|L_2 - L_1| = 0$, which implies that $L_1 = L_2$. Hence, every convergent sequence has a singular unique limit. □

Proposition 7. If $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} y_n = M$, then $\lim_{n\to\infty} (x_n + y_n) = L + M$.

Proof. Let $\varepsilon > 0$, since $\{x_n\}_{n=1}^{\infty}$ converges to L, there exists an $N_1 \in \mathbb{N}$, such that for all $n \geq N$,

$$|x_n - L| < \frac{\varepsilon}{2}.$$

Similarly, since $\{y_n\}_{n=1}^{\infty}$ converges to M, there exists an $N_2 \in \mathbb{N}$, such that for all $n \geq N_2$,

$$|y_n - M| < \frac{\varepsilon}{2}.$$

Take $N = \max\{N_1, N_2\}$, then for all $n \geq N$,

$$\begin{aligned} |x_n + y_n - (L + M)| &= |(x_n - L) + (y_n - M)| \\ &\leq |x_n - L| + |y_n - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

Hence, $\lim_{n\to\infty} (x_n + y_n) = L + M$. □

Theorem 1 (The Squeeze Theorem)**.** Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be sequences such that $\forall n \in \mathbb{N}$,

$$a_n \leq b_n \leq c_n.$$

Suppose that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Then $\lim_{n\to\infty} b_n = L$.

Proof. Take $\varepsilon > 0$. Then by convergence of a_n, there exists an $N_1 \in \mathbb{N}$, such that for all $n \geq N_1$,

$$|a_n - L| < \varepsilon \iff L - \varepsilon < a_n.$$

Similarly, by convergence of c_n to L, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$|c_n - L| < \varepsilon \iff c_n < L + \varepsilon.$$

Take $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, by boundedness of b_n by a_n and c_n,

$$\begin{aligned} a_n \leq b_n \leq c_n &\implies L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon \\ &\implies L - \varepsilon < b_n < L + \varepsilon \\ &\implies |b_n - L| < \varepsilon. \end{aligned}$$

Hence there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|b_n - L| < \varepsilon$, and consequently,

$$\lim_{n\to\infty} b_n = L.$$

□

Proposition 8. If $\{x_n\}_{n=1}^{\infty}$ converges then $\{|x_n|\}_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} |x_n| = \left|\lim_{n\to\infty} x_n\right|. \tag{7}$$

Proof. Reverse triangle inequality. Will complete later □

Definition 6 (Monotonicity)**.** A sequence is *monotone increasing* if $\forall n \in N$, $x_n \leq x_{n+1}$. A sequence is *monotone decreasing* if $\forall n \in \mathbb{N}$, $x_{n+1} \leq x_n$.

Theorem 2 (Monotone Convergence Theorem)**.** A monotone sequence is bounded if and only if it is convergent. Furthermore,

- If the sequence is monotone increasing,

  $$\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} x_n$$

  ,
- or if the sequence is monotone decreasing,

  $$\lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n.$$

Proof. Suppose WLOG that x_n is monotone increasing. Assume that it is bounded and set $L = \sup\limits_{n\in\mathbb{N}} x_n$.

For $\varepsilon > 0$, there exists $x_0 \in \{x_n : n \in \mathbb{N}\}$, such that

$$L - x_0 < \varepsilon.$$

Hence there exists an $N \in \mathbb{N}$, such that for all $n \geq N$,

$$L - \varepsilon < x_N \leq x_{N+1} \leq x_{N+2} \leq \dots$$

So for all $n \geq N$,

$$L - \varepsilon < x_n$$
$$L - x_n < \varepsilon.$$

By definition $|x_n| \leq L$, $\forall n \in \mathbb{N}$, so

$$|x_n - L| = L - x_n < \varepsilon.$$

Hence $\lim_{n\to\infty} x_n = L$. The proof for monotone decreasing sequences is analogous. The reverse direction (convergence implies boundedness) was also proved in Proposition 2.7. □

2.3 Subsequences

Definition 7 (Subsequences)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and $n_1 < n_2 < n_3 < \dots < n_i \in \mathbb{N}$. Then the sequence $\{x_{n_i}\}_{i=1}^{\infty}$ is called a subsequence of x_n.

Example. $\{(-1)^n\}_{n=1}^{\infty}$ could have the subsequences, $\{x_{n_i}\}_{i=1}^{\infty}$ for $n_i = 2i$, $x_{n_i} = 1$, or $\{x_{n_i}\}_{i=1}^{\infty}$ for $n_i = 2i - 1$, so $x_{n_i} = -1$.

Proposition 9 (Sequence Subsequence Limit Equivalence)**.** If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence then every subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ is also convergent with

$$\lim_{n\to\infty} x_n = \lim_{i\to\infty} x_{n_i}.$$

Proof. By induction, (proof is complicated try it later). □

Lemma 1. Every sequence $\{x_n\}_{n=1}^{\infty}$ has a monotone subsequence.

Proof. Take $x_1, x_2, \dots$, call an index n a *peak index* if $x_n \geq x_k$ $\forall k \geq n$, and we call x_n a peak. Now there are two cases,

1. Either there are *infinitely* many peaks, or
2. a *finite* number of peaks.

Case 1 - Infinitely many peaks: Suppose n_1 is the first peak index, then n_2 is the next peak index and so on, so that $n_1 < n_2 < \dots$.

Then by definition, $x_{n_1} \geq x_{n_2} \geq \dots$. Hence we have constructed a monotone decreasing subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ in terms of peaks.

Case 2 - Finitely many peaks: Let N be the last peak index, so $\forall n > N$, x_n is not a peak. Take $n_1 = N + 1$, then there exists an $n_2 > n_1$ with $x_{n_2} > x_{n_1}$. Similarly, there exists an $n_3 > n_2$ with $x_{n_3} > x_{n_2}$. We can continue this recursive process continuously to construct a monotone increasing subsequence $\{x_{n_i}\}_{i=1}^{\infty}$. □

Theorem 3 (Bolzano-Weierstrass Theorem). Every bounded sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$.

