Chapter 2: Permutations & Combinations

Combinatorics is the study of the combinations of objects, how to count them, & how to make algorithms to describe them.

The addition principle says that given pairwise disjoint sets $A_1, \ldots, A_n$, the size of their union is

$$\left|\bigcup_{i=1,n} A_i\right| = \sum_{i=1,n} |A_i|$$

overlaping sets use inclusion-exclusion discussed later

The multiplication principle says that given one set S where $|S|=s$ & another set T where $|T|=t$, the number of ways to pick an element from S & then T is $s \cdot t$.

The subtraction principle says that if we have set S which is the disjoint union of R & T, that is $S = R \sqcup T$, If $|S|=s$ & $|R|=r$, then $|T| = |S| - |R| = s-r$. This is derived from the addition principle.

The division principle says that if set S is a finite set w/ k subsets which partition S, each of which having exactly r elements, then

$$\frac{|S|}{r} = k$$

See circular permutations later

Def: Permutation

Picking items where order matters is called a permutation.
Picking r elements from a set S, where $|S|=n$ has

$$P(n,r) = {}_nP_r = (n)(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$ possibilities

Note: $P(n,r)=0$ iff $r>n$.

Def: Combination

Picking items where order does NOT matter is called a combination. This is also commonly called an r-subset.
Picking r elements from a set S where $|S|=n$ has

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$ possibilities.

or $C(n,r)$

Note: ${}_nC_r = 0$ if $r>n$

\# Using the Principles

The addition principle is best used to partition a set exhaustively into managable parts.

The multiplication principle is applied when you are taking one element from set A & another from set B. It is a consequence of the addition principle

The subtraction principle is used when you can "strike out" possibilities of some larger/less specific version of what you're trying to count.

If we know how many elements are in a partition & how many elements there are in total, we can find how many partitions there are.

Sets & Multiset

Def:

A set is an unordered collection of elements w/o repetition. (This is a very informal definition.)

Def:

A multiset is a set where elements are allowed to repeat. There can be infinite repetitions.
We denote a multiset the same as w/ sets but w/ repetition numbers before each element.

Examples:

Let M be a multiset w/ 3 a's, 1 b, 2 c's, & 4 d's.

$M = \{3 \cdot a,\ 1 \cdot b,\ 2 \cdot c,\ 4 \cdot d\}$

We say M has 4 types of elements. We say M has repetition numbers $(3, 1, 2, 4)$

Let M be a multiset w/ ∞ a's & c's but 2 & 4 b's & d's respectively

$M = \{\infty \cdot a,\ 2 \cdot b,\ \infty \cdot c,\ 4 \cdot d\}$

Example:

How many odd 4 digit numbers are there w/ distinct digits?

Each position can be one of

Units: 1, 3, 5, 7, 9
Tens: 0..9
Hundreds: 0..9
Thousands: 1..9

By picking units, thousands, tens, hundreds we get

5 (1s)
$\cdot 8$ (1000s)
$\cdot 8$ (10s)
$\cdot 7$ (100s)
$= 2240$

If we picked in a different order, says 1000s → 100s → 10s → 1s, this would be much more difficult b/c choice of 1s depends on choice of previous.

Also, notice order is arbitrary for me.

Example:

How many ways are there to order the alphabet such that no two vowels are adjacent?

The easiest way to solve this problem is to first think about ordering the consonants & then putting vowels in the gaps b/w the consonants.

There are 21! ways to organize the consonants. There are 22 slots for vowels b/w & around the consonants so there are $P(22, 5) = \frac{22!}{17!}$ ways to organize the vowels in those slots, giving us

$$21! \cdot \frac{22!}{17!} \text{ possibilities.}$$

Example:

How many 7 digit numbers are there w/ distinct digits taken from 1..9 where 5 & 6 are not adjacent

There are $P(9,7)$ numbers w/ distinct digits 1..9. There are 6 ways to have a 5 followed by a 6 & 6 ways to have 6 followed by 5 meaning 12 ways for 6 & 5 to be adjacent. There are $P(7,5)$ choices for other digits, meaning there are $12 \cdot P(7,5)$ of those total numbers where 6 & 5 are adjacent, giving us

$$P(9,7) - 12 \cdot P(7,5) = \frac{9!}{2!} - 12 \cdot \frac{7!}{2!} \text{ possibilities}$$

Permutations

The above example is a linear permutation where there is a distinct start & end which are not adjacent.

In a circular permutation there is no distinct start/end & all numbers are adjacent.

Thm:

The number of circular r-permutations of a set of n elements is

$$\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}$$

(Aka how many ways to sort r elements from a set of n into a circle)

Example:

How many necklaces can be made w/ 20 beads, each of a different color, where each necklace has 20 beads.

There are $\frac{P(20,20)}{20} = \frac{20!}{20} = 19!$ possible arrangements of beads. Since each necklace can be turned over w/o changing the arrangement of beads there are $\frac{19!}{2}$ total necklaces.

Combinations

See the definition of combinations from earlier.

Example:

Given 25 points on a plane w/ no 3 colinear, how many lines can you draw? How about triangles?

$\binom{25}{2} = \frac{25!}{2!23!}$ lines & $\binom{25}{3} = \frac{25!}{3!22!}$

Example:

How many 8 letter "words" are there w/ 3, 4, or 5 vowels?

There are $\binom{8}{3}$ slots for vowels in the 3 vowel case & 3 vowel slots to fill & 5 consonant slots to fill, giving us

$\binom{8}{3} 5^3 21^5$ 3-vowel "words"

Similarly

$\binom{8}{4} 5^4 21^4$ 4-vowel "words" & $\binom{8}{5} 5^5 21^3$ 5-vowel "words"

Summing these up gives us our answer.

Thm:

For $0 \le r \le n$ $\binom{n}{r} = \binom{n}{n-r}$.

Thm: Pascal's Formula

$\forall n, k \in \mathbb{Z}$ where $1 \le k \le n-1$

$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Pf:

Consider some set S w/ n elements. Distinguish some element in S & name it x.

We partition the set X of k-subsets of S into parts A & B. A contains the k-subsets which don't contain x & B contains those which do. By the addition principle

$\binom{n}{k} = |A| + |B| = |X|$.

The k-subsets in A are exactly the k-subsets of $S \setminus \{x\}$ thus

$$|A| = \binom{n-1}{k}$$

Every k-subset in B can be obtained by adjoining x to a $(k-1)$-subset of $S \setminus \{x\}$, thus

$$|B| = \binom{n-1}{k-1}$$

$$\text{So } \binom{n}{k} = |A| + |B| = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Thm:

$\forall n \in \mathbb{Z} \quad n \geq 0$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

You can see this b/c every combination is a choice of whether an element goes in the set or not.

Permutations of Multisets

I multisets we do not distinguish b/w the repeated values.

Thm:

Let S be a multiset w/ objects of k different types, where each object is repeated infinitely.

The number of r-permutations of S is k^r.

Note: This also holds if there are just $\geq r$ objects for each type.

Thm:

Let S be a set w/ k types of objects w/ finite repetition numbers $n_1, \ldots, n_k$ respectively. Let $|S| = n = \sum_{i=1}^{k} n_i$.
The number of permutations of S is

$$\frac{n!}{n_1! \cdots n_k!}$$

Example:

The number of permutations of the word MISSIAPI is

$$\frac{11!}{1!4!4!2!}$$

b/c the multiset is $\{1\cdot M, 4\cdot I, 4\cdot S, 2\cdot P\}$.

