1.1 Background

- Differential Equation (DE): Equation containing some derivative either ordinary (ODE) or partial (PDE)
  - ODE: $\frac{d^3y}{dt^2} + \sin(t)\, y = t^4$ ← order = 3
  - PDE: $\frac{\partial y}{\partial s} + s^2 \frac{\partial y}{\partial t} = s + t$ ← squiggly ∂'s

※ $\frac{dy}{dx} \Rightarrow$ y is dependent variable, x is independent

- Order of DE: Highest derivative present
- Linear ODE: ODE where we have functions of independent variable times some derivative of the dependent variable

$$A_n(x)\frac{d^ny}{dx^n} + A_{n-1}(x)\frac{d^{n-1}y}{dx} + \dots A_1(x)\frac{dy}{dx} + A_0(x)y = B(x)$$

- Non-Linear ODE: Not linear ODE.
- Independent & Dependent Variables:

$\frac{dy}{dx} \Rightarrow$ y is dependent (on x), x is independent

Words to DEs

The rate of change of mass A of salt is proportional to the square of mass present at time t.

$$\frac{dA}{dt} = kA^2, \quad k \in \mathbb{R}$$

1.2 Solutions & Initial Value Problems

Goals:

- Show that something isn't a solution to a DE
- Given an initial value problem (IVP) determine if a unique solution exists.

Solutions

- Solution to DE: Function $y(x)$ which satisfies (i.e. yields correct results) for the given DE on $a < x < b$.
  - Explicit Solution: Solution solved for dependent variable. ← preferred
  - Implicit Solution: Solution not solved for dependent variable.
- Candidate Solution: Solution we're testing.

To validate a solution you take its derivatives, plug into the DE, & solve.

Implicit Differentiation Review ← Used for verifying implicit solutions

$$\frac{d}{dx}[x] = 1, \quad \frac{d}{dx}[x^3] = 3x^2$$

$$\frac{d}{dx}[y] = \frac{dy}{dx}, \quad \frac{d}{dx}[y^3] = 3y^2\frac{dy}{dx}$$

Essentially, you're doing $\frac{d}{dy}[\ldots] \cdot \frac{dy}{dx}$, i.e. take the derivative normally & multiply by $\frac{dy}{dx}$.

Initial Value Problems (IVPs)

- Initial Value Problem: Find the solution to an nth order DE w/ n initial conditions. *Solutions don't neccessarily need to be unique.*

To validate these solutions, you do the same you did earlier, but also validate the initial conditions. Normally check initial conditions first b/c they're easy.

Solutions to IVPs don't necessarily need to be unique. But, we'd like to be able to determine which are unique & which aren't.

(On test, you must always show checking the initial conditions for credit.)

<u>Theorem:</u> Existence & Uniqueness Theorem (E&UT)

Given IVP $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$. (Partial's are used here.)

If f & $\frac{\partial f}{\partial y}$ are continuous functions in a rectangle R in the xy-plane that contains (x_0, y_0) in interior, then the IVP has a unique solution defined on interval I where I is contained w/in R.

Otherwise, either there is no solution or the solutions aren't unique.

make sure to mention continuity!

(We won't prove this here. Proof in chapter 13.)

* Note: Since y is dependent on x, we normally only define the interval for x like

$a < x < b$ ← This also shows we need a continuous range containing x_0

1.3 Direction Fields (or Slope Fields)

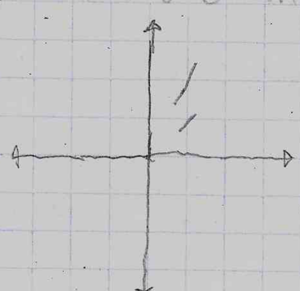
(On test, you always have to talk about continuity)

<u>Goals:</u>
- Determine nature of solution to 1st order ODE. ← Helps in understanding & also when we can't get an explicit solution.

$\frac{dy}{dx} = f(x,y)$ gives us the slope of the solution at all points (x,y). When we graph this (by drawing arrows at points in the direction of slope) we call it a <u>direction field</u>.

$\frac{dy}{dx} = xy$

x	y	xy
0	y	0
1	1	1
1	2	2



This technique of plotting random points takes FOREVER (if you're not a computer), use the method below if possible.

(*Slope fields give us <u>families</u> of solutions to DEs.)

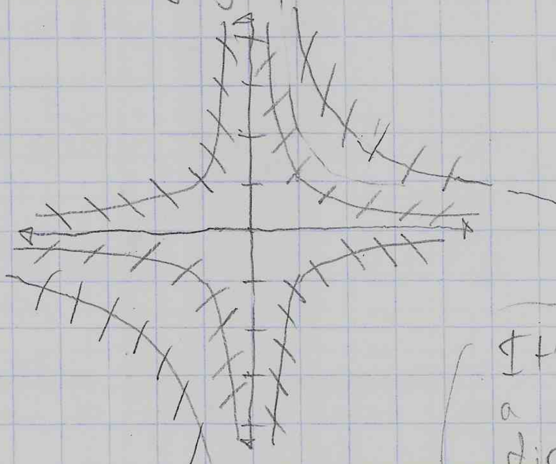
Method of Isocline

A technique to make creating direction fields easier.

- **Isocline** (or level curves): Set of points in xy-plane where all the solutions have the same slope.

Isoclines are useful b/c they give you (a description of) a bunch of points.

$\frac{dy}{dx} = xy$



Let $\frac{dy}{dx} = xy = 0$.
$\Rightarrow x = 0$ or $y = 0$

Let $\frac{dy}{dx} = xy = 1$
$\Rightarrow y = 1/x$

Let $\frac{dy}{dx} = xy = 2$
$\Rightarrow y = 2/x$

Let $\frac{dy}{dx} = xy = -1$
$\Rightarrow y = -1/x$

It's still best to use a computer to draw a direction field.

Using Slope Fields

The reason we want to make slope fields in the first place is to help us solves DEs (& IVPs).

For IVPs, go to the given point & draw a curve following the slope field from that point. This is the solution.

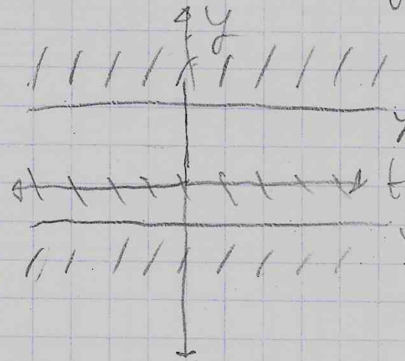
pg 32-34 Phase Lines

Goal:
- Sketch condensed direction field for autonomous DE
- Classify equilibria

- **Autonomous DE**: DE whose independent variable doesn't appear explicitly.
  $\frac{dy}{dt} = f(y)$.
- **Equilibrium Solution**: Constant solution making $\frac{dy}{dt} = 0$. nothing changes!
  $y = c \Rightarrow \frac{dy}{dt} = 0$ (remember derivative of constant is always zero)
  - Stable Equilibrium / Sinks: Attracts neighboring solutions.
  - Sources: Repels neighboring solutions.
  - Nodes: Neither.

Draw direction field of $\frac{dy}{dt} = (y-2)(y+1)$

Equilibrium Solutions: $y=2$, $y=-1$



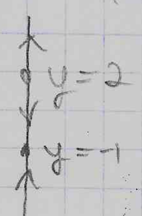
$y > 2 \Rightarrow \frac{dy}{dt} > 0$; $y=3 \Rightarrow \frac{dy}{dt} = 4$

$-1 < y < 2 \Rightarrow \frac{dy}{dt} < 0$; $y=0 \Rightarrow \frac{dy}{dt} = -2$

$y < -1 \Rightarrow \frac{dy}{dt} > 0$; $y=-2 \Rightarrow \frac{dy}{dt} = 4$

* Note: Since $\frac{dy}{dt}$ above is autonomous, t never mattered meaning all vertical lines were identical. A phase line take advantage of this by only drawing one vertical strip w/ arrows parallel for direction.

For $\frac{dy}{dt} = (y-2)(y+1)$ above



This is like a sign chart from calc. 1.

Or a test do something like

$y > 2 \Rightarrow \frac{dy}{dt} > 0$

$-1 < y < 2 \Rightarrow \frac{dy}{dt} < 0$

...

Using the phase line, we can see $y=2$ is a source & $y=-1$ is a source.

As $t \to \infty$, what happens to solution $y(0) = -2$? It tends to $y=-1$.

For $y(0) = 2$? It stays at $y=2$.

For $y(0) = 4$? It tends to $+\infty$.

2.2 Separables DEs

- Separable DE: DE which can be rewritten as product of two functions $f(x)$ & $g(y)$.

Separable DEs allow us to do the following

Our first technique for solving DEs!

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x) \qquad g(y) \neq 0$$

$G'(y) \equiv 1/g(y)$ ← Triple bars mean define LHS to be RHS

$F'(x) \equiv f(x)$

$$G'(y)\frac{dy}{dx} = F'(x)$$

Recognize we have the derivative of the outside (G) times the derivative of the inside (y) on the LHS. This is the chain rule!

$$\frac{d}{dx}\left[G(y)\right] = F'(x)$$

We now integrate wrt x to get our general solution.

$$\boxed{G(y) = F(x) + C}$$

Technique: ← This uses the above
- Separate y's & x's to get
  $\frac{dy}{g(y)} = f(x)\,dx$ ← * This is an abuse of notation, treating $\frac{dy}{dx}$ like a fraction. We can do this b/c of the formal proof above since this gives us the above
- Integrate
  $\int \frac{dy}{g(y)} = \int f(x)\,dx$ ← Don't forget to add +C (to RHS)
- Solve for y (if possible).
- Check for any solutions lost when $g(y)=0$ (b/c we divided by $g(y)$)

Solve IVP!

$$\frac{dy}{dx} = (1+y^2)\tan(x), \quad y(0)=\sqrt{3}$$

$$\int \frac{dy}{1+y^2} = \int \tan(x)\,dx$$

Since $1+y^2$ can't be 0, we can't lose solutions.

$$\tan^{-1}(y) = \int \frac{\sin(x)}{\cos(x)}\,dx$$ ← u sub time

$u = \cos(x)$

$du = -\sin(x)\,dx$

Sin $\frac{\sqrt{3}}{2}$

$\tan^{-1}(y) = -\int \frac{1}{u}\,du$

$\tan^{-1}(y) = -\ln|u| + C$

$\tan^{-1}(y) = -\ln|\cos(x)| + C$

$\boxed{y = \tan(-\ln|\cos(x)| + C)}$ ← General Solution

$y(0) = \tan(-\ln|\cos(0)| + C) = \sqrt{3}$

$-\ln|1| + C = \pi/3$

$C = \pi/3$

$$\boxed{y = \tan(-\ln|\cos(x)| + \pi/3)}$$

2.3 Linear Equations

Sometimes you get separable linear equations!

Recall 1st order linear equation: $\boxed{a_1(x)\frac{dy}{dx} + a_0(x)\,y = b(x)}$

To illustrate the technique of solving these kinds of equations, we manipulate the above equation.

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = b(x)$$

$\div\ 1/a_1(x)$

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}\,y = \frac{b(x)}{a_1(x)}$$

$$P(x) \equiv \frac{a_0(x)}{a_1(x)}$$

$$Q(x) \equiv \frac{a_0(x)}{a_1(x)}$$

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)\,y}$$

We call the boxed form standard form & we no longer care about a_1, a_0, or b.
Here's how you solve a standard form equation.

Multiply both sides by $\mu(x)$ s.t. $\mu(x)P(x) = \mu'(x)$.

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)\,y = \mu(x)Q(x)$$

$$\underbrace{\mu(x)\frac{dy}{dx} + \mu'(x)\,y}_{\text{product rule}} = \mu(x)Q(x)$$

$$\frac{d}{dx}[\mu(x)\,y] = \mu(x)Q(x)$$

$$\mu(x)\,y = \int(\mu(x)Q(x)dx) + C \leftarrow \text{don't forget!}$$

$$y = \frac{\int(\mu(x)Q(x)dx) + C}{\mu(x)} \leftarrow C \text{ can't absorb } \mu(x) \text{ b/c } C \text{ is a constant \& } \mu(x) \text{ isn't.}$$

Since $\mu(x)P(x) = \mu'(x)$, $P(x) = \frac{\mu'(x)}{\mu(x)} = \frac{d}{dx}[\ln(\mu(x))]$. We use this to find $\mu(x)$.

$$\Rightarrow \int P(x)dx = \ln(\mu(x))$$

$$\mu(x) = e^{\int P(x)dx + C'}$$

$$\mu(x) = K e^{\int P(x)dx}$$

$$y = \frac{\int(\cancel{K}e^{\int P(x)dx}Q(x)dx) + C}{\cancel{K}e^{\int P(x)dx}}$$

$$\boxed{y = \frac{\int e^{\int P(x)dx}Q(x)\,dx) + C}{e^{\int P(x)dx}}}$$

$$\boxed{\text{OR } \quad y = \frac{\int \mu(x)Q(x)dx + C}{\mu(x)} \quad \text{where } \mu(x) = e^{\int P(x)dx}}$$

* Note: you don't need a $+C$ w/ $\int P(x)dx$ b/c it cancels out, as shown

Examples:

- Solve $x\frac{dy}{dx} + 2y = x^{-3}$

$$\frac{dy}{dx} + \frac{2}{x}y = x^{-4}$$

$P(x) = 2/x$

$Q(x) = x^{-4}$

$\mu(x) = e^{\int P(x)dx}$

$\int P(x)dx = 2\ln|x|$

$= e^{2\ln|x|}$

$= x^2$ ← the ± from abs value cancels

$$y = \frac{\int \mu(x)\, Q(x)\, dx + C}{\mu(x)}$$

$$= \frac{\int x^2 x^{-4} dx + C}{x^2}$$

$$= \frac{\int x^{-2} dx + C}{x^2}$$

$$\boxed{y = \frac{-x^{-1} + C}{x^2}}$$

$x^2\frac{dy}{dx} + x^2\frac{2}{x}y = x^{-4}x^2 = x^{-2}$

$\frac{d}{dx}[x^2 y] = x^{-2}$

$x^2 y = \int x^{-2} dx$

$x^2 y = -x^{-1} + C$

$y = \frac{-x^{-1} + C}{x^2}$

- Solve the IVP $\frac{dy}{dx} - 2y = e^{2t}\sin(t)$

$$\frac{dy}{dx} - 2y = e^{2t}\sin(t)$$

$P(t) = -2$

$Q(t) = e^{2t}\sin(t)$

$\mu(t) = e^{\int P(t)dt} = e^{-2t}$

$e^{-2t}\frac{dy}{dx} - e^{-2t}\cdot 2y = e^{-2t}e^{2t}\sin(t)$

$\frac{d}{dx}[e^{-2t}y] = \sin(t)$

$e^{-2t}y = \int \sin(t)\, dt$

$e^{-2t}y = -\cos(t) + C$

$y = e^{2t}(-\cos(t) + C)$

$y(0) = 3$

$\cancel{e^{2(0)}}(-\cos(0) + C) = 3$

$-1 + C = 3$

$C = 4$

$$\boxed{y = e^{2t}(-\cos(t) + 3)}$$

Existence & Uniqueness

For 1st order linear equations, we solve for $\frac{dy}{dx}$ as before.

$$\frac{dy}{dx} + P(x)y = Q(x)$$
$$\frac{dy}{dx} = -P(x)y + Q(x)$$

Now $f(x,y) = -P(x)y + Q(x)$. Since we have this specific form though, we only need <u>$P(x)$ & $Q(x)$ to be cont. at & around x_0</u>, as shown below.

$$f(x,y) = -P(x)y + Q(x)$$
$$\frac{\partial f}{\partial y} f(x,y) = -P(x)$$

<u>Examples:</u>

Does there exist a unique solution to $\frac{dy}{dt} = 2y + e^{2t}\sin(t)$ @ $y(0) = 3$?

$P(t) = 2$ is continuous everywhere

$Q(t) = e^{2t}\sin(t)$ is continuous everywhere

$\therefore$ E&U thm guarantees unique solution.

OR $f(x,y) = 2y + e^{2t}\sin(t)$ cont. everywhere

$\frac{\partial y}{\partial t} f(x,y) = 2$ cont. everywhere

3.2 Mixing Problems

In Calc 2 we had tanks @ volume equilibrium. W/ 1st order linear equations, we can solve non-volume equilibrium problems.

Let $x(t)$ be amount of substance tank at time t.
$\frac{dx}{dt}$ be rate of change of amount of substance.

Using basic physical knowledge

$$\frac{dx}{dt} = (\text{input rate}) - (\text{output rate}) = (\text{flow in})(\text{concentration in}) - (\text{flow out})(\text{concentration out})$$

$$\frac{dx}{dt} = F_i C_i - F_o C_o$$

$F :: \frac{\text{volume}}{\text{time}}$

$C :: \frac{\text{amount}}{\text{volume}}$

* Note: In all cases here, since we assume well-mixed substance, $C_o = \frac{x(t)}{\text{volume}}$

When $F_i = F_o$, the volume of fluid doesn't change, but concentration will.

<u>Examples:</u>

- Brine solution of salt flows at constant rate 8 L/min into large tank which initially held 100 L of brine w/ 0.5 kg of salt. The solution is well-mixed & flow out occurs at the same rate. If the concentration of brine entering is 0.05 kg/L, determine mass of salt at t min. When will concentration reach 0.02 kg/L?

$F_i = F_o = 8$ L/min, $C_i = 0.05$ kg/L, $C_o = x(t)/100$ kg/L

$x(0) = 0.5$ kg

$\frac{dx}{dt} = F_i C_i - F_o C_o = (8)(0.05) - (8)\left(\frac{x}{100}\right)$

this would be different if volume isn't const.

$\frac{dx}{dt} = \frac{2}{5}(1 - \frac{1}{5}x)$ ← We could solve as separable or 1st order linear. We choose separable.

$\int \frac{dx}{1-\frac{1}{5}x} = \int \frac{2}{5}\,dt$

$u = 1 - \frac{1}{5}x$

$du = -\frac{1}{5}dx \Rightarrow dx = -5du$

$-5\int \frac{du}{u} = \frac{2}{5}t + C$

$-5\ln|u| = \frac{2}{5}t + C$

$\ln|1-\frac{1}{5}x| = -\frac{2}{25}t + C$

$1 - \frac{1}{5}x = \pm e^{-\frac{2t}{25} + C}$

$-\frac{1}{5}x = Ke^{-\frac{2t}{25}} - 1$

$x = Ke^{-\frac{2t}{25}} + 5$

$x(0) = K\cancel{e^0} + 5 = 0.5$

$K = -4.5$

$\therefore \boxed{x = -4.5e^{-\frac{2t}{25}} + 5}$

Find $C = 0.02$ kg/L. $C = \frac{x \text{ kg}}{100 \text{ L}} \Rightarrow x = 2$ kg.

$2 = -4.5e^{-\frac{2t}{25}} + 5$

$4.5e^{-\frac{2t}{25}} = 3$

$e^{-\frac{2t}{25}} = 3/4.5$

$-\frac{2}{25}t = \ln(3/4.5)$

$\boxed{t = -\frac{25}{2}\ln(3/4.5) \approx 5.07 \text{ min}}$

$\boxed{\lim_{t\to\infty} x(t) = 5 - 4.5e^{-\frac{2}{25}t} = 5 \text{ kg}}$

Now, we can use a similar method for when $F_i \neq F_o$. (i.e. volume is changing). However, it won't be separable.

