Chapter 1: Real Numbers (Axiomatically)

<u>Axioms</u> are the fundamental assumptions a field of math makes.

In this class we'll be using real numbers $\mathbb{R}$ (& similar structures). These structures are called <u>fields</u> & must obey the <u>field axioms</u>.

<u>Def:</u> Field Axioms
A field F has two binary operations addition $(+)$ & multiplication $(\cdot)$ w/ the following properties (axioms)

i) $+$ & $\cdot$ are both commutative.
ii) $+$ & $\cdot$ are both associative.
iii) $+$ & $\cdot$ have identities. We call $+$'s identity 0 & $\cdot$'s 1.
iv) $+$ & $\cdot$ have inverses. For $+$, the inverse of x is $-x$ & for $\cdot$ it is $\frac{1}{x}$. That is
$x+(-x)=0$ & $x\cdot\frac{1}{x}=1$
v) $\cdot$ distributes over $+$. That is
$x\cdot(y+z) = xy + xz$

<u>Def:</u>
On top of addition & multiplication, we define subtract $(-)$ & division $(\div)$ as
$x-y = x+(-y)$ & $x \div y = \frac{x}{y} = x\cdot\frac{1}{y}$

However, the field axioms don't fully describe the real numbers. For example $\mathbb{Z}_2 = \{0,1\}$ is a (finite!) field.

To (more) fully describe the reals, we must have the <u>order axiom</u>.

<u>Def:</u> Order Axiom
There exists a set $\mathbb{R}^+ \subset \mathbb{R}$ w/ the following properties

i) Closure under $+$ & $\cdot$
$\forall x,y \in \mathbb{R}^+$ $x+y\in\mathbb{R}^+$ & $x\cdot y\in\mathbb{R}^+$.

ii) Given $a\in\mathbb{R}$, either a is positive, a is zero, or a is negativ[e]
Only one must/can be true.

W/ the order axiom, we can define a total ordering on $\mathbb{R}$.

Def: Total Ordering

We define a relation $\leq$ on $\mathbb{R}$ as follows. $\forall x, y \in \mathbb{R}$

$\forall x, y \in \mathbb{R}$, $x \leq y$ is true iff $x - y \in \mathbb{R}^+ \cup \{0\}$.

The other relations are defined similarly.

Remark:

We won't prove this but $\leq$ (& $\geq$) have the following properties

i) Reflexivity: $x \leq x \quad \forall x \in \mathbb{R}$
ii) Antisymmetry: $x \leq y$ & $y \leq x \Rightarrow x = y \quad \forall x, y \in \mathbb{R}$
iii) Transitivity: $x \leq y$ & $y \leq z \Rightarrow x \leq z \quad \forall x, y, z \in \mathbb{R}$

} This only makes $(\mathbb{R}, \leq)$ a partially ordered set

The relations $<$ & $>$ are merely transitive

Thm: Law of Trichotomy

$\forall a, b \in \mathbb{R}$ one & only one of the following hold

- $a < b$
- $a = b$
- $a > b$

This is what makes $(\mathbb{R}, \leq)$ a totally ordered set.

Thm:

We can define some ways ordering interacts w/ arithmetic. We won't prove these here.

Consider $a, b, c, d \in \mathbb{R}$.

i) $a > b$ & $c \geq d \Rightarrow a + c > b + d$
ii) $a > b > 0$ & $c \geq d > 0 \Rightarrow ac > bd$
iii) $a > b$ & $c < 0 \Rightarrow ac < bc$

Def:

The absolute value converts any $\mathbb{R}$ to some $\mathbb{R}^+$.

$$\forall a \in \mathbb{R} \quad |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Now we have order but still don't have irrational numbers. (For example rational numbers $\mathbb{Q}$ satisfy the field axioms & order axiom)

Note that $\mathbb{Z}_2$ doesn't satisfy the order axioms b/c there's no sence of disjoint positives & negatives. It does satisfy the field axioms tho.

The natural numbers $\mathbb{N}$ don't obey the field axioms but do they order axiom. Likewise the integers work. Also $\mathbb{R}^+$.

We will need the least upper bound axiom to fully describe the reals.

Def: Least Upper Bound / Supremum

Let $K \subseteq \mathbb{R}$ & $x \in \mathbb{R}$.

x is an upper bound for K if $x \geq k \; \forall k \in K$.

If x is an upper bound for K & $x \leq u$ for all upper bounds u of K, then x is the least upper bound or supremum of K. We say

$$x = \sup K$$

If $x = \sup K$ & $x \in K$, then x is the greatest element of K.

Def: Greatest Lower Bound / Infimum

This is the same as the above but as a lower bound & we denote the infimum as $\inf K$.

Thm:

Let B be a non-empty subset of $\mathbb{R}$, the $b \in \mathbb{R}$ be an upper bound for B. Then

$$\left.\begin{aligned} &b = \sup(B) \\ &\Leftrightarrow \forall \varepsilon \in \mathbb{R}^+ \; \exists x \in B \text{ st } |x - b| < \varepsilon \\ &\Leftrightarrow \forall \varepsilon \in \mathbb{R}^+ \; \exists x \in B \text{ st } x \in (b - \varepsilon, b] \end{aligned}\right\} \text{Like epsilon-delta proofs}$$

A similar theorem exists for non-empty sets bounded below.

Def: Least Upper Bound Axiom

Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

This is called the least upper bound property.

Thm: Greatest Lower Bound

Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

This is called the greatest lower bound property.

Thm:

$\forall x \in \mathbb{R} \; \exists n \in \mathbb{N}$ st $n > x$

$\forall \varepsilon \in \mathbb{R}^+ \; \exists n \in \mathbb{N}$ st $1/n < \varepsilon$

Thm: Existence of Square Roots
$\forall a \in \mathbb{R}^+ \ \exists x \in \mathbb{R}$ st $x^2 = a$.

Pf:
TODO: Theorem 1.4.8

This theorem/proof shows that irrational numbers exist.

Thm: Nested Interval Theorem
Every nested sequence of non-empty closed intervals in $\mathbb{R}$ has a non-empty intersection.

Let (a_i) & (b_i) be sequences of real numbers where
$\forall i \in \mathbb{N}$ $a_i \leq b_i$ & $\forall i \in \mathbb{N}$ $a_i \leq a_{i+1}$ & $b_i \geq b_{i+1}$
Then
$\bigcap_{i \in \mathbb{N}} [a_i, b_i] \neq \emptyset$

Chapter 2: Measuring Distances

Def:

A metric space is a set w/ some sense of distance b/w elements. Ex: $\mathbb{Z}$, $\mathbb{R}^n$

Formally, it is a non-empty set & function $d: X \times X \to \mathbb{R}$ where $\forall a, b, c \in X$

(a) Positive: $d(a, b) \geq 0$
(b) Positive Definite: $d(a, b) = 0 \Leftrightarrow a = b$
(c) Symmetric: $d(a, b) = d(b, a)$
(d) Triangle Inequality: $d(a, c) \leq d(a, b) + d(b, c)$

d is called the distance function or metric.