Proof. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. Then by Lemma 1, $\{x_n\}_{n=1}^{\infty}$ has a monotone subsequence $\{x_{n_i}\}_{i=1}^{\infty}$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, $\{x_{n_i}\}_{i=1}^{\infty}$ must also be bounded. $\{x_{n_i}\}_{i=1}^{\infty}$ is both bounded and monotone , so by the Monotone Convergence Theorem, $\{x_{n_i}\}_{i=1}^{\infty}$ is convergent. Thus all bounded sequences $\{x_n\}_{n=1}^{\infty}$ have a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$. □

2.4 Cauchy Sequences

Definition 8 (Cauchy Sequences). A sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence (or just called Cauchy), if $\forall \varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $k, \ell \geq N$,

$$|x_k - x_\ell| < \varepsilon.$$

Remark. Effectively this just means that we need terms to become aritrarily closer to each other as we get further along in the sequence.

Work on a figure for this one later.

Example. We can show that $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is a Cauchy Sequence: Set $\varepsilon > 0$, and $N = \left\lceil \frac{2}{\varepsilon} \right\rceil > \frac{2}{\varepsilon}$. Then for all $k, \ell \geq N$, we have

$$\begin{aligned} \left|\frac{1}{k} - \frac{1}{\ell}\right| &\leq \left|\frac{1}{k}\right| + \left|\frac{1}{\ell}\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Example. The sequence $\{(-1)^n\}_{n=1}^{\infty}$ is not a Cauchy Sequence: Let $k \geq N$ be even, with $\ell = k + 1$ (consecutive odd number after k). Then

$$\begin{aligned} \left|(-1)^k + (-1)^l\right| &= |1 + (-1)| \\ &= 2 \geq \varepsilon, \ \forall \varepsilon \leq 2. \end{aligned}$$

Hence $\{(-1)^n\}_{n=1}^{\infty}$ is not Cauchy.

Proposition 10. If a sequence is Cauchy then it is bounded.

Proof. Assume that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence. Let $\varepsilon > 0$ (in this case we take $\varepsilon = 1$). Take $N \in \mathbb{N}$ such that $\forall k, \ell \geq N$,

$$|x_k - x_\ell| < 1. \tag{8}$$

By the Reverse Triangle Inequality,

$$\begin{aligned} |x_k - x_\ell| \leq ||x_k| - |x_\ell|| \leq |x_k - x_\ell| < 1, \\ \implies |x_k| - |x_\ell| < 1, \\ |x_k| < 1 + |x_\ell| . \end{aligned}$$

Let $\ell = N$. Then we see that $\forall k \geq N$,

$$|x_k| < 1 + |x_\ell| \iff |x_k| < \varepsilon + |x_N| . \tag{9}$$

Set

$$M = \max\{\underbrace{|x_1|, |x_2|, \ldots, |x_{N-1}|}_{\text{bound terms } 1 \leq n < N}, \underbrace{1 + |x_N|}_{\text{by (9)}}\}.$$

Thus $|x_n| \leq M$, $\forall n \in \mathbb{N}$. ☐

Theorem 4 (Convergent Sequences are Cauchy)**.** A sequence is Cauchy if and only if it converges.

Remark. That is, every Cauchy sequence is convergent, and every convergent sequence is Cauchy.

Proof. First proving that Convergent $\implies$ Cauchy: Take $\varepsilon > 0$, and suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a limit L.

Then there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|x_n - L| < \frac{\varepsilon}{2}.$$

Then for all $k, \ell \geq N$,

$$\begin{aligned} |x_k - x_\ell| &= |x_k - L + L - x_\ell| \\ &\leq |x_k - L| + |L - x_\ell| \text{ (by } \Delta\text{–Ineq)} \\ &= |x_k - L| + |x_\ell - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence $\{x_n\}_{n=1}^{\infty}$ is also a Cauchy Sequence.

Now proving that Cauchy $\implies$ Convergence: Assume that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence. Then by Proposition 10 it is also Bounded. By Bolzano-Weierstrass, every bounded sequence has a convergent subsequence, $\{x_{n_i}\}_{i=1}^{\infty}$. Let the limit of this subsequence be L.

Note. Bolzano-Weierstrass implies that every bounded sequence has a convergent subsequence. But, this means also that if a subsequence has a limit L, then the original sequence must also adopt the limit L if it converges. So we just need to show that the original sequence also converges to L using the convergent subsequence.

Take $\varepsilon > 0$. Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence, there exists an $N_1 \in \mathbb{N}$, so that $\forall k, \ell \geq N_1$,

$$|x_k - x_\ell| < \frac{\varepsilon}{2}. \tag{10}$$

By convergence of $\{x_{n_i}\}_{i=1}^{\infty}$, there exists an $N_2 \in \mathbb{N}$ so that $\forall i > N_2$,

$$|x_{n_i} - L| < \frac{\varepsilon}{2}. \tag{11}$$

Take $N = \max\{N_1, N_2\}$. Now suppose that $i > N$ (so we can use the term $x_{n_{N+1}}$, while knowing that $n_i \geq N$). Then $\forall n \geq N$,

$$\begin{aligned} |x_n - L| &= \left|x_n - x_{n_{N+1}} + x_{n_{N+1}} - L\right| \\ &\leq \left|x_n - x_{n_{N+1}}\right| + \left|x_{n_{N+1}} - L\right|, \text{ (by } \Delta\text{–Ineq)} \\ &< \underbrace{\frac{\varepsilon}{2}}_{\text{from (10)}} + \underbrace{\frac{\varepsilon}{2}}_{\text{from (11)}} = \varepsilon. \end{aligned}$$

Hence, Cauchy sequences are convergent.