$F_i = 6 \quad C_i = 0.2$

$F_o = 8 \quad C_o = \frac{x(t)}{V(t)} = \frac{x}{200-2t}$

$V(0) = 200 \quad V(t) = V(0) + (F_i - F_o)t = 200 - 2t$

$x(0) = 0.005 \cdot V(0) = 1$

In the problem, since we're given percent, we're just getting bogus units for x but that's okay.

$$\frac{dx}{dt} = (6)(0.2) - (8)\left(\frac{x}{200-2t}\right) = 1.2 - \frac{4x}{100-t}$$ "Formulate the IVP"

Notice that this equation is not separable. We therefore must do 1st order linear. ~~We~~

We rewrite the equation in standard form.

$$\frac{dx}{dt} + \frac{4}{100-t}x = 1.2$$

no $+C$ b/c coefficients cancel

$$\mu(t) = e^{\int P(t)dt} = e^{-4\ln|100-t|} = (100-t)^{-4}$$

$$P(t) = \frac{4}{100-t}$$

$$Q(t) = 1.2$$

$$\frac{d}{dt}[\mu(t)x] = 1.2\mu(t)$$

*Recall we multiply by $\mu(t)$, convert $\mu(t)P(t)$ into $\mu'(t)$, & then use the product rule

From construction of μ.

$$\frac{d}{dt}\left[(100-t)^{-4}x\right] = 1.2(100-t)^{-4}$$

Integrate

$$(100-t)^{-4}x = \int 1.2(100-t)^{-4}dt$$

$$(100-t)^{-4}x = 1.2\cdot\left(-\tfrac{1}{3}(100-t)^{-3}\cdot -1\right) + C$$

$$(100-t)^{-4}x = 0.4(100-t)^{-3} + C$$

$$x = (100-t)^4(0.4(100-t)^{-3} + C)$$

$$\boxed{x = 0.4(100-t) + C(100-t)^4}$$

$$x(0) = 0.4(100-0) + C(100-0)^4 = 1$$

$$40 + 100^4C = 1$$

$$C = \frac{-39}{100^4}$$

$$\boxed{x = 0.4(100-t) - \frac{39}{100^4}(100-t)^4}$$

When will concentration be 10%?

$$0.1 = \frac{x(t)}{V(t)} = \frac{0.4(100-t) - \frac{39}{100^4}(100-t)^4}{200-2t}$$

$$0.2 = \frac{0.4(100-t) - \frac{39}{100^4}(100-t)^{4\,3}}{100-t}$$

$$0.2 = 0.4 - \frac{39}{100^4}(100-t)^3$$

$$\frac{39}{100^4}(100-t)^3 = 0.6$$

$$(100-t)^3 = \frac{100^4\cdot 0.6}{39}$$

$$100-t = \sqrt[3]{\left(\frac{100^4}{39}\right)0.6}$$

$$\boxed{t = 100 - \sqrt[3]{\left(\frac{100^4}{39}\right)0.6} \approx 19.96 \text{ min}}$$

Population problems

We can also model population growth/decline as a first order linear DE (when growth is proportional & there's an external force).

Fish Example

Initial mass is 7 million tons. Natural growth rate proportionality constant is 2x/year. Fishing removes 15 million tons / year.

Let $x(t)$ be mass of fish in million tons at year t after the start.

$x(0) = 7$

$\frac{dx}{dt} = 2x - 15$ ← Treat as separable.

$\int \frac{dx}{x - 15/2} = \int 2\, dt$

$\ln|x - 15/2| = 2t + C$

$|x - 15/2| = Ke^{2t}$

$\underline{x = Ke^{2t} + 15/2}$ ← K absorbs the +/-

$x(0) = Ke^{0} + 15/2 = 7$

$K = -1/2$

$$\boxed{x = -\frac{1}{2}e^{2t} + \frac{15}{2}}$$

$x = -\frac{1}{2}e^{2t} + 15/2 = 0$

$e^{2t} = 15$

$t = \frac{\ln(15)}{2} \approx 1.354$ years

What fishing rate is sustainable?

$\frac{dx}{dt} = 2x - a = 0$

$a = 2x$

$x(0) = 7$

$\Rightarrow \boxed{a = 14}$

3.3 Heating & Cooling

Def: Newton's Law of Heating/Cooling
Rate of change of temperature is proportional to the difference of outside temperature $M(t)$ & inside temperature $T(t)$.

$T(t)$ = temp. of object
$M(t)$ = temp. of surroundings

$\frac{dT}{dt} = K(M(t) - T(t))$ ← go towards $M(t)$

Suppose $M(t)$ is constant @ M_0. We solve the separable DE.

$\frac{dT}{dt} = K(M_0 - T(t))$

$\frac{dT}{dt} = -K(T - M_0)$

$\int \frac{dT}{T - M_0} = \int -K\,dt$

$\ln|T - M_0| = -Kt + C$

$|T - M_0| = e^{-Kt+C}$

$T - M_0 = Ae^{-Kt}$

$T = Ae^{-Kt} + M_0$

Suppose $T(0) = T_0$.

$T(0) = Ae^0 + M_0 = T_0$

$A = T_0 - M_0$

$\therefore \boxed{T = M_0 + (T_0 - M_0)e^{-Kt}}$

Def: Time Constant for Heating/Cooling
An arbitrary measure of how quickly/slowly temperature changes.
It is $1/K$ & is the time for $T_0 - M_0$ to change to $\frac{T_0 - M_0}{e}$

$t = 0: (T_0 - M_0)e^0$

$t = 1/K: (T_0 - M_0)e^{-K \cdot 1/K}$

Example:

$M_0 = 12°C$ When will T be $16°C$?

$T_0 = 21°C$

$1/K = 3$ hr

$K = 1/3$ $1/hr$ ← messy unit

$T = 12 + (21 - 12)e^{-1/3 t} = 16$

$9e^{-1/3 t} = 4$

$e^{-1/3 t} = 4/9$

$-1/3 t = \ln(4/9)$

$t = -3\ln(4/9) \approx 2.43$ hr

2.4 Exact d.e.'s

implicit solution!

Goal: Solve d.e. that arise from curves like $F(x,y)=C$

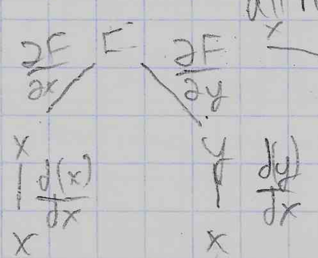
We'll manipulate it a bit now to justify the method.

Given $F(x,y)=C$ & $y=y(x)$

More 1st order linear equations :o!

$\frac{dF}{dx}=0$ b/c const.

Now we do chain rule for another form of $\frac{dF}{dx}$.



all rooted at x

$$\underline{\frac{dF}{dx}=\frac{\partial F}{\partial x}\frac{d(x)}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}}=F_x\overset{1}{\cancel{\frac{d(x)}{dx}}}+F_y\frac{dy}{dx}$$

$$\frac{dF}{dx}=0=F_x+F_y\frac{dy}{dx}$$

Let $M(x,y)=F_x$ & $N(x,y)=F_y$

$$\boxed{\begin{gathered}M(x,y)+N(x,y)\frac{dy}{dx}=0\\ \text{or}\\ M(x,y)\,dx+N(x,y)\,dy=0\end{gathered}}$$

Thm:

Let M, N, M_y, & N_x be continuous in rectangular region R where $R=\{(x,y) \mid a<x<b,\ c<y<d\}$.

$M(x,y)dx+N(x,y)dy=0$ is an <u>exact d.e.</u> in R iff $M_y(x,y)=N_x(x,y)$ for all $(x,y)\in R$.

<u>Justification</u>

Recall $F_x=M$, $F_y=N$.

$M_y=F_{xy}$ & $N_x=F_{yx}$.

$F_{xy}=F_{yx}$ by Clairaut's theorem when F is continuous along region.

Examples:

Determine if equation is exact & find the solution if it is.

Exact?

$(e^x\sin(y)-3x^2)\,dx+(e^x\cos(y)+y^{-2/3}/3)\,dy$

$M=e^x\sin(y)-3x^2$

$N=e^x\cos(y)+\frac{y^{-2/3}}{3}$

Is $M_y \overset{?}{=} N_x$? In other words, does $F_{xy}=F_{yx}$

$M_y=e^x\cos(y)$

$N_x=e^x\cos(y)$

$M_x=N_y \Rightarrow$ this is an exact d.e. $\Rightarrow F(x,y)=C$.

Find Solution

Recall $F_x=M$ & $F_y=N$.

$F_x=e^x\sin(y)-3x^2$

$F_y=e^x\cos(y)+\frac{y^{-2/3}}{3}$

{She actually doesn't care how you do it that much. You can do it in your head}

You could solve this in your head, but to illustrate we'll do it fully here.

$F_x=e^x\sin(y)-3x^2$

↓ integrate wrt x

$F=e^x\sin(y)-x^3+g(y)$

↓ diff wrt y

$F_y=e^x\cos(y)+g'(y)$ ← If the test for exactness fails, this fails here.

↓ combine w/ other F_y

$F_y=\cancel{e^x\cos(y)}+g'(y)=\cancel{e^x\cos(y)}+\frac{y^{-2/3}}{3}$ ← Things should match up here.

$g'(y)=\frac{y^{-2/3}}{3}$

↓ integrate

$g(y)=y^{1/3}+C'$ ← gets absorbed by C in $F(x,y)=C$

$\therefore F=e^x\sin(y)-x^3+y^{1/3}+C'$

$\therefore F(x,y)=e^x\sin(y)-x^3+y^{1/3}=C$

It's possible to have inexact d.e.'s made exact by multiplying by some combo of x & y. This does mean you can lose solutions. You must check these.

* Note: We can & will have IVPs w/ these.

4.2 Homogeneous Linear Equation

Def: Linear 2nd Order, Constant Coefficient Diff Eq

$ay''+by'+cy=f(t)$ ← like a quadratic!

(t ← or x)

where

$a\neq 0$ ← otherwise not 2nd order

a,b,c are const.

When $f(t)=0$, we call these homogeneous.

$ay''+by'+cy=0$

Thm: *Important!

If y_1 & y_2 are solutions to $ay'' + by' + cy = 0$
c_1 & c_2 are constants

Then $y = c_1 y_1 + c_2 y_2$ is also a solution.

This lets you synthesize a solution

Proof

We show $y = c_1 y_1 + c_2 y_2$ solves $ay'' + by' + cy$ (i.e. makes it 0)

$$a(c_1 y_1'' + c_2 y_2'') + b(c_1 y_1' + c_2 y_2') + c(c_1 y_1 + c_2 y_2)$$
$$= ac_1 y_1'' + bc_1 y_1' + cc_1 y_1$$
$$+ ac_2 y_2'' + bc_2 y_2' + cc_2 y_2$$
$$= c_1(ay_1'' + by_1' + cy_1)$$
$$+ c_2(ay_2'' + by_2' + cy_2)$$

Since y_1 & y_2 are solutions, $ay_1'' + by_1' + cy_1 = 0$ & likewise for y_2.

$\therefore c_1 \cdot 0 + c_2 \cdot 0 = 0$

$\therefore y = c_1 y_1 + c_2 y_2$ is also a sol.

Def: Linear Independent

y_1 & y_2 are linearly independent if $ay_1(t) \neq y_2(t)$ (i.e. they're not constant multiples)

Def: Linear Dependent

y_1 & y_2 are linearly dependent if $ay_1(t) = y_2(t)$ for some a (i.e. they are constant multiples)

Examples:

$y_1 = 3e^{2t}$, $y_2 = e^t$ are linearly independent.

$y_1 = 3e^{2t}$, $y_2 = -7e^{2t}$ are linearly dependent. $(a = -7/3)$

Thm:

Given 2 linearly independent solutions y_1 & y_2 to $ay'' + by' + cy = 0$,
The general solution is

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

where c_1 & c_2 are arbitrary constants

They need to be linearly independent or else they'd combine & their constants would mush together.

Solving Linear Homogeneous d.e.'s w/ theorem:

We try to solve $ay'' + by' + cy = 0$ w/ $y = e^{rt}$

$y = e^{rt}$, $y' = re^{rt}$, $y'' = r^2 e^{rt} \Rightarrow ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$

We factor out e^{rt} to get

$e^{rt}(ar^2+br+c)=0$

Here $\boxed{ar^2+br+c=0}$ is the characteristic/auxillary equation.

This gives us $r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ & their are cases familiar to determinants in earlier classes

Case 1: $b^2-4ac>0$

We have two distinct roots & thus 2 r's.

$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$, $r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$

This gives us two linearly independent solutions $y_1=e^{r_1t}$ & $y_2=e^{r_2t}$

∴ Therefore the general solution is $\boxed{y=c_1e^{r_1t}+c_2e^{r_2t}}$

Case 2: $b^2-4ac=0$

We have a repeated root & thus 1 r.

$r=\frac{-b}{2a}$

This gives us one linearly independent solution (or two linearly dependent solutions)

$y_1=c_1e^{rt}$

Our y_2 is $y_2=c_2te^{rt}$. y_1 & y_2 are trivially linearly independent, so now we show y_2 is a solution. (i.e. $ay_2''+by_2'+cy_2=0$)

$y_2=te^{rt}$, $y_2'=e^{rt}+tre^{rt}$, $y_2''=re^{rt}+re^{rt}+tr^2e^{rt}=2re^{rt}+tr^2e^{rt}$

$ay_2''+by_2'+cy_2$

$=a(2re^{rt}+tr^2e^{rt})+b(e^{rt}+tre^{rt})+c(te^{rt})$

$=2are^{rt}+atr^2e^{rt}+be^{rt}+btre^{rt}+cte^{rt}$

$=e^{rt}(2ar+atr^2+b+btr+ct)$

$=e^{rt}(t(\underbrace{ar^2+br+c}))+2ar+b)$

Since ar^2+br+c is our characteristic equation & r solves it, $ar^2+br+c=0$

$=e^{rt}(2ar+b)$

Recall $r=-b/2a$

$=e^{rt}(2a(-b/2a)+b)$

$=e^{rt}(-b+b)$

$=e^{rt}(0)$

$=0$

∴ y_2 is a solution to $ay''+by'+cy=0$

∴ The general solution is $\boxed{y=c_1e^{rt}+c_2te^{rt}}$

Case 3: $b^2 - 4ac < 0$

We have complex roots.

We'll rewrite the quadratic equation to help us get a nice answer.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a}$$

$$= \frac{-b \pm \sqrt{4ac - b^2}\sqrt{-1}}{2a}$$

$$= \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i$$

$$\left.\begin{aligned} \alpha &\equiv \frac{-b}{2a} \\ \beta &\equiv \frac{\sqrt{4ac - b^2}}{2a} \end{aligned}\right\} \text{handy for simplifying}$$

$$\equiv \alpha \pm \beta i$$

$$\therefore r_1 = \alpha + \beta i, \; r_2 = \alpha - \beta i$$

$$\therefore y = c_1 e^{(\alpha + \beta i)t} + c_2 e^{(\alpha - \beta i)t}$$

$$= c_1 e^{\alpha t + \beta i t} + c_2 e^{\alpha t - \beta i t}$$

$$= c_1 e^{\alpha t} e^{\beta i t} + c_2 e^{\alpha t} e^{-\beta i t}$$

$$= e^{\alpha t}\left(c_1 e^{\beta i t} + c_2 e^{-\beta i t}\right)$$

Recall Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$$y = e^{\alpha t}\left(c_1(\cos(\beta t) + i\sin(\beta t)) + c_2(\cos(-\beta t) + i\sin(-\beta t))\right)$$

$$= e^{\alpha t}\left(c_1 \cos(\beta t) + c_1 i \sin(\beta t) + c_2 \cos(-\beta t) + c_2 i \sin(-\beta t)\right)$$

use even & odd properties

$$= e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \cos(\beta t) + c_1 i \sin(\beta t) - c_2 i \sin(\beta t)\right)$$

$$= e^{\alpha t}\left((c_1 + c_2)\cos(\beta t) + (c_1 i - c_2 i)\sin(\beta t)\right)$$

$$\left.\begin{aligned} c_1' &= c_1 + c_2 \\ c_2' &= c_1 i - c_2 i \end{aligned}\right\} \text{they're arbitrary constants anyway. let's clean them up}$$

$$= e^{\alpha t}\left(c_1' \cos(\beta t) + c_2' \sin(\beta t)\right)$$

In conclusion, the general solution if $r = \alpha \pm \beta i$ is

$$\boxed{y = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)}$$

Example: Case 1

solve $y''-y'-2y=0$

$a=1,\ b=-1,\ c=-2$

characteristic equation: $ar^2+br+c=0$

$r^2-r-2=0$

$(r-2)(r+1)=0$

$r_1=2,\ r_2=-1$

$\Rightarrow y_1=e^{2t},\ y_2=e^{-t}$

$\therefore y=c_1y_1+c_2y_2=c_1e^{2t}+c_2e^{-t}$

Example: Case 2

solve $y''+6y'+9y=0$

$r^2+6r+9=0$ ← characteristic eqn.

$(r+3)^2=0$

$r=-3$

$\Rightarrow y_1=e^{-3t},\ y_2=te^{-3t}$

$\therefore y=c_1e^{-3t}+c_2te^{-3t}$

Example: Case 1 IVP

Solve $y''+2y'-8y=0$ where $y(0)=3$ & $y'(0)=-12$

$r^2+2r-8=0$

$(r+4)(r-2)=0$

$r_1=-4,\ r_2=2$

$\Rightarrow y=c_1e^{-4t}+c_2e^{2t}$ $\quad y'=-4c_1e^{-4t}+2c_2e^{2t}$

$y(0)=c_1+c_2=3 \rightarrow 4c_1+4c_2=12$

$y'(0)=-4c_1+2c_2=-12$

$0c_1+6c_2=0$

$c_2=0$

$y(0)=c_1+0=3$

$c_1=3$

$\boxed{y=3e^{-4t}}$

Example: Case 3 IVP

solve $y''+2y'+17y=0$ where $y(0)=1$ & $y'(0)=-1$

$r^2+2r+17=0$

$r=\dfrac{-2\pm\sqrt{4-4(1)(17)}}{2}$

$=\dfrac{-2\pm\sqrt{4(1-17)}}{2}$

$=\dfrac{-2\pm2\sqrt{-16}}{2}$

$=-1\pm4i$

$\alpha=-1,\ \beta=4$ ← could be -4 b/c constant absorbs sign

$y = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t)) = e^{-t}(c_1\cos(4t) + c_2\sin(4t))$
$y' = -e^{-t}(c_1\cos(4t) + c_2\sin(4t)) + e^{-t}(-4c_1\sin(4t) + c_2\cos(4t))$

$y(0) = \cancel{e^0}(c_1\cancel{\cos(0)} + \cancel{c_2\sin(0)}) = 1$
$c_1 = 1$

$y'(0) = -\cancel{e^0}(c_1\cancel{\cos(0)} + \cancel{c_2\sin(0)}) + \cancel{e^0}(\cancel{-4c_1\sin(0)} + 4c_2\cancel{\cos(0)}) = -1$
$-c_1 + 4c_2 = -1$
$-1 + 4c_2 = -1$
$4c_2 = 0$
$c_2 = 0$

$\therefore \boxed{y = e^{-t}\cos(4t)}$

Boundary Value Problems (BVPs) are different than IVPs b/c IVPs give you multiple values at one point (i.e. $y(0)=1$ & $y'(0)=-1$). Meanwhile BVPs give you different points (normally w/ the same function) (i.e. $y(0)=2$ & $y(\pi/6)=-(2)$.
- IVPs always have one unique solution.
- BVPs may have none, one, or infinitely many.

4.4 Method of Undetermined Coefficients

non-homogenous, 2nd order, linear d.e.

Goal: Find solutions to $ay'' + by' + cy = f(t)$ where $f(t) \neq 0$ & $f(t)$ is sine, cosine, an exponential, or a polynomial.

The related homogenous d.e. $ay'' + by' + cy = 0$ is called the complementary equation.

Undetermined coefficients is basically an educated guess.

Thm: Complementary Solution & Particular Solution
The general solution to $ay'' + by' + cy = f(t)$ is
$y = y_c + y_p$

where
y_c is the general solution to the complementary equation ($ay'' + by' + cy = 0$) ← sometimes y_H b/c homogenous
y_p is any particular solution to $ay'' + by' + cy = f(t)$

Pf:

Suppose y is any solution to $ay'' + by' + cy = f(t)$. We show $y - y_p$ is solution to $ay'' + by' + cy = 0$

$a(y'' - y_p'') + b(y' - y_p') + c(y - y_p)$
$= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p)$
$= f(t) - f(t)$ ← by definition of y_p
$= 0$

$\therefore$ $y - y_p$ solves $ay'' + by' + cy = 0$

We now have $y - y_p = y_c$, giving us $y = y_c + y_p$. □

Cool! This theorem doesn't help us w/ finding y_p at all. This is where the method of undetermined coefficients comes in.

Example: $f(t)$ is polynomial

Solve $y'' + 4y' + 4y = 12t$

Step 1: Find y_c

Solve $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$r = -2$

$y_c = c_1 e^{-2t} + c_2 t e^{-2t}$

Step 2: Find y_p

Notice $f(t)$ is a polynomial of degree 1. This means the solution will be a polynomial of degree 1. (Match the degree)

$y_p = At + B$, $y_p' = A$, $y_p'' = 0$ ← Look! Undetermined coefficients

We now find A & B.