The elements of X are called points.

Example: Trivial Metric

For any non-empty set X define d by

$$d(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$$

This is clearly positive, positive definite, symmetric, & w/ some work satisfies the triangle inequality.

Example: Standard Metric

$d(a, b) = |a - b|$

a) $|a-b| \geq 0$ by def
b) $|a-b| = 0 \Leftrightarrow a = b$
c) $|a-b| = |-(b-a)| = |b-a|$
d) $|a-b| + |b-c| \geq |a| - |b| + |b| - |c|$ *
$\geq |a| - |c|$ by 1.3.8

Def: Euclidean Metric

$\mathbb{R}^n$ space has the following metric d where $a = [a_1, \dots a_n]$ & $b = [b_1, \dots b_n]$

$$d(a, b) = \|a - b\| = \sqrt{(a_1 - b_1)^2 + \dots + (a_n - b_n)^2}$$

Pf:

It's easy to show a-c for the Euclidean metric. Let's prove the triangle inequality.

Consider some $a, b, c \in \mathbb{R}^n$ where $a = (a_1, \dots a_n)$, $b = (b_1, \dots b_n)$, & $c = (c_1, \dots c_n)$.

We need to show
$d(a,c) \le d(a,b) + d(b,c)$
or equivalently
$d(a,c)^2 \le (d(a,b) + d(b,c))^2 = d(a,c)^2 + 2d(a,b)d(b,c) + d(b,c)^2$

First let's expand $d(a,c)^2$

$$d(a,c)^2 = \sum_{i=1}^{n} (a_i - c_i)^2$$
$$= \sum_{i=1}^{n} (a_i - b_i + b_i - c_i)^2$$
$$= \sum_{i=1}^{n} \left((a_i - b_i)^2 + 2(a_i - b_i)(b_i - c_i) + (b_i - c_i)^2\right)$$
$$= \sum_{i=1}^{n} (a_i - b_i)^2 + 2\sum_{i=1}^{n} (a_i - b_i)(b_i - c_i) + \sum_{i=1}^{n} (b_i - c_i)^2$$

We almost have what we want. We just need to show that

$$\sum_{i=1}^{n} (a_i - b_i)(b_i - c_i) \le d(a,b)d(b,c) = \left(\sum_{i=1}^{n} (a_i - b_i)^2\right)^{1/2} \left(\sum_{i=1}^{n} (b_i - c_i)^2\right)^{1/2}$$

To do this, we prove the stronger statement

$$\left|\sum_{i=1}^{n} (a_i - b_i)(b_i - c_i)\right| \le \left(\sum_{i=1}^{n} (a_i - b_i)^2\right)^{1/2} \left(\sum_{i=1}^{n} (b_i - c_i)^2\right)^{1/2}$$

This statement can be proven by the Cauchy-Swartz Inequality where $u = a - b$ & $v = b - c$. Thus the triangle inequality is shown by the lemma. $\square$

Lemma: Cauchy-Swartz Inequality

$$\left|\sum_{i=1}^{n} u_i v_i\right| \le \left(\sum_{i=1}^{n} (u_i)^2\right)^{1/2} \left(\sum_{i=1}^{n} (v_i)^2\right)^{1/2} \quad \forall u, v \in \mathbb{R}^n$$

This can be seen as $u \cdot v \le \|u\|\|v\|$. And $u \cdot v = \|u\|\|v\|\cos(\theta)$.

Equality holds iff v is a scalar multiple of u. That is $v = tu$ for some $t \in \mathbb{R}$.

Pf:
TODO.

Chapter 3: Sets & Limits

Open Sets

Def: Balls

Let (X, d) be a metric space.
Let $a \in X$ & $r \in \mathbb{R}^+$.

The open ball of radius r centered at a is
$B_r(a) = \{x \in X \mid d(a, x) < r\}$

The closed ball of radius r centered at a is
$C_r(a) = \{x \in X \mid d(a, x) \leq r\}$.

Example:

Consider the trivial metric space (X, d) where

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

$B_{1/2}(a) = \{a\}$ $\quad$ $B_1(a) = \{a\}$ $\quad$ $B_5(a) = X$
$C_{1/2}(a) = \{a\}$ $\quad$ $C_1(a) = X$ $\quad$ $C_5(a) = X$

Def:

Let (X, d) be a metric space.
A subset $U \subseteq X$ is called an open subset of X or an open set if it is the union of open balls.

Thm:

Let (X, d) be a metric space w/ subset $U \subseteq X$.

U is an open set
$\Leftrightarrow \forall a \in U \; \exists r > 0$ st $B_r(a) \subseteq U$
$\Leftrightarrow \forall a \in U \; \exists r > 0$ st for some $x \in X$ $d(x, a) < r \Rightarrow x \in U$

Thm:

Every union of open sets is open & every finite intersection of open sets is open.

Pf:

Every open set is the union of open balls so their union is also a union of open balls & thus an open set.

Consider open sets $U_1, \ldots, U_n \subseteq X$.
Take $x \in \bigcap_{i \in [n]} U_i$. Then $x \in U_i \; \forall i \in [n]$. Since each U_i is open $\exists r_1, \ldots, r_n \in \mathbb{R}^+$ st $B_{r_i}(x) \subseteq U_i$.

Let $r = \min(r_1, \dots, r_n)$ (note $r>0$). Then $\forall i \in [n]$ $B_r(x) \subseteq B_{r_i}(x) \subseteq U_i$.

Thus $B_r(x) \subseteq \bigcap_{i \in [n]} U_i$ & this set must be open.

Exercise: 3.1.9

This proof (& theorem) doesn't work for an infinite intersection of open sets w/ the minimum is replaced w/ an infimum b/c you can make the radius arbitrarily small so you can make the infimum 0, which is invalid.

$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$ is not open even tho its the union of open sets?

Boundedness of Open Sets

Bounds as exist in ordered sets (e.g. $\mathbb{R}$) don't extend neatly to metric spaces b/c there's no real sense of above/below. Instead we must talk edges being finite (in a sense).

Def:

Let (X, d) be a metric space w/ $S \subseteq X$. Let's define the diameter of S.