Thus, all convergent sequences are Cauchy sequences, and all Cauchy sequences are convergent. □

2.5 Upper and Lower Limits

Definition 9. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. Define $a_n = \sup\{x_k : k \geq n\}$, and $b_n = \inf\{x_k : k \geq n\}$. Suppose the limits exist, we define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n, \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n.$$

Add figure here

Proposition 11. Let $\{x_n\}_{n=1}^{\infty}$ be bounded, then $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ exist.

Proof. The proof for lim sup:

$$a_{n+1} = \sup\{x_k : k \geq n+1\} \leq \sup\{x_k : k \geq n\} = a_n,$$

that is, $\{x_k : k \geq n+1\} \subseteq \{x_k : k \geq n\}$.

Since $a_{n+1} \leq a_n$, a_n is monotone decreasing and bounded.

So, by the monotone convergence theorem, $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$ exists. The proof is analogous for $\liminf x_n$. $\square$

3 Series

3.1 Basic Definitions

Definition 10 (Series)**.** Consider an infinite, ordered sequence $\{x_n\}_{n=1}^{\infty} = (x_1, x_2, \dots)$, which can consist of numbers, functions, matrices, vectors and so on. Then we call the sum of all these objects, a *series*, written as:

$$\sum_{n=1}^{\infty} x_n \iff x_1 + x_2 + \dots$$

Definition 11 (Partial Sums)**.** The *partial sum* of a series, is the sum of the first k-number of terms of the series, written as:

$$S_k = \sum_{n=1}^{k} x_n \iff S_k = x_1 + x_2 + \dots + x_k.$$

Note. Sometimes partial sums are also written in the form, $S_k = \sum\limits_{1\leq n\leq k} x_n$.

Remark. Since $S_k = \sum_{n=1}^{k} x_n$, we can write

$$x_k = S_k - S_{k-1} = (x_1 + x_2 + \dots + x_{k-1} + x_k) - (x_1 + x_2 + \dots + x_{k-1}).$$

This fact is often very useful for proofs.

Definition 12 (Convergence of a Series)**.** Suppose that have the series $\sum_{n=1}^{\infty} x_n$, with partial sums, $S_k = \sum_{n=1}^{k} x_n$. Consider the sequence of partial sums,

$$\{S_k\}_{k=1}^{\infty} = (x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3, \dots)$$

A series is said to be *convergent*, if $\{S_k\}_{k=1}^{\infty}$ converges; in which case,

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} S_k \iff \lim_{k\to\infty} \sum_{n=1}^{k} x_n.$$

Note. Similarly, if $\{S_k\}_{k=1}^{\infty}$ diverges, then $\sum_{n=1}^{\infty} x_n$ also diverges.

Remark. We will get to an alternative, and more refined definition of convergence of series in terms of the *Cauchy Criterion* later on.

Proposition 12. Let $r \in (-1, 1) \subset \mathbb{R}$, then the *geometric series*

$$\sum_{n=0}^{\infty} ar^n \text{ converges, and } \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Note. This also means that the geometric series is a valid *Maclaurin Series* for the function $f(x) = \frac{a}{1-x}$, for $x \in (-1, 1)$.

Proof. We will first show that the formula for the partial sums of the geometric series is explicity given as

$$S_k = \frac{a\left(1 - r^{k+1}\right)}{1-r}.$$

Suppose $r \in (-1, 1)$. Then

$$\begin{aligned} S_k &= a + ar + ar^2 + \cdots + ar^k, \\ rS_k &= ar + ar^2 + \cdots + ar^{k+1}. \end{aligned}$$

Subtracting rS_k from S_k,

$$\begin{aligned} S_k - rS_k &= (a + ar + ar^2 + \cdots + ar^k) - (ar + ar^2 + \cdots + ar^k + ar^{k+1}) \\ S_k - rS_k &= a - ar^{k+1} \\ (1-r)S_k &= a\left(1 - r^{k+1}\right) \\ S_k &= \frac{a\left(1 - r^{k+1}\right)}{1-r}. \end{aligned}$$

Remember that $L = \sum_{n=0}^{\infty} x_n = \lim_{k\to\infty} S_k$. So, we take the limit of S_k as $k \to \infty$,

$$\begin{aligned} L &= \sum_{n=0}^{\infty} ar^n \\ &= \lim_{k\to\infty} \frac{a\left(1 - r^{k+1}\right)}{1-r} \\ &= \lim_{k\to\infty} \frac{a}{1-r} - \lim_{k\to\infty} \frac{r^{k+1}}{1-r} \\ &= \frac{a}{1-r} - \frac{1}{1-r} \lim_{k\to\infty} r^{k+1}. \end{aligned}$$

Note. To show that $\lim_{k\to\infty} \frac{r^{k+1}}{1-r} = 0$, it requires more sequence proofs than I am unwilling to do. But with sequence techniques it can be shown that for $r \in (-1, 1)$, $\lim_{k\to\infty} r^{k+1} = 0$.

With that,

$$
\begin{aligned}
L &= \frac{a}{1-r} - \lim_{k\to\infty} \frac{r^{k+1}}{1-r} \\
&= \frac{a}{1-r}.
\end{aligned}
$$

$\square$

Definition 13. A series, $\sum_{n=1}^{\infty} x_n$ is called Cauchy if the sequence of partial sums is a Cauchy Sequence.

Remark. Remember when we found earlier that a sequence is Cauchy if and only if it is convergent? Well indeed, for a series is convergent if and only if it's partial sums are Cauchy (and in turn convergent).