Note that in a multiset w/ two types, a_1 & a_2, of objects w/ sizes n_1 & n_2 where $n = n_1 + n_2$, the number of permutations is

$$\frac{n}{n_1! n_2!} = \frac{n!}{n_1!(n-n_1)!} = \binom{n}{n_1}.$$

Thm:

Let $n_1, \dots, n_k \in \mathbb{Z}^+$ & $n = n_1 + \dots + n_k$. The number of ways to partion a set of n objects into k labeled boxes where box 1 has n_1 objects, ..., & box k has n_k objects is

$$\frac{n!}{n_1! \cdots n_k!}$$

If the boxes are unlabeled & $n_1 = \dots = n_k$ then it is

$$\frac{n!}{k!\, n_1! \cdots n_k!}$$

there are $k!$ ways to assign labels

Thm:

Let there be n rooks of k colors where there are n_1 rooks of the 1st color, ..., & n_k rooks of the kth color.
The number of ways to arrange these rooks on an $n \times n$ board so that no rook can attach another is

$$\underbrace{n!}_{\text{ways to place rooks}} \underbrace{\frac{n!}{n_1! \cdots n_k!}}_{\text{ways to assign colors}}$$

Thm:

Consider a multiset S w/ n elements. If $r < n$ there is no simple formula to find the number of r-permutations of S.

Example:

Consider multiset $S = \{3\cdot a, 2\cdot b, 4\cdot c\}$. We find the number of 8-permutations of S.

To do this we partion S into 3 parts

1) 8-permutations of $\{2\cdot a, 2\cdot b, 4\cdot c\}$ $\quad \frac{8!}{2!2!4!} = 420.$

2) 8-permutations of $\{3\cdot a, 1\cdot b, 4\cdot c\}$ $\quad \frac{8!}{3!2!4!} = 280.$

3) 8-pemutations of $\{3\cdot a, 2\cdot b, 3\cdot c\}$ $\quad \frac{8!}{3!2!3!} = 560.$

Thus the number of 8-permutations of S is $420 + 280 + 560 = 1260.$

Combinations of Multisets

Selecting r elements from a multiset results in another multiset, we call it <u>r-submultisets</u> or more usually <u>r-combinations</u>.

<u>Thm:</u>

Let S be a multiset w/ k kinds of objects each w/ an infinite repetition number.
The number of r combinations of S is

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$ equal by property of binomial coefficient

Theorem holds if each repetition number $n_1, \ldots, n_k \geq r$.

<u>Example:</u>

A bakery has 8 varieties of donuts.
How many different options are there for a box of 12 doughnuts?

$$\binom{12+8-1}{12} = \binom{19}{12} = \binom{19}{7}$$ (Note: $12+7=19$)
← duh

<u>Example:</u>

How many non-decreasing sequences of length r can you make w/ terms from $1, \ldots, k$?

You can form a nondecreasing sequence by picking a combination of r numbers & then arranging them in increasing order so the answer is

$$\binom{r+k-1}{r}$$

<u>Coro:</u>

We can extend the earlier theorem when we need at least c of of each kind of element. In that case the number of combinations is

$$\binom{r-ck+k-1}{r-ck}$$

This comes as an extension b/c the proof of the earlier theorem boiled down to finding the number of non-negative integral solutions to

$$x_1 + \ldots + x_k = r$$

To transform this to account for minimum amounts we do a change of variable

$y_i = x_i - c \quad \forall i$

Thus our equation is now

$y_1 + \dots + y_k = r - ck$

__Example:__

Using the donut shop w/ 8 varieties, How many options is there for a box of 12 if you get one of each variety?

$r = 12, \ k = 8, \ c = 1$

$$\binom{r - kc + k - 1}{r - kc} = \binom{12 - 8 + 8 - 1}{12 - 8} = \binom{11}{4}$$

Finite probability

All finite probability can be reduced to counting possibilities, which is exactly what we've been doing.

Consider an __experiment__ $\mathcal{E}$ w/ a finite set of outcomes S. We assume each outcome in the __sample space__ S is equally likely, that is the experiment is __random__. If $S = \{s_1, \dots, s_n\}$ has n outcomes then the __probability__ of each outcome is $1/n$ & we write

$\text{Prob}(s_i) = 1/n = 1/|S| \quad i = 1 \dots n$

An __event__ E is a subset of outcomes in the sample space S (normally given descriptively). If $|E| = k$ then

$\text{Prob}(E) = \frac{k}{n} = \frac{|E|}{|S|}$

Note: $\text{Prob}(E) \in [0, 1] \quad \forall E \subseteq S$
& $\text{Prob}(E) = 0 \Leftrightarrow E = \emptyset$
& $\text{Prob}(E) = 1 \Leftrightarrow E = S$

__Example:__

Consider the random experiment $\mathcal{E}$ of flipping a coin twice. The sample space for $\mathcal{E}$ is

$S = \{(T, T), (T, H), (H, T), (H, H)\}$

The event E that you get 1 head is

$E = \{(T, H), (H, T)\}$

$P((H, T)) = \frac{1}{|S|} = 1/4 \qquad P(E) = \frac{|E|}{|S|} = \frac{2}{4} = 1/2$

__Remark:__

For any event $E \subseteq S$

$\text{Prob}(E) \in [0, 1]$
$\text{Prob}(E) = 0 \Leftrightarrow E = \emptyset$
$\text{Prob}(E) = 1 \Leftrightarrow E = S$

We call $E = \emptyset$ the impossible event & $E = S$ the __guaranteed event__.

Example:

Let $n \in \mathbb{N}$. Suppose we choose a sequence $i_1, \ldots, i_n$ from integers b/w $1, \ldots, n$ at random. (Note that $|S| = n^n$ b/c n choices n times.)

What is the probability that (i_n) is a permutation of $1, \ldots, n$?
There are $n!$ different permutations of $1, \ldots, n$ so $|E| = n!$ so

$$\text{Prob}(E) = \frac{|E|}{|S|} = \frac{n!}{n^n}$$

What is the probability that (i_n) contains exactly $n-1$ different integers?
There is 1 repeated integer, 1 missing, & $n-2$ other integers. There are n possibilities for the repeated one, $n-1$ for the missing, & then there are $(n-2)!$ permutations for the other integers. There are n positions for the repeated integer & we choose 2 so $\binom{n}{2}$ possibilities. Then we put the permutation of the other integers in the remaining positions in order. So

$$|E| = n(n-1)\binom{n}{2}(n-2)! = n(n-1)\frac{n!}{2!(n-2)!}(n-2)! = \frac{n!^2}{2(n-2)!}$$

$$\Rightarrow \text{Prob}(E) = \frac{|E|}{|S|} = \frac{1}{n^n}\left(\frac{n!^2}{2(n-2)!}\right)$$

Example:

5 rooks are placed in non-attacking positions on an 8×8 board. What is the probability the rooks are on rows 1-5 & colums 4-8?

Our sample space S is given by choosing 5 rows & 5 unordered columns & then permuting the colums to give us pairs of rows & cols

$$|S| = \binom{8}{5}^2 \cdot 5! = \frac{8!^2}{3!^2 5!}$$

Our event E has size $5!$ b/c there are $5!$ ways to place 5 non-attacking rooks on a 5 by 5 board. Using the same logic we used for $|S|$

$$|E| = \binom{5}{5}^2 \cdot 5! = 5!$$

Thus

$$\text{Prob}(E) = \frac{|E|}{|S|} = \frac{3!^2 5!^2}{8!^2}$$

Chapter 3: Pigeonhole Principle

Def:

The **pigeonhole principle** (informally) states that if $n+1$ objects are distributed in n boxes, at least one box has 2+ objects.

Example:

Among 13 people at least 2 of their birthday's have the same month.

Coro:

If n objects are put in n boxes & no box is empty, then every box has exactly 1 object. Likewise if no box has 2+ objects.