$0 + 4(A) + 4(At + B) = 12t$

$4A + 4At + 4B = 12t$

$(4A)t + (4A + 4B) = 12t$

$4A = 12$ $\quad$ $4A + 4B = 0$

$A = 3$ $\quad$ $B = -A$

$B = -3$

$\therefore y_p = 3t - 3$

Now that we know y_c & y_p, we have

$$\boxed{y = y_c + y_p = (c_1 e^{-2t} + c_2 t e^{-2t}) + (3t - 3)}$$

Example: $f(t)$ is sine or cosine

Solve $y'' + 4y' + 4y = 169\cos(3t)$

Recall $y_c = c_1 e^{-2t} + c_2 t e^{-2t}$ from earlier

We now find y_p

Since $f(t)$ is sine or cosine of $3t$, the solution will be of the form

$y_p = A\cos(3t) + B\sin(3t)$, $y_p' = -3A\sin(3t) + 3B\cos(3t)$, $-9A\cos(3t) - 9B\sin(3t)$

We now find A & B (on the next page).

$$(-9A\cos(3t) - 9B\sin(3t)) + 4(-3A\sin(3t) + 3B\cos(3t))$$
$$+ 4(A\cos(3t) + B\sin(3t)) = 169\cos(3t)$$
$$(-9A + 12B + 4A)\cos(3t) + (-9B - 12A + 4B)\sin(3t) = 169\cos(3t)$$

$-5A + 12B = 169$ $\qquad$ $-5B - 12A = 0$

$-5A - {}^{144}/_{5}A = 169$ $\qquad$ $B = -{}^{12}/_{5}A$

$-{}^{169}/_{5}A = 169$

$A = -5 \longrightarrow B = 12$

$y_p = -5\cos(3t) + 12\sin(3t)$

Now w/ y_p & y_c, we find

$$\boxed{y = (c_1 e^{-2t} + c_2 t e^{-2t}) + (-5\cos(3t) + 12\sin(3t))}$$

Example: $f(t)$ is exponential

Solve $y'' + 4y' + 4y = e^{4t}$

Recall $y_c = c_1 e^{-2t} + c_2 t e^{-2t}$ from earlier

We now find y_p:

Since $f(t)$ is an exponential, the solution will be of the form

$y_p = Ae^{4t}$, $y_p' = 4Ae^{4t}$, $y_p'' = 16Ae^{4t}$

$16Ae^{4t} + 4(4Ae^{4t}) + 4(Ae^{4t}) = e^{4t}$

$36Ae^{4t} = e^{4t}$

$A = 1/36$

$y_p = \frac{1}{36}e^{4t}$

Now w/ y_p & y_c, we find

$$\boxed{y = (c_1 e^{-2t} + c_2 t e^{-2t}) + {}^{1}/_{36}\, e^{4t}}$$

Example: Uh-oh Exponential

Solve $y'' + 4y' + 4y = 7e^{-2t}$

Recall $y_c = c_1 e^{-2t} + c_2 t e^{-2t}$ from earlier.

Now, using the method above, we'd get $y_p = Ae^{-2t}$, but that is linearly dependent w/ y_c.

To make it not linearly dependent, we multiply by t some number of times (as we did when we got repeated roots earlier).

Normally we'd just multiply by t but that won't work b/c it'd still be linearly dependent. We therefore multiply by t^2 in this case, giving us

$y_p = Ae^{-2t}t^2$
$y_p' = -2Ae^{-2t} \cdot t^2 + Ae^{-2t} \cdot 2t = 2Ae^{-2t}(-t^2+t)$
$y_p'' = -4Ae^{-2t}(-t^2+t) + 2Ae^{-2t}(-2t+1) = 2Ae^{-2t}(-2(-t^2+t)+(-2t+1))$
$= 2Ae^{-2t}(2t^2-2t-2t+1)$
$= 2Ae^{-2t}(2t^2-4t+1)$

$2Ae^{-2t}(2t^2-4t+1) + 4(2Ae^{-2t}(-t^2+t)) + 4(Ae^{-2t}t^2) = 7e^{-2t}$
$\cancel{4Ae^{-2t}t^2} - \cancel{8Ae^{-2t}t} + 2Ae^{-2t} - \cancel{8Ae^{-2t}t^2} + \cancel{8Ae^{-2t}t} + \cancel{4Ae^{-2t}t^2} = 7e^{-2t}$
$2Ae^{-2t} = 7e^{-2t}$
$A = 7/2$

$y_p = 7/2\, e^{-2t}t^2$

Now w/ y_p & y_c, we find
$y = (c_1e^{-2t} + c_2te^{-2t}) + (7/2e^{-2t} t^2)$

Summary of Undetermined Coefficients

Given d.e. $ay'' + by' + cy = f(t)$ to solve.

Find $y_c = c_1y_1 + c_2y_2$.

Here's a summary on how to find y_p.

$f(t)$	y_p
at^n	$A_nt^n + \dots + A_0$
$a\cos(bt)$ or $a\sin(bt)$	$A\cos(bt) + B\sin(bt)$
ae^{bt}	Ae^{bt}

You can also trivially do combinations by combining the y_p's in the same way. For example

$f(t) = ae^{bt}\cos(ct) \implies e^{bt}(A\cos(ct) + B\sin(bt))$ We can exclude this constant b/c A & B absorb them

If y_p is linearly dependent w/ y_c, multiply y_t by t until the issue resolves.

Superposition Principle

Def: Superposition Principle

Let y_1 be a solution to $ay''+by'+cy=f(t)$
& y_2 be a particular solution to $ay''+by'+cy=g(t)$.

$y_P = y_1 + y_2$ is a particular solution to $ay''+by'+cy=f(t)+g(t)$

This allows us to split up the right hand side more easily.

Pf: Superposition Principle

$$\begin{aligned} & ay_P''+by_P'+cy_P = \\ &= a(y_1''+y_2'')+b(y_1'+y_2')+c(y_1+cy_2) \\ &= (ay_1''+by_1'+cy_1)+(ay_2''+by_2'+cy_2) \\ &= f(t)+g(t) \quad \square \end{aligned}$$

Example:

Find form of particular solution to $y''+4y=6+\sin(2t)$

We first find the complementary solution y_c w/ the characteristic equation b/c we need l.i. of y_c & y_p.

Find y_c:

$$\begin{aligned} & r^2+4=0 \\ & r=\pm 2i \\ & \alpha=0,\ \beta=2 \\ & y_c = e^{\alpha t}(A\cos(\beta t)+B\sin(\beta t)) = A\cos(2t)+B\sin(2t) \end{aligned}$$

Find y_p:

We split this into y_{P1} & y_{P2}, which solve $y''+4y=6$ & $y''+4y=\sin(2t)$ respectively.

$y_{P1} = A$

$\cancel{y_{P2} = B\cos(2t)+C\sin(2t)}$ ↯ y_{P2} is l.d. w/ y_c, so multiply by t

$y_{P2} = (B\cos(2t)+C\sin(2t))\,t$

We have now found the form of y_{P1} & y_{P2}. We could find the constants A, B, & C by solving their respective solutions but eh

Variation of Parameters

Goal: Find y_p when undetermined coefficients doesn't work for $ay''+by'+cy=f(t)$

* Note: This also works whenever undetermined coefficients would b/c it's stronger.

We will now essentially derive/do variation of parameters.

Derive Variation of Parameters

Given $ay''+by'+cy=f(t)$ & a general solution $y_c=c_1y_1+c_2y_2$.

Redefine c_1 & c_2 as functions v_1 & v_2

$y_p=v_1(t)y_1+v_2(t)y_2=v_1y_1+v_2y_2$ for brevity

Now, take the derivative assuming $\boxed{v_1'y_1+v_2'y_2=0}$

$y_p'=v_1'y_1+v_1y_1'+v_2'y_2+v_2y_2'=v_1y_1'+v_2y_2'$

Take the derivative again to get y_p''

$y_p''=v_1'y_1'+v_1y_1''+v_2'y_2'+v_2y_2''$

We now solve $ay''+by'+cy=f(t)$ w/ y_p

$a(v_1'y_1'+v_1y_1''+v_2'y_2'+v_2y_2'')+b(v_1y_1'+v_2y_2')+c(v_1y_1+v_2y_2)=f(t)$

Now we distribute & group the v's together

$v_1(ay_1''+by_1'+cy_1)+v_2(ay_2''+by_2'+cy_2)+av_1'y_1'+av_2'y_2'=f(t)$

Recall y_1 & y_2 were solutions to the complementary equation $ay''+by'+cy=0$

$v_1(0)+v_2(0)+av_1'y_1'+av_2'y_2'=0$

$$\boxed{v_1'y_1'+v_2'y_2'=\frac{f(t)}{a}}$$

Summary: Variation of Parameters

Given $ay''+by'+cy=f(t)$

i) Find $y_c=c_1y_1+c_2y_2$

ii) $y_p=v_1y_1+v_2y_2$

iii) Find v_1 & v_2 using $v_1'y_1+v_2'y_2=0$ & $v_1'y_1'+v_2'y_2'=\frac{f(t)}{a}$ ← from $ay''+by'+cy$

Example:

Find general solution to $y''+y=\sec^3(t)$.

We can't use undetermined coefficients (what would general form of $\sec^3(t)$ be?)
Use variation of parameters

Find y_c:

$r^2+1=0$

$r=\pm i$

$\alpha=0,\ \beta=1$

$$\boxed{y_c=c_1\cos(t)+c_2\sin(t)}$$

Find y_p: $y_p = v_1 \overbrace{\cos(t)}^{y_1} + v_2 \overbrace{\sin(t)}^{y_2}$

$v_1' y_1 + v_2' y_2 = 0$

$v_1' \cos(t) + v_2' \sin(t) = 0$

$v_1' = -v_2' \dfrac{\sin(t)}{\cos(t)}$

$v_1' y_1' + v_2' y_2' = \dfrac{f(t)}{a}$

$v_1'(-\sin(t)) + v_2'(\cos(t)) = \dfrac{\sec^3(t)}{1}$ $\quad 1y'' + y = \sec^3(t)$

$-v_2'\left(\dfrac{\sin(t)}{\cos(t)}\right)(-\sin(t)) + v_2' \cos(t) = \sec^3(t)$

$v_2' \dfrac{\sin^2(t)}{\cos(t)} + v_2' \cos(t) = \sec^3(t)$

$v_2' \sin^2(t) + v_2' \cos^2(t) = \sec^2(t)$

$v_2'(\sin^2(t) + \cos^2(t) = \sec^2(t)$

$\underline{v_2' = \sec^2(t)}$

$v_1' = -\sec^2(t) \dfrac{\sin(t)}{\cos(t)} = \underline{-\sec^2(t)\tan(t)}$

$v_2 = \int v_2' \, dt = \int \sec^2(t) dt$ no +C b/c we want ~~a~~ particular solution, not the most general one

$\underline{v_2 = \tan(t)}$

$v_1 = \int -\sec^2(t) \tan(t) \, dt$ ← could also do u-substitution here, but Kurtz didn't so I didn't

$= \int -\dfrac{\sin(t)}{\cos^3(t)} \, dt$

$u = \cos(t)$

$du = -\sin(t) dt$

$= \int \dfrac{du}{u^3}$

$= \int u^{-3} dt$

$= -\tfrac{1}{2} u^{-2}$

$= -\tfrac{1}{2} \cos^{-2}(t)$

$\underline{v_1 = -\tfrac{1}{2} \sec^2(t)}$

$\therefore \boxed{y_p = -\tfrac{1}{2} \sec^2(t) \cos(t) + \tan(t) \sin(t)}$

Now that we have y_p & y_c, we find

$y = y_c + y_p = c_1 \cos(t) + c_2 \sin(t) - \tfrac{1}{2} \sec^2(t) \cos(t) + \tan(t) \sin(t)$

4.9 Free Mechanical Vibrations.

Goal: Find position function of vibrating spring.

We'll now show how we can derive $y(t)$, the position of a spring at time t

Derive: Free Mechanical Vibrations w/o Damping (Simple Harmonic Motion)

y = position, y' = velocity, y'' = acceleration.

Recall Newton's 2nd Law: $F_{net} = ma$. (w/ constant mass & no relativity) In terms of y, we have

$F_{net} = my''$.

Recall Hooke's Law: $F_{spring} = -ky$, (assuming non-excessive compression/extension) where $k>0$ is some stiffness constant.

$F_{spring} = -ky$.

Recall Friction: $F_{friction} = -by'$, where $b>0$ is some friction constant

$F_{friction} = -by'$.

We also have some external force F_{ext}.

Putting the spring, friction, & external force together we have

$$F_{spring} + F_{friction} + F_{ext} = F_{net}$$

$$-ky - by' + F_{ext} = my''$$

$$my'' + by' + ky = F_{ext}.$$

This is a 2nd order linear ODE. Nice!

For right now, we'll consider the case w/ no damping force (i.e. friction) & no external force, meaning $b=0$ & $F_{ext}=0$, giving us

$$my'' + ky = 0$$

$$y'' + k/m\, y = 0.$$

We now solve this

$$r^2 + k/m = 0$$

$$r^2 = -k/m$$

$$r = \pm\sqrt{-k/m}$$

$$r = \pm\sqrt{k/m}\, i$$ ← note that $k>0$ & $m>0$

We define our angular frequency (ω) to be $\omega = \sqrt{k/m}$, meaning

$$\omega = \sqrt{k/m},$$

meaning

$$r = \pm\omega i$$

$$\alpha = 0, \quad \beta = \omega.$$

This gives us

$$\boxed{y = c_1\cos(\omega t) + c_2\sin(\omega t)}$$

When we have this, we have <u>simple harmonic motion</u>.

In this kind of system we have the following properties:

- Amplitude $= \sqrt{c_1^2 + c_2^2}$ ($A =$ Amplitude)
- $\tan(\phi) = c_1/c_2$, the quadrant of ϕ is given by signs of c_1 & c_2. where ϕ is the phase angle.
- Period $= \frac{2\pi}{\omega}$
- Natural Frequency $= \frac{\omega}{2\pi}$

We can also rewrite our $y = c_1\cos(\omega t) + c_2\sin(\omega t)$ as
$\boxed{y = A\sin(\omega t + \phi)}$
where $c_1 = A\sin(\phi)$ & $c_2 = A\cos(\phi)$ & you use a trig identity

Example: Simple Harmonic Motion

$my'' + by' + ky = F_{ext}$
$m = 2$ kg
$b = 0$ N/(m/s)
$k = 50$ N/m
$y(0) = -1/4$ m, $y'(0) = -1$ m/s

$2y'' + 50y = 0$
$y'' + 25y = 0$
↓ characteristic eqn
$r^2 + 25 = 0$
$r^2 = -25$
$r = \pm 5i$
$\alpha = 0$, $\beta = 5$
$y = c_1\cos(5t) + c_2\sin(5t)$
$y' = -5c_1\sin(5t) + 5c_2\cos(5t)$
↓ initial conditions
$y(0) = c_1\cancel{\cos(0)} + \cancel{c_2\sin(0)} = -1/4$
$c_1 = -1/4$
$y'(0) = \cancel{-5c_1\sin(0)} + 5c_2\cancel{\cos(0)} = -1$
$c_2 = -1/5$
↓ together
$\boxed{y = -\tfrac{1}{4}\cos(5t) - \tfrac{1}{5}\sin(5t)}$ ← Equation of Motion of Mass

For fun we'll rewrite y in the form $y = A\sin(\omega t + \phi)$ ← on test we won't need

$A = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1/4)^2 + (-1/5)^2} = \frac{\sqrt{41}}{20}$
$\omega = \sqrt{k/m} = \sqrt{50/2} = \sqrt{25} = 5$
$\tan(\phi) = c_1/c_2 = \frac{-1/4}{-1/5} = 5/4$
$\phi = \tan^{-1}(5/4) - \pi$ ← put it in 3rd quadrant b/c $c_1 < 0$ & $c_2 < 0$.

$\boxed{y = \frac{\sqrt{41}}{20}\sin(5t + \tan^{-1}(5/4) - \pi)}$

When does $y=0$?

$y=\frac{\sqrt{41}}{20}\sin(5t+\tan^{-1}(5/4)-\pi)=0$
$\sin(5t+\tan^{-1}(5/4)-\pi)=0$
$5t+\tan^{-1}(5/4)-\pi=0$
$t=\frac{\pi-\tan^{-1}(5/4)}{5}\approx 0.449\text{ s}$

~~Derive~~: Damping Free Mechanic Vibration

We again have $my''+by'+ky=F_{ext}$, but in this case $b\neq 0$ & $F_{ext}=0$, giving us

$r=\frac{-b\pm\sqrt{b^2-4mk}}{2m}$.

This breaks into our 3 familiar cases

Case 1: $b^2-4mk>0$
Overdamping, no oscillations. $y=c_1e^{r_1t}+c_2e^{r_2t}$

Case 2: $b^2-4mk=0$
Critical damping, no oscillations. Converges to equilibrium fastest.
$y=c_1e^{rt}+c_2te^{rt}$

Case 3: $b^2-4mk<0$
Underdamping, has oscillations w/ decreasing amplitude.
$y=e^{\alpha t}(c_1\cos(\beta t)+c_2\sin(\beta t))$

* $\alpha<0$ b/c $m>0$, $b>0$, $k>0$ so it's always decreasing amplitude.
all strict!

Example: Damped Spring
$my''+by'+ky=F_{ext}$
$m=20$ kg
$b=140$ N/(m/s)
$k=200$ N/m
$F_{ext}=0$ N
$y(0)=0.25$ m, $y'(0)=-1$ m/s
$20y''+140y'+200y=0$
$y''+7y'+10y=0$
↓ characteristic eqn
$r^2+7r+10=0$
$(r+5)(r+2)=0$
$r=-5$, $r=-2$ ← Overdamping b/c 2 distinct roots
$y=c_1e^{-5t}+c_2e^{-2t}$
$y'=-5c_1e^{-5t}-2c_2e^{-2t}$
↓ solve IVP
$y(0)=c_1+c_2=0.25$
$y'(0)=-5c_1-2c_2=-1$

$-3c_2=-0.5$
$c_2=-1/6$

$c_1+c_2=1/4$
$c_1=1/4+1/6$
$c_1=5/12$

$$\boxed{y = \tfrac{1}{12}e^{-5t} - \tfrac{1}{6}e^{-2t}}$$

When does it reach equilibrium?

$y(t) = \frac{1}{6}e^{-5t} + \frac{1}{12}e^{-2t} = 0$

This never reaches equilibrium! ($x+y=0$ when $x>0$ & $y>0$).

When $t \Rightarrow \infty$, $y=0$

This is overdamping b/c we have 2 unique, real roots & never reach equilibrium. You could also plug in & show $b^2 - 4mk > 0$.

4.10 Forced Mechanical Vibrations

Recall: $my'' + by' + ky = F_{ext}$ From previous section.

Now $F_{ext} \neq 0$. This means we have a non-homogenous d.e. & thus $y = y_c + y_p$.

If we have damping (i.e. $b \neq 0$), then

$\lim_{t \to \infty} y_c = 0$

& therefore

$\lim_{t \to \infty} y = y_p$.

Otherwise, its the same style of problem & solution.

Example: Steady State Solution

$m = 8$ kg

$b = 3$ N/(m/s)

$k = \frac{F(0)}{y(0)} = \frac{mg}{1.96}$ ← From the initial condition & Hooke's Law $F=ky$

$= \frac{8(9.8)}{1.96} = 40$ N/m

$F_{ext} = \cos(2t)$

↓

$8y'' + 3y' + 40y = \cos(2t)$

Since we only need to find the steady state solution (y_p), we just need to show y_c is not linearly dependent on y_p.

$8r^2 + 3r + 40 = 0$

$r = \frac{-3 \pm \sqrt{9 - 4(8)(40)}}{2(8)}$

We can already see $\alpha = -3/16$, so y_c is not linearly dependent w/ $\cos(2t)$.