Proposition 13 (Cauchy Criterion)**.** The series $\sum_{n=1}^{\infty} x_n$ is convergent if and only if there exists an $N \in \mathbb{N}$, so that $\forall n \geq N$ and $\forall k > n$,

$$
\left| \sum_{i=n+1}^{k} x_i \right| < \varepsilon \iff |S_k - S_n| = \left| \sum_{i=1}^{k} x_i - \sum_{i=1}^{n} x_i \right| < \varepsilon.
$$

Proof. Take $\varepsilon > 0$, and suppose $N \in \mathbb{N}$ exists so that for all $n \geq N$ and $k > n$ (so $k > n \geq N$),

$$
\left| \sum_{i=n+1}^{k} x_i \right| < \varepsilon.
$$

But, for $k > n$, we know that $k \geq n+1$, and that x_k is further along in the series than x_n. This is equivalent to noticing

$$
\sum_{i=1}^{k} x_i = x_1 + x_2 + \cdots + x_n + \cdots + x_k.
$$

This implies,

$$
\begin{aligned}
\sum_{i=1}^{k} x_i - \sum_{i=1}^{n} x_i &= (x_1 + x_2 + \cdots + x_n + \cdots + x_k) - (x_1 + x_2 + \cdots + x_n) \\
&= x_{n+1} + \cdots + x_k \\
&= \sum_{i=n+1}^{k} x_i.
\end{aligned}
$$

But remember by the definition of partial sums, $\sum_{i=1}^{k} x_i = S_k$ and $\sum_{i=1}^{n} x_i = S_n$. Then,

$$\begin{aligned}\left|\sum_{i=n+1}^{k} x_i\right| < \varepsilon \quad &\implies \left|\sum_{i=1}^{k} x_i - \sum_{i=1}^{n} x_i\right| < \varepsilon \\ &\qquad\quad |S_k - S_n| < \varepsilon \\ &\implies S_k \text{ is Cauchy.}\end{aligned}$$

$\square$

Proposition 14 (Divergence Criterion)**.** $\sum_{n=1}^{\infty} x_n$ converges $\implies \lim_{n\to\infty} x_n = 0$.

Alternitavely, the contrapositive (which is a lot more useful), states:

If $\lim_{n\to\infty} x_n \neq 0$ or is undefined, then $\sum_{n=1}^{\infty} x_n$ diverges.

Include a figure for this

Note. Be very careful with this. The original statement: $\sum_{n=1}^{\infty} x_n$ convergence implies $\lim_{n\to\infty} x_n = 0$, can only be used if you ***know*** that $\sum_{n=1}^{\infty} x_n$ converges. Similarly, you can't take the converse either: $\lim_{n\to\infty} x_n = 0$ ***does not*** imply that $\sum_{n=1}^{\infty} x_n$ converges. You can only use it to prove that a series diverges if there is a non-zero or non-existent limit.

Example (The Harmonic Series)**.** Consider $\sum_{n=1}^{\infty} \frac{1}{n}$.

We can see that $\lim_{n\to\infty} \frac{1}{n} = 0$. However the Harmonic Series, is pretty much the standard example of a divergent series. We will be able to prove this later with the integral test. I will show an example here: Firstly, $\sum_{n=1}^{\infty} x_n$ converges if and only if

$$\lim_{k\to\infty} \int_1^k x_n \,\mathrm{d}n < \infty$$

For the harmonic series, $x_n = \frac{1}{n}$, so

$$\begin{aligned}\lim_{k\to\infty} \int_1^k x_n \,\mathrm{d}n &= \lim_{k\to\infty} \int_1^{\infty} \frac{1}{n} \,\mathrm{d}n \\ &= \lim_{k\to\infty} \ln k - \ln(1) \\ &= \ln\left(\lim_{k\to\infty} k\right) \to \infty.\end{aligned}$$

So by the integral test, the Harmonic Series, $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. We shall prove that the integral test holds more formally later.

Let this be a cautionary tale, $\lim_{n\to\infty} x_n = 0$ ***does not*** imply that $\sum_{n=1}^{\infty} x_n$ converges, but we can use $\lim_{n\to\infty} x_n \neq 0$ or undefined to check if $\sum_{n=1}^{\infty} x_n$ will diverge instead.

Now we shall prove Proposition 13:

Proof. Take $\varepsilon > 0$. Then suppose that $\sum_{n=1}^{\infty} x_n$ converges. By the Cauchy criterion, this implies that it is Cauchy. Hence, there exists an $N \in \mathbb{N}$, so that $\forall n \geq N$,

$$|x_{n+1} - 0| = |x_{n+1}| = \left| \sum_{i=n+1}^{n+1} x_i \right| < \varepsilon.$$

So for $n \geq N + 1$,

$$|x_n - 0| < \varepsilon \iff \lim_{n \to \infty} x_n = 0$$

$\square$

3.2 Absolute Convergence

Proposition 15. If $x_n \geq 0$, $\forall n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence of partial sums is bounded above (i.e., $\exists M$ such that $\forall k \in \mathbb{N}$, $|S_k| \leq M$)

Remark. If $x_n \geq 0$, then this implies that S_k is monotone increasing. If $x_n \geq 0$, then $x_1 \leq x_1 + x_2$, and so on, so

$$S_k = \sum_{n=1}^{k} x_n \leq \sum_{n=1}^{\infty} x_n.$$

If $\sum_{n=1}^{\infty} x_n$ doesn't converge, then $S_k < \infty$, and is unbounded above. If S_k is bounded above and is monotone increasing because $x_k \geq 0$, then by the monotone convergence theorem, S_k converges and so does $\sum_{n=1}^{\infty} x_n$.

Definition 14 (Absolute Convergence)**.** A series is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} |x_n|$ converges.

Definition 15 (Conditional Convergence)**.** A series is said to be *conditionally convergent* if $\sum_{n=1}^{\infty} x_n$ is convergent but $\sum_{n=1}^{\infty} |x_n|$ does not converge.

Proposition 16. If a series $\sum_{n=1}^{\infty} x_n$ converges absolutely, then $\sum_{n=1}^{\infty} x_n$ also converges.