We can state this formally w/ functions

Def:

Formally, the **pigeonhole principle** is this.
Let X & Y be finite sets w/ function $f: X \to Y$.

If $|X| > |Y|$, then f is not one-to-one.
If $|X| = |Y|$, & f is onto, then f is one-to-one.
If $|X| = |Y|$ & f is one-to-one, then f is onto.

Example:

Consider the integer sequence $a_1, \dots, a_m$.]
We know $\exists k, l \in \mathbb{Z}$ where $0 \leq k < l \leq m$ st the subsequence $a_k + a_{k+1} + \dots + a_l$ is divisible by m.
That is there exist consecutive a's in the sequence $a_1, \dots, a_m$ whose sum is divisible by m.

To show this, consider the m sums
$$a_1, \quad a_1 + a_2, \quad \dots, \quad a_1 + \dots + a_m$$

If any of these sums is divisible by 1 then the conclusion holds. Since

Thus we assume that each of these sums has a non-zero remainder equal to one of $1, \dots, m-1$. Since there are m sums & $m-1$ remainders, we know two sums have the same remainder.

Therefore we have two integers $k < l$ st $a_1 + \dots + a_k$ & $a_1 + \dots + a_l$ have remainder r when divided by m

$a_1 + \dots + a_k = bm + r$ $\quad$ $a_1 + \dots + a_l = cm + r$

Subtracting these we find

$a_{k+1} + \dots + a_l = (c-b)m$

is divisible by m.

$\square$

Example: Chinese Remainder Theorem

Let m & n be relatively prime positive integers. Let $a, b \in \mathbb{Z}$ where $0 \le a \le m-1$ & $0 \le b \le n-1$.

Then, there exists an $x \in \mathbb{Z}^+$ st the remainder when x is divided by m is a & the remainder when x is divided by n is b. That is

$x = pm + a = qn + b$ for some $p, q \in \mathbb{Z}$.

Consider the n integers

$a, \ m+a, \ 2m+a, \ \dots, \ (n-1)m + a$

Each of these numbers has remainder a when divided by m.

Suppose for contradiction that two have the same remainder r when divided by n. Let the two numbers be $im + a$ & $jm + a$ where $0 \le i < j \le n-1$.

Then there are integers q_i & q_j st

$im + a = q_i n + r$ & $jm + a = q_j n + r$

Subtracting these equations we get

$(j-i)m = (q_j - q_i)n$.

This equation says n is a factor of $(j-i)m$. Since n & m are coprime they have no factors, so n is a factor of $j-i$.

However $0 \le i < j \le n-1 \Rightarrow 0 < j-i \le n-1$ means n cannot be a factor of $j-i$.

We thus conclude that each n integers has a distinct remainder divided by n. Thus for any $b \in \{0, \dots, n-1\}$ we know $\exists p \in \mathbb{Z}$ where $0 \le p \le n-1$ st

$x = pm + a$

has remainder b when divided by n. So for some q

$x = qn + b$

Thus $x = pm + a$ & $x = qn + b$ & x has the required properties. $\square$

Thm: Strong Pigeonhole Principle

Let $q_1, \dots, q_n$ be positive integers. If $q_1 + \dots + q_n - n + 1$ objects are distributed into n boxes, then either

box 1 contains at least q_1 objects,

box 2 contains at least q_2 objects,

$\dots$, or

box n contains at least q_n objects.

Pf:

Suppose we distribute $q_1 + \dots + q_n - n + 1$ objects among n boxes.

If for each $i \in \{1, \dots, n\}$ the ith box contains fewer than q_i objects then the total number of objects in all boxes does not exceed

$$(q_1-1)+\dots+(q_n-1) = q_1+\dots+q_n-n$$

which is fewer objects than we have. So at least one box has q_i objects.

Coro:

Let $n, r \in \mathbb{N}$. If $n(r-1)+1$ objects are distributed into n boxes then at least one box contains r or more objects.

We can generalize the pigeonhole principle into Ramsey's theorem.

Informally, Ramsey's theorem states than in a group of 6 either there are 3 pairs of acquainted people or 3 pairs of unaquainted people.

More abstractly, we write

$K_6 \to K_3, K_3$ (K_6 arrows K_3, K_3)

By K_n we mean a set of n objects & their possible pairs.

K_1 K_2 K_3 (aka triangle) K_4

We distinguish b/w acquainted & unacquainted by color edges red & blue accordingly. 3 pairs of unacquainted people now means a K_3 w/ blue edges.

Therefore $K_6 \to K_3, K_3$ is the assertion that there is always a red or blue triangle in any K_6.
This falls from the pigeonhole principle.

We now state the formal Ramsey's Theorem.

Thm: Ramsey's Theorem

If $m \geq 2$ & $n \geq 2$ are integers, then there is a $p \in \mathbb{N}$ such that
$K_p \to K_m, K_n$

Further, if $K_p \to K_m, K_n$ then $K_q \to K_m, K_n$ $\forall q \geq p$.

The Ramsey number $r(m,n)$ is the smallest such $p \in \mathbb{N}$.
Note that red & blue can be interchanged freely so
$r(m,n) = r(n,m)$.

Rem:

- $r(2,n) = r(n,2) = n$. ← Trivial Ramsey's number

If we color the edges of K_n either blue or red, then either some edge is red (so we have a red K_2) or all are blue (giving us a blue K_n). So $r(2,n) \le n$.

If we color K_{n-1} all blue we have neither a red K_2 or blue K_n. So $r(2,n) > n-1$.

Thm:

For all integers $m \ge 3$ & $n \ge 3$,

$$r(m,n) \le r(m-1,n) + r(m,n-1)$$

&

$$r(m,n) \le \binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}$$

Rem:

$r(3,3) = 6$
$r(3,4) = 9$
$r(3,5) = 14$
$r(3,6) = 18$
$r(3,7) = 23$
$r(3,8) = 28$
$r(3,9) = 36$
$40 \le r(3,10) \le 43$

$r(4,4) = 18$
$r(4,5) = 25$
$35 \le r(4,6) \le 41$

$43 \le r(5,5) \le 49$
$58 \le r(5,6) \le 87$
$102 \le r(6,6) \le 165$

We can generalize Ramsey's theorem to k colors. So

$r(n_1, \dots, n_k) = p$

$\Rightarrow K_p \to K_{n_1}, \dots, K_{n_k}$ & $K_{p-1} \not\to K_{n_1}, \dots, K_{n_k}$

We can further generalize this where instead of pairs of points (i.e. edges) to sets/groups of t points.

$K_p^t \to K_{n_1}^t, \dots, K_{n_k}^t$

Thm: Generalized Ramsey's Theorem

Given integers $t \ge 2$ & $q_1, \dots, q_k \ge t$, there exists integer p such that

$K_p^t \to K_{q_1}^t, \dots, K_{q_k}^t$

The smallest such p is Ramsey's number

$r_t(q_1, \dots, q_k)$

Note: When $t=1$, Ramsey's theorem is the strong form of the pigeonhole principle. In general, little is known about $r_t(q_1, \dots, q_k)$ but we know

$r_t(q_1, \dots, q_k) = r_t(t, q_1, \dots, q_k)$ & order doesn't matter

Chapter 4: Generating Permutations & Combinations

Partial Orders & Other Relations

Def:

Let X be a set. A <u>relation</u> R on X is a subset of $X \times X$, that is ordered pairs of X. Essentially it is a list of all pairs for which the relation holds.

We write aRb iff $(a,b) \in R$ & $a \not R b$ iff $(a,b) \notin R$.