Here we'll do undetermined coefficients to find y_p

$y_p = A\cos(2t) + B\sin(2t)$, $y' = -2A\sin(2t) + 2B\cos(2t)$, $y'' = -4A\cos(2t) - 4A\sin(2t)$

$8(-4A\cos(2t) - 4B\sin(2t))$
$+ 3(-2A\sin(2t) + 2B\cos(2t))$
$+ 40(A\cos(2t) + B\sin(2t)) =$
$= 2\cos(2t)$

$(-32A + 6B + 40A)\cos(2t) + (-32B - 6A + 40B)\sin(2t) = \cos(2t)$

$8A + 6B = 1$ $\qquad -6A + 8B = 0$
$4A + 3B = 1/2$ $\qquad B = 3/4 A$
$4A + 9/4 A = 1/2$
$16A + 9A = 2$
$\underline{A = 2/25}$ $\qquad \underline{B = 3/50}$

$\boxed{y_p = 2/25 \cos(2t) + 3/50 \sin(2t) \text{ is the steady state solution.}}$

Example: Whole Thing

$F_g = mg$
$\Rightarrow 2 = m(32)$ ← pounds are a unit of force!
$\underline{m = 1/16 \text{ slug}}$

$F_{sp} = ku$
$\Rightarrow 2 = k(1/2)$
$\underline{k = 4 \text{ lb/ft}}$

$\underline{F_{ext} = 55\sin(3t)}$

$\underline{y(0) = 1/4 \text{ ft}, \quad y'(0) = 0 \text{ ft/s}}$ ← just released

$1/16\, y'' + 4y = 55\sin(3t)$

We find y_c

$1/16\, r^2 + 4 = 0$
$r^2 + 64 = 0$
$r = \pm 8i$
$\alpha = 0, \beta = 8$
$\underline{y_c = c_1\cos(8t) + c_2\sin(8t)}$

We find y_p

$y_p = A\cos(3t) + B\sin(3t)$, $y_p'' = -9A\cos(3t) - 9B\sin(3t)$

$1/16(-9A\cos(3t) - 9B\sin(3t)) + 4(A\cos(3t) + B\sin(3t)) = 55\sin(3t)$

$-9/16 A + 4A = 0$ $\qquad -9/16 B + 4B = 55$
$\underline{A = 0}$ $\qquad 1/16 B(-9 + 64) = 55$
$\qquad 1/16 B(55) = 55$
$\qquad \underline{B = 16}$

$\underline{y_p = 16\sin(3t)}$

$y = c_1\cos(8t) + c_2\sin(8t) + 16\sin(3t)$
$y(0) = c_1\cos(0) + \cancel{c_2\sin(0)} + \cancel{16\sin(0)} = 1/4$
$c_1 = 1/4$
$y' = -8c_1\sin(8t) + 8c_2\cos(8t) + 48\cos(3t)$
$y'(0) = \cancel{-8c_1\sin(0)} + 8c_2\cos(0) + 48\cos(0) = 0$
$c_2 = -48/8 = -6$

$$\boxed{y = \tfrac{1}{4}\cos(8) - 6\sin(8t) + 16\sin(3t)}$$

7.2 Intro to Laplace Transforms

Goal: Use a Laplace transform to solve d.e. by converting a d.e. into an algebraic system. ← useful when RHS discontinuous

Method:
- Take Laplace of both sides of d.e.
- Solve for $\mathcal{L}(y)$
- Use partial fractions to simplify if neccessary
- Take inverse Laplace.

Def: Laplace Transform

Let $f(t)$ be a function $[0,\infty) \to \mathbb{R}$.
The Laplace transform $\mathcal{L}\{f\}$ is defined as

$$F(s) = \mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt$$

where the domain of $F(s)$ is all the values of s for which the integral exists.

We go from function of t to s

* Note: $\int_0^\infty e^{-st} f(t)\,dt$ is an improper integral. We really do

$$\int_0^\infty e^{-st} f(t)\,dt = \lim_{n\to\infty} \int_0^n e^{-st} f(t)\,dt$$

Example: Simple Laplace's

$f(t) = 1$

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\,dt$$

$$= \lim_{n\to\infty} \int_0^n e^{-st}\,dt \quad \leftarrow \text{mandatory!}$$

$$= \lim_{n\to\infty} \left[-\tfrac{1}{s} e^{-st}\right]_0^n$$

$$= \lim_{n\to\infty} \left(-\tfrac{1}{s} e^{-sn} + \tfrac{1}{s}\cancel{e^0}\right)$$

For what values s does this exist?

$s \neq 0$ b/c divide by zero

Let's isolate the important bit

$\lim_{n\to\infty} e^{-sn}$

This converges to 0 when $s>0$. Otherwise it diverges.

This means

$\mathcal{L}\{1\} \lim_{n\to\infty} -\frac{1}{s} e^{-sn} + \frac{1}{s}, \quad s>0$

$\therefore \boxed{\mathcal{L}\{1\} = \frac{1}{s}, \quad s>0}$

On tests, we'll be given a table of Laplace transforms b/c they're kinda a pain.

Example:

$f(t) = e^{at}$, $a \in \mathbb{R}$. Find $\mathcal{L}\{f\}$.

$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \underbrace{e^{at}}_{f(t)} dt$

$= \int_0^\infty e^{(a-s)t} dt$

$= \lim_{n\to\infty} \int_0^n e^{(a-s)t} dt$

$= \lim_{n\to\infty} \left[\frac{1}{a-s} e^{(a-s)t}\right]_0^n$

$= \lim_{n\to\infty} \left[\frac{1}{a-s} e^{(a-s)n} - \frac{1}{a-s}\right]$

We know $s \neq a$ & $a-s<0 \Rightarrow s>a$

$\Rightarrow \mathcal{L}\{e^{at}\} = \lim_{n\to\infty} \frac{1}{a-s} e^{(a-s)n} - \frac{1}{a-s}, \quad s>a$

$\boxed{\mathcal{L}\{e^{at}\} = \frac{-1}{a-s}, \quad s>a}$ ← Previous problem was case where $a=0$

Example:

$f(t) = \begin{cases} 1-t & ; 0<t<1 \\ 0 & ; 1<t \end{cases}$. Find $\mathcal{L}\{f\}$.

$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\, dt$

We split up the integral into the cases for f

$\mathcal{L}\{f\} = \int_0^1 e^{-st}(1-t)\,dt + \cancel{\int_1^\infty e^{-st}(0)\,dt}$

$= \int_0^1 e^{-st}(1-t)\,dt$

$u = 1-t \qquad v = -\frac{1}{s} e^{-st} dt$

$du = -dt \qquad dv = e^{-st} dt$

$= (1-t)(-\frac{1}{s}e^{-st}) - \int_0^1 -\frac{1}{s}e^{-st} \cdot -dt$

$= (1-t)(-\frac{1}{s}e^{-st}) - \int_0^1 \frac{1}{s}e^{-st} dt$

$= -\frac{1}{s}e^{-st}(1-t) + \frac{1}{s^2}e^{-st} \Big|_0^1$

$= (\cancel{-\frac{1}{s}e^{-s}(1-1)} + \frac{1}{s^2}e^{-s})$

$- (-\frac{1}{s}e^{0}(1-0) + \frac{1}{s^2}e^{0})$

$= \boxed{\frac{1}{s^2}e^{-s} + \frac{1}{s} - \frac{1}{s^2}}$

Now we find the domain: $s \neq 0$

$\boxed{\mathcal{L}\{f\} = \frac{1}{s^2}e^{-s} + \frac{1}{s} - \frac{1}{s^2}, \quad s \neq 0}$

Properties of Laplace Transform

If we can prove some properties about the Laplace transform, that'll make our lives easier using it.

Thm: Linearity of Laplace

Let f, f_1, f_2 be functions whose transforms exist for $s > \alpha$ & $c \in \mathbb{R}$ be a constant.

$$\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}.$$

$$\mathcal{L}\{cf\} = c\mathcal{L}\{f\}.$$

Proof:

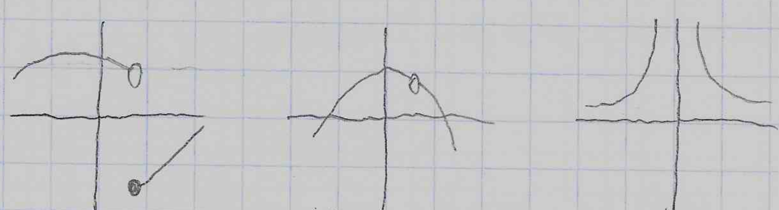
$$\mathcal{L}\{f_1 + f_2\} = \int_0^\infty e^{-st}(f_1 + f_2)\,dt = \int_0^\infty e^{-st} f_1\,dt + \int_0^\infty e^{-st} f_2\,dt = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}.$$

$$\mathcal{L}\{cf\} = \int_0^\infty e^{-st} cf\,dt = c\int_0^\infty e^{-st} f\,dt = c\mathcal{L}\{f\}.$$

These drop out of the linearity of integrals!

Def: Piecewise Continuous

A function is piecewise continuous if its only discontinuities are jump discontinuities.



Jump Discontinuity Holes Infinite

Def: Exponential Order

A function f is exponential order α if there exists positive constants T & M such that

$$|f(t)| \leq Me^{\alpha t} \text{ for all } t \geq T.$$

Basically, is $f \in O(t)$.

Thm: Existence of Transform

If $f(t)$ is piecewise continuous on $[0, \infty)$ & of exponential order α, then

$\mathcal{L}\{s\}$ exists for $s > \alpha$.

Example:

$\mathcal{L}\{t^3 - te^t + e^{4t}\cos(t)\}$ find.

$\mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$

$\because \mathcal{L}\{t^n\} = \frac{n!}{t^{n+1}}$

$\mathcal{L}\{-te^t\} = -\frac{1}{(s-1)^2}$ case where $a=0$

$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

$\mathcal{L}\{e^{4t}\cos(t)\} = \frac{s-4}{(s-4)^2+1}$

$\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$

Putting this together

$$\mathcal{L}\{t^3 - te^t + e^{4t}\cos(t)\} = \frac{6}{s^4} - \frac{1}{(s-1)^2} + \frac{s-4}{(s-4)^2+1}$$

Thm: Translation in s

If the laplace transform $\mathcal{L}\{f\} = F(s)$ exists for $s > \alpha$

then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ for $s > \alpha$.

Pf:

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{(-s+a)t} f(t)\,dt = \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

This is the standard definition but $s \equiv s-a$. Therefore

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

Thm: Derivatives & the Laplace

Let $f(t)$ be cont. on $[0,\infty)$ & $f'(t)$ be piecewise cont. on $[0,\infty)$.

If $f(t)$ & $f'(t)$ are of exponential order α, then
$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$ for $s > \alpha$.

Pf:

$\mathcal{L}\{f'\} = \int_0^\infty e^{-st} f'(t)\,dt$

$= \lim_{n\to\infty} \int_0^n e^{-st} f'(t)\,dt$

$u = e^{-st}$ $\quad v = f(t)$

$du = -se^{-st}\,dt$ $\quad dv = f'(t)\,dt$

$= \lim_{n\to\infty}\left(e^{-st} f(t)\Big|_0^n - \int_0^n f(t)(-se^{-st})\,dt \right)$

$= \lim_{n\to\infty}\left(e^{-sn} f(n) - f(0) + s\int_0^n e^{-st} f(t)\,dt \right)$

$= 0 - f(0) + s\mathcal{L}\{f\}$ goes to 0 b/c $f(t)$ is of exponential order α

$= \boxed{s\mathcal{L}\{f\} - f(0)}$

We can extend this theorem to 2nd derivatives.

Thm: 2nd Derivative & the Laplace.

Let $f(t)$ & $f'(t)$ be cont on $[0,\infty)$ & $f''(t)$ be piecewise cont. on $[0,\infty)$.

$$\mathcal{L}\{f''\} = \mathcal{L}\{(f')'\}$$
$$= s\mathcal{L}'\{f'\} - f'(0)$$
$$= s(s\mathcal{L}\{f\} - f(0)) - f'(0)$$
$$= s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$

* Note: This could be taken further.

Thm:

Let $F(s) = \mathcal{L}\{f\}$ & assume $f(t)$ is piecewise cont. on $[0,\infty)$ & of exponential order α.

Then $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n F}{ds^n}$ for $s > \alpha$.

Pf:

If $F(s) = \mathcal{L}\{f\}$ then $\dfrac{dF}{ds} = \dfrac{d}{ds}\mathcal{L}\{f\}$

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$
$$= \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t)\right)dt \leftarrow \text{Leibniz's rule!}$$
$$= \int_0^\infty -te^{-st} f(t)\,dt$$
$$= -\int_0^\infty e^{-st}\left(t f(t)\right)dt$$
$$= -\mathcal{L}\{tf(t)\}$$

This pattern continues trivially for all powers n of t.

7.4 Inverse Laplace Transform

Goal: Undo Laplace transform.

<u>Def:</u> Inverse Laplace

Given a function $F(s)$.

If there is a function $f(t)$ cont. on $[0,\infty)$ & satisfies $L\{f\}=F$ then $f(t)$ is the <u>inverse Laplace transform of F.</u>

We write

$$f=\mathcal{L}^{-1}\{F\}.$$

Look at the tables & be careful! It's harder than Laplace.

<u>Example:</u>

- $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

  Recall $\mathcal{L}\{\sin(bt)\}=\frac{b}{s^2+b^2}$

  $$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}=\boxed{\sin(2t)}$$

- $\mathcal{L}^{-1}\left\{\frac{4}{s^2+9}\right\}$

  If we had $b=3$, we would get it wrong. We must therefore scale the result by ~~a constant~~ multiple

  $$\mathcal{L}^{-1}\left\{\frac{4}{s^2+9}\right\}\underset{\text{linearity}}{=}\frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}=\boxed{\frac{4}{3}\sin(3t)}$$

- $\mathcal{L}^{-1}\left\{\frac{3}{(2s+5)^3}\right\}$

  This looks close-ish to $\mathcal{L}\{t^n e^{at}\}=\frac{n!}{(s-a)^{n+1}}$ where $n=2$

  $$\mathcal{L}\{t^2e^{at}\}=\frac{2!}{(s-a)^3}=\frac{2!}{\frac{1}{8}(2s-2a)^3}=\frac{16}{(2s-2a)^3}$$

  $$-2a=5$$
  $$a=-5/2$$

  $$\mathcal{L}\{t^2e^{-5/2t}\}=\frac{16}{(2s+5)^3}$$

  Now we just scale appropriately

  $$\mathcal{L}^{-1}\left\{\frac{3}{(2s+5)^3}\right\}=\frac{3}{16}\mathcal{L}^{-1}\left\{\frac{16}{(2s+5)^3}\right\}=\boxed{\frac{3}{16}t^2e^{-5/2t}}$$

Method of Partial Fractions w/ Inverse Laplace

Goal: Simplify rational function, so it's easier to take inverse Laplace transform.

Recall: $q(x)$ is a rational function if it can be written as
$q(x) = \frac{f(x)}{g(x)}$ where $f, g \in P$.

Doing partial fraction decomposition splits into 3 cases, where $P(s)$ & $Q(s)$ are polynomials, for function $\frac{P(s)}{Q(s)}$

Case 1:

$Q(s)$ can be written as product of distinct linear factors.

$$\frac{P(s)}{Q(s)} = \frac{P(s)}{(s-r_1)(s-r_2)\dots(s-r_n)}$$

The partial fraction decomposition is thus

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \dots + \frac{A_n}{s-r_n}$$

Example:

Find $\mathcal{L}^{-1}\left\{\frac{s+11}{(s-1)(s+3)}\right\} = \frac{A}{s-1} + \frac{B}{s+3}$

We multiply by $(s-1)(s+3)$ to remove denominator.

$s+11 = A(s+3) + B(s-1)$

$s+11 = (A+B)s + (3A-B)$

$A+B=1 \quad 3A-B=1$

$4A=12$

$A=3$

$B=-2$

$$\mathcal{L}^{-1}\left\{\frac{s+11}{(s-1)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s-1} - \frac{2}{s+3}\right\}$$

Recall $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$$\mathcal{L}^{-1}\left\{\frac{3}{s-1} - \frac{2}{s+3}\right\} = \boxed{3e^{t} - 2e^{-3t}}$$

Case 2:

$Q(s)$ has repeated linear factors

$$\frac{P(s)}{Q(s)} = \frac{P(s)}{(s-r)^n}$$ can be mix

The partial fraction decomposition is thus

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r} + \frac{A_2}{(s-r)^2} + \dots + \frac{A_n}{(s-r)^n}$$

Example:

Find $\mathcal{L}^{-1}\{F\}$ where $F(s) = \dfrac{7s^3-2s^2-3s+6}{s^3(s-2)}$

$$F(s) = \frac{7s^3-2s^2\ 3s+6}{s^3(s-2)} = \frac{A}{s-2} + \frac{B}{s} + \frac{C}{s^2} + \frac{D}{s^3}$$

We multiply by $s^3(s-2)$ to remove denominator

$$7s^3-2s^2\cdot 3s+6 = As^3 + Bs^2(s-2) + Cs(s-2) + D(s-2)$$
$$7s^3-2s^2\ 3s+6 = (A+B)s^3 + (-2B+C)s^2 + (-2C+D)s + (-2D)$$

$A+B=7$ $\quad$ $-2B+C=-2$ $\quad$ $-2C+D=-3$ $\quad$ $-2D=6$

$\underline{A=6}$ $\quad$ $\underline{B=1}$ $\quad$ $-2C=0$ $\quad$ $\underline{D=-3}$

$\underline{C=0}$

$$F(s) = \frac{6}{s-2} + \frac{1}{s} - \frac{3}{s^3}$$

$$\mathcal{L}^{-1}\{F\} = \mathcal{L}\left\{\frac{6}{s-2} + \frac{1}{s} - \frac{3}{s^3}\right\} = \boxed{6e^{2t}+1-3/2\,t}$$

Case 3:

$Q(s)$ has complex roots / quadratic factors.

$$\frac{P(s)}{Q(s)} = \frac{P(s)}{((s-\alpha)^2+\beta^2)^m}$$

In most problems we do, $m=1$

The partial fraction decomposition is thus

$$\frac{P(s)}{Q(s)} = \frac{A_1(s-\alpha)+B_1\beta}{(s-\alpha)^2+\beta^2} + \frac{A_2(s-\alpha)+B_2\beta}{((s-\alpha)^2+\beta^2)^2} + \dots + \frac{A_m(s-\alpha)+B_m\beta}{((s-\alpha)^2+\beta^2)^m}$$

We can simplify this if $m=1$.

$$\frac{P(s)}{Q(s)} = \frac{A(s-\alpha)}{(s-\alpha)^2+\beta^2} + \frac{B_1\beta}{(s-\alpha)^2+\beta^2}$$

When we find the inverse Laplace, we get

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \mathcal{L}^{-1}\left\{\frac{A(s-\alpha)}{(s-\alpha)^2+\beta^2}\right\} + \mathcal{L}^{-1}\left\{\frac{B\beta}{(s-\alpha)^2+\beta^2}\right\} = e^{\alpha t}\cos(\beta t) + e^{\alpha t}\sin(\beta t)$$

Example:

Find $\mathcal{L}^{-1}\{F\}$ where $F(s) = \dfrac{7s^2 - 41s + 84}{(s-1)(s^2-4s+13)}$.

Notice that $s^2 - 4s + 13$ doesn't factor. We rewrite $s^2 - 4s + 13$ to be in the form $(s-\alpha)^2 + \beta$ by completing the square.

$s^2 - 4s + 13$
$= s^2 - 4s + 4 + 9$
$= (s-2)^2 + (\pm 3)^2$
$\therefore \alpha = -2,\ \beta = 3$ ← choose positive for simplicity

We thus rewrite F to

$$F(s) = \frac{7s^2 - 41s + 84}{(s-1)((s-2)^2 + 3^2)}$$

Now the partial fraction decomposition is

$$F(s) = \frac{A}{s-1} + \frac{B(s-2) + C3}{(s-2)^2 + 3^2}.$$

We find the constants.

$A(s^2 - 4s + 13) + (B(s-2) + 3C)(s-1) = 7s^2 - 41s + 84$

← recall $(s-2)^2 + 3^2 = s^2 - 4s + 13$

$As^2 - 4As + 13A + (Bs - 2B + 3C)(s-1) = 7s^2 - 41s + 84$

$As^2 - 4As + 13A + (Bs^2 - 2Bs + 3Cs) + (-Bs + 2B - 3C) = 7s^2 - 41s + 84$

$(A+B)s^2 + (-4A - 2B + 3C - B)s + (13A + 2B - 3C) = 7s^2 - 41s + 84$

$$\begin{cases} A + B = 7 \\ -4A - 3B + 3C = -41 \\ 13A + 2B - 3C = 84 \end{cases}$$

$\{9A - B = 43$

$10A = 50$

$\underline{A = 5}$

$\underline{B = 2}$

$-20 - 6 + 3C = -41$
$3C = -15$
$\underline{C = -5}$

We now have

$$F(s) = \frac{5}{s-1} + \frac{2(s-2) - 5(3)}{(s-2)^2 + 3^2}$$

We now find

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2(s-2)}{(s-2)^2 + 3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-5(3)}{(s-2)^2 + 3^2}\right\}$$

$$= \boxed{5e^{t} + 2e^{2t}\cos(3t) - 5e^{2t}\sin(3t)}$$

7.5 Solving IVPs w/ Laplace Transform

Method:

1) Take laplace of both sides of d.e.
2) Solve for $\mathcal{L}\{y\}$
3) Use partial fractions if neccessary to simplify
4) Take inverse laplace.