If $S = \emptyset$, we say the diameter of S is 0
$\text{diam}(S) = 0$.

If $S \neq \emptyset$, then the diameter of S is infinite iff
$\{d(a,b) \mid a, b \in S\}$
is not bounded above.
It is a real number if the supremum exists
/ $\text{diam}(S) = \sup\{d(a,b) \mid a, b \in S\}$.
Here we say S has finite diameter.

Thm:

Let (X, d) be a metric space w/ $S \subseteq X$. The following statements about S are equivalent

S has finite diameter
$\Leftrightarrow \exists a \in X$ & $r > 0$ st $S \subseteq B_r(a)$ ← you can encompass S w/ a ball
$\Leftrightarrow \forall a \in X$ $\exists r > 0$ st $S \subseteq B_r(a)$

Def:

Let (X, d) be a metric space. Subspace $S \subseteq X$ is bounded if one of the above is true.

Thm:

Let $S \subseteq \mathbb{R}$. S is bounded as a subset of the metric space $(\mathbb{R}, |\cdot|)$ iff $\exists K \in \mathbb{R}$ st $|s| \leq K$ $\forall s \in S$.

The following is a review of sequences in prep for future work.

Sequences: A Formal Review

Informally a sequence is an infinite string of elements in order (i.e. there's a first element then a second, etc.) For example
- 1, 0, 1, 00, 1, 000, 1, ...
- 1, 1/2, 1/3, 1/4, ...

Def:
Let A be a non-empty set. Any function $s: \mathbb{N} \to A$ is a sequence in A. We refer/write the sequence as
$s(1), s(2), s(3), \dots$ OR $s_1, s_2, s_3, \dots$ OR (s_n)

Def:
Given sequence (a_n) in S, we define the range of (a_n) as the set of elements in (a_n)
$$\text{Ran}(a_n) = \{ a_n \in S \mid n \in \mathbb{N} \}$$
The difference is the range is unordered & has no duplicates.

Def:
Let (s_i) be a sequence in set A.

(s_i) is a sequence of distinct terms iff $s_i \neq s_j$ $\forall i, j \in \mathbb{N}$ $i \neq j$.

(s_i) is a constant sequence iff $\exists a \in A$ st $s_i = a$ $\forall i \in \mathbb{N}$.

Def: Sequences in Ordered Sets
Let (s_i) be a sequence in a totally ordered set $(A, \leq)$.

(s_i) is increasing iff $\forall n, m \in \mathbb{N}$ $n \leq m$ $s_n \leq s_m$.
(s_i) is strictly increasing iff $\forall n, m \in \mathbb{N}$ $n < m$ $s_n < s_m$.
(s_i) is decreasing iff $\forall n, m \in \mathbb{N}$ $n \leq m$ $s_n \geq s_m$.
(s_i) is strictly decreasing iff $\forall n, m \in \mathbb{N}$ $n < m$ $s_n > s_m$.
} you can negate to flip

(s_i) is monotonic if it is either increasing or decreasing.

(s_i) is bounded from below iff $\exists b \in A$ st $\forall i \in \mathbb{N}$ $b \leq s_i$.
Here we call b the lower bound for (s_i).

(s_i) is bounded from above iff $\exists c \in A$ st $\forall i \in \mathbb{N}$ $s_i \leq c$.
Here we call c the upper bound for (s_i).

(s_i) is <u>bounded</u> if it is bounded from below & above.

<u>Thm:</u>
Let (n_i) be a strictly increasing sequence of natural numbers.
Then $i \leq n_i \ \forall i \in \mathbb{N}$. ← This is useful for subsequences

Subsequences

Sometimes we want to create a sequence using only some of an original sequence. (e.g. we may want every other term in the sequence)

For example given an original sequence 1, 2, 3, 4, 5, 6, ... we may want every other term to give us the even numbers 2, 4, 6, ...

<u>Def:</u>
Let (s_i) be a sequence in set A. If (n_i) is a strictly increasing sequence in $\mathbb{N}$ then the sequence
$s_{n_1}, s_{n_2}, s_{n_3}, \ldots$
is a <u>subsequence</u> of (s_i). We denote this subsequence as (s_{n_i}).

Note that using our earlier theorem we know that every element in the subsequence is in the correct relative order.

<u>Thm:</u>
Let (s_i) be a sequence in set A.

If (s_i) is constant, every subsequence of (s_i) is constant.

If (s_i) has distinct terms, every subsequence of (s_i) does too.

Subsequences also preserve all of the other properties of sequences in ordered sets!

<u>Thm:</u>
The following properties are preserved on subsequences

1) being constant
2) having distinct terms
3) (strictly) increasing / decreasing
4) monotonic
5) bounded above/below
6) bounded

For many properties, we can describe them as "eventually" ___ which means after some initial $i \in \mathbb{N}$ terms the property holds.

↑ We later formalize this w/ tails

Convergences of Sequences

The idea w/ converge / limits of sequences is the distance b/w the values of a sequence & some fixed value get closer & closer, eventually smaller than any fixed number.

Def:

The sequence (a_i) in metric space (X, d) <u>converts</u> to the <u>limit</u> $x \in X$ iff $\forall \varepsilon \in \mathbb{R}^+ \ \exists N \in \mathbb{N}$ such that

$$d(a_n, x) < \varepsilon \quad \forall n > N$$

When (a_n) converges to x we write

$$\lim_{n\to\infty} a_n = x \qquad \text{OR} \qquad a_n \to x$$

Def:

Let (a_n) be a seauence in metric space (X, d).
Let $N \in \mathbb{N}$. Then the subsequence of (a_n) w/ the terms

$$a_N, a_{N+1}, a_{N+2}, \dots$$

is called the <u>Nth tail of the sequence</u> & written

$$(a_n)_{n \geq N} \quad \text{or} \quad (a_n)_{n=N}^{\infty}$$

Def:

A seauence (a_n) is called eventually constant iff it has a tail which is constant.

Thm:

Let (a_i) be a seauence in metric space X. Then the following are equivalent

1) (a_i) converges to a
2) For all open sets U containing a, $\exists N \in \mathbb{N}$ st $\forall n > N$ $a_n \in U$.
3) $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ st if $n > N$ then $a_n \in B_\varepsilon(a)$.

Pf:

$1 \Rightarrow 2$:

Let U be an open set containing U. ~~Then~~
Then $\exists \varepsilon > 0$ st $B_\varepsilon(a) \subseteq U$.
The fact that (a_i) converges to a tells us that $\exists N \in \mathbb{N}$ st $\forall n > N$ $d(a_n, a) < \varepsilon$. ~~But~~
But then $\forall n > N$, $a_n \in U$.