Def:

For some relation R on set X we say R is
- <u>reflexive</u> iff $aRa \ \forall a \in X$
- <u>irreflexive</u> iff $a \not R a \ \forall a \in X$
- <u>symmetric</u> iff $aRb \Rightarrow bRa \ \forall a,b \in X$
- <u>antisymmetric</u> iff $aRb \Rightarrow b \not R a \ \forall a,b \in X$ where $a \neq b$
  or aRb & $bRa \Rightarrow a = b \ \forall a,b \in X$
- <u>transitive</u> iff aRb & $bRc \Rightarrow aRc \ \forall a,b,c \in X$

Example:

$\subseteq$ is reflexive, transitive, & antisymmetric. (Partial Order)

$\subset$ is irreflexive, transitive, & antisymmetric. (Strict Partial Order)

Def:

A <u>partial order</u> (denoted $\leq$) is reflexive, transitive & antisymmetric.

A <u>strict partial order</u> (denoted $<$) is irreflexive, transitive, & antisymmetric.

Def:

A set X w/ partial order $\leq$ is called a <u>partially ordered set</u> or <u>poset</u> denoted $(X, \leq)$.

Def:

Let R be a relation on set X. We say $x, y \in X$ are <u>comparable</u> iff either xRy or yRx. Otherwise they are <u>incomparable</u>.

Def:

A partial order R on set X is a <u>total order</u> iff every pair of elements is comparable.

Each permutation $a_1, \ldots, a_n$ of n elements can be identified w/ a total ordering where $a_i \le a_j$ $\forall i, j \in \{1, \ldots, n\}$ where $i \le j$.

Thm:

Let X be a finite set w/ n elements. There is a 1-to-1 correspondence b/w the total orders on X & the permutations on X. In particular the number of orders is $n!$

Thm:

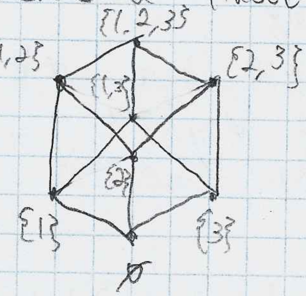
Let $(X, <)$ a strictly partially ordered set. We say a is covered by b or a covers b iff $a < b$ & there is no x st $a < x < b$. We write this as

$$a <_c b.$$

We can construct Hasse diagrams by putting elements that directly below those that cover them & connecting them by lines.

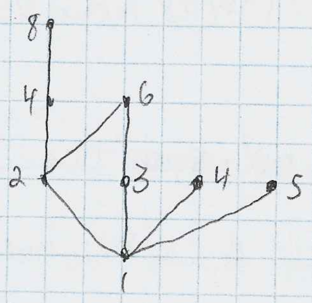
Example:

Here's a Hasse diagram for $(\{1,2,3\}, \emptyset)$.



Example:

Here's a divisor of 1-8 & the "is a divisor of" relation



Def:

Consider some partial orders $\le_1$ & $\le_2$ on set X. $\le_2$ is called an extension of $\le_1$ iff

$a \le_1 b \Rightarrow a \le_2 b$ $\forall a, b \in X$. (& $\exists a, b \in X$ where a & b are uncomparable by $\le_1$ but comparable by $\le_2$.)

A linear extension is an extension of $(X, \le)$ st all elements are comparable & $\le$ still agrees w/ the extension. It is done by fixing a certain semi-arbitrary order of elements.

Thm:

An equivalence relation (denoted $\sim$) is a reflexive, symmetric, & transitive relation.

Def

Def:

Consider some equivalence relation $\sim$ on X.
The equivalence class of $a \in X$ is all elements equivalent to a, that is
$$[a] = \{ x \in X \mid x \sim a \}.$$

Thm:

Let $\sim$ be an equivalence relation on X. The distinct equivalence classes partition X into non-empty parts.

Equivalently, given any partition of X in non-empty parts, there is an equivalent equivalence relation.

Generating Subsets

We want an efficient way to generate an r-subset from a set of n elements.

We do this using a lexicographic ordering, which requires the set itself to be ordered. Then you order each element in priority going 1st, 2nd, 3rd, ... Shorter sets are first all else equal. We also agree to write subsets' elements in increasing order.

Thm:

Consider $S = \{1, \dots, n\}$.

The first r-subset of S is $1, 2, \dots, r$.
The last is $(n-r+1), (n-r+2), \dots, n$.

Now for the middle assume $a_1, \dots, a_r \neq (n-r+1), \dots, n$.
Let k be the largest integer such that $a_k < n$ & $a_k + 1$ is different from each $a_{k+1}, \dots, a_r$. Then the immediate sucessor to the current ordering is
$$a_1, \dots, a_{k-1}, (a_k + 1), (a_k + 2), \dots, (a_k + r - k + 1)$$

Example:
Generate the 2-subsets of $\{1, 2, 3, 4\}$.

1, 2	2, 3
1, 3	2, 4
1, 4	3, 4

Thm:
The r-subset $a_1, \ldots, a_r$ or $\{1, \ldots, n\}$ is in place number
$$\binom{n}{r} - \binom{n-a_1}{r} - \binom{n-a_2}{r-1} - \cdots - \binom{n-a_{r-1}}{2} - \binom{n-a_r}{1}$$

Generating Arbitrary Subsets/Combinations ← I did this in the wrong order

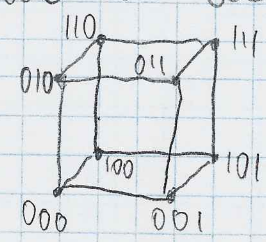
The method here is to (arbitrarily) order the set S. Then, we make a table where each column represents the inclusion of a particular element & each row represents a combination of those elements.

Once you have this table, fill in the 2^n rows by incrementing the n-bit binary number from $0 - 2^n - 1$.

Example:
Generate all combinations of $\{1, 2, 3, 4\}$.

4	3	2	1	
0	0	0	0	$\Rightarrow \emptyset$
0	0	0	1	$\Rightarrow \{1\}$
0	0	1	0	$\Rightarrow \{2\}$
0	0	1	1	$\Rightarrow \{1, 2\}$
		...		
1	1	0	0	$\Rightarrow \{3, 4\}$
1	1	0	1	$\Rightarrow \{1, 3, 4\}$
1	1	1	0	$\Rightarrow \{2, 3, 4\}$
1	1	1	1	$\Rightarrow \{1, 2, 3, 4\}$

We can actually identify the n-bit bitstrings or n-tuples w/ n-dimensional unit cubes!



An algorithm generating all binary n-tuples by only changing 1 bit at a time is called a <u>Gray code of order n</u>. If it is possible to change 1 more bit at the end to reach the beginning is called <u>cyclic</u>.

We can inductively construct Gray codes of any order $n \geq 1$ by reflecting the Gray code & appending a 0 on the 1st half & 1 on the 2nd.

These Gray codes are called <u>reflected</u> gray codes & they are guaranteed to be cyclic.

Chapter 5: The Binomial Coefficients

We can generate binomial coefficients w/ Pascal's triangle.

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \end{array} \iff \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

That is every new entry is the sum of its 2 parents plus two 1s on the frontier. This comes from the following properties

$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1 \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \text{ where } 0 \le k \le n.$$

Pascal's Formula

Recall: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Also note

$$\binom{n}{k} = \binom{n}{n-k} \qquad \& \qquad \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$

Def:

The counting numbers (i.e. the naturals) are

$$\binom{n}{1} = n$$

(Number of points stacked in a n-tall line in 1D)

The triangle numbers are

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

(Number of dots stacked in a n-tall triangle in 2D)

The tetrahedral numbers are

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3}$$

(Number of balls stacked in a n-tall tetrahedron in 3D.)