Example:

$y''-4y'+5y=4e^{3t}$ where $y(0)=2,\ y'(0)=7$

We take the laplace of both sides.

$$\mathcal{L}\{y''\}+\mathcal{L}\{-4y'\}+\mathcal{L}\{5y\}=\mathcal{L}\{4e^{3t}\}$$
$$(s^2\mathcal{L}\{y\}-sy(0)-y'(0))-4(s\mathcal{L}\{y\}-y(0))+5\mathcal{L}\{y\}=\frac{4}{s-3}$$
$$(s^2-4s+5)\mathcal{L}\{y\}-2s-7+8=\frac{4}{s-3}$$
$$(s^2-4s+5)\mathcal{L}\{y\}=\frac{4}{s-3}+2s-1$$
$$\mathcal{L}\{y\}=\frac{\frac{4}{s-3}+2s-1}{s^2-4s+5}$$
$$\mathcal{L}\{y\}=\frac{4+(2s-1)(s-3)}{(s-3)(s^2-4s+5)}$$
$$\mathcal{L}\{y\}=\frac{4+2s^2-7s+3}{(s-3)(s^2-4s+5)}$$
$$\mathcal{L}\{y\}=\frac{2s^2-7s+7}{(s-3)((s-2)^2+1^2)}$$

We simplify $\mathcal{L}\{y\}$ w/ partial fractions

$$\mathcal{L}\{y\}=\frac{2s^2-7s+7}{(s-3)((s-2)^2+1^2)}=\frac{A}{s-3}+\frac{B(s-2)+C(1)}{(s-2)^2+1^2}$$
$$2s^2-7s+7=A(s^2-4s+5)+(Bs-2B+C)(s-3)$$
$$2s^2-7s+7=As^2-4As+5A+Bs^2-2Bs+Cs-3Bs+6B-3C$$
$$2s^2-7s+7=(A+B)s^2+(-4A-2B+C-3B)s+(5A+6B-3C)$$

$1A+1B=2$

$-4A-5B+1C=-7$

$5A+6B-3C=7$

$0+0-2C=-2$

$\underline{C=1}$

$A+C=3$

$\underline{A=2}$

$\underline{B=0}$

This gives us

$$\mathcal{L}\{y\} = \frac{2}{s-3} + \frac{0(s-2)+1(1)}{(s-2)^2+1^2}$$

$$\mathcal{L}\{y\} = \frac{2}{s-3} + \frac{1}{(s-2)^2+1^2}$$

This is the same thing we would get w/ undetermined coefficients, unsurprisingly.

We now take the inverse laplace of both sides, giving us

$$\boxed{y = 2e^{3t} + e^{2t}\sin(t)}$$

as the solution.

Example:

Solve $y''-2y'+17y=0$ w/ laplace transforms, where $y(0)=0$ & $y'(0)=4$

This could be solved trivially w/ the complementary equation, but this is for practice.

$$y''-2y'+17y=0$$

$\downarrow$ laplace

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2(s\mathcal{L}\{y\} - y(0)) + 17\mathcal{L}\{y\} = 0$$

$$(s^2-2s+17)\mathcal{L}\{y\} - sy(0) - y'(0) + 2y(0) = 0$$

$$(s^2-2s+17)\mathcal{L}\{y\} - 4 = 0$$

$$\mathcal{L}\{y\} = \frac{4}{s^2-2s+17} = \frac{4}{(s-1)^2+4^2}$$

Actually, this is in a good form already, so you don't need to simplify w/ partial fractions.

Now we do the partial fraction decomposition of $\mathcal{L}\{y\}$.

$$\mathcal{L}\{y\} = \frac{A(s-1)+B(4)}{(s-1)^2+4^2} = \frac{4}{(s-1)^2+4^2}$$

$As - A + 4B = 4$

$A=0$

$-A+4B=4$

$B=1$

$$\mathcal{L}\{y\} = \frac{4}{(s-1)^2+4^2}$$

Take the inverse laplace to get the solution

$$\boxed{y = e^{t}\sin(4t)}$$

Example:

Solve $w''-2w'+w=6t-2$ where $w(-1)=3$ & $w'(-1)=7$.

For laplace transforms, we need $y(0)$ to be our initial conditions, so we replace t w/ $t-1$ & use y instead of w to remind ourselves. That is, $y(t)=w(t-1)$

$$y''(t-1) - 2y'(t-1) + y(t-1) = 6(t-1)-2$$

$$y''-2y'+y = 6t-8$$

$\downarrow$ Take laplace

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2(s\mathcal{L}\{y\} - y(0)) + \mathcal{L}\{y\} = \frac{6}{s^2} - \frac{8}{s}$$

$$(s^2-2s+1)\mathcal{L}\{y\} - s(3) - (7) + 2(3) = \frac{6}{s^2} - \frac{8}{s}$$

$$(s^2-2s+1)\mathcal{L}\{y\} = 3s + 1 - \frac{8}{s} + \frac{6}{s^2}$$

$$s^2(s^2-2s+1)\mathcal{L}\{y\} = 3s^3 + s^2 - 8s + 6$$

$$\mathcal{L}\{y\} = \frac{3s^3+s^2-8s+6}{s^2(s-1)^2}$$

Partial Fractions:

$$\frac{3s^3+s^2-8s+6}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$3s^3+s^2-8s+6 = As(s^2-2s+1) + B(s^2-2s+1) + Cs^2(s-1) + Ds^2$$

$$3s^3+s^2-8s+6 = A(s^3-2s^2+s) + B(s^2-2s+1) + C(s^3-s^2) + Ds^2$$

$$3s^3+s^2-8s+6 = (A+C)s^3 + (-2A+B-C+D)s^2 + (A-2B)s + B \leftarrow$$ could skip easily

$A+C=3$ $\quad$ $-2A+B-C+D=1$ $\quad$ $A-2B=-8$ $\quad$ $\underline{B=6}$

$\underline{C=-1}$ $\quad$ $-8+6+1+D=1$ $\quad$ $\underline{A=4}$

$\underline{D=2}$

$$\mathcal{L}\{y\} = \frac{4}{s} + \frac{6}{s^2} - \frac{1}{s-1} + \frac{2}{(s-1)^2}$$

$\downarrow \mathcal{L}^{-1}$

$$\underline{y} = 4 + 6t - e^t + 2te^t$$

Now we solve for w using $y(t) = w(t-1) \Rightarrow w(t) = y(t+1)$

$$\boxed{w = 4 + 6(t+1) - e^{t+1} + 2(t+1)e^{t+1}} \leftarrow$$ could simplify but eh

Example: No Undetermined Coefficients!

Solve $ty'' - ty' + y = 2$ where $y(0)=2$ & $y'(0)=-1$.

$$\mathcal{L}\{ty''\} - \mathcal{L}\{ty'\} + \mathcal{L}\{y\} = \frac{2}{s}$$

Recall $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f\})$

$$\mathcal{L}\{ty''\} = (-1)\frac{d}{ds}(\mathcal{L}\{y''\})$$

$$= -\frac{d}{ds}\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right)$$

$$= -\left(2s\mathcal{L}\{y\} + s^2\frac{d}{ds}\mathcal{L}\{y\} - y(0)\right)$$

$\mathcal{L}\{y\}$ may be function of s

$L \equiv \mathcal{L}\{y\}$

$$= -2sL - s^2L' + 1$$

$$\mathcal{L}\{ty'\} = -\frac{d}{ds}(\mathcal{L}\{y'\})$$
$$= -\frac{d}{ds}\left(s\mathcal{L}\{y\} - y(0)\right)$$
$$= -\left(\mathcal{L}\{y\} + s\frac{d}{ds}\mathcal{L}\{y\}\right)$$
$$L = \mathcal{L}\{y\}$$
$$= -L - sL'$$

$$(-2sL - s^2L' + 2) - (-L - sL') + L = \frac{2}{s}$$
$$-2sL - s^2L' + 2 + L + sL' + L = \frac{2}{s}$$
$$(s - s^2)L' + (2 - 2s)L = \frac{2}{s} - 2$$

Note that this is a 1st order linear d.e.

$$\frac{dy}{dx} + P(x)y = Q(x) \leftarrow \text{standard form}$$

We put this in standard form

$$L' + \frac{(2-2s)}{(s-s^2)}L = \frac{2/s - 2}{(s-s^2)} \cdot \frac{s}{s}$$
$$L' + \frac{2(1-s)}{s(1-s)}L = \frac{2(1-s)}{s^2(1-s)}$$
$$L' + \frac{2}{s}L = \frac{2}{s^2}$$
$$P(x) = 2/s$$
$$\mu(x) = e^{\int P(x)dx} = e^{2\ln|s|} = s^2$$
$$\frac{d}{ds}\left[L \cdot \mu(x)\right] = \frac{2}{s^2} \cdot \mu(x)$$
$$\frac{d}{ds}(L \cdot s^2) = \frac{2}{s^2} \cdot s^2 = 2$$
$$L \cdot s^2 = 2s + C$$
$$L = \frac{2}{s} + \frac{C}{s^2}$$
$$\mathcal{L}\{y\} = \frac{2}{s} + \frac{C}{s^2}$$
$$\downarrow \mathcal{L}^{-1}$$
$$y = 2 + Ct$$

Now we solve for the initial conditions

$$y(0) = 2 + C(0) = 2$$
$$2 = 2 \checkmark$$
$$y'(0) = \underline{C = -1}$$

$$\boxed{y = 2 - t}$$

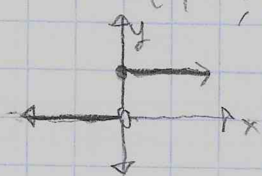
7.6 Transforms of Discontinuous Functions

Goal:

Solve IVPs that arise when describing electric circuits w/ on-off switches.

Def: Unit Step Function / Heaviside Function

$$u(t) = \begin{cases} 0 & ; \ t<0 \\ 1 & ; \ t \geq 0 \end{cases}$$



This means $u(t-a) = \begin{cases} 0 & ; \ t-a<0 \Leftrightarrow t<a \\ 1 & ; \ t-a \geq 0 \Leftrightarrow t \geq a \end{cases}$

We can represent discontinuous functions using unit step functions.

Example: Simple Function

Represent $f(t)$ w/ unit step functions where

$$f(t) = \begin{cases} 2 & ; \ t<4 \\ 6t & ; \ 4 \leq t < 7 \\ 8 & ; \ 7 \leq t < 9 \\ 0 & ; \ 9 \leq t \end{cases}$$

$$\begin{aligned} f(t) = \ & 2 \\ & + (6t-2)\,u(t-4) \\ & + (8-6t)\,u(t-7) \\ & + (0-8)\,u(t-9) \end{aligned}$$

As you can see above, the general way to transform a function $f(t)$ where

$$f(t) = \begin{cases} f_0(t) & ; \ a_1 < t \\ f_1(t) & ; \ a_1 \leq t < a_2 \\ \dots & \dots \\ f_n(t) & ; \ a_n \leq t \end{cases}$$

← index at 1 b/c barriers are b/w

is by doing

$$\begin{aligned} f(t) = \ & f_0(t) \\ & + (f_1(t) - f_0(t))\,u(t-a_1) \\ & + \dots \\ & + (f_n(t) - f_{n-1}(t))\,u(t-a_n) \end{aligned}$$

We do this transformation so that we can do the Laplace transform of piecewise functions because

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

which can be seen below.

$$\mathcal{L}\{u(t-a)\}$$
$$= \int_0^\infty e^{-st} u(t-a)\,dt$$
$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$
$$= \int_a^\infty e^{-st}\,dt$$
$$= \lim_{n\to\infty} \int_a^n e^{-st}\,dt$$
$$= \lim_{n\to\infty} -\tfrac{1}{s} e^{-st}\Big|_a^n$$
$$= \lim_{n\to\infty} -\tfrac{1}{s} e^{-sn} + \tfrac{1}{s} e^{-sa}$$
$$= \underline{\tfrac{1}{s} e^{-sa}}$$

Thm: Translation in t ← the above on steroids

Let $F(s) = \mathcal{L}\{f\}$.

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(t)$$

This is normally used when taking the inverse Laplace.

Pf:

$$\mathcal{L}\{f(t-a)\,u(t-a)\}$$
$$= \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt$$
$$= \int_0^a e^{-st} f(t-a)(0)\,dt + \int_a^\infty e^{-st} f(t-a)(1)\,dt$$
$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

$v = t-a \Rightarrow t = v+a$

$dv = dt$

$$= \int_0^\infty e^{-s(v+a)} f(v)\,dv$$ ← bounds go back to $\int_0^\infty$ b/c $v(a) = a-a = 0$

$$= \int_0^\infty e^{-sv} e^{-sa} f(v)\,dv$$

$$= e^{-sa}\int_0^\infty e^{-sv} f(v)\,dv$$ ← variables have changed, but it's the same

$$= e^{-sa}\,\mathcal{L}\{f\}$$
$$= \underline{e^{-sa}\,F(s)}$$

* Note: If we have $g(t) = f(t-a) \Leftrightarrow f(t) = g(t+a)$, then

$$\mathcal{L}\{g(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}.$$

Example:

Find $\mathcal{L}\{g\}$ where

$$g(t) = \begin{cases} 0 & ; 0 < t < 1 \\ 2 & ; 1 < t < 2 \\ 1 & ; 2 < t < 3 \\ 3 & ; 3 < t \end{cases}$$

$$g(t) = 0$$
$$+ (2-0)\,u(t-1)$$
$$+ (1-2)\,u(t-2)$$
$$+ (3-1)\,u(t-3)$$
$$= 2u(t-1) - u(t-2) + 2u(t-3)$$

$$\mathcal{L}\{g\} = \frac{2e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{2e^{-3s}}{s}$$

Matrixes & Vectors

Goal: Understand different types & properties of matrixes.

I already covered a lot of this in MA 405, so these notes will be brief.

Def: Singular matrix
Matrix w/ no inverse.

Def: Diagonal Matrix
Square matrix where all entries off main diagonal is 0.

$$A = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}. \quad A \in M_{n \times n} \text{ where } a_{ij} = 0 \text{ if } i \neq j.$$

This isn't discussed here

Def: Determinant
The determinant of a $A \in M_{n \times n}$ describes how the linear transformation scales the volume of the space.

For $A \in M_{2 \times 2}$,

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For a $A \in M_{3 \times 3}$

For each column, multiply the top value by the determinant of the sub-matrix w/o that column & top row. Sum them together w/ alternating signs.

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

To generalize the algorithm where $A \in M_{n \times n}$:

For each column of A:
  Let A' be the submatrix where the current column & top row are eliminated.
  Multiply the top value in the current column by the determinant of A'.
Sum the above w/ alternating signs.

Thm:

Let $A \in M_{n\times n}$.

The following statements are equivalent

If A^{-1} exists, $A\bar{x}=\bar{0}$ is solved by $\bar{x}=A^{-1}\bar{0} \Rightarrow \bar{x}=\bar{0}$.

i) A is singular (no inverse),
ii) $\det(A)=0$,
iii) $A\bar{x}=\bar{0}$ has non-trivial solutions ($\bar{x}\neq 0$), &
iv) The columns & rows of A form a linearly dependent set.

Example: Inverse & Determinant

Is $A=\begin{bmatrix}1&0&2\\0&3&-1\\-1&2&1\end{bmatrix}$ invertible? Use the determinant.

$\det(A) = 1(3\cdot 1-{-1}\cdot 2)-\cancel{0(0\cdot 1 - -1\cdot -1)}+2(\cancel{0}\cdot 2-3\cdot -1)$
$= 5+6$
$=11$

Yes, A is invertible b/c $\det(A)\neq 0$.

9.4 Linear Systems in Normal Form

Def: Normal Form

A system of n linear d.e.s is in normal form if it is expressed as

$$\bar{x}'(t)=A(t)\bar{x}(t)+\bar{f}(t)$$

If $\bar{f}(t)=\bar{0}$, then the system is homogeneous.

Otherwise, it is non-homogeneous.

Any stem of ODEs can be written in normal form.

Example:

Write $\begin{cases} r'(t)=2r(t)+\sin(t) \\ \theta'(t)= r(t)-\theta(t)+1\end{cases}$ as a system in normal form.

We want

$$\begin{bmatrix} r\\ \theta\end{bmatrix}'=\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} r\\ \theta\end{bmatrix}+\begin{bmatrix} f \end{bmatrix}.$$

To rewrite the given system, we do

$$\begin{bmatrix} r\\ \theta\end{bmatrix}'=\begin{bmatrix} 2&0\\1&-1\end{bmatrix}\begin{bmatrix} r\\ \theta\end{bmatrix}+\begin{bmatrix} \sin(t)\\ 1\end{bmatrix}$$

Example:

Write $\begin{cases} \frac{dx}{dt} = t^2 x - y - z + 1 \\ \frac{dy}{dt} = e^t z + 5 \\ \frac{dz}{dt} = tx - y + 3z - e^t \end{cases}$ in normal form.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} t^2 & -1 & -1 \\ 0 & 0 & e^t \\ t & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ -e^t \end{bmatrix}$$

We can also use this method for higher order d.e.s by making them into a system.

Example: Higher Order d.e.

Solve $\frac{d^3y}{dt^3} - \frac{dy}{dt} + y = \cos(t)$

We solve for the highest order derivative

$$\frac{d^3y}{dt^3} = \frac{dy}{dt} - y + \cos(t)$$

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} & & \\ & & \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} + \begin{bmatrix} \\ \\ \cos(t) \end{bmatrix}$$

We need to fill out the rest, but we can just solve for $\frac{dy}{dt}$ b/c then we'd have it in terms of $\frac{d^3y}{dt^3} = y'''$, which we can't express.

To make progress, realize $y' = y'$ & $y'' = y''$. We express this in the problem.

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos(t) \end{bmatrix}$$

Example: More Higher Order

Write $\frac{d^4w}{dt^4} + w = t^2 \Leftrightarrow \frac{d^4w}{dt^4} = -w + t^2$ in normal form.

$$\begin{bmatrix} w \\ w' \\ w'' \\ w''' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ w' \\ w'' \\ w''' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{bmatrix}$$

$w' = w'$
$w'' = w''$
$w''' = w'''$

Def: Linear (In)Dependence

A set of vectors $\bar{x}_1, \dots, \bar{x}_m$ is <u>linearly dependent on interval I</u> if there exists constants $c_1, \dots, c_m$ such that $c_1\bar{x}_1 + \dots + c_m\bar{x}_m = \bar{0}$ for all t in I.

Otherwise, $\bar{x}_1, \dots, \bar{x}_m$ are <u>linear independent.</u>

Def: Fundamental Solution Set

The set of solutions $\{\bar{x}_1, \dots, \bar{x}_n\}$ that are linearly independent is called the <u>fundamental solution set.</u>

9.5 Homogeneous Linear Systems w/ Constant Coefficients

Goal: Find solution to $\bar{x}'(t) = A\bar{x}(t)$ where $A \in M_{n\times n}$ is a constant coefficient matrix.

We will try the exponential $\bar{x}(t) = e^{rt}\bar{u}$ where $\bar{u}$ is a constant vector.

$$\bar{x}(t) = e^{rt}\bar{u}$$
$$\bar{x}'(t) = re^{rt}\bar{u}$$
$$x'(t) = A\bar{x}(t)$$
$$re^{rt}\bar{u} = Ae^{rt}\bar{u}$$
$$r\bar{u} = A\bar{u}$$
$$\bar{0} = A\bar{u} - r\bar{u}$$
$I \in M_{n\times n}$ is identity matrix
$$\underline{\bar{0} = (A - rI)\bar{u}}$$

Def: Eigenvalues & Eigenvectors

Recall earlier theorem.

Let $A \in M_{n\times n}$. The <u>eigenvalues</u> of A are numbers r for $(A - rI)\bar{u} = \bar{0}$ has at least one non-trivial solution. That is $(A - rI)$ is l.d or $\det(A - rI) = \bar{0}$.

The corresponding non-trivial solutions $\bar{u}$ are called the <u>eigenvectors of A associated w/ r.</u>

How do we find the eigenvalues & eigenvectors? We use the <u>characteristic eqn</u> $\det(A - rI) = 0$.