$2 \Rightarrow 3$:

Assume that given any open set U containing a $\exists N \in \mathbb{N}$ st $\forall n > N$ $a_n \in U$. Fix $\varepsilon > 0$. B/c $B_\varepsilon(x)$ is open & contains a $\exists N \in \mathbb{N}$ st $\forall n > N$ $a_n \in B_\varepsilon(a)$.

$3 \Rightarrow 1$:

Fix $\varepsilon > 0$. Choose $N \in \mathbb{N}$ st $n > N \Rightarrow a_n \in B_\varepsilon(a)$. This means that if $n > N$, $d(a_n, a) < \varepsilon$, so $a_n \to \varepsilon$.

Sequences in $\mathbb{R}$

Sequences in $\mathbb{R}$ have additional structure as $\mathbb{R}$ has many properties.

Thm:

Let (a_n) be a sequence of real numbers that converges to $a \in \mathbb{R}$. Let Let $k \in \mathbb{R}$.

If $a_n \geq k$ $\forall n \in \mathbb{N}$, then $a \geq k$.
If $a_n \leq k$ $\forall n \in \mathbb{N}$, then $a \leq k$.

We can not state the some theorem w/ $>$ & $<$ b/c if $a_n \to a = k$ but $a_n < a$ or $a_n > a$ $\forall n \in \mathbb{N}$.

Thm:

Let $(a_n) \to a \in \mathbb{R}$. Let $k \in \mathbb{R}$. where $0 < k < |a|$.

If $0 < k < |a|$, then $\exists N \in \mathbb{N}$ st $\forall n > N$ $|a_n| \geq k$.
If $|a| < k$, then $\exists N \in \mathbb{N}$ st $\forall n > N$ $a_n \leq |a_n| < k$.

Thm:

Let $(a_n) \to a \in \mathbb{R}$ & $(b_n) \to b \in \mathbb{R}$.

If $a_n \leq b_n$ $\forall n \in \mathbb{N}$, then $a \leq b$
If $a_n \geq b_n$ $\forall n \in \mathbb{N}$, then $a \geq b$

Thm:

Every bounded sequence of real numbers has a convergent subsequence.

Thm:

Let $S \subseteq \mathbb{R}$ w/ least upper bound ℓ. Then there exists a sequence (a_n) in S st $(a_n) \to \ell$. Likewise for greatest lower bound.

Sequence Arithmetic & Convergence

Thm:

Let $(a_n) \to a \in \mathbb{R}$ & $(b_n) \to b \in \mathbb{R}$. Let $k \in \mathbb{R}$ be some real number.

$(a_n + b_n) \to a + b$
$(a_n - b_n) \to a - b$
$(a_n b_n) \to ab$
$(k a_n) \to ka$
$\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$ given $b \neq 0$ & $b_n \neq 0$ $\forall n \in \mathbb{N}$

More on Sequences & Limits

Thm: Tuples & Sequences

Consider the sequence (a_k) in $\mathbb{R}^n$ where $a_k = (a_{1k}, \ldots, a_{nk})$. Then

$(a_k) \to b = (b_1, \ldots, b_n) \in \mathbb{R}^n$
$\Leftrightarrow (a_{jk})_{k=1}^{\infty} \to b_j \in \mathbb{R}$ $\forall j \in \{1, \ldots, n\}$.

Epsilonics is the informal name given to the "game" of finding a $N \in \mathbb{N}$ that works for all $\varepsilon \geq 0$. It's called this to evoke the idea of your enemy picking an ε & you picking N. You want to algorithmically pick N based on ε ideally (or at least show it exists).

Thm:

Let (a_n) & (b_n) be sequences of real numbers where $0 \leq a_n \leq b_n$. Then $b_n \to 0 \Rightarrow a_n \to 0$.

Thm:

Let r be a real number. Then

$$\lim_{r \to \infty} r^n = \begin{cases} 0 & \text{if } r \in (-1,1) \\ 1 & \text{if } r = 1 \\ \text{DNE} & \text{if } r \leq 1 \\ \infty & \text{if } r > 1 \end{cases}$$

Thm:

Let x be a real number. Then

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0$$

Thm: Bernoulli's Inequality

Let $x \in \mathbb{R}$ where $x > -1$. Let $n \in \mathbb{N}$. Then

$$(1+x)^n \geq 1 + nx$$

Thm:

$$\lim_{n \to \infty} n^{1/n} = 1$$

Thm:

Let $c \in \mathbb{R}$ where $c > 1$. Then

$$\lim_{n \to \infty} c^{1/n} = 1.$$

Limit Points

Informally, limit points are points for which a limit can reach. This includes all the points inside & on the edge of an open set.

The eyes aren't limit points, but the circle & mouth (both points inside & on the edge of the mouth) are limit points.

Thm:

Let X be a metric space (not a discrete metric space?) w/ $S \subseteq X$. Let $x \in X$. Then

$\exists (a_n)$ in $S \setminus \{x\}$ which converges to x
$\Leftrightarrow \exists (a_n)$ in S which converges to x
$\Leftrightarrow \forall r > 0$ $B_r(x)$ contains infinitely many points of S
$\Leftrightarrow \forall r > 0$ $B_r(x)$ contains a point in $S \setminus \{x\}$

Def:

Let X be a metric space w/ $S \subseteq X$ & $x \in X$. We say x is a <u>limit point</u> of S iff it satisfies one/any of the above conditions.

Closed Set

Informally a closed set includes its boundary/limit points.

Def:

Let X be a metric space w/ $C \subseteq X$. C is a <u>closed subset</u> of X or <u>closed set</u> if C contains all the limit points.

Thm:

Let X be a metric space w/ $C \subseteq X$. Then C is close

C is closed
$\Leftrightarrow$ If (a_n) is a sequence in C converging to x, then $x \in C$
$\Leftrightarrow$ C contains all the limits of its convergent sequences

Remark:

Limit Point of S in S? Interior points of open set. (Inside line, plane, etc.)

Limit Point of S not in S? Boundary points of open set.

Not Limit Point of S in S? Isolated points.

Not Limit Point of S not in S? Distant points.

Thm:

Let X be a metric space w/ $S \subseteq X$. S is closed iff S^c is open.

Pf:

Suppose S is a closed set. Consider $x \in S^c$. By the definition of a closed set we know there cannot be a sequence in S converging to x.
Thus by 3.5 #1a there is an open ball $B_\varepsilon(x)$ w/ no points in S.
But then $B_\varepsilon(x) \subseteq S^c$, so S^c is open set.