Proof. $\sum_{n=1}^{\infty} |x_n|$ is Cauchy since it converges. So, take $\varepsilon > 0$, and there exists an $N \in \mathbb{N}$ such that $\forall k \geq N$, and $\forall n > k$,

$$\left| \sum_{i=k+1}^{n} |x_i| \right| < \varepsilon.$$

Then,

$$\underbrace{\left| \sum_{i=k+1}^{n} x_i \right| \leq \sum_{i=k+1}^{n} |x_i|}_{\text{(by generalised }\Delta\text{-Ineq)}} = \left| \sum_{i=k+1}^{n} |x_i| \right|$$

Consequently, this implies that

$$\left|\sum_{i=k+1}^{n} x_i\right| < \varepsilon$$

Hence, $\sum_{n=1}^{\infty} x_n$ converges. □

Example. The Alternating Harmonic Series: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln(2)$ and is convergent. However, as covered earlier, $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right|$ is the Harmonic Series which is divergent. Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

> **Note.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln(2)$ can be confirmed through the Maclaurin series for $\ln(x+1)$,
>
> $$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

3.3 Series Convergence Tests

Proposition 17 (Series Comparison Test)**.** Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be series such that $\forall n \in \mathbb{N}$ $0 \leq x_n \leq y_n$. Then,

1. If $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ converges, and
2. if $\sum_{n=1}^{\infty} x_n$ diverges, then $\sum_{n=1}^{\infty} y_n$ diverges.

Proof. First proving 1: Suppose that $\sum_{n=1}^{\infty} y_n$ converges. Let the partial sums of $\sum_{n=1}^{\infty} y_n$ be given by Y_k and the partial sums of $\sum_{n=1}^{\infty} x_n$ be given by S_k. Suppose that $\forall n \in \mathbb{N}$, $0 \leq x_n \leq y_n$. Then,

$$Y_k \leq Y_k + y_{k+1} = Y_{k+1} \implies Y_k \leq Y_{k+1}.$$

Similarly,

$$S_k \leq S_k + x_{k+1} = S_{k+1} \implies S_k \leq S_{k+1}.$$

Hence the sequences of partial sums for $\sum_{n=1}^{\infty} y_n$ and $\sum_{n=1}^{\infty} x_n$ are monotone increasing. Also, because $x_n \leq y_n$, then $S_k \leq Y_k$.

Since $\sum_{n=1}^{\infty} y_n$ converges, $Y_k \leq \sum_{n=1}^{\infty} y_n$ (Y_k is bounded). So,

$$S_k \leq Y_k \leq \sum_{n=1}^{\infty} y_n \implies S_k \leq \sum_{n=1}^{\infty} y_n.$$

Since S_k is bounded above and $x_n \geq 0$ $\forall n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} x_n$ must converge by Proposition 14. □

Proposition 18 (p-series test)**.** For $p \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Remark. The p-series test is just a specific case of the series comparison test using the Harmonic series ($p = 1$).

Example. For $p = 0$, we have $\sum_{n=1}^{\infty} 1 \to \infty$.

For $p = 1$, we have $\sum_{n=1}^{\infty} \frac{1}{n} \to \infty$ (Harmonic Series).

For $p \in (0, 1]$,

$$0 \leq \frac{1}{n} \leq \frac{1}{n^p},$$

so the series comparison test dictates that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

For $p = 2$, we have $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (can be confirmed with Fourier Series).

Proposition 19 (Ratio Test)**.** Let $\sum_{n=1}^{\infty} x_n$ be a series with $x_n \neq 0$, $\forall n \in \mathbb{N}$. Let

$$L = \lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right|.$$

Then, if $L < 1$, $\sum_{n=1}^{\infty} x_n$ converges absolutely.

If $L > 1$ or L doesn't exist, then $\sum_{n=1}^{\infty} x_n$ is divergent.

Remark. If $L = 1$, the test is inconclusive and you should look for a different test.

Proposition 20 (Root Test)**.** Let $\sum_{n=1}^{\infty} x_n$ be a series. Define,

$$L = \limsup_{n \to \infty} |x_n|^{\frac{1}{n}}.$$

Then, if $L < 1$, $\sum_{n=1}^{\infty} x_n$ converges absolutely.

If $L > 1$ or L doesn't exist, then $\sum_{n=1}^{\infty} x_n$ is divergent

Remark. Similarly to the ratio test, if $L = 1$, the test is inconclusive and you should look for a different test.

Proposition 21 (Alternating Series Test)**.** Let $\{x_n\}_{n=1}^{\infty}$ be monotone-decreasing, with $x_n > 0$ and $\lim_{n \to \infty} x_n = 0$. Then $\sum_{n=1}^{\infty} (-1)^n x_n$ converges.

Note. I'm not going to write about rearrangements here.

I think it's important to understand however that if you rearrange an absolutely convergent series then it won't change the convergence or limit of the series. However the same cannot always be said for series which are conditionally convergent. The alternating harmonic series being a classic example, which can be rearranged to produce any limit.

4 Functions

4.1 Preliminary Definitions

Definition 16. A function, f, is defined as the *mapping* of the elements one subset onto another subset, denoted $f : X \to Y$, where $X \subseteq Y$.

Remark. Typically, we work with real-valued functions, where $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$. For most functions we work with, we will have $f : X \to \mathbb{R}$.

4.2 Limits with Functions

Definition 17 (The ε–δ limit)**.** Let $c \in \mathbb{R}$, and f be defined as $f : \mathbb{R} \setminus \{c\} \to \mathbb{R}$. We say that $\lim\limits_{x \to c} f(x) = L$ if, $\forall \varepsilon > 0$, there exists a $\delta > 0$, such that

$$0 < |x - c| < \delta \quad \implies \quad |f(x) - L| < \varepsilon.$$

Remark. Basically what this means, is that we should be able to find some distance around $x = c$, (δ), such that we can guarantee the values of the function in this region, are of a certain distance (ε) away from the limit L.

If we are *always* able to find a region where the function is a certain distance away from the limit, no matter the distance, then as the distance between the function and the limit gets smaller, so too should the distance between x-values around c.

This definitely needs a figure

Example. Consider $\lim_{x \to 3} \frac{x-1}{2} = 1$.