Thm:

Let $n \in \mathbb{N}$ & $x, y \in \mathbb{R}$ (really any ring). Then

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \dots + \binom{n}{n-1} x y^{n-1} + y^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Coro:

Let $n \in \mathbb{N}$ & $x \in \mathbb{R}$. Then

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Rem:
$k\binom{n}{k} = n\binom{n-1}{k-1}$. This falls from the factorial definition of binom.

$$\binom{n}{0} + \binom{n}{2} + \dots = \binom{n}{1} + \binom{n}{3} + \dots = 2^{n-1}$$

We obtain this from the binomial theorem & $(1-1)^n$.

$$(1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots = 0 \Rightarrow \binom{n}{0} + \binom{n}{2} + \dots = \binom{n}{1} + \binom{n}{3} + \dots$$

& $\binom{n}{0} + \dots + \binom{n}{n} = 2^n$

Thm:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Thm:
The binomial coefficients are said to be a unimodal distribution.
That is for any $n \in \mathbb{N}$

$$\binom{n}{0} < \binom{n}{1} < \dots < \binom{n}{n/2} \quad \& \quad \binom{n}{n/2} > \dots > \binom{n}{n-1} > \binom{n}{n} \text{ where } n \text{ even.}$$

$$\binom{n}{0} < \binom{n}{1} < \dots < \binom{n}{(n-1)/2} = \binom{n}{(n+1)/2} > \dots > \binom{n}{n-1} > \binom{n}{n} \text{ where } n \text{ odd.}$$

Coro:
For any $n \in \mathbb{N}$ the largest binomial coefficient is

$$\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$$

The Multinomial Theorem

The binomial theorem gave us an expansion for $(x+y)^n$. The multinomial theorem is a generalization to expand
$(x_1 + \dots + x_t)^n$

Def:
A multinomial coefficient is calculated as follows for any $n_1, \dots, n_t \in \mathbb{N}^0$

$$\binom{n}{n_1 \, \dots \, n_t} = \frac{n!}{n_1! \cdots n_t!} \quad \text{where } n = n_1 + \dots + n_t$$

Rem:
In multiset notation, the binomial coefficient becomes

$$\binom{n}{k} = \binom{n}{k \;\; n-k}$$

Thm:
Just as we had a Pascal's formula for binomial coefficients, we have it for multinomial coefficients

$$\binom{n}{n_1 \, \dots \, n_t} = \binom{n-1}{n_1 - 1 \, \dots \, n_t} + \dots + \binom{n}{n_1 \, \dots \, n_t - 1}$$

Thm: Multinomial Theore

Let $n_1, \ldots, n_t \in \mathbb{N}^0$ where $n = n_1 + \ldots + n_t$. Then for all $x_1, \ldots, x_t \in R$

$$(x_1 + \ldots + x_t)^n = \sum_{\substack{n_i \\ \text{combs}}} \binom{n}{n_1 \cdots n_t} x_1^{n_1} + \ldots + x_t^{n_t}$$

Rem:

$$\binom{n}{n_1 \cdots n_t} = \binom{n}{n_1}\binom{n-n_1}{n_2} \cdots \binom{n-n_1-\ldots-n_{t-1}}{n_{t-1}}$$

Chains & Antichains & Sperner's Theorem

Def:

Let S be a set of n elements. An antichain of S is a collect A of subsets of S where no subset is contained inside another.

Example:

$S = \{a, b, c, d\}$

$A = \{\{a, b\}, \{b, c, d\}, \{a, d\}, \{a, c\}\}$

Def:

One way to create an antichain A_k of S where $|S| = n$ & $0 \leq k \leq n$ is to let A_k be all k-subsets of S.

Every k-subset contains an element not in the other, otherwise they would be the same.

Def:

Let S be a set of n elements. A collection C of subsets of S is a chain provided

$\forall A_1, A_2 \in C \quad A_1 \neq A_2 \Rightarrow A_1 \subset A_2$ or $A_2 \subset A_1$.

Example:

$\emptyset \subset \{2, 3, 5\} \subset \{1, 2, 3, 5\}$

Def:

A chain C in S is called a maximal chain iff it contains of each possible size. Formally if $S = \{1, \ldots, n\}$ a maximal chain is

$A_0 = \emptyset \subset A_1 \subset \ldots \subset A_n$ where $|A_i| = i$.

Rem:

There is a 1-1 correspondance b/w permutations & maximal chains. Consider the ith element of the permutation to be the element in A_i not in A_{i-1}. That is the element added at that link.

Thm:

Let S be a set of n elements. Any antichain in S has at most $\binom{n}{\lfloor n/2 \rfloor}$ sets.

Pf:

Let $\mathcal{A}$ be an antichain. We count the number β of ordered pairs (A, C) where $A \in \mathcal{A}$ & C is a maximal chain containing A.

Since each maximal chain contains at most one subset of the antichain $\mathcal{A}$, we know β is at most the number of maximal chains, that is

$$\beta \leq n!$$

Now consider one subset $A \in \mathcal{A}$. We know if $|A| = k$ there are at most $k!(n-k)!$ maximal chains C containing A. Let α_k be the number of subsets in the antichain $\mathcal{A}$ of size k so that $|\mathcal{A}| = \sum_{k=0}^{n} \alpha_k$. Thus

$$\beta = \sum_{k=0}^{n} \alpha_k \, k!(n-k)!$$

Since $\beta \leq n!$ we calculate

$$\sum_{k=0}^{n} \alpha_k \, k!\,(n-k)! \leq n!$$

$$\Rightarrow \sum_{k=0}^{n} \alpha_k \frac{k!\,(n-k)!}{n!} = \sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} \leq 1$$

By our earlier corollary, $\binom{n}{k}$ is at a maximum when $k = \lfloor n/2 \rfloor$, so

$$|\mathcal{A}| = \sum_{k=0}^{n} \alpha_k \leq \binom{n}{\lfloor n/2 \rfloor}$$

Note: We will later prove something stronger than this.

More on Partially Ordered Sets (POSETS)

Previously, we discussed chains & antichains on the poset $P(X)$ of all subsets of X. We can extend that notion to any poset.

Def:

Let $(X, \leq)$ be a finite poset.

An antichain is a subset A of X where no pair of elements is comparable.
A chain is a subset A of X where every pair of elements is comparable.

In other words, a chain is a totally ordered subset of X. (We normally write chains in order as such.)

Thm:

For any antichain A & chain C

$$|A \cap C| \leq 1$$

Example:

Consider poset $X = \{1, \dots, 10\}$ under $|$ "is divisible by".

$\{4, 6, 7, 9, 10\}$ is an antichain of size 5 since no no integer in the set is divisible by another (i.e. they are all coprime)

$\{1, 2, 4, 8\}$ is a chain of size 4.

There are no size 6 antichains or size 5 chains

Rem:

Recall the minimal element of a poset is a x st $x \leq a \ \forall a$.
Likewise the maximal element is a y st $a \leq y \ \forall a$.

The minimal elements of a poset form a chain as do the maximal elements.

Thm:

Let $(X, \leq)$ be a finite poset. Let r be the size of the largest chain.
Then X can be partioned into r but no fewer antichains
(Smallest chain partion is r)

Thm: Dilworth's Theorem

Let $(X, \leq)$ be a finite poset. Let m be the size of the largest antichain.
Then X can be partioned into m but no fewer antichains.
(Smallest antichain partion is m)

Example:

Let $S = \{1, \dots, n\}$. Consider the poset $P(S)$. By our earlier theorem, the largest size of an antichain for $P(S)$ is the largest binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$. Each chain will contain exactly one subset of size $\binom{n}{\lfloor n/2 \rfloor}$

$n=1$:

$\emptyset \subset \{1\}$

$n=2$:

$\emptyset \subset \{2\} \subset \{3\}$ & $\{2\}$

$n=3$:

$\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\}$ & $\{2\} \subset \{2,3\}$ & $\{3\} \subset \{1,3\}$

For this kind of poset to get the next set of chains, go thru every chain & create 2 new chains (or 1 if they result in the same chain)

i) Attach at the end the subset obtained by appending n to the last subset of the chain.

ii) Append n to all subsets of the chain & then delete the last one.