Thm:

Every intersection of closed sets is closed & every finite union of closed sets is closed.

Pf:

Let X be a metric space. Suppose $\{C_\alpha\}_{\alpha \in \Lambda}$ is a collection of closed subsets of X. We show $\bigcap_{\alpha \in \Lambda} C_\alpha$ is a closed set.

This follows from the fact that the union of open sets is open. By DeMorgan's
$(\bigcap_{\alpha \in \Lambda} C_\alpha)^c = \bigcup_{\alpha \in \Lambda} (C_\alpha)^c$

Thus the complement of $\bigcap_{\alpha \in A} C_\alpha$ is a union of open sets (that is an open set), so $\bigcap_{\alpha \in A} C_\alpha$ is a closed set.

By similar logic the union of $C_1, \dots, C_n$ has a complement which is the intersection of finitely many open sets (& so is open), so we conclude $C_1 \cup \dots \cup C_n$ is open.

<u>Def:</u>

Let $S \subseteq X$ where X is a metric space. The intersection of all closed subsets of X that contain S is called the <u>closure of S</u> denote $\bar{S}$.

<u>Rem:</u>

Let S be a subset of metric space X. We know

1) $\bar{S}$ exists.
2) $\bar{S}$ is a subset of every closed set that contains S.
3) $\bar{S}$ is a closed set.
4) S is closed iff $S = \bar{S}$.
5) Let $x \in X$. The following are equivalent
   i) There exists a sequence in S converging to x. ← Easy if $x \in S$.
   ii) Every open ball around x contains a point in S.
   iii) For every open set U containing x, $U \cap S \neq \emptyset$.
   (i)–(iii): 3.5 #1a
   iv) $x \in \bar{S}$.

All metric spaces are kinda closed b/c outside points aren't meaningful :)

<u>Thm:</u>

Let S be a subset of metric space X. Then $\bar{S} = S \cup \{x \in X \mid x \text{ is a limit point of } S\}$.

<u>Pf:</u>

$\subseteq$:

Suppose that $x \notin S$ & x is not a limit point of S. By the earlier remark part 5, we know $x \notin \bar{S}$.

$\supseteq$:

If $x \in S$, then trivially $x \in S \subseteq \bar{S}$.

If x is a limit point of S, then there exists a sequence (a_n) in S converging to x. If K is any closed set containing S, then (a_n) is a sequence in K. Because K is closed, $x \in K$. That is x is in every closed set K containing S, so $x \in \bar{S}$.

We closures defined, we arrive at a meaningful definition for the interior & boundary of a set.

Def:

Let X be a metric space w/ $S \subseteq X$. A point $x \in S$ is called an <u>interior point</u> iff there is an open ball about x contained w/in S.

The set of all interior points of S is called the <u>interior</u> of S & denoted $Int(S)$.

Thm:

Let X be a metric space w/ $S \subseteq X$. Then we know

i) $Int(S)$ is an open subset of X

ii) $S' \subseteq Int(S)$ $\forall$ open subsets S' of X where $S' \subseteq S$.

iii) The union of all open subsets of X contained in S is equal to $Int(S)$.

iv) S is open iff $S = Int(S)$

v) $Int(S) = (\overline{S^c})^c$.

Def:

Let X be a metric space w/ $S \subseteq X$. The <u>boundary</u> of S is $\overline{S} \cap \overline{S^c} = \overline{S} \setminus Int(S)$.

We denote it by $\partial(S)$.

Thm:

Let X be a metric space w/ $S \subseteq X$. Then we know

i) $\partial(S)$ is a closed set.

ii) $x \in \partial(S)$ iff $\forall r > 0$ $B_r(x) \cap S \neq \emptyset$ & $B_r(x) \cap S^c \neq \emptyset$

iii) S is closed iff $\partial(S) \subseteq S$.

iv) S is open iff $S \cap \partial(S) = \emptyset$ (i.e. S & $\partial(S)$ are disjoint)

Series

Def:

Let (a_n) be a sequence of real numbers. We define a new sequence (s_n) based on (a_n) as

$$s_n = \sum_{i=0}^{n} a_i \quad \forall n \in \mathbb{N}.$$

This sequence denoted $\sum_{n=0}^{\infty} a_n$ is called a <u>series</u> & its <u>terms</u> are those of (a_n).

The terms of (s_n) are called <u>partial sums</u> of the series.

ambiguous notation !!

If $(s_n) \to L \in \mathbb{R}$, we say the <u>series</u> $\sum_{n=0}^{\infty} a_n$ <u>converges</u>. & write $\sum_{n=0}^{\infty} a_n = L$.

If (s_n) doesn't converge, the series diverges.

Thm:

Let (a_n) be a sequence of positive real numbers. Then we know

i) The sequence of partial sums of $\sum_{n=0}^{\infty} a_n$ is strictly increasing.

ii) The series $\sum_{n=0}^{\infty} a_n$ converges iff the sequence of partial sums is bounded above

iii) If $\sum_{n=0}^{\infty} a_n = S \in \mathbb{R}$, then $\sum_{n=0}^{N} a_n < S$ $\forall N \in \mathbb{N}$.

iv) If $\sum_{n=0}^{\infty} a_n$ diverges, then the partial sums tend to $+\infty$.

There are some special types of series.

Def:
Let $a \in \mathbb{R}$ be a non-zero real number & $r \in \mathbb{R}$ be any real. ~~Th~~ Then the series
$$\sum_{i=0}^{\infty} ar^i$$
is called a geometric series w/ common ratio r & leading term a.

Thm:
A geometric series converges iff $|r|<1$. Further, when $|r|<1$
$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$$

In general, it is difficult to find the sum of a series, so we'll focus just on whether it converges or not.

Notice that you never state the limit

Thm: Cauchy Criterion for Series Convergence ($\mathbb{R}$)
Let (a_n) be a sequence of real numbers.
$\sum_{n=1}^{\infty} a_n$ converges if & only if $\forall \varepsilon>0\ \exists N \in \mathbb{N}$ st
$$\left|\sum_{i=m}^{n} a_i\right| < |\varepsilon| \quad \forall n,m \in \mathbb{N} \text{ w/ } n>m>N$$

This isn't useful by itself but is used to make other tests

Let's use the Cauchy Criterion to create the N^{th} term test for divergence.

Intuitively, for a series to converge, its partial sums must eventually not change very much / essentially not at all, that is we must be adding smaller & smaller numbers. If we're not, then we diverge.