Proof: Let $\varepsilon > 0$, and set $\delta = \varepsilon$. Then, suppose that $\forall x$, $0 < |x - 3| < \delta$, so

$$\left| \frac{x-1}{2} - 1 \right| = \frac{|x-3|}{2} < \frac{\varepsilon}{2} < \varepsilon.$$

Remark. The way we worked out to set $\delta = \varepsilon$ is through some working out. Our goal is to always get $|f(x) - L|$ to be some expression in terms of $|x - c|$, so that we can pick some δ in terms of ε, that will end up smaller than ε. In this case we had $|f(x) - L| = \frac{1}{2}|x - 3|$. We should notice that this is related to our predicate, $0 < |x - 3| < \delta$. We can then see that this means, $\frac{1}{2}|x-3| < \frac{\delta}{2}$. So if we want $\frac{\delta}{2} < \varepsilon$, picking $\delta = \varepsilon$ would work well.

Example. Show that for $m,\ b \in \mathbb{R}$, $\lim_{x\to c} mx + b = mc + b$.

Our predicate is $0 < |x-c| < \delta$. We now want to rearrange $|(mx+b)-(mc+b)|$ to be in terms of $|x-c|$. So,

$$\begin{aligned} |(mx+b)-(mc+b)| &= |m(x-c)| \\ &= |m|\,|x-c|\,. \end{aligned}$$

From our predicate, we have $|x-c| < \delta$, so, we want to show that

$$|m|\,|x-c| < |m|\,\delta \le \varepsilon.$$

Take $\delta = \frac{\varepsilon}{|m|+1}$, (we have $|m|+1$ in the denominator so that for $|m| = 0$, δ is still defined).

Then,

$$\begin{aligned} |(mx+b)-(mc+b)| &= |m|\,|x-c| \\ &< |m|\,\frac{\varepsilon}{|m|+1} \\ &< |m|\,\frac{\varepsilon}{|m|} \\ &< \varepsilon. \end{aligned}$$

Example. Show that $\lim_{x\to 4} x^2 = 16$.

Remember, that we want $|f(x)-L|$ in terms of $|x-c|$; for this question, we should then notice that we can factorise $|f(x)-L| = |x^2-16|$,

$$\left|x^2-16\right| = |(x+4)(x-4)| = |x+4|\,|x-4|\,.$$

Now if we want to find a δ that works here, we can't just pick $\delta = \frac{\varepsilon}{|x+4|+1}$. Remember, $0 < |x-4| < \delta$, must be true for all x, meaning we need to pick a δ that is *independent of* x. This means that we need some way to establish an upper bound on $|x+4|$.

The best way to do this is to suppose that we briefly assume that $\delta = 1$, since we are working around $x \to 4$, we can safely assume that we're working in the interval $3 < x < 5$, so that $|x-4| < 1$.

Well, if $0 < |x-4| < 1$, then

$$\begin{aligned} 0 < |x-4| < 1 &\implies x < 5 \\ &\implies x + 4 < 9. \end{aligned}$$

Okay, so now we can write that

$$\left|x^2-16\right| = |x+4|\,|x-4| < |9|\,|x-4| = 9\,|x-4|\,.$$

Then for $0 < |x-4| < \delta$, and $9|x-4| < \varepsilon$, we can see that we should have $\delta = \frac{\varepsilon}{9}$. Since our earlier upper bound on $|x+4|$ was derived for $\delta = 1$, and $\delta = \frac{\varepsilon}{9} > 1$ for $\varepsilon > 9$ (which is possible), we need to take the minimum of these two deltas to guarantee our inequality always works.

Hence our final choice is $\delta = \min\left(1, \frac{\varepsilon}{9}\right)$.

Note. In the lectures a different approach (and perhaps more rigorous one) was used with the reverse triangle inequality, so that

$$|x| - 4 \leq ||x| - 4| \leq |x-4| < 1$$

This approach is a lot more helpful for $x \to c$, when $c < 0$, as it deals with the possibility of $x < 0$ which my method above doesn't.

Remark. The general strategy for limits of quadratics (and higher polynomials too), is to factorise $|f(x) - L|$ into some expression $|g(x)|\,|x-c|$, where $g(x)$ is some other polynomial.

The goal is to then find an upper bound on $|g(x)|$ by making an assumption such as $\delta = 1$ or something similar, and then proceeding with the working out to find the other delta.

Your final delta will be the minimum of the two found earlier,minimum, $\delta = \min(1, \delta(\varepsilon))$ (where $\delta(\varepsilon)$ is the δ you find after applying the upper bound on $|g(x)|$.)

Proposition 22. If a function attains a limit at a point, then it is unique.

Proof. By contradiction, suppose that there exists two limits L_1 and L_2, such that $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$. Using a similar techinque as the one used to prove that the limit of a sequence is unique, you can suppose that $0 < |x-c| < \delta_1$ implies $|f(x) - L_1| < \frac{\varepsilon}{2}$, and $0 < |x-c| < \delta_2$ implies $|f(x) - L_2| < \frac{\varepsilon}{2}$. Take $\delta = \min(\delta_1, \delta_2)$ and write $|L_2 - L_1| = |L_2 - f(x) + f(x) - L_1|$, from there the triangle inequality can be used to get $|L_2 - L_1| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which in turn implies that $L_2 = L_1$, completing the proof. (**I'll expand on this later**) $\square$

Definition 18 (Sequential Limits)**.** We say that $\lim\limits_{x\to c} f(x) = L$, if for all sequences $\{x_n\}_{n=1}^{\infty}$ converging to c, then $\lim\limits_{n\to\infty} f(x_n) = L$.

Proposition 23. The sequential limit and ε–δ limit definitions are equivalent.

Proof. This proof will be completed by showing that ε–$\delta \implies$ Sequential Limit, and that $\neg\, \varepsilon$–$\delta \implies \neg$ Sequential Limit.

Part 1: ε–δ limit implies the sequential limit.