Def:

A chain partition of the subsets of $\{1, \ldots, n\}$ is a symmetric chain partition iff

i) Each subset in a chain has 1 more element than the preceding subset

ii) The size of the first subset in a chain plus the last subset in the chain equals n. (In a chain w/ one subset, then it is both the first & last so it's size is $n/2$.)

The number of chains in a symmetric chain partition is $\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$.

Inclusion Exclusion Principle

The inclusion-exclusion principle gives us a way to calculate arbitrary unions' cardinalities.

Thm:

The number of objects in the set S that have neither property P_1 or P_2 are

$$|\bar{A}_1 \cap \bar{A}_2| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|$$

For having no properties $P_1, \dots, P_m$ it is

$$|\bar{A}_1 \cap \dots \cap \bar{A}_m| = |S| - \left(\sum |A_i|\right) + \left(\sum |A_i \cap A_j|\right) - \left(\sum |A_i \cap A_j \cap A_k|\right) + \dots + (-1)^m |A_1 \cap \dots \cap A_m|.$$

Note: The number of terms grows as

$$\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \dots + \binom{m}{m} = 2^m$$

Coro:

The number of objects in set S that have at least one property $P_1, \dots, P_m$ are

$$|A_1 \cup \dots \cup A_m| = \left(\sum |A_i|\right) - \left(\sum |A_i \cap A_j|\right) + \left(\sum |A_i \cap A_j \cap A_k|\right) + \dots + (-1)^{m+1} |A_1 \cap \dots \cap A_m|.$$

We know this b/c

$$\overline{A_1 \cup \dots \cup A_m} = \bar{A}_1 \cap \dots \cap \bar{A}_m \quad \& \quad |\bar{C}| = |S| - |C|$$

Multisets & Inclusion-Exclusion

Recall:

The number of r-combinations a multiset w/ k-distinct objects, each w/ repetition numbers $\geq r$ is

$$\binom{r+k-1}{r}$$

Rather than stating specific formula, the textbook just lists examples, so that's what I'll do.

Example:

Determine the number of 10-combinations of the multiset $T = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.

We shall apply inclusion-exclusion on $T^* = \{10 \cdot a, 10 \cdot b, 10 \cdot c\}$.

Let P_1 be the property that a 10-combination has more than 3 a's, P_2 be more than 4 b's, & P_3 be more than 5 c's. Define A_1, A_2, A_3 as previously.

We wish to find $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$. By the inclusion-exclusion principle

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$$

We find the parts now.

$$|S| = \binom{10+3-1}{10} = \binom{12}{10}$$

To find A_1 since we know a must occur 4 times, this is the same as the 6-combinations of T^* (w/ the a's added later)

$$|A_1| = \binom{6+3-1}{6} = \binom{8}{6}$$

Similarly for A_2 & A_3

$$|A_2| = \binom{5+3-1}{5} = \binom{7}{5} \quad \& \quad |A_3| = \binom{4+3-1}{4} = \binom{6}{4}$$

For $A_1 \cap A_2$ we need 4 a's & 5 b's, so we find the 1 combinations

$$|A_1 \cap A_2| = \binom{1+3-1}{1} = \binom{3}{1} = 3$$

For $A_1 \cap A_3$ we need 4 a's & 6 c's. This is 10 elements so

$$|A_1 \cap A_3| = 1$$

For $A_2 \cap A_3$ & $A_1 \cap A_2 \cap A_3$ we need more than 10 elements so

$$|A_2 \cap A_3| = |A_1 \cap A_2 \cap A_3| = 0$$

And so

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = \binom{12}{10} - \left(\binom{8}{6} + \binom{7}{5} + \binom{6}{4}\right) + (3+1+0) - 0 = 6$$

Derangements

If we remove 8 sparkplugs from a V8, how many ways can we put the plugs back such that no plug goes into the cylinder whence it came.

<u>Def</u>:

Let $X = \{1, \ldots, n\}$. A <u>derangement</u> of X is a permutation $i_1 i_2 \ldots i_n$ such that $i_k \neq k$ $\forall k = 1..n$.

The <u>derangement number</u> D_n is the number of derangements of $\{1, \ldots, n\}$.

Thm:

For $n \geq 1$

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}\right]$$

Rem:

From alternative infinite series, we conclude e^{-1} & $D_n/n!$ differ by less than $\frac{1}{(n+1)!}$

In fact D_n is the integer closest to $n!/e$.

Rem:

$\frac{D_n}{n!}$ is the ratio of number of derangements & total permutations.

Consider the event E that a random permutation is a derangement. Then

$$Prob(E) = \frac{D_n}{n!} \approx \frac{1}{e} \qquad \& \quad |E| = D_n.$$

Example:

If there are n men & n women at a party, how many pairs can there be for the first dance? What about the second if they all must switch partners?

First Dance: $n!$
Second Dance: D_n

Permutations w/ Forbidden Positions $\leftarrow$ Generalization of derangements

Def:

Let $X_1, \dots, X_n$ be possibly empty subsets of $\{1, \dots, n\}$.
We denote the set of all permutations $i_1, \dots, i_n$ of $\{1, \dots, n\}$ such that $i_k \notin X_k$ $\forall k = 1, \dots, n$ by
$P(X_1, \dots, X_n)$.

We denote the number of permutations by
$p(X_1, \dots, X_n)$ by $|P(X_1, \dots, X_n)|$.

Example:

Let $n=5$ & $X_1=\{1,4\}$, $X_2=\{3\}$, $X_3=\emptyset$, $X_4=\{1,5\}$, $X_5=\{2,5\}$.

Then $P(X_1,\dots,X_5)$ has a 1-1 correspondance w/ the placements of 5 non-attacking rooks on the board w/ the given forbidden positions.

	1	2	3	4	5
1	X			X	
2			X		
3					
4	X				X
5		X			X

Chapter 7: Recurrence Relations & Generating Functions

Fibonacci Numbers

Sometimes w/ counting problems we get recurrence relations, where the value at an input depends on / is defined by a previous value. If we can write these in closed form we call them generating functions.

Example:

Consider an arithmetic & geometric series w/ some initial term h_0 & constant q_0.

Arithmetic: $h_0, h_0+q, h_0+2q, \ldots$

Geometric: $h_0, qh_0, q^2h_0, \ldots$

The recurrent & general definition of h_n is

Arithmetic: $h_n = h_{n-1} + q$ $\qquad h_n = h_0 + nq$

Geometric: $h_n = qh_{n-1}$ $\qquad h_n = q^n h_0$

Example:

Consider the Fibonacci sequence f_n: 1, 1, 2, 3, 5, 8, ...

$f_n = f_{n-1} + f_{n-2}$ $\quad n \geq 2$

$f_0 = 0$ & $f_1 = 1$

Example:

The partial sums of the Fibonacci sequence

$s_n = \sum_{i=0}^{n} f_i = f_{n+2} - 1$

We show this by induction. When $n=0$

$s_0 = f_0 = 0 = 1-1 = f_2 - 1$

When $n \geq 1$, suppose $s_n = f_{n+2} - 1$. We find s_{n+1}

$s_{n+1} = (f_0 + \ldots + f_n) + f_{n+1}$

$= s_n + f_{n+1}$

$= f_{n+2} - 1 + f_{n+1}$

$= f_{n+3} - 1$ ✓

Thus this holds for all $n \geq 0$.