Thm: Nth Term Test
Let (a_n) be a sequence of real numbers. If $\sum_{n=0}^{\infty} a_n$ converges, $a_n \to 0$
The contrapositive, $a_n \not\to 0 \Rightarrow \sum_{n=0}^{\infty} a_n$ diverges.

Note: The converse of this isn't necessarily true.

Example:
Consider the harmonic series $a_n = 1/n$. Altho $a_n \to 0$, $\sum_{n=0}^{\infty} a_n$ diverges.
You can see this by grouping the blocks into groups w/ lengths of a power of two
$$1 + \underbrace{1/2}_{1/2} + \underbrace{1/3 + 1/4}_{>2 \cdot 1/4} + \underbrace{1/5 + 1/6 + 1/7 + 1/8}_{>4 \cdot 1/8} \dots$$

The following theorem exemplifies this phenomenon.

<u>Thm:</u> Grouping Lemma
Let (a_n) be a decreasing sequence of positive real numbers.
Then $\sum_{n=0}^{\infty} a_n$ converges if & only if $\sum_{n=0}^{\infty} 2^{n-1} a_{2^n}$ converges.

This grouping lemma establishes a convergence criterion for a whole class of series called p-series.

<u>Coro:</u> p-Series
Let $p \in \mathbb{R}^+$.
$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$ & diverges iff $p \leq 1$.

Let's go back & show why geometric series $\sum_{n=0}^{\infty} ar^i$ converge to $\frac{a}{1-r}$

<u>pf:</u>
Consider a geometric series w/ $a \in \mathbb{R}$ $a \neq 0$ & $r \in \mathbb{R}$.
$\sum_{i=0}^{\infty} ar^i$

When $r = 1$, then $\sum_{i=0}^{\infty} ar^i = \sum_{i=0}^{\infty} a$ clearly diverges by the Nth term test.
Since $(a) \not\to 0$, we know that $\sum_{i=0}^{\infty} a$ diverges.

When $r \neq 1$, we simplify the partial sums S_n as such

$$S_n = a + ar + \dots + ar^n$$
$$= \frac{a(1 + r + \dots + r^n)(1-r)}{(1-r)}$$
$$= \frac{a(1 - r^{n+1})}{(1-r)}$$

Now, if $r^{n+1} \to 0$, then we have $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$ as we want.
Indeed, we know $r^{n+1} \to 0$ when $|r| < 1$.
(We won't prove that right now)

Thus when $|r| \geq 1$, the series diverges. When $|r| < 1$, the series converges to $\frac{a}{1-r}$.

You'd assume that a series whose sign alternates would be "more likely" to converge than one whose doesn't. You'd be right & this is captured in the following theorem.

<u>Thm:</u>
Let (a_n) be a sequence of real numbers.
If $\sum_{n=0}^{\infty} |a_n|$ converges then so does $\sum_{n=0}^{\infty} a_n$, but the converse is not always true.

Def:

Let (a_n) be a sequence of real numbers where $\sum_{n=0}^{\infty} a_n$ converges.

We say $\sum_{n=0}^{\infty} a_n$ <u>converges absolutely</u> iff $\sum_{n=0}^{\infty} |a_n|$ converges.

We say $\sum_{n=0}^{\infty} a_n$ <u>converges conditionally</u> iff $\sum_{n=0}^{\infty} |a_n|$ diverges.

Def:

A series of real numbers where are the even terms are positive & odd terms negative (or vice versa) is called an <u>alternating series.</u>

Thm: Alternating Series Test

Let $\sum_{n=0}^{\infty} a_n$ be an alternating series of real numbers. If $(|a_n|)$ is monotonically decreasing & $a_n \to 0$, then $\sum_{n=0}^{\infty} a_n$ converges.

Example:

Consider the sequence $(a_n) = 1, -1, 1, -1, \ldots$

The series $\sum_{n=0}^{\infty} a_n$ does not converge b/c you can group the terms in the following valid ways

$$\sum_{n=0}^{\infty} a_n = \underbrace{1-1}+\underbrace{1-1}+\underbrace{1-1}+\ldots = 0$$

$$\sum_{n=0}^{\infty} a_n = 1 \underbrace{-1+1}\underbrace{-1+1}\underbrace{-1+1}-\ldots = 1$$

Example:

Consider the alternating harmonic sequence

$(a_n) = 1, -1/2, 1/3, -1/4, \ldots$

By the alternating series test we know the <u>alternating harmonic series</u> converges

$$\sum_{n=0}^{\infty} a_n = 1 - 1/2 + 1/3 - 1/4 + \ldots$$

Since $\sum_{n=0}^{\infty} |a_n|$ diverges, we have simple example of a series that <u>converges conditionally.</u>

We can get further insight about convergence/divergence by <u>comparing series.</u>

These are called <u>extrinsic tests</u> b/c they rely on knowledge of the convergence of another series.

(R)

Thm: Comparison Test

Let (a_n) & (b_n) be sequences of real numbers where $0 \le a_n \le b_n \ \forall n \in \mathbb{N}$. Then we know

$$\sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges} \ \& \ \sum_{n=1}^{\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ diverges}$$

You can think of this intuitively as upper & lower bounds clamping you down or blowing up respectively.

Note: Both sequences must be positive. If both sequences were negative then the condition must be reversed.

Thm: Limit Comparison Test

Let $(a_n), (b_n) \in (\mathbb{R}^+)$ where $\lim_{n\to\infty} \frac{a_n}{b_n} = K \in \mathbb{R}^+$. Then we know

$$\sum_{n=0}^{\infty} a_n \text{ converges} \Leftrightarrow \sum_{n=0}^{\infty} b_n \text{ converges}$$

Pf:

Suppose that Σa_n converges. We show Σb_n converges. Chose $N \in \mathbb{N}$ st

$$\left|\frac{a_n}{b_n} - K\right| < \frac{K}{2} \quad \forall n > N$$

Thus we know

$$0 < K - \frac{K}{2} < \frac{a_n}{b_n} \Rightarrow 0 < \frac{K}{2} b_n < a_n \Rightarrow \sum_{n=0}^{\infty} \frac{K}{2} b_n \text{ converges by the comparison test}$$

So it follows by problem II.1 #1 that

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{2}{K} \frac{K}{2} b_n$$

Recall: The Grouping Lemma

Suppose $(a_n) \in (\mathbb{R}^+)$ is a decreasing sequence. Then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=0}^{\infty} 2^{n-1} a_{2^n}$ converges.

Pf:

Suppose $(a_n) \in (\mathbb{R}^+)$ is a decreasing sequence of positive real numbers. Let (S_n) be the partial sums of (a_n).