Suppose that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall x$, $0 < |x-c| < \delta$ implies that $|f(x) - L| < \varepsilon$.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \neq c$, and $\forall \varepsilon^* > 0$, $\exists N \in \mathbb{N}$, such that $\forall n \geq N$, $|x_n - c| < \varepsilon^*$.

Take $\varepsilon > 0$, and $\varepsilon^* = \delta$. Then there exists an $N \in \mathbb{N}$, such that $\forall n \geq N$,

$$0 < |x_n - c| < \varepsilon^* \iff |x_n - c| < \delta,$$

then by assumption,

$$|f(x_n) - L| < \varepsilon.$$

Part 2: The negation of ε–δ implies the negation of the sequential limit.

Assume that $\exists \varepsilon > 0$, such that $\forall \delta > 0$, $\exists x$ such that $0 < |x - c| < \delta$ implies $|f(x) - L| \geq \varepsilon$.

Consider the sequence $\{\delta_n\}_{n=1}^{\infty} = \frac{1}{n}$. Now constructing $\{x_n\}_{n=1}^{\infty}$ using δ_n, suppose that $\exists \varepsilon > 0$ such that $\forall n$, $\exists \{x_n\}_{n=1}^{\infty} \neq c$ with $0 < |x_n - c| < \delta_n$, and $|f(x_n) - L| \geq \varepsilon$.

Consider an arbitrary sequence $\{x_n\}_{n=1}^{\infty}$ that fits these assumptions, so that we have $\forall n$ $0 < |x_n - c| < \delta_n = \frac{1}{n}$. Then

$$0 < |x_n - c| < \frac{1}{n} \implies c - \frac{1}{n} < x_n < c + \frac{1}{n}.$$

This implies that $\lim_{n\to\infty} x_n = c$, but by our earlier assumption of the negation of ε–δ, $|f(x_n) - L| \geq \varepsilon$. So by our construction of $\{x_n\}_{n=1}^{\infty}$, we have found a sequence which satisfies the negation of the sequantial limit, and so the negation of ε–δ implies the negation of the sequential limit. $\square$

Example. $\lim\limits_{x\to 0} \sin\left(\dfrac{1}{x}\right)$ does not exist.

Consider the sequence $x_n = \frac{2}{(2n+1)\pi}$, noting that $\lim_{n\to\infty} x_n = 0$, but $\forall n$, $x_n \neq 0$.

Then $\sin\left(\frac{1}{x_n}\right) = (-1)^n$. In this case $\limsup_{n\to\infty} \sin\left(\frac{1}{x_n}\right) = 1$, while we also have $\liminf_{n\to\infty} \sin\left(\frac{1}{x_n}\right) = -1$. These limits aren't equal, and so $\sin\left(\frac{1}{x}\right)$ diverges for $x \to 0$.

Example. $\lim\limits_{x\to 0} x \sin\left(\dfrac{1}{x}\right) = 0$.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $\forall n$, $x_n \neq 0$ and $\lim_{n\to\infty} x_n = 0$.

Then

$$\left| x_n \sin\left(\frac{1}{x_n}\right) - 0 \right| = |x_n| \left| \sin\left(\frac{1}{x_n}\right) \right| \leq |x_n|$$

By assumption, $x_n \to 0$, so $|x_n| \to 0$ and hence $f(x_n) \to 0$.

Corollary. If $f(x) \to L_1$ and $g(x) \to L_2$ as $x \to c$, and $f(x) \leq g(x)$ for all x, then $L_1 \leq L_2$.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \to c$. Then we can define the sequences $a_n = f(x_n)$ and $b_n = g(x_n)$ and $a_n \leq b_n$ for all n. Hence $L_1 \leq L_2$ (shown in earlier sequence theorems). $\square$

Corollary. Suppose that $f(x) \leq g(x) \leq h(x)$, with $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = \ell$. Then $\lim_{x\to c} g(x)$ exists with $\lim_{x\to c} g(x) = L$.

Remark. Similarly, all of the standard limits properties for sequences, such as preservation of addition, subtraction, division and multiplication of sequences all apply to functions.

4.3 Continuity

Definition 19 (Continuity)**.** Let f be a function on a real subset of $X \subseteq \mathbb{R}$, and $f : X \to \mathbb{R}$. Then for $c \in X$, the function f is *continuous at* c if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x$, $0 < |x - c| < \delta$ implies that $|f(x) - f(c)| < \varepsilon$.

In other words, the limit of $f(x)$ at $x = c$, should be equal to $f(c)$.

Definition 20. Let x be in some neighbourhood about c. Take $x = c + \Delta x$, and so $f(x) = f(c + \Delta x)$. Then $x \to c$ is equivalent to $\Delta x \to 0$, and f is continuous at c if

$$\lim_{\Delta x\to 0} f(c + \Delta x) = f(c) \iff \lim_{\Delta x\to 0} [f(c + \Delta x) - f(c)] = 0$$

Definition 21. A function f is said to be *everywhere continuous* if for $f : X \to \mathbb{R}$, then $\forall c \in X$, $\lim_{x\to c} f(x) = f(c)$.

Remark. Sometimes when it is said that a 'function is continuous' it is meant that a function is *everywhere continuous*, which means that the function is continuous everywhere on the real subset that it is defined for.

An example of where this could matter is a function like $f(x) = \ln(x)$, which is said to be continuous. However, we know that $f(x)$ for $x < 0$ is undefined, so we might wonder, why is it said to be continuous if $f(x)$ doesn't exist for $x < 0$?

This is where the distinction between continuous everywhere and continuous is important, as we can say that $f(x) = \ln(x)$ is continuous, as long as we remember that is is standard to have $X = (0, \infty)$, and $f : X \to \mathbb{R}$.

A similar situation is $f(x) = \sqrt{x}$, where by not defining the function properly as $f : [0, \infty) \to \mathbb{R}$, we might expect that saying 'the square root is a continuous function' would imply that $f(x)$ would be continuous at $x = -1$, when this is not the case.