Let's try to write the Fibonacci sequence like a geometric series.

Consider the Fibonacci recurrence relation written as

$f_n - f_{n-1} - f_{n-2} = 0 \quad \forall n \geq 2.$

We want to find some q such that $f_n = q^n$. (Note that we say the first term is 1)

This q satisfies the Fibonacci recurrence relation iff $\forall n \geq 2$
$q^n - q^{n-1} - q^{n-2} = 0$ or equivalently $q^{n-2}(q^2 - q - 1) = 0$.

Since we assume $q \neq 0$, we find q thru the roots of
$q^2 - q - 1 = 0$.

Using the quadratic formula we find
$q_1 = \frac{1+\sqrt{5}}{2}$ & $q_2 = \frac{1-\sqrt{5}}{2}$

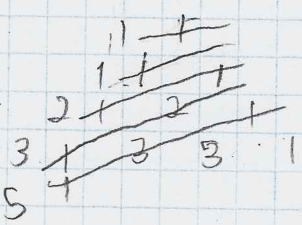
so
$f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ & $f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$ satisfy the recurrence relation.

Since the recurrence relation is linear & homogenous, we know
$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ $\forall c_1, c_2 \in \mathbb{R}$ satisfy the recurrence.

Solving the initial values $f_0 = 0$ & $f_1 = 1$, we find the direct formula for the Fibonacci numbers
$$\underline{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

Our conclusion actually works for any initial conditions $f_0 = a$ & $f_1 = b$ given the following system is non-singular
$c_1 + c_2 = a$
$c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = b$

<u>Thm:</u>
The sums of the binomial coefficients along the diagonals of Pascal's triangle running upward from the left are Fibonacci numbers



More formally
$$f_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \dots + \binom{n-t}{t-1} \quad \text{where } t = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Generating Functions

<u>Def:</u>
Let $h_0, h_1, \dots$ be an infinite sequence of numbers. It's <u>generating function</u> is the infinite power series
$g(x) = h_0 + h_1 x + \dots + h_n x^n + \dots$

This can be extended to finite sequences by concidering an infinite sequence of 0s following the last term.

Example:

Consider an infinite sequence of 1s.

$$g(x) = 1 + x + \dots + x^n + \dots$$

This generating function is the sum of a geometric series, so

$$g(x) = \frac{1}{1-x}.$$

Example:

Let $k \in \mathbb{N}$ & let the sequence $h_0, h_1, \dots$ be defined by letting h_n equal the number of non-negative integral solutions to

$$e_1 + \dots + e_k = n.$$

From Chapter 3 we know

$$h_n = \binom{n+k-1}{k-1} \quad n \geq 0$$

Thus the generating function is

$$g(x) = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

From chapter 5 we know this means

$$g(x) = \frac{1}{(1-x)^k}$$

Recall the derivation from this

$$\left.\begin{aligned} \frac{1}{(1-x)^k} &= \left(\frac{1}{1-x}\right) \cdots \left(\frac{1}{1-x}\right) \\ &= (1+x+\dots) + \dots + (1+x+\dots) \end{aligned}\right\} k \text{ times}$$

$$= \left(\sum_{e_1=0}^{\infty} x^{e_1}\right) \cdots \left(\sum_{e_k=0}^{\infty} x^{e_k}\right)$$

Example:

Let h_n denote the number of non-negative integral solutions to the equation

$$3e_1 + 4e_2 + 2e_3 + 5e_4 = n.$$

Find the generating function $g(x)$ for $h_0, h_1, \dots$

We introduce a change of variables

$$f_1 = 3e_1 \quad f_2 = 4e_2 \quad f_3 = 2e_3 \quad f_4 = 5e_5.$$

So now we are finding the number of solutions to
$f_1 + f_2 + f_3 + f_4 = n$
where f_1 is a multiple of 3, f_2 of 4, f_3 of 2, & f_4 of 5.

Now we find

$$g(x) = [1+x^3+x^6+\dots](1+x^4+\dots)[1+x^2+x^4+\dots](1+x^5+\dots)$$
$$= \left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)$$

Thm:

Let $n \in \mathbb{N}$. Then

$$g_n(x) = 1(1+x)(1+x+x^2)\cdots(1+x+\dots+x^{n-1}) = \frac{\prod_{j=1}^{n}(1-x^j)}{(1-x)^n}$$

Exponential Generating Function

Exponential generating functions where the terms are considered w/ respect to the monomials.

$$1,\ x,\ \frac{x^2}{2!},\ \dots,\ \frac{x^n}{n!},\ \dots$$

which arise in the Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So given a sequence $h_0, h_1, \dots$ the exponential generating function is

$$g^{(e)}(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$$

Thm:

Let $S = \{n_1 \cdot a_1, \dots, n_k \cdot a_k\}$ be a multiset. Let h_n be the n-permutations of S. Then the exponential generating function $g^{(e)}(x)$ for $h_0, h_1, \dots$ is

$$g^{(e)}(x) = f_{n_1}(x) \cdots f_{n_k}(x)$$

where

$$f_{n_i}(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n_i}}{n_i!}$$

Solving Linear Homogenous Recurrence Relations

Def:

Let (h_n) be a sequence of numbers. We say (h_n) satisfies a linear recurrence relation of order k iff $\exists\, a_1, \dots, a_k$ w/ $a_k \neq 0$ & quantity b_n such that

$$h_n = a_1 h_{n-1} + \dots + a_k h_{n-k} + b_n \qquad \text{(Note } a_1, \dots, a_k \text{ may depend on } n\text{.)}$$

Example:

The sequence of derangement numbers (D_n) satisfies the following recurrence relation

$$\underbrace{D_n = nD_{n-1} + (-1)^n}_{\text{order } 1} \quad \& \quad \underbrace{D_n = (n-1)D_{n-1} + (n-1)D_{n-2}}_{\text{order } 2}$$

Example:

The Fibonacci numbers (f_n) are a linear recurrence of order 2.

Def:

We say the recurrence relation is <u>homogeneous</u> iff $b_n = 0$ always & has constant coefficients if $a_1, \dots, a_k$ are constants.

Def:

Since the recurrence relation doesn't kick in until $n = k$, we define initial values $h_1, \dots, h_{k-1}$.

Example:

Consider the differential equation
$$y'' - 5y' + 6y = 0.$$

We find solutions $y = e^{qx}$, $y' = qe^{qx}$, $y'' = q^2 e^{qx}$.

So it follows $y = e^{ax}$ is a solution iff
$$q^2 e^{qx} - 5qe^{qx} + 6e^{ax} = 0$$
Since $e^{ax} \neq 0$ ever we cancel to get
$$q^2 - 5q + 6 = 0$$
This has roots $q = 2$ & $q = 3$.

Thm:

Let q be a non-zero number. Then $h_n = q^n$ is a solution to the recurrence relation
$$h_n - a_1 h_{n-1} - \dots - a_k h_{n-k} = 0 \qquad a_k \neq 0.$$
w/ constant coefficients iff q is a root of
$$x^k - a_1 x^{k-1} - \dots - a_k = 0.$$

If the polynomial has k distinct roots $q_1, \dots, q_k$ then
$$h_n = c_1 q_1^n + \dots + c_k q_k^n$$
is the general solution.