Let $(b_n) = 2^{n-1} a_{2^n}$ & (t_n) be the partial sums of (b_n).

$\Rightarrow$: Suppose Σa_n converges to $M \in \mathbb{R}$. We need to show Σb_n converges.

We claim $t_n < M \ \forall n \in \mathbb{N}$. If this is true, then Σb_n converges by theorem H.1.2 part 2

Consider some t_n

$$\begin{aligned} t_n &= \underbrace{\tfrac{1}{2} a_1}_{b_0} + \underbrace{a_2}_{b_1} + \underbrace{a_4 + a_4}_{b_2} + \underbrace{a_8 + a_8 + a_8 + a_8}_{b_3} + \dots + \underbrace{\overbrace{a_{2^n} + \dots + a_{2^n}}^{2^{n-1}}}_{b_n} \\ &< a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{2^{n-1}+1} + \dots + a_{2^n} \\ &= S_{2^n} \\ &< M \quad \text{(by theorem H.1.2 part 3)} \end{aligned}$$

Thus since $t_n < M \ \forall n \in \mathbb{N}$, we know Σb_n converges.

$\Leftarrow$: Suppose $\sum b_n$ converges to L. We claim $s_n < s_{2^n} \underset{\le}{<} 2L$ $\forall n \in \mathbb{N}$.
We know $2^n > n$ for all $n \in \mathbb{N}$, so $s_{2^n} > s_n$ by theorem H.1.2 part 1.

$$s_{2^n} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots$$

Wrong! $\rightarrow$ $\cancel{<}$ $\underbrace{\tfrac{1}{2}a_1}_{b_0} + \underbrace{a_2}_{b_1} + \underbrace{a_4 + a_4}_{b_2} + \underbrace{a_8 + a_8 + a_8 + a_8}_{b_3} + \ldots$

$$< \underbrace{a_1}_{2b_0} + \underbrace{a_2 + a_2}_{2b_1} + \underbrace{a_4 + a_4 + a_4 + a_4}_{2b_2} + \underbrace{a_8 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8}_{2b_3} + \ldots$$

$$= 2t_n$$
$$< 2L$$

We have thus shown that $s_n < s_{2^n} < 2L$ $\forall n \in \mathbb{N}$ so we know by theorem H.1.2 that s_n converges & so does $\sum a_n$.

(This is actually problem H.1 #5a which is homework !!)

Tests from Geometric Series

Recall a geometric series is defined by $\sum_{n=0}^{\infty} r^n$. Notice that

$$\frac{r^{n+1}}{r^n} = r \qquad \& \qquad \sqrt[n]{r^n} = r.$$

That is the ratio b/w adjacent terms & the nth root of the nth term is constant.

Recall we know that $\sum_n r^n$ converges iff $|r| < 1$.
Using the ratio fact about geometric series we can get intuition about the ratio test.

Thm: Ratio Test
Let $(a_n) \in (\mathbb{R} \setminus \{0\})$.

The series $\sum a_n$ converges absolutely if
$$\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$$

The series $\sum a_n$ diverges if
$$\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$$

Note: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists then lim sup & lim inf become normal limits

Example:

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Notice how the ratio test says nothing

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{1/(n+1)^2}{1/n^2}\right| = \lim \left|\frac{n^2}{(n+1)^2}\right| = 1 \;\; \ddot\frown$$

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$. Again, nothing

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{1/(n+1)}{1/n}\right| = \lim \left|\frac{n}{n+1}\right| = 1 \;\; \ddot\frown$$

Now again $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(-1)^{n+1}/(n+1)}{(-1)^n/n}\right| = \lim \left|\frac{-n}{n+1}\right| = 1 \;\; \ddot\frown$$

Rem:

In general, if $\lim \left|\frac{a_{n+1}}{a_n}\right| < 1$, then we know the difference in the partial sums approaches 0, which isn't sufficient to know $\sum a_n$ converges in general.

Using the intuition from the roots for geometric series, that is $\sqrt[n]{r^n} = r$ we get the root test.

Thm: Root Test

Let $(a_n) \in (\mathbb{R})$.

If $\limsup \sqrt[n]{|a_n|} < 1$, then $\sum a_n$ converges absolutely.

If $\liminf \sqrt[n]{|a_n|} > 1$, then $\sum a_n$ diverges.

Note: The root test is strictly more powerful than the ratio test but more difficult in general. Also both are imperfect. Both are weaker for divergence than the Nth term test.

Thm:

This formalizes our previous note. Let $(a_n) \in (\mathbb{R} \setminus \{0\})$. If the ratio test applies, then so does the root test. That is

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \limsup \sqrt[n]{|a_n|} < 1$$

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \liminf \sqrt[n]{|a_n|} > 1.$$

Limit Superior & Inferior

Def:

Let $(a_n) \in (\mathbb{R})$ be a bounded sequence of real numbers.

The upper sequence of (a_n) is defined as for each $n \in \mathbb{N}$ $\overline{a_n} = \sup_{k \geq n} a_k$.

Limit Superior & Inferior

Def:

Let $(a_n) \in (\mathbb{R})$ be a bounded sequence of real numbers.

The upper sequence of (a_n) is defined as for each $n \in \mathbb{N}$, $\overline{a_n} = \sup_{k \geq n} a_k$. ← decreases

The lower sequence of (a_n) is defined as for each $n \in \mathbb{N}$, $\underline{a_n} = \sup_{k \geq n} a_k$. ← increases

The upper & lower sequences of bounded real sequences (a_n) are by definition monotonic & bounded & therefore are convergent.

Def:

Let $(a_n) \in (\mathbb{R})$ be a bounded sequence of real numbers.

Then $\lim_{n \to \infty} \overline{a_n}$ is called the limit superior (or lim sup) of (a_n) & is denoted $\limsup a_n$.

Then $\lim_{n \to \infty} \underline{a_n}$ is called the limit inferior (or lim inf) of (a_n) & is denoted $\liminf a_n$.

Note: Every bounded sequence has a lim sup & lim inf even if the sequence itself has no limit. It's often as good to consider lim sup / lim inf rather than just lim.

Chapter 4, 5, 6: Limits, Continuity, & Completeness

The least upper bound axiom of the reals guarantees completeness, essentially filling in the holes in the rationals.