<u>Example:</u>

Find eigenvectors & eigenvalues of
$$A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}.$$

We find $A - rI$

$$A - rI = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6-r & -3 \\ 2 & 1-r \end{bmatrix}$$

We find $|A - rI| = \det(A - rI)$

$$\begin{vmatrix} 6-r & -3 \\ 2 & 1-r \end{vmatrix} = 0$$
$$(6-r)(1-r) - (-3)(2)$$
$$6 - 7r + r^2 + 6 = 0$$
$$r^2 - 7r + 12$$
$$(r-3)(r-4) = 0$$

$r_1 = 3$, $r_2 = 4$

We have thus found our eigenvalues $r_1 = 3$ & $r_2 = 4$, giving

$$\begin{bmatrix} 6-r_1 & -3 \\ 2 & 1-r_1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} = A - r_1 I$$

$$\begin{bmatrix} 6-r_2 & -3 \\ 2 & 1-r_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} = A - r_2 I$$

Note that these are linearly dependent since $A - rI = 0$.

We find eigenvectors for r_2, $\bar{u}_2$

$$\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = 0$$

$2u_a - 3u_b = 0$

$u_a = 3/2\, u_b$

for simplicity

Choose $u_b = 2$, giving us $u_a = 3$. This gives us eigen vectors

$\bar{u}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} s$ for $s \in \mathbb{R}$. (& $s \neq 0$) ← no trivial eigen vectors

s is just some scalar

We find eigenvectors for r_1, u_1

$$\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = 0$$

$3u_a - 3u_b = 0$

$u_a = u_b$

Choose $u_a = 1 = u_b$. This gives us eigen vectors

$\bar{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s$ for $s \in \mathbb{R}$ (& $s \neq 0$).

would be trivial

<u>Thm:</u> Case 1 (Unique Eigenvalues)

Suppose $A_{n \times n}$ constant matrix has n linearly independent eigenvectors, $\bar{u}_1, \dots, \bar{u}_n$.

Let r_i be the eigenvalue associated w/ $\bar{u}_i$.

The fundamental solution set for $\bar{x}' = A\bar{x}$ is

$\{e^{r_1 t}\bar{u}_1, \dots, e^{r_n t}\bar{u}_n\}$

& the general solution is

$\bar{x} = c_1 e^{r_1 t} u_1 + \dots + c_n e^{r_n t} u_n$. ← Notice we don't include constant scalars s b/c c_i absorbs it.

Example:

Using the previous example of $\bar{x}' = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}\bar{x}$, the general solution is

$$\bar{x} = c_1 e^{4t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example:

Find general solution to $\bar{x}' = A\bar{x}$ where $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}$.

We first find the eigenvalues by solving $|A - rI| = 0$.

$$|A - rI| = 0$$

$$\begin{vmatrix} -1-r & 1 & 0 \\ 1 & 2-r & 1 \\ 0 & 3 & -1-r \end{vmatrix} = 0$$

$$(-1-r)\big((2-r)(-1-r) - (1)(3)\big) - 1\big(1(-1-r) - \cancel{(1)(0)}\big) + \cancel{0\big((1)(3) - (2-r)(0)\big)} = 0$$

$$(-1-r)(-2-r+r^2-3) + 1 + r = 0$$

$$-(1+r)(r^2-r-5) + (1+r) = 0$$

$$-(1+r)(r^2-r-5-1) = 0$$

$$-(1+r)(r^2-r-6) = 0$$

$$-(1+r)(r-3)(r+2) = 0$$

$$r_1 = -1 \qquad r_2 = 3 \qquad r_3 = -2$$

We find eigenvectors for r_1

$$(A - r_1 I)\bar{u}_1 = \bar{0}$$

$$(A + I)\bar{u}_1 = \bar{0}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 0 \end{bmatrix}\bar{u}_1 = \bar{0}$$

This shouldn't be invertible (& it isn't)

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 3R_2} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} & & s & \end{array} \end{array}$$

Fix column 3 ($\bar{u}_{1,3}$) to be some arbitrary constant & solve the rest

$$\bar{u} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s \text{ for } s \in \mathbb{R}.$$

We find eigenvectors for r_2.

$$(A - r_2 I)\bar{u}_2 = \bar{0}$$

$$(A - 3I)\bar{u}_2 = \bar{0}$$

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -4 \end{bmatrix}\bar{u}_2 = \bar{0}$$

$$\left[\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2,\ R_1 + 4R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right] \xrightarrow[R_3 + R_2]{R_2 \cdot -\frac{1}{3}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 + R_2} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} & & s & \end{array} \end{array}$$

$$\bar{u}_2 = \begin{bmatrix} \frac{1}{3}s \\ \frac{4}{3}s \\ s \end{bmatrix}$$ multiply to make it pretty

$$\bar{u}_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} s \text{ for } s \in \mathbb{R}.$$

We find eigenvectors for r_3.

$(A - r_3 I)\bar{u}_3 = \bar{0}$

$(A + 2I)\bar{u}_3 = \bar{0}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 3 & 1 \end{bmatrix} \bar{u}_3 = \bar{0}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] R_2 - R_1 \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \begin{matrix} R_2 \cdot \frac{1}{3} \\ R_3 - R_2 \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
s

$$\bar{u}_3 = \begin{bmatrix} \frac{1}{3}s \\ -\frac{1}{3}s \\ s \end{bmatrix}$$

$$\bar{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} s \text{ for } s \in \mathbb{R}.$$

Thus the general solution to $\bar{x}' = A\bar{x}$ is

$$\boxed{\bar{x} = c_1 e^{-t}\bar{u}_1 + c_2 e^{3t}\bar{u}_2 + c_3 e^{-2t}\bar{u}_3}$$ ← On test, write out $\bar{u}_1, \bar{u}_2, \bar{u}_3$

<u>Thm</u>: Real Symmetric Matrix

Let $A \in M_{n \times n}$. A is a real symmetric matrix if it has real values that satisfy $A^T = A$.

If $A^T = A$, then A is symmetric about its main diagonal.

Thm: Real Symmetric Matrix & Eigenvectors

If $A \in M_{n\times n}$ is a real symmetric matrix then there always exist n linearly independent eigen vectors.

! If A is ~~not~~ a real symmetric matrix, we might not have linearly independent eigenvectors.

Example:

Find the general solution to $\bar{x}' = A\bar{x}$ where $A = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix}$.

*Note: A is a real symmetric matrix

We find the eigen values by solving $|A - rI| = 0$

$$\begin{vmatrix} -7-r & 0 & 6 \\ 0 & 5-r & 0 \\ 6 & 0 & 2-r \end{vmatrix}$$

$(-7-r)((5-r)(2-r) - \cancel{0}) = 0$

~~$- 0(0(2-r) - 0(6))$~~

$+ 6(\cancel{0} - (5-r)(6))$

$(-7-r)(5-r)(2-r) - 36(5-r) = 0$

$(5-r)(-14 - 2r + 7r + r^2 - 36) = 0$

$(5-r)(r^2 + 5r - 50) = 0$

$(5-r)(r+10)(r-5) = 0$

$-(5-r)^2(r+10) = 0$

$r_1 = r_2 = 5 \quad r_3 = -10$

We find eigenvectors for $r_1 = r_2$

$(A - 5I)\bar{u} = \bar{0}$

$$\left[\begin{array}{ccc|c} -12 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & -3 & 0 \end{array}\right] 2R_3 + R_1 \quad \left[\begin{array}{ccc|c} -12 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \cdot -\tfrac{1}{12} \quad \left[\begin{array}{ccc|c} 1 & 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$s \quad t$

$\bar{u} = \begin{bmatrix} -\tfrac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -\tfrac{1}{2} \\ 0 \\ 1 \end{bmatrix} t$ ← Notice how we have 2 independent vectors. We break these up into 2 eigenvectors. We must pick linearly independ eigenvectors.

$\bar{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s$ for $s \in \mathbb{R}$.

$\bar{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} t$ for $t \in \mathbb{R}$.

We find eigenvectors for r_3

$(A + 10I)\bar{u} = \bar{0}$

$$\left[\begin{array}{ccc|} 3 & 0 & 0 \\ 0 & 15 & 0 \\ 6 & 0 & 12 \end{array}\right] \begin{array}{l} R_1 \cdot \tfrac{1}{3} \\ R_2 \cdot \tfrac{1}{15} \\ R_3 - R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

s

$\bar{u} = \begin{bmatrix} -2s \\ 0 \\ s \end{bmatrix}$

$\bar{u}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} s$ For $s \in \mathbb{R}$.

This gives us the general solution to $\bar{x}' = A\bar{x}$ to be

$$\bar{x} = c_1 e^{5t}\bar{v}_1 + c_2 e^{5t}\bar{u}_2 + c_3 e^{-10t} u_3$$ ← On test, write out $\bar{v}_1, \bar{v}_2, \bar{v}_3$

Don't multiply by t b/c $\bar{v}_1$ & $\bar{v}_2$ already make them linearly independent.

Example:

Solve $\bar{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \bar{x}$ where $\bar{x}(0) = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$.

We find eigenvalues:

$|A - rI| = 0$

↓ given

$(r+1)^2(r-2) = 0$

$r_1 = r_2 = -1 \quad r_3 = 2$

We find eigenvectors for $r_1 = r_2$

$(A + I)\bar{u} = \bar{0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} \quad \begin{array}{c} \begin{array}{cccc} & s & t & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\bar{v} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$$

$\bar{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s$ for $s \in \mathbb{R}$.

$\bar{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$ for $t \in \mathbb{R}$.

We find eigenvectors for r_3

$(A - 2I)\bar{v} = \bar{0}$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right] \begin{matrix} \\ R_2 - R_1 \\ R_3 + 2R_1 \end{matrix} \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \begin{matrix} \\ R_2 \cdot -\frac{1}{3} \\ R_3 + R_2 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} R_1 - R_1 \\ \\ \\ \end{matrix} \quad \begin{array}{c} \begin{array}{cccc} & & s & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\bar{u} = \begin{bmatrix} s \\ s \\ s \end{bmatrix}.$$

$$\bar{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} s \text{ for } s \in \mathbb{R}.$$

This gives us the general solution to $\bar{x}' = A\bar{x}$ as

$$\bar{x} = c_1 e^{-t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We now solve the IVP

$$\bar{x}(0) = c_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 \cdot -1 \\ \\ R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & -1 & 2 & -1 \end{array}\right] R_3 + R_2 \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 3 & 3 \end{array}\right] R_3 \cdot \tfrac{1}{3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$c_3 = 1$
$c_2 = 4 - c_3 = 3$
$c_1 = 1 - \overset{3}{c_2} + c_3 = 1 - 3 + 1 = -1$

Thus,

$$\boxed{\bar{x} = -e^{-t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 3e^{t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

Could multiply everything through & combine into 1 vector, but eh.

We could apply our newfound ability to solve w/ vectors, we can model interconnected tanks by modeling each tank as an element of the vector.

9.6 Complex Eigenvalues

Goal: Find general solution to $\bar{x}' = A\bar{x}$ when we have complex eigenvalues $r = \alpha \pm \beta i$.
This gives us
$r_1 = \alpha + \beta i$ $\quad$ $r_2 = \alpha - \beta i$ ← eigenvectors
$\bar{z}_1 = \bar{a} + \bar{b} i$ $\quad$ $\bar{z}_2 = \bar{a} - \bar{b} i$ ← eigenvalues

We know $\bar{w}_1(t) = e^{r_1 t}\bar{z}_1$ is a solution to $\bar{x}' = A\bar{x}$, but that has complex values we don't want. We simplify that now

$\bar{w}_1 = e^{r_1 t}\bar{z}_1$
$= e^{\alpha t} e^{\beta i t}(\bar{a} + \bar{b} i)$

Recal Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$\bar{w}_1 = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\bar{a} + \bar{b} i)$

$\bar{w}_1 = e^{\alpha t}(\cos(\beta t)\bar{a} + i\cos(\beta t)\bar{b} + i\sin(\beta t)\bar{a}i + \overset{-1}{i^2}\sin(\beta t)\bar{b})$
$= (e^{\alpha t}\cos(\beta t)\bar{a} - e^{\alpha t}\sin(\beta t)\bar{b}) + (e^{\alpha t}\cos(\beta t)\bar{b} + e^{\alpha t}\sin(\beta t)\bar{a})i$

We now split the equation to make it easier to read & understand
$\bar{x}_1 = e^{\alpha t}\cos(\beta t)\bar{a} - e^{\alpha t}\sin(\beta t)\bar{b}$
$\bar{x}_2 = e^{\alpha t}\cos(\beta t)\bar{b} + e^{\alpha t}\sin(\beta t)\bar{a}$ And to get 2 solutions later!
$\bar{w}_1 = \bar{x}_1 + \bar{x}_2 i$

Recall that $\bar{w}_1' = A\bar{w}_1$ b/c $\bar{w}_1$ is a solution to $\bar{x}' = A\bar{x}$.
Therefore

$\bar{x}_1' + \bar{x}_2' i = A(\bar{x}_1 + \bar{x}_2 i) = A\bar{x}_1 + A\bar{x}_2 i$

Using pattern matching, we get the following constraints
$\bar{x}_1' = A\bar{x}_1$
$\bar{x}_2' = A\bar{x}_2$

This means $\bar{x}_1$ & $\bar{x}_2$ are linearly independent solutions to $\bar{x}' = A\bar{x}$!

This gives us the general solution if $A \in M_{2\times 2}$ as
$\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2$
where
$\bar{x}_1 = e^{\alpha t}\cos(\beta t)\bar{a} - e^{\alpha t}\sin(\beta t)\bar{b}$ &
$\bar{x}_2 = e^{\alpha t}\cos(\beta t)\bar{b} + e^{\alpha t}\sin(\beta t)\bar{a}$
for
complex eigenvalue $r = \alpha \pm \beta i$ &
complex eigenvector $\bar{z} = \bar{a} + \bar{b}i$.

Note we only use $r_1 = \alpha + \beta i$ & $\bar{z}_1 = \bar{a} + \bar{b}i$. If we used r_2 & $\bar{z}_2$, we'd get l.d. sols.

Example: Basic, no IV:
Find general solution to $\bar{x}' = A\bar{x}$ where
$$A = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}.$$

We first find the eigenvalues by solving $\det(A - rI) = 0$.
$$\begin{vmatrix} -2-r & -5 \\ 1 & 2-r \end{vmatrix} = 0$$
$-(2+r)(2-r) - (-5)(1) = 0$
$-4 + r^2 + 5 = 0$
$r^2 + 1 = 0$
$r = \pm i$
$\alpha = 0, \quad B = 1$ ← pick the easy one (i.e. non-negative)

This means our eigenvalues are $r_1 = 0+1i$ & $r_2 = 0-1i$.

We now find the eigenvectors by solving $(A - r_1 I)\bar{u} = 0$.

$$\begin{bmatrix}-2-i & -5\\ 1 & 2-i\end{bmatrix}\bar{u}=\bar{0}$$

I solve w/ Gaussian elimination (she'll give advice for solving on test).

$$\left[\begin{array}{cc|c}-2-i & -5 & 0\\ 1 & 2-i & 0\end{array}\right] \rightleftarrows \left[\begin{array}{cc|c}1 & 2-i & 0\\ -2-i & -5 & 0\end{array}\right] R_1\cdot -1 \quad \left[\begin{array}{cc|c}1 & 2-i & 0\\ 2+i & 5 & 0\end{array}\right] R_2\cdot(2-i)$$

$$\left[\begin{array}{cc|c}1 & 2-i & 0\\ 5 & 5(2-i) & 0\end{array}\right] R_2 - 5R_1 \quad \left[\begin{array}{cc|c}1 & 2-i & 0\\ 0 & 0 & 0\end{array}\right]$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad s$$

$$\bar{u} = \begin{bmatrix}-2+i\\ 1\end{bmatrix}s = \begin{bmatrix}-2\\ 1\end{bmatrix}s + \begin{bmatrix}1\\ 0\end{bmatrix}si$$

This gives us $\bar{a} = \begin{bmatrix}-2\\ 1\end{bmatrix}$ & $\bar{b} = \begin{bmatrix}1\\ 0\end{bmatrix}$. We now find $\bar{x}$ using our earlier eqns...

$$\bar{x}_1 = e^{\alpha t}(\cos(\beta t)\bar{a} - \sin(\beta t)\bar{b}) = \cancel{e^{0t}}\left(\cos(t)\begin{bmatrix}-2\\ 1\end{bmatrix} - \sin(t)\begin{bmatrix}1\\ 0\end{bmatrix}\right)$$

$$\bar{x}_2 = e^{\alpha t}(\cos(\beta t)\bar{b} + \sin(\beta t)\bar{a}) = \cancel{e^{0t}}\left(\cos(t)\begin{bmatrix}1\\ 0\end{bmatrix} + \sin(t)\begin{bmatrix}-2\\ 1\end{bmatrix}\right)$$

$$\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2$$

$$\boxed{\bar{x} = c_1\left(\cos(t)\begin{bmatrix}-2\\ 1\end{bmatrix} - \sin(t)\begin{bmatrix}1\\ 0\end{bmatrix}\right) + c_2\left(\cos(t)\begin{bmatrix}1\\ 0\end{bmatrix} + \sin(t)\begin{bmatrix}-2\\ 1\end{bmatrix}\right)}$$

← Could simplify to vector, but not neccessary

Example: IVP!

Solve $\bar{x}' = \begin{bmatrix}1 & 0 & -1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{bmatrix}\bar{x}$. The eigenvalues are $r_1 = 2$, $r_2 = 1+i$, $r_3 = 1-i$, where $\bar{x}(0) = \begin{bmatrix}-2\\ 2\\ 1\end{bmatrix}$

We find the eigenvector(s) for r_1

$$(A - 2I)\bar{u} = \bar{0}$$

$$\begin{bmatrix}-1 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & -1\end{bmatrix}\bar{u} = \bar{0}$$

$$\left[\begin{array}{ccc|c}-1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & -1 & 0\end{array}\right]\begin{matrix}R_1\cdot -1\\ \\ R_3 + R_1\end{matrix} \quad \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -2 & 0\end{array}\right]\begin{matrix}R_1 + \frac{1}{2}R_3\\ \\ R_3\cdot -\frac{1}{2}\end{matrix} \quad \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad s$$

$$\bar{u} = \begin{bmatrix}0\\ s\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}s$$

$$\bar{x}_1 = e^{rt}\bar{u} = e^{2t}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

We find eigenvectors using r_2

$(A - r_2 I)\bar{u} = \bar{0}$

$$\begin{bmatrix} -i & 0 & -1 \\ 0 & 1-i & 0 \\ 1 & 0 & -i \end{bmatrix} \bar{u} = \bar{0} \qquad (\alpha = 1,\ \beta = 1)$$

$$\left[\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 0 & 1-i & 0 & 0 \\ 1 & 0 & -i & 0 \end{array}\right] \begin{array}{l} R_1 \cdot i \\ R_3 \cdot 1/1-i \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -i & 0 \end{array}\right] R_3 - R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

s

$$\bar{u} = \begin{bmatrix} si \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} s = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\bar{a}} s + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{b}} si$$

$$\bar{x}_2 = e^{\alpha t}(\cos(\beta t)\bar{a} - \sin(\beta t)\bar{b}) = e^t\left(\cos(t)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \sin(t)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$\bar{x}_3 = e^{\alpha t}(\cos(\beta t)\bar{b} + \sin(\beta t)\bar{a}) = e^t\left(\cos(t)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \sin(t)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

We now find the general solution

$$\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + c_3\bar{x}_3 = c_1 e^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^t\left(\cos(t)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \sin(t)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + c_3 e^t\left(\cos(t)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \sin(t)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\bar{x} = \begin{bmatrix} -c_2 e^t \sin(t) + c_3 e^t \cos(t) \\ c_1 e^{2t} \\ c_2 e^t \cos(t) + c_3 e^t \sin(t) \end{bmatrix}$$

We plug in to solve the IVP

$$\bar{x}(0) = \begin{bmatrix} c_3 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{\bar{x} = \begin{bmatrix} -e^t \sin(t) - 2e^t \cos(t) \\ 2e^{2t} \\ e^t \cos(t) - 2e^t \sin(t) \end{bmatrix}}$$

4.1 Non-Homogeneous Linear Systems

Goal: Find a general solution to $\bar{x}' = A\bar{x} + \bar{f}$.

We start this section by extending <u>undetermined coefficients</u>.

<u>Example:</u>

Solve $\bar{x}' = A\bar{x} + \bar{f}$ where $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ & $\bar{f} = \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix}$.

For brevity, we are given the complementary solution, but it is found using the same techniques we've already done.