Thm:

Let $q_1, \dots, q_t$ be distinct roots of the following characteristic equation of the linear homogeneous recurrence relation w/ constant coefficients
$$h_n = a_1 h_{n-1} + \dots + a_k h_{n-k} \qquad a_k \neq 0.$$

If q_i is a s_i-fold root of the equation, the part of the general solution of this reccurrence relation for q_i is
$$H_n^{(i)} = c_1 q_i^n + c_2 n q_i^n + \dots + c_{s_i} n^{s_i - 1} q_i^n$$
$$= (c_1 + c_2 n + \dots + c_{s_i} n^{s_i - 1}) q_i^n.$$

Thm:

Let (h_n) be a sequence satisfying
$$h_n + c_1 h_{n-1} + \dots + c_k h_{n-k} = 0 \qquad c_k \neq 0$$

Then the generating function $g(x)$ is the form
$$g(x) = \frac{p(x)}{q(x)}$$
where $q(x)$ is a k-degree polynomial w/ non-zero constant term & p is a polynomial of $\leq k$-degree.

Nonhomogeneous Recurrence Relations

In general, non-homogeneous equations are harder to solve

Example:

Consider the tower of Hanoi problem. The number of moves required for n disks satisfies the recurrence relation
$$h_n = 2h_{n-1} + 1 \qquad \& \qquad h_0 = 0.$$

That is you solve the bigger problem by moving $n-1$ disks to the middle (h_{n-1}), moving the bottom disk, then moving $n-1$ disks again.

By finding this we find
$$h_n = 2^{n-1} + \dots + 2 + 1 = 2^n - 1.$$
Here is the work for this
$$\begin{aligned} h_n &= 2h_{n-1} + 1 \\ &= 2(2h_{n-2} + 1) + 1 \\ &= 2^2 h_{n-2} + 2 + 1 \\ &= 2^2(2h_{n-3} + 1) + 2 + 1 \\ &= 2^3 h_{n-3} + 2^2 + 2 + 1 \\ &\;\;\vdots \\ &= 2^{n-1} + \dots + 1 \\ &= 2^n - 1. \end{aligned}$$

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Chapter 8: Special Counting Sequences

Catlan Numbers

Def:

The Catlan sequence $C_0, C_1, \ldots$ defined by
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

For example,

$C_0 = 1$ $\quad$ $C_4 = 14$

$C_1 = 1$ $\quad$ $C_5 = 42$

$C_2 = 2$ $\quad$ $C_6 = 132$

$C_3 = 5$ $\quad$ $C_7 = 429$

Thm:

The number of sequences $a_1, \ldots, a_{2n}$ are the numbers that can be formed by n $+1$'s & n -1's whose partial sums are always positive. equals the nth Catalan number
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

This theorem is a more generalized version of more specific problems.

Example:

Suppose $2n$ people are lined up for a 50¢ theater. n people have 50¢ & n people have 1\$ & will need change.

How many ways can people line up such that the theater never runs out of change? Assume individuals are indistinguishable beyond their payment method.

This is the nth Catalan number C_n.

Thm: The Catalan numbers (C_n) have recurrence relation
$$C_n = \frac{4n-2}{n+1} C_{n-1}$$

Pf:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1} \cdot \frac{(2n)!}{n!\,n!} \quad \& \quad C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} = \frac{1}{n} \cdot \frac{(2n-2)!}{(n-1)!\,(n-1)!}$$

Dividing C_n/C_{n-1} gives us

$$\frac{C_n}{C_{n-1}} = \frac{1}{n+1}\cdot\frac{(2n)!}{n!\,n!}\cdot\frac{n}{1}\,\frac{(n-1)!\,(n-1)!}{(2n-2)!} = \frac{\cancel{n}}{n+1}\cdot\frac{(2\cancel{n})(2n-1)}{\cancel{n}\cdot\cancel{n}} = \frac{4n-2}{n+1}$$

Then we just solve for C_n

$$C_n = \frac{4n-2}{n+1}\,C_{n-1}.$$

<u>Def:</u>

The pseudo-Catalan numbers $C_1^*, C_2^*, \ldots$ are defined by

$C_n^* = n!\,C_{n-1}$ ← normal Catalan number

We can simplify this using our recurrence relation to get

$$C_n^* = n!\,\frac{4n-6}{n}\,C_{n-2} = (n-1)!\,(4n-6)\,C_{n-2} = (4n-6)\,C_{n-1}^*$$

& $C_n^* = 1$

We also get the direct expression

$$C_n^* = (n-1)!\binom{2n-2}{n-1} = \frac{(2n-2)!}{(n-1)!}$$

Partition Numbers

<u>Def:</u>

A <u>partition of $n \in \mathbb{N}$</u> is a representation of n as a sum of positive integers called <u>parts</u>. We sometimes write

$$\lambda = n^{a_n} \ldots 1^{a_1}$$

where a_i is the number of parts equal to i. We omit numbers where $a_i = 0$.

<u>Example:</u>

Here are the partitions of 5

5^1, $4^1 1^1$, $3^1 2^1$, $3^1 1^2$, $2^2 1^1$, $2^1 1^3$, 1^5

<u>Def:</u>

Let p_n be the number of partitions for $n \in \mathbb{N}$ & let $p_0 = 1$. The <u>partition sequence</u> is the numbers

$$(p_n) = p_0, p_1, \ldots$$

<u>Thm:</u>

p_n is the number of solutions using non-negative integers $a_1, \ldots, a_n$ of the equation

$$n a_n + \ldots + 2a_2 + 1a_1 = n.$$

<u>Def:</u>

The <u>Ferrers diagram</u> of a partition represents it as dots.

For example for $10 = 4^1 2^2 1^2$

(4) • • • •
(2) • •
(2) • •
(1) •
(1) •

Thm:

Let $n, r \in \mathbb{N}$ where $r \le n$. Let $p_n(r)$ be the number of partitions of n in which the largest part is r. Let $q_n(r)$ be the number of partitions of $n-r$ where no part is greater than r. Then
$$p_n(r) = q_n(r).$$

Pf:

Consider a partion of n w/ largest part r. Remove r & you have a partition of $n-r$ where no part is greater than r.

Conversely, consider a partion of $n-r$ w/ no part greater than r. Add r & you get a partition of n where the largest part is r.

Def:

The conjugate of partion λ of number $n \in \mathbb{N}$ is the partion λ^* obtained by flipping the rows & columns of the diagram of λ.

For example

$\lambda = 4^1 2^2 1^2$ $\qquad$ $\lambda^* = 5^1 3^1 1^2$

Note: $\lambda = (\lambda^*)^*$.

A partition is a self-conjugate partion iff $\lambda = \lambda^*$.

Thm:

Let $n \in \mathbb{N}$. Let p_n^s be the number of self-conjugate partitions of n & let p_n^t be the number of partitions of n in distinct odd parts. Then
$$p_n^s = p_n^t.$$

Thm:

Let $n \in \mathbb{N}$. Let p_n^o be the number of partitions of n into odd parts & p_n^d into distinct parts. Then
$$p_n^o = p_n^d.$$

Pf:

Given duplicate odd parts, combine them until distinct. Given distinct even parts, split them until odd.

<u>Thm:</u>

$$\sum_{n=0}^{\infty} P_n x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Let P_n be the set of all partitions of $n \in \mathbb{N}$. Then there is a natural partial order of P_n.

Consider two partitions of n

$\lambda = n_1 + \dots + n_k \quad (n_1 \geq \dots \geq n_k \geq 0)$ & $\mu = m_1 + \dots + m_k \quad (m_1 \geq \dots \geq m_k \geq 0)$.

We say <u>λ is majorized by μ</u>, or alternatively <u>μ majorizes λ</u> & write $\lambda \leq \mu$ iff $n_1 + \dots + n_i \leq m_1 + \dots + m_i \quad \forall i = 1, \dots, k$

This majorization relation is reflexive, antisymmetric, & transitive & thus a partial order.

We can extend the majorization partial order to get a total order called lexicographic order.