Conclusion: when we say a function is *continuous everywhere* or just *continuous*, it is important to properly define the function to avoid confusion with a function being continuous on all of $\mathbb{R}$, instead of a subset of $\mathbb{R}$.

Example. Show that $f(x) = \sin x$ is a continuous function (here we mean $\forall x \in \mathbb{R}$).

Let $c \in \mathbb{R}$, and we shall show that $\lim_{\Delta x\to 0} [f(c + \Delta x) - f(c)] = 0$, $\forall c \in \mathbb{R}$.

We have that

$$
\begin{aligned}
f(c+\Delta x) - f(c) = \sin(c+\Delta x) - \sin(c) &= 2\sin\left(\frac{c+\Delta x - c}{2}\right)\cos\left(\frac{c+\Delta x + c}{2}\right) \\
&= 2\sin\left(\frac{\Delta x}{2}\right)\cos\left(c+\frac{\Delta x}{2}\right).
\end{aligned}
$$

Since $|\cos x| \leq 1$ and $|\sin x| \leq |x|$ for all x, we have

$$
\begin{aligned}
|f(c+\Delta x) - f(c)| &= \left|2\sin\left(\frac{\Delta x}{2}\right)\cos\left(c+\frac{\Delta x}{2}\right)\right| \\
&\leq 2\frac{|\Delta x|}{2} = \Delta x.
\end{aligned}
$$

Since $\lim_{\Delta x\to 0} |\Delta x| = 0$, we have by the Squeeze theorem $\forall c \in \mathbb{R}$,

$$
\lim_{\Delta x\to 0} [f(c+\Delta x) - f(c)] = 0.
$$

Example. Show that $f(x) = \frac{1}{x}$ is continuous for $f : (0,\infty), \to \mathbb{R}$.

Let $c \in (0,\infty)$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence with $x_n \in (0,\infty)$, and $\lim_{n\to\infty} x_n = c$. Then

$$
\begin{aligned}
f(c) = \frac{1}{c} &= \frac{1}{\lim_{n\to\infty} x_n} \\
&= \lim_{n\to\infty} \frac{1}{x_n} = \lim_{n\to\infty} f(x_n).
\end{aligned}
$$

Hence f is continuous at c for all $c \in (0,\infty)$.

Proposition 24. All polynomials $p : \mathbb{R} \to \mathbb{R}$ are continuous.

Proof. Let the polynomial of degree d, be given by $p(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, for $a_d \in \mathbb{R}$, $d \in \mathbb{Z}^+$.

Take $c \in \mathbb{R}$ and define the sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = c$. Then

$$
\begin{aligned}
f(c) &= a_d c^d + a_{d-1}c^{d-1} + \cdots + a_1 c + a_0 \\
&= a_d \left[\lim_{n\to\infty} x_n\right]^d + a_{d-1}\left[\lim_{n\to\infty} x_n\right]^{d-1} + \cdots + a_1\left[\lim_{n\to\infty} x_n\right] + a_0 \\
&= \lim_{n\to\infty}\left[a_d x_n^d + a_{d-1}x_n^{d-1} + \cdots + a_1 x_n + a_0\right] \\
&= f(x_n).
\end{aligned}
$$

Hence f is continuous at c, $\forall c \in \mathbb{R}$. $\square$

Proposition 25. The composition of continuous functions is also continuous. That is, if $f : X \to Y$ and $g : Y \to Z$ are continuous, with $X, Y, Z \subseteq \mathbb{R}$, with f being continuous at every point $c \in X$, and g cotninuous at every point in $f(c) \in Y$, then $(g \circ f)(x) : X \to Z$, and $(g \circ f)(x) = g(f(x))$ is continuous at every point $c \in X$.

Corollary (The Algebraic Limit Theorem)**.** Any standard operation ($+$, $-$, $\times$, $\div$) on continuous functions, is also continuous.

Remark. A consequence of this, is that for any continuous functions f and g at a point c, $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$.

4.4 The Properties of Continuous Functions

Continuous functions satisfy three important properties which we shall cover. These being,

1. **Boundedness**(Weierstrass's First Theorem)
2. **The Extreme Value Theorem**
3. **The Intermediate Value Theorem**

Lemma 2. A continuous function $f : [a, b] \to \mathbb{R}$ is bounded.

Proof. To-do later (proof in lecture 11) $\square$

Theorem 5 (Weierstrass's First Theorem)**.** All continuous functions defined on a bounded subset are bounded

Proof. To-do later (proof in lecture 12) $\square$

Remark. It is important to specify that $f : [a, b] \to \mathbb{R}$ and not $f : \mathbb{R} \to \mathbb{R}$, or $f : (a, b] \to \mathbb{R}$ (or $[a, b)$). As boundedness of continuous functions only holds on bounded and closed intervals. An example is $f(x) = \ln(x)$, as f is continuous for all $x \in (0, e)$, but even though $\lim_{x\to c} f(e) = 1$, $\lim_{x\to 0^+} f(x) = -\infty$. So for this property to hold, a function must also be *continuous at it's endpoints*.

Theorem 6 (Extreme Value Theorem)**.** If $f : [a, b] \to \mathbb{R}$, is continuous ($\forall x \in [a, b]$), then it attains a maximum and a minimum in $[a, b]$.

So, there eixsts a point $c \in [a, b]$ such that $f(x) \leq f(c)$, $\forall x \in [a, b]$.

Similarly, there exists a point $c^* \in [a, b]$ such that $f(c^*) \leq f(x)$, $\forall x \in [a, b]$.

Theorem 7 (The Intermediate Value Theorem)**.** If $f : [a, b] \to \mathbb{R}$, is continuous, then there exists a point $c \in (a, b)$ such that $\min(f(a), f(b)) < f(c) < \max f(a), f(b)$.

Corollary. A very useful consequence of the Intermediate Value Theorem (IVT) is that if $\min(f(a), f(b)) < 0$ and $\max(f(a), f(b)) > 0$, then there exists a point c such that $f(c) = 0$. This is a special case of the IVT which is a little easier to prove.