$\bar{x}_c = c_1 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

We find $\bar{x}_p$ by first matching the form of the particular

$\bar{x}_p = \bar{a}t + \bar{b} \Rightarrow \bar{x}_p' = \bar{a}$

$\downarrow$ plug in

$\bar{a} = A(\bar{a}t + \bar{b}) + \bar{f}$

$\bar{a} = A\bar{a}t + A\bar{b} + \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix}$

$\bar{a} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\bar{a}t + \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\bar{b} + \begin{bmatrix} -1 \\ -4 \end{bmatrix}t + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

We now pattern match using the t's

$\bar{a} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\bar{b} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$\bar{0} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\bar{a} + \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\bar{a} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ Gaussian elimination

Could find A^{-1} & multiply!

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & 1 & 4 \end{array}\right] \xrightarrow{R_2 - 4R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & 0 \end{array}\right] \xrightarrow[R_2 \cdot -1/3]{R_1 + 1/3 R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$

$\bar{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\bar{b} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\bar{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 4 & 1 & 2 \end{array}\right] \xrightarrow{R_2 - 4R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -3 & -6 \end{array}\right] \xrightarrow[R_2 \cdot -1/3]{R_1 + 1/3 R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right]$

$\bar{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

This gives us $\bar{x}_p$

$\bar{x}_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} t \\ 2 \end{bmatrix}$

This gives us the general solution

$$\boxed{\bar{x} = c_1 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} t \\ 2 \end{bmatrix}}$$

Example: Large form of particular

What is form of particular if $\bar{f} = \begin{bmatrix} t \\ e^t \\ t^2 \end{bmatrix}$?

$\bar{x}_p = \bar{a}t^2 + \bar{b}t + \bar{c} + e^t\bar{d}$

Example:

Find $\bar{x}_p$ if $\bar{x}' = A\bar{x} + \bar{f}$ where $A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ & $\bar{f} = \begin{bmatrix} 6e^t \\ 2 \end{bmatrix}$.

$\bar{x}_p = e^t\bar{a} + \bar{b} \Rightarrow \bar{x}'_p = e^t\bar{a}$

$\Downarrow$

$e^t\bar{a} = A(e^t\bar{a} + \bar{b}) + \begin{bmatrix} 6e^t \\ 2 \end{bmatrix}$

$e^t\bar{a} = e^t A\bar{a} + A\bar{b} + e^t\begin{bmatrix} 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\bar{a} = A\bar{a} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ $\qquad$ $\bar{0} = A\bar{b} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

I find A^{-1} to speed my work

$\underset{A}{\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right]} \begin{matrix} R_1 \cdot \frac{1}{2} \\ R_2 - R_1 \end{matrix} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \begin{matrix} R_1 + \frac{1}{2}R_2 \\ R_2 \cdot -1 \end{matrix} \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & -1 \end{array}\right]$ A^{-1}

$A\bar{b} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$\bar{b} = A^{-1}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \bar{b}$

$\bar{a} = A\bar{a} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 + a_2 \\ 2a_1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$\begin{bmatrix} a_1 + a_2 \\ 2a_1 - a_2 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$

$3a_1 = -6$

$a_1 = -2$

$a_2 = -4$

$\bar{a} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

$$\boxed{\bar{x}_p = e^t\begin{bmatrix} -2 \\ -4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

Now, we will adapt <u>variation of parameters.</u>

Let $\bar{X}(t)$ be a fundamental matrix for $\bar{x}' = A\bar{x}$.

Recall: The fundamental matrix
$\bar{X} = [e^{r_1 t}\bar{u}_1 \ \cdots \ e^{r_n t}\bar{u}_n]$ ← $\bar{u}_1 \cdots \bar{u}_n$ are vectors! (columns)

A general solution to $\bar{x}' = A\bar{x}$ is $\bar{X}\bar{c}$. ← This is like the complementary solution (b/c $\bar{x}' = A\bar{x}$ is homogeneous)

For variation of parameters, our particular solution is

$$\bar{x}_p = \bar{X}\bar{v}(t) \Rightarrow \bar{x}_p' = \bar{X}'\bar{v}(t) + \bar{X}\bar{v}'(t)$$

We plug these into the original differential equation

$$\bar{X}'\bar{v} + \bar{X}\bar{v}' = A\bar{X}\bar{v} + \bar{f}$$

Recall we know $\bar{X}' = A\bar{X}$ b/c the general solution to $\bar{x}' = A\bar{x}$. This allows us to simplify

$$\cancel{\bar{X}'\bar{v}} + \bar{X}\bar{v}' = \cancel{A\bar{X}\bar{v}} + \bar{f}$$
$$\bar{X}\bar{v}' = \bar{f}$$

Now we can trivially solve for $\bar{v}'$ which will allow us to find $\bar{v}$ using integration, allowing us to find the particular equation.

* Note: We know the fundamental matrix $\bar{X}$ is always invertible b/c it's columns are linearly independent.

$$\bar{v}' = \bar{X}^{-1}\bar{f}$$
$$\bar{v} = \int \bar{X}^{-1}\bar{f} \qquad \bar{x}_c = \bar{X}\bar{c}$$
$$\boxed{\bar{x}_p = \bar{X}\int \bar{X}^{-1}\bar{f}}$$

Then, we find the general solution as we normally would

$$\bar{x} = \bar{x}_c + \bar{x}_p$$

How do we find $\bar{X}^{-1}$? You could do the linear algebra way, but you can also use an equation to find the inverse of $A \in M_{2\times 2}$. This is helpful when the matrix is complicated (e.g. uses functions).

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \underbrace{\frac{1}{ad-bc}}_{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

<u>Example:</u>
Find general solution to $\bar{x}' = A\bar{x} + \bar{f}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ & $\bar{f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Given $\bar{x}_c = c_1 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$ ← On the test we may have to find. You find using the earlier method w/ eigenvalues & eigenvectors & the 3 cases

theres a e^{0t} here b/c $r = 0 \pm i$ is the eigenvalue

Using $\bar{x}_c$ we find X to be

$$X = \begin{bmatrix} \sin(t) & -\cos(t) \\ \cos(t) & \sin(t) \end{bmatrix}$$

Undetermined coefficients would probably be easier b/c form of particular is $\begin{bmatrix} a \\ b \end{bmatrix}$ b/c f is constant

We now find X^{-1}

$$X^{-1} = \frac{1}{\det(X)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\sin^2(t) + \cos^2(t)} \begin{bmatrix} \sin(t) & \cos(t) \\ -\cos(t) & \sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} \sin(t) & \cos(t) \\ -\cos(t) & \sin(t) \end{bmatrix}$$

We find the particular solution using $\bar{x}_p = X \int X^{-1} f$

$$\bar{x}_p = X \int X^{-1} f$$

$$X^{-1} f = \begin{bmatrix} \sin(t) & \cos(t) \\ -\cos(t) & \sin(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}$$

$$\bar{x}_p = X \int \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}$$

$$\bar{x}_p = X \begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix}$$ ← No $+C$ b/c we want only particular

$$\bar{x}_p = \begin{bmatrix} \sin(t) & -\cos(t) \\ \cos(t) & \sin(t) \end{bmatrix} \begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\cos(t)\sin(t) + \cos(t)\sin(t) \\ -\cos^2(t) - \sin^2(t) \end{bmatrix}$$

$$\bar{x}_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

This gives us the general solution of

$$\bar{x} = \bar{x}_c + \bar{x}_p = c_1 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Example:

Solve IVP $\bar{x}' = A\bar{x} + \bar{f}$ where

$$A = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix},$$

$$F = \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix}, \ \&$$

$$\bar{x}(0) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

For brevity, we're given $\bar{x}_c$

$$\bar{x}_c = c_1 e^{t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\-3\end{bmatrix}$$

This gives us the fundamental matrix

$$\mathbf{X} = \begin{bmatrix} e^{t} & e^{-t} \\ -e^{t} & -3e^{-t}\end{bmatrix}$$

$$\mathbf{X}^{-1} = \frac{1}{\det(\mathbf{X})}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix} = \frac{1}{-3+1}\begin{bmatrix} -3e^{-t} & -e^{-t} \\ e^{t} & e^{t}\end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}e^{-t} & \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} & -\frac{1}{2}e^{t}\end{bmatrix}$$ ← could leave scalar on outside

We find $\bar{x}_p$

$$\bar{x}_p = \mathbf{X}\int \mathbf{X}^{-1} f$$

$$\mathbf{X}^{-1} f = \begin{bmatrix} \frac{3}{2}e^{-t} & \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} & -\frac{1}{2}e^{t}\end{bmatrix}\begin{bmatrix} 2e^{t} \\ 4e^{t}\end{bmatrix} = \begin{bmatrix} 3+2 \\ -e^{2t} - 2e^{2t}\end{bmatrix}$$ ← Shouldn't be done in separate block

$$= \begin{bmatrix} 5 \\ -3e^{2t}\end{bmatrix}$$

$$\bar{x}_p = \mathbf{X}\int \begin{bmatrix} 5 \\ -3e^{2t}\end{bmatrix}$$

$$\bar{x}_p = \begin{bmatrix} e^{t} & e^{-t} \\ -e^{t} & -3e^{-t}\end{bmatrix}\begin{bmatrix} 5t \\ -\frac{3}{2}e^{2t}\end{bmatrix}$$

$$\bar{x}_p = \begin{bmatrix} 5te^{t} - \frac{3}{2}e^{t} \\ -5te^{t} + \frac{9}{2}e^{t}\end{bmatrix}$$ ← this makes sense b/c we have e^{t} & a te^{t} to make $\bar{x}_p$ linearly independent of $\mathbf{X}$.

This gives us the general solution

$$\bar{x} = \bar{x}_c + \bar{x}_p = c_1 e^{t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}5te^{t} - \frac{3}{2}e^{t} \\ -5te^{t} + \frac{9}{2}e^{t}\end{bmatrix}$$

We solve the IVP

$$\bar{x}(0) = c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\-3\end{bmatrix} + \begin{bmatrix}-\frac{3}{2}\\ \frac{9}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_2 - \frac{3}{2} \\ -c_2 - 3c_2 + \frac{9}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

$c_1 + c_2 = 2$ $\quad -c + 3c_2 = -4$

$-c_1 + 3c_2 = -4$

$-2c_2 = -2$

$c_2 = 1$

$c_1 = 1$

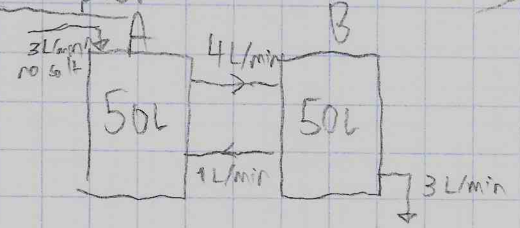
$$\boxed{\bar{x} = e^{t}\begin{bmatrix}1\\-1\end{bmatrix} + e^{-t}\begin{bmatrix}1\\-3\end{bmatrix} + \begin{bmatrix}5te^{t} - \frac{3}{2}e^{t} \\ -5te^{t} + \frac{9}{2}e^{t}\end{bmatrix}}$$

Interconnected Tanks

Recall for simpler system $\frac{dx}{dt} = F_i C_i - F_o C_o$ where x = mass of thing.

It is strongly recommended that you use a picture to form your equations b/c the word problems are long & hard to parse.

Example: Entire IVP



We need the volume of fluid in each tank to remain constant for the methods in this class. (Gives us coefficient matrix as function) However, the tanks can be different volume.

Write x_1 & x_2 as salt in A & B respectively.
Write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Initial Conditions:

$x_1(0) = 2.5$ kg $\Rightarrow x(0) = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix}$
$x_2(0) = 0$ kg

$$\frac{dx_1}{dt} = F_i C_i - F_o C_o = \left(\overbrace{\cancel{(3\ L/min)(0\ kg/L)}}^{\text{external}} + \overbrace{(1\ L/min)\left(\frac{x_2\ kg}{50\ L}\right)}^{B}\right) - (4\ L/min)\left(\frac{x_1\ kg}{50\ L}\right)$$

$$= \frac{x_2}{50} - \frac{4x_1}{50}$$

could combine at start

$$\frac{dx_2}{dt} = F_i C_i - F_o C_o = (4\ L/min)\left(\frac{x_1\ kg}{50\ L}\right) - \left(\overbrace{(1\ L/min)\left(\frac{x_2\ kg}{50\ L}\right)}^{\text{to A}} + \overbrace{(3\ L/min)\left(\frac{x_2}{50L}\right)}^{\text{to outsid}}\right)$$

$$\frac{dx_2}{dt} = \frac{4x_1}{50} - \frac{4x_2}{50}$$

We rewrite the above as a matrix

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -4/50 & 1/50 \\ 4/50 & -4/50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ ← pure water coming in (this is $\vec{F}$)

$$\boxed{\vec{x}' = \begin{bmatrix} -4/50 & 1/50 \\ 4/50 & -4/50 \end{bmatrix} \vec{x}}$$ ← "Formulate the IVP"
← don't simplify b/c it makes future work harder

We now solve the homogenous d.e.

We find eigenvalues by solving $\det(A - rI) = 0$

$$0 = \begin{vmatrix} -4/50 - r & 1/50 \\ 4/50 & -4/50 - r \end{vmatrix}$$
$$= (-4/50 - r)^2 - 4/50^2$$
$$= 16/50^2 + 8/50\, r + r^2 - 4/50^2$$
$$= r + 8/50\, r + 12/50^2$$
$$= (r + 6/50)(r + 2/50) = 0$$
$$r_1 = -6/50 \quad r_2 = -2/50$$

We find eigenvector using r_1

$(A - r_1 I)\bar{u}_1 = \bar{0}$

$$\left[\begin{array}{cc|c} 2/50 & 1/50 & 0 \\ 4/50 & 2/50 & 0 \end{array}\right] \begin{array}{l} R_1 \cdot 50/2 \\ R_2 - 2R_1 \end{array} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

s

$$\bar{u}_1 = \begin{bmatrix} -1/2 s \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} s \quad \text{(for some s)}$$

We find eigenvector using r_2

$(A - r_2 I)\bar{v}_2 = \bar{0}$

$$\left[\begin{array}{cc|c} -2/50 & 1/50 & 0 \\ 4/50 & -2/50 & 0 \end{array}\right] \begin{array}{l} R_1 \cdot -50/2 \\ R_2 + 2R_1 \end{array} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

s

$$\bar{v}_2 = \begin{bmatrix} 1/2 s \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} s \quad \text{(for some s)}$$

This gives us the general solution

$$\bar{x} = c_1 e^{-4/50 t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-2/50}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We now plug in to solve the IVP

$$\bar{x}(0) = c_1\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix}$$

$-c_1 + c_2 = 2.5$

$2c_1 + 2c_2 = 0$

$4c_2 = 5$

$c_2 = 5/4$

$c_1 = -5/4$

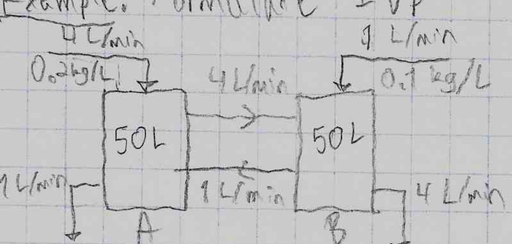
$$\boxed{\bar{x} = -5/4 e^{-4/50 t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 5/4 e^{-2/50 t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

We can rewrite this into x_1 & x_2 if we want (or the problem asks)

$$\boxed{\begin{aligned} x_1 &= 5/4 e^{-4/50 t} + 5/4 e^{-2/50 t} \\ x_2 &= -5/2 e^{-4/50 t} + 5/2 e^{-2/50 t} \end{aligned}}$$

$\lim_{t\to\infty} x_1 = \lim_{t\to\infty} x_2 = 0$, which makes sense b/c pure water is flowing in

Example: Formulate IVP



Write x_1 & x_2 as salt in A & B respectively.

Write $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Initial Values: $x_1(0) = 0$, $x_2(0) = 0.5$

$$\frac{dx_1}{dt} = F_i C_i - F_o C_o$$

$$= \left((4\ L/min)(0.2\ kg/L) + (1\ L/min)\left(\frac{x_2\ kg}{50\ L}\right)\right) - \left((4\ L/min)\left(\frac{x_1\ kg}{50\ L}\right) + (1\ L/min)\left(\frac{x_1\ kg}{50\ L}\right)\right)$$

$$= 0.8 + \frac{x_2}{50} - \frac{5x_1}{50}$$

$$\frac{dx_2}{dt} = F_i C_i - F_o C_o$$

$$= \left((1\ L/min)(0.1\ kg/L) + (4\ L/min)\left(\frac{x_1\ kg}{50\ L}\right)\right) - \left((1\ L/min)\left(\frac{x_2\ kg}{50\ L}\right) + (4\ L/min)\left(\frac{x_2\ kg}{50\ L}\right)\right)$$

$$= 0.1 + \frac{4x_1}{50} - \frac{5x_2}{50}$$

We rewrite the above as a matrix equation

$$\underbrace{\bar{x}'}_{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'} = \begin{bmatrix} -5/50 & 1/50 \\ 4/50 & -5/50 \end{bmatrix} \underbrace{\bar{x}}_{=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} + \begin{bmatrix} 0.8 \\ 0.1 \end{bmatrix}$$

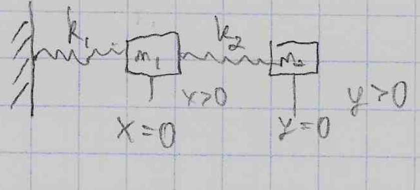
$$\bar{x}(0) = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

5.6 Coupled Mass-Spring Systems ← You can solve these w/ other methods (not taught here)

We'll on undamped (no friction) & homogenous (no external forces) systems.

Recall how we used $mx'' + bx' + kx = F_{ext}$ to model springs earlier. (0 here)

Let's model a simple coupled mass-spring system.



Only think about springs connected to your mass!

Recall Newton's 2nd law $F_{net} = ma$. We use this to model the motion of our springs

Mass 1: $F_{net} = m_1 a$ $\quad x > 0 \Rightarrow$ spring 1 is stretched, so m_1 pulled left

$F_{net} = m_1 a$

$m_1 x'' = -k_1 x + k_2(y - x)$ $\quad (y-x) > 0 \Rightarrow$ spring 2 stretched, so m_1 pulled right

Mass 2:

$m_2 y'' = -k_2 (y-x)$ ← $y-x>0 \Rightarrow$ spring 1 stretched & pulls m_2 to left

← opposite of what m_1 feels!

We now have 2 homogeneous diff eq's as linear equations of x & y. That means we can express them as

$\vec{u}' = A\vec{u}$

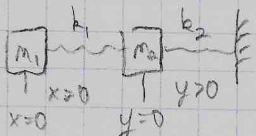
where

$$\vec{u} = \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}.$$

We then do our normal homogenous matrix diff eqs. ↙ or system of homogenous linear equations.

Example: 2 Springs, 2 Masses, 1 Wall

$m_1 = 1$ kg	$x(0) = -1$	$y(0) = 0$
$m_2 = 2$ kg	$x'(0) = 0$	$y'(0) = 0$
$k_1 = 4$ N/m		
$k_2 = 10/3$ N/m		



Imagine just moving 1 at a time

$m_1 x'' = k_1(y-x)$ $\quad y-x>0 \Rightarrow k_1$ stretched so x to right

← opposite of m_1. Also thing

$m_2 y'' = -k_1(y-x) - k_2 y$ ← $y>0 \Rightarrow k_2$ compressed, so y to left

We now plug in our values & solve for x'' & y''

$1x'' = 4(y-x) = 4y - 4x$

$2y'' = -4(y-x) - \frac{10}{3}y \Rightarrow y'' = -2(y-x) - \frac{5}{3}y = -\frac{11}{3}y + 2x$

We now set up our equations to be of the form $\vec{u}' = A\vec{u}$

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -\frac{11}{3} & 0 \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$$

← just say $x' = x'$ & $y' = y'$

We now solve this system using our earlier techniques.

We solve $\det(A - rI) = 0$.

$$0 = \begin{vmatrix} -r & 1 & 0 & 0 \\ -4 & -r & 4 & 0 \\ 0 & 0 & -r & 1 \\ 2 & 0 & -\frac{11}{3} & -r \end{vmatrix}$$

$$= -r \begin{vmatrix} -r & 4 & 0 \\ 0 & -r & 1 \\ 0 & -\frac{11}{3} & -r \end{vmatrix} - 1 \begin{vmatrix} -4 & 4 & 0 \\ 0 & -r & 1 \\ 2 & -\frac{11}{3} & -r \end{vmatrix}$$

$= -r(-r(r^2 + \frac{11}{3}) \cancel{-4(0)}) - 1(-4(r^2 + \frac{11}{3}) - 4(0-2))$

$= r^2(r^2 + \frac{11}{3}) - (-4(r^2 + \frac{11}{3}) + 8)$

$= r^4 + \frac{11}{3}r^2 + 4r^2 + \frac{44}{3} - 8$

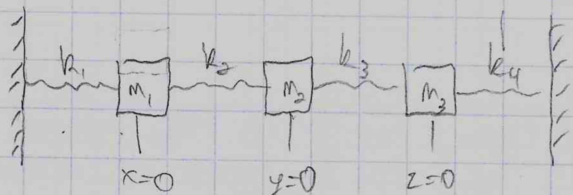
$\downarrow$ Wolfram Alpha (or Friends)

$$\therefore \quad = \tfrac{1}{3}(r^2+1)(3r^2+20) = 0$$

$$\therefore \quad r = \pm i \qquad r = \pm \tfrac{20}{3}$$

Now you'd find the eigenvectors & find the general solution.

Example: 4 springs, 3 masses, 2 walls



Right is positive. If this isn't stated, assume right is positive.

$m_1 = m_2 = m_3 = m$

$k_1 = k_2 = k_3 = k_4 = k$

m_1: $m_1 x'' = -k_1 x + k_2(y-x)$ $\quad\therefore$ think through directions!

m_2: $m_2 y'' = -k_2(y-x) + k_3(z-y)$

m_3: $m_3 z'' = -k_3(z-y) - k_4(z)$

Now we simplify everything & solve for positions

$$x'' = \frac{-2kx + ky}{m}$$

$$y'' = \frac{kx - 2ky + kz}{m}$$

$$z'' = \frac{ky - 2z}{m}$$

Now we set up our equation $\vec{v}' = A\vec{v}$

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ z' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{2k}{m} & 0 & k/m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ k/m & 0 & -\frac{2k}{m} & 0 & \frac{k}{m} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k}{m} & 0 & -\frac{2k}{m} & 0 \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ z' \end{bmatrix}$$

That's big. Let's not do it.

5.4 Introduction to Phase Plane

Goal: Use phase plane to analyze first order autonomous systems.

We use this to understand the diff eq when we cannot solve it.

Recall an anonymous d.e. is one where the independent variable doesn't appear explicitly. For example,

$$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y)$$

Def: Phase Plane, Trajectory & Phase Portrait

If $x(t), y(t)$ is a solution to the above eqn. for t in the interval I, then a plot in the xy-plane $x=x(t)$, $y=y(t)$ for t in I, together w/ arrows indicating the direction of increasing t is the <u>trajectory</u> of the eqn.

In this context, we call the xy-plane the <u>phase plane</u> & a set of representative trajectories from the phase plane the <u>phase portrait</u>.

We can find trajectories w/o knowing the solution pair $x(t), y(t)$ by solving the <u>phase plane equation</u>

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Def: Critical/Equilibrium Point

A point (x_0, y_0) where $\underbrace{f(x_0,y_0)=0}_{\text{i.e. } \frac{dx}{dt}=0}$ & $\underbrace{g(x_0,y_0)=0}_{\text{i.e. } \frac{dy}{dt}=0}$ is called the <u>critical or equilibrium point</u> of the system

$$\frac{dx}{dt} = f(x,y) \quad \& \quad \frac{dy}{dt} = g(x,y)$$

<u>Example:</u>

Find the critical point set for the system & solve the phase plane equation for

$$\frac{dx}{dt} = y^2 - 3y + 2 \quad \& \quad \frac{dy}{dt} = (x-1)(y-2) = 0$$

We find the critical points for $\frac{dx}{dt}$ & $\frac{dy}{dt}$ independently & union them.

For $\frac{dx}{dt}$:

$$\frac{dx}{dt} = y^2 - 3y + 2 = (y-1)(y-2) = 0 \Rightarrow y=1 \text{ or } y=2$$

For $\frac{dy}{dt}$:

If $y=1$: $\frac{dy}{dt} = (x-1)(1-2) = 0 \Rightarrow x = 1$

<u>Critical Point: $(1,1)$</u>

If $y=2$: $\frac{dx}{dt} = (x-1)(2-2) = 0 \Rightarrow x = c \quad c \in \mathbb{R}$ <u>Critical Point: $(c, 2)$</u>

We now have the critical points set
$\{(1,1)\} \cup \{(x,2) \mid x \in \mathbb{R}\}$.

We now solve the phase plane equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(x-1)\cancel{(y-2)}}{(y-1)\cancel{(y-2)}}$$ ← not guaranteed

$$\frac{dy}{dx} = \frac{x-1}{y-1}$$ ← Separable d.e.!

$$\int y-1\,dy = \int x-1\,dx$$

$$\boxed{\tfrac{1}{2}y^2 - y = \tfrac{1}{2}x^2 - x + C}$$ ← Implicit solution

We clean up the implicit solution to make it nicer to draw

$$y^2 - 2y = x^2 - 2x + C$$

$$(y^2 - 2y) - (x^2 - 2x) = C$$

$$((y-1)^2 \cancel{+1}) - ((x-1)^2 \cancel{+1}) = C$$

$$\boxed{(y-1)^2 - (x-1)^2 = C}$$ Cleaner implicit solution.

This is a family of hyperbolas!

Example:

Solve phase plane equation & sketch several representative trajectories w/ their flow arrows, where

$$\frac{dx}{dt} = -8y \quad \& \quad \frac{dy}{dt} = 18x$$

We also find the critical points that make $\frac{dx}{dt} = 0$ & $\frac{dy}{dt} = 0$.

Critical Point: $(0,0)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{18x}{-8y}$$ ← Solve phase plane equation

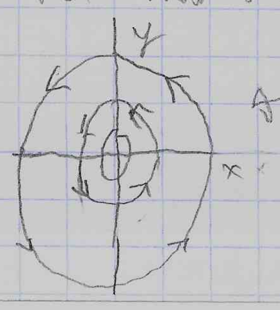
$$\int -8y\,dy = \int 18x\,dx$$

$$-4y^2 = 9x^2 + C$$

$$\boxed{-4y^2 - 9x^2 = C}$$

Family of elipses!

We'll know draw several trajectories



← $(0,0)$ is called a center in this case & is stable.

If $\frac{dx}{dt} > 0$, arrows point right, else left.

If $\frac{dy}{dt} > 0$, arrows point up, else up.

$\frac{dx}{dt} = -8y > 0$ when $y < 0$ $\quad$ $\frac{dx}{dt} = -8y < 0$ when $y > 0$

$\frac{dy}{dt} = 18x > 0$ when $x > 0$ $\quad$ $\frac{dy}{dt} = 18x < 0$ when $x < 0$

Def: Nullclines

Nullclines are places where $\frac{dy}{dt} = 0$ or $\frac{dx}{dt} = 0$.

Nullclines break up the phase plane into regions of different qualitative behavior.

Def: Critical / Equilibrium Points

Critical/Equilibrium points occur when $\frac{dy}{dt} = \frac{dx}{dt} = 0$ (i.e. where nullclines cross). We classify them as ...

- stable if all trajectories flow toward the point. (Centers / concentric ellipses are always stable.)

stronger than → stable
- asymptotically stable if it attracts its neighboring trajectories as $t \to \infty$.
- unstable if trajectories flow away from the point. (Saddle points are always unstable.)

Example:

Solve the phase plane equation, classify critical points, & draw trajectories.

$\frac{dx}{dt} = y(1+x^2+y^2) = 0 \Rightarrow y = 0$

$\frac{dy}{dt} = x(1+x^2+y^2) = 0 \Rightarrow x = 0$

(0, 0) is equilibrium point.

$\frac{dy}{dx} = \frac{x\cancel{(1+x^2+y^2)}}{y\cancel{(1+x^2+y^2)}} = \frac{x}{y}$ ← Equation of phase plane

We solve the phase plane equation as separable d.e.

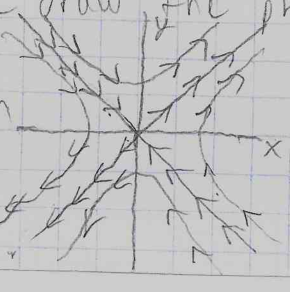
$\int y\,dy = \int x\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$

$y^2 - x^2 = 0$ ← hyperbola!

We draw the phase plane equation in the xy-plane

This could be done in 3d! (if we had x & y)



if $C = 0$, $y^2 = x^2 \Rightarrow y = \pm x$

if $C = 1$, $y^2 = x^2 + 1$ ← vertical hyperbolas

if $C = -1$, $x^2 = y^2 + 1$ ← horizontal hyperbolas

$\frac{dx}{dt} > 0$ when $y > 0$ (arrows flow right)

$\frac{dy}{dt} > 0$ when $x > 0$ (arrows flow up)

We update the old image

$(0,0)$ is a saddle point & thus unstable. (Some flow away, some flow towards)

12.2 Linear Systems in the Plane

no critical point directly adjacent

<u>Thm:</u>

Assume the origin is an isolated critical point for the linear system

$$\frac{dx}{dt} = ax + by \quad \& \quad \frac{dy}{dt} = cx + dy \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a, b, c, d \in \mathbb{R}$ & $ad - bc \neq 0$ (i.e. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$).

Let r_1 & r_2 be roots to the characteristic equation, $|A - rI| = 0$.

That is, r_1 & r_2 are eigenvalues of A.

The stability of the critical point & the classification of the origin as a critical point depends on r_1 & r_2 as follows.

(Type will be given, stability won't ...)

- Distinct & Positive
  - Type: improper node
  - Stability: Unstable
    $\bar{x} = c_1 e^{r_1 t} \bar{u}_1 + c_2 e^{r_2 t} \bar{u}_2 \to \infty$ as $t \to \infty$ b/c $r_1, r_2 > 0$.
- Distinct & Negative
  - Type: improper node
  - Stability: Asymptotically stable
    $\bar{x} = c_1 e^{r_1 t} \bar{u}_1 + c_2 e^{r_2 t} \bar{u}_2 \to 0$ as $t \to \infty$ b/c $r_1, r_2 < 0$
- Opposite Sign
  - Type: saddle point
  - Stability: Unstable
    $\bar{x} = c_1 e^{r_1 t} \bar{u}_1 - c_2 e^{r_2 t} \bar{u}_2 \to ?$ is undecidable
- Equal & Positive
  - Type: proper or improper node
  - Stability: Unstable (see Distinct & Positive)

- Equal & Negative
  - Type: improper or proper node
  - Stability: asymptotically stable (see Distinct & Negative)
- $r = \alpha \pm \beta i$, $\alpha > 0$
  - Type: spiral point
  - Stability: unstable
    $\bar{x} = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t)) \to \infty$ as $t \to \infty$ b/c $\alpha > 0$
- $r = \alpha \pm \beta i$, $\alpha < 0$
  - Type: spiral point
  - Stability: asymptotically stable
    $\bar{x} = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t)) \to 0$ as $t \to \infty$ b/c $\alpha < 0$
- $r = \pm \beta i$
  - Type: center
  - Stability: stable

<u>Def</u>: Proper Node

A critical point is a proper node if every direction thru the origin defines a trajectory.

<u>Def</u>: Improper Node

A critical point is an improper node if almost all trajectories have the same tangent line at the origin.

<u>Example</u>: Improper Node

Sketch the phase plane & classify critical points of

$x' = 3x$ A
$y' = 6y$.

Let's do it w/ the old method first.

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{6y}{3x}$$

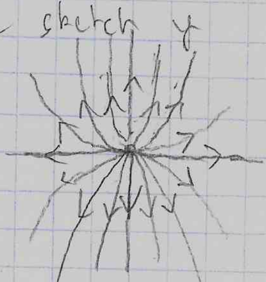
$$\int \frac{dy}{y} = \int \frac{2\,dx}{x}$$

$$\ln|y| = 2\ln|x| + C$$

$$y = e^{2\ln|x| + C}$$

$$y = K x^2$$

We sketch



$x' > 0$ when $x > 0$ so right when $x > 0$

$y' > 0$ when $y > 0$

We can see than critical point $(0,0)$ is <u>unstable</u> & <u>improper</u> node. trajectories tangent line is $x = 0$

Now let's use our new technique.

$$\begin{matrix} x' = 3x \\ y' = 6y \end{matrix} \Rightarrow \begin{matrix} x' = ax + by \\ y' = cx + dy \end{matrix} \Rightarrow A = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

We find the eigenvalues of A

$$0 = \begin{vmatrix} 3-r & 0 \\ 0 & 6-r \end{vmatrix}$$

$$= (3-r)(6-r) = 0$$

$$r_1 = 3 \quad r_2 = 6$$

We could also solve the equation!! :)

We have distinct & positive roots (eigenvalues), so it is an <u>unstable</u> & <u>improper node</u>.

<u>Example:</u> Proper Node

Classify critical point & draw the phase plane of

$$\frac{dx}{dt} = 3x \quad \& \quad \frac{dy}{dt} = 3y$$

Trivially the critical point is at $(0,0)$.

Now we solve the phase plane equation ($\frac{dy}{dx} = 0$)

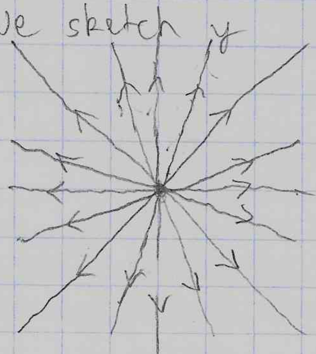
$$\frac{dy}{dx} = \frac{3y}{3x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C$$

$$y = Kx$$

We sketch



$\frac{dx}{dt} > 0$ when $x > 0$

$\frac{dy}{dt} > 0$ when $y > 0$

We can see critical point $(0,0)$ is unstable & proper node.

Now let's use our new technique

$$\frac{dx}{dt} = 3x \qquad \Rightarrow A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$
$$\frac{dy}{dt} = 3y$$

We find the eigenvalues of A

$$0 = \begin{vmatrix} 3-r & 0 \\ 0 & 3-r \end{vmatrix}$$

$$= (3-r)^2 = 0$$

$$r_1 = r_2 = 3$$

We have equal & positive roots (eigenvalues), so it is a proper or improper node & unstable

Not good on test since we can solve the phase plane eqn!!

Classifying Critical Points not at Origin

Suppose we find a critical point at (a, b). We essentially shift the equation over to put it at $(0,0)$.

We define

$x = u + a$,

$y = v + b$,

$\frac{dx}{dt} = \frac{du}{dt}$, &

$\frac{dy}{dt} = \frac{dv}{dt}$.

Example:

Find & classify critical point where

$\frac{dx}{dt} = -3x - y - 4$ &

$\frac{dy}{dt} = 29x + y + 30$.

We find the critical point (x, y) where $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

$-3x - y - 4 = 0$

$29x + y + 30 = 0$

$26x + 26 = 0$

$x = -1$

$y = -1$

We have critical point $(\overset{a}{-1}, \overset{b}{-1})$. We define x & y

$x = u + a = u - 1$

$y = v + b = v - 1$

We now plug in our x & y into $\frac{dx}{dt}$ & $\frac{dy}{dt}$ to get $\frac{du}{dt}$ & $\frac{dv}{dt}$.

$\frac{du}{dt} = -3(u-1) - (v-1) - 4 = -3u - v$

$\frac{dv}{dt} = 29(u-1) + (v-1) + 30 = 29u + v$

Constants always disappear for linear system but not always for almost linear system.

Now we plug these new equations into the matrix & resume our method as before

$\frac{du}{dt} = -3u - v$ $\Rightarrow A = \begin{bmatrix} -3 & -1 \\ 29 & 1 \end{bmatrix}$

$\frac{dv}{dt} = 29u + v$

We now find the eigenvalues

$0 = \begin{vmatrix} -3-r & -1 \\ 29 & 1-r \end{vmatrix}$

$= (-3-r)(1-r) + 29$

$= -3 + 2r + r^2 + 29$

$= r^2 + 2r + 26 = 0$

$$\therefore\ r = \frac{-2 \pm \sqrt{4 - 4(26)}}{2}$$

We can see we get a negative discriminant, so we know our roots match $r = \alpha \pm \beta i,\ \alpha < 0$

$(-1,-1)$ is an asymptotically stable spiral point.

12.3 Almost Linear

Def: Almost Linear System

Let the origin $(0,0)$ be a critical point of the autonomous system

$$x'(t) = ax + by + F(x,y)$$
$$y'(t) = cx + dy + G(x,y)$$

where

- a, b, c, d are constants,
- F, G are continuous about the origin,
- $ad - bc \neq 0$ (i.e. matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible).

The corresponding linear system is where $F = G = 0$.

The system is said to be almost linear near the origin iff

$$\lim_{\sqrt{x^2+y^2}\to 0} \frac{F(x,y)}{\sqrt{x^2+y^2}} = \lim_{\sqrt{x^2+y^2}\to 0} \frac{G(x,y)}{\sqrt{x^2+y^2}} = 0$$

distance from origin

Thm: Almost Linear Systems

If we have almost linear system w/ r_1 & r_2 roots to the characteristic equation ($F = G = 0$), then we use the following chart to determine the type of critical point

Same as previous table (mostly)

Roots	Type	Stability
distinct, +	improper node	unstable
distinct, −	improper node	asymptotically stable
opposite signs	saddle point	unstable
equal, +	proper, improper, or saddle	unstable
equal, −	proper, improper, or saddle	asymptotically stable
$r = \alpha \pm \beta i$ $\alpha > 0$	spiral pt	unstable
$r = -\alpha \pm \beta i$ $\alpha > 0$	spiral pt	asymptotically stable
$r = \pm \beta i$	center or spiral	indeterminate

Positive ⇒ unstable
Negative ⇒ asymp. stable

Example:

Show the given system is almost linear near origin & determine type & stability of critical point (@ origin)

$$\frac{dx}{dt} = -2x + 2xy$$
$$\frac{dy}{dt} = x - y + x^2$$

We find F & G

$F(x,y) = 2xy$
$G(x,y) = x^2$

We show limits hold

$F(x,y)$ $\lim_{\sqrt{x^2+y^2} \to 0} \frac{2xy}{\sqrt{x^2+y^2}}$ convert to polar

$= \lim_{r \to 0^+} \frac{2r\cos(\theta) r\sin(\theta)}{r} =$

$= \lim_{r \to 0^+} 2r\cos(\theta)\sin(\theta) = 0$ ✓

$\lim_{\sqrt{x^2+y^2} \to 0} = \lim_{r \to 0^+}$

$G(x,y)$ $\lim_{\sqrt{x^2+y^2} \to 0} \frac{x^2}{\sqrt{x^2+y^2}}$

$= \lim_{r \to 0^+} \frac{r^2\cos^2(\theta)}{r}$

$= \lim_{r \to 0^+} r\cos^2(\theta) = 0$ ✓

Our coefficient matrix: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}$$

Since A is a triangular matrix, the roots to $\det(A - rI) = 0$ are

$r_1 = -2,\ r_2 = -1$

Roots are negative & distinct, so it is asymptotically stable improper node.

What if our critical point isn't at the origin? Then we shift it to the origin!

Example:

Find & classify all critical points to

$$\frac{dx}{dt} = 4 - xy$$
$$\frac{dy}{dt} = x - y$$

We trivially solve the system

$\frac{dy}{dt} = x - y = 0 \Rightarrow x = y$

$\frac{dx}{dt} = 4 - xy = 4 - x^2 \Rightarrow x = \pm 2$

$\therefore$ (2, 2) & (-2, -2) are our critical points

We classify $(\overset{a}{2},\overset{b}{2})$ by shifting it to origin

$x = u + a \qquad y = v + b$

$x = u + a \qquad y = v + 2$

$$\frac{du}{dt} = 4 - (u+2)(v+2) = 4 - uv - 2u - 2v - 4$$

$= -uv - 2u - 2u$

$$\frac{dv}{dt} = (u+2) - (v+2) = u - v$$

A this point, we'd have to show this is almost linear, but we assume it here for brevity.

Our coefficient matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A = \begin{bmatrix} -2 & -2 \\ 1 & -1 \end{bmatrix}$$

We find roots to $p(r) = \det(A - rI)$

$$\begin{vmatrix} -2-r & -2 \\ 1 & -1-r \end{vmatrix}$$

$= r^2 + 3r + 2 + 2$

$= r^2 + 3r + 4$

$$r = \frac{-3 \pm \sqrt{3^2 - 4(4)}}{2}$$

$$= \frac{-3 \pm \sqrt{-7}}{2}$$

Here we have $r = -\alpha \pm \beta i$, so $(2,2)$ is an asymptotically stable spiral point.

We follow the same procedure for $(-2,-2)$...