Intuitively, we know convergent sequences get closer & closer to their limit. This also means that the points themselves get closer & closer, which is the Cauchy criterion. We use this fact in the notion of a Cauchy Sequences

Def:

Let $(a_n) \in (X)$ where X is a metric space. We say (a_n) is a Cauchy sequence iff $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ st
$d(a_n, a_m) < \varepsilon \quad \forall n, m > N$

Thm:

Let $(a_n) \in (X)$ where X is metric space.

i) (a_n) converges $\Rightarrow$ (a_n) is a Cauchy sequence.

ii) (a_n) is a Cauchy sequence $\Rightarrow$ every subsequence of (a_n) is a Cauchy sequence.

iii) (a_n) is a Cauchy sequence $\Rightarrow$ (a_n) is bounded

iv) (a_n) is a Cauchy sequence w/ a convergent subsequence $\Rightarrow$ (a_n) converges

Why do we care about Cauchy Sequences? They let us generalize the idea of completeness from the least upper bound axiom to any metric space.

Note: ~~N~~

Not all Cauchy sequences converge. Consider
$3, \; 3.1, \; 3.14, \; 3.141, \; 3.1415, \ldots$

In $\mathbb{R}$ this converges to π.
In $\mathbb{Q}$ this does not converge b/c of a "hole" at π.

Def:

A metric space X is complete iff every Cauchy sequence in X converges to an element of X.

Thm:
$\mathbb{R}$ & $\mathbb{R}^n$ are complete.

Continuity at Points

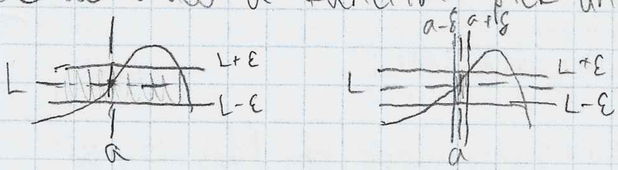
Def:
A function f is continuous at $a \in \mathbb{R}$ iff $\lim_{x \to a} f(x) = f(a)$.

What is this limit point tho?

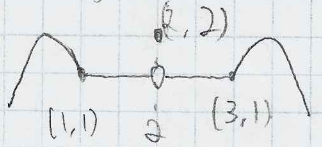
Def:
Let (X, d_X) & (Y, d_Y) be metric spaces w/ $K \subseteq X$. Let $f: K \to Y$ be a function
Let $a \in X$ be a limit point of K

We say that the limit of f as x goes to a is L iff
For all $\varepsilon > 0$
there exists $\delta > 0$
such that $x \in K$ & $0 < d_X(x, a) < \delta \Rightarrow d_Y(f(x), L) < \varepsilon$

Example:
Here we draw a function, pick an arbitrary ε, & then find an appropriate δ.



Note: We require $0 < d_X(x, a)$ for x to ensure $x \neq a$. However, we allow $d_Y(f(x), L) = 0$ to allow flat functions. Consider the following



Normally as $\varepsilon \to 0$, $\delta \to 0$ but here we never need a δ smaller than 1.

Thm:
Let (X, d_X) & (Y, d_Y) be metric spaces w/ $K \subseteq Y$. Let $f: K \to Y$ be a function & $a \in X$ be a limit point of K.

The limit of f as x approaches a is unique.
That is if $\lim_{x \to a} f(x) = L$ & $\lim_{x \to a} f(x) = M$, then $L = M$.

Thm:
Consider $X, Y, K, f,$ & a as above. Then the following are equivalent
i) $\lim_{x \to a} f(x) = L \in Y$

ii) For all sequences of distinct points (x_n) in X, $x_n \to a \Rightarrow f(x_n) \to L$.

Note: We require a to be a limit point of K b/c we need points x arbitrarily close to a so we can satisfy
$0 < d_X(x, a) < \delta$.

Continuous Functions

<u>Def</u>: Local Definition of Continuity
Let X & Y be metric spaces w/ $f: X \to Y$.
We say f is continuous iff it is continuous at every $a \in X$.

<u>Thm</u>:
Let (X, d_x) & (Y, d_y) be metric spaces w/ $f: X \to Y$ & $a \in X$.
The following conditions are equivalent.
i) f is continuous at a
ii) For all $\varepsilon > 0$ there exists a $\delta > 0$ st
$d_x(x, a) < \delta \Rightarrow d_y(f(x), a))$
iii) For all sequences (x_n) in X converging to a, we know
$(f(x_n))$ converges to a.

<u>Thm</u>:
Let X, Y, Z be metric spaces w/ continuous functions
$f: X \to Y$ & $g: Y \to Z$.
Then $g \circ f$ is a continuous function.

Global def doesn't discuss f at any points
&

<u>Thm</u>: Global Definition of Continuity
Let X & Y be metric spaces w/ $f: X \to Y$.
f is continuous if & only if the inverse image of every open subset of X is an open set of Y.

<u>Pf</u>:
$\Rightarrow$ Let V be an open subset of Y.
Consider $x \in f^{-1}(V)$ (so $f(x) \in V$).
B/c V is open, $\exists\, \varepsilon > 0$ st $d_y(z, f(x)) < \varepsilon \Rightarrow z \in V$ for all $z \in Y$.

B/c f is continuous, we know $\exists\, \delta > 0$ such that for all $y \in X$,
$d_x(y, x) < \delta \Rightarrow d_y(f(y), f(x)) < \varepsilon$.
Let $z = f(y)$. Since $d_y(z, f(x)) = d_y(f(y), f(x)) < \varepsilon$, we know $z = f(y) \in V$.
Therefore $y \in f^{-1}(V)$.

Since we know that for all $y \in X$, $d_x(y, x) \Rightarrow y \in f^{-1}(V)$, we conclude $f^{-1}(V)$ is open.

$\Leftarrow$ Let $f: X \to Y$ be a function. Suppose that for all open sets $V \subseteq Y$, $f^{-1}(V)$ is an open subset of X.

Let $x \in X$, $y = f(x)$, & fix some $\varepsilon > 0$. Then $B_\varepsilon(y)$ is an open subset of Y so $f^{-1}(B_\varepsilon(y))$ is an open subset of X.

Note that $x \in f^{-1}(B_\epsilon(y))$.

Since $f^{-1}(B_\epsilon(y))$ is open & $x \in f^{-1}(B_\epsilon(y))$, we know $\exists\, \delta > 0$ st $B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$.
That is every point in X that lies w/in a distance δ of x is sent by f to a point in Y that lies w/in a distance ϵ of $y = f(x)$.
This is the definition of continuous.

Def:
Let X & Y be metric spaces w/ $f: X \to Y$. & $S \subseteq X$.
f is uniformly continuous on S iff $\forall \epsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(a,b) < \delta \Rightarrow d_Y(f(a), f(b)) \quad \forall a, b \in S$$

If f is uniformly continuous on X itself we just say f is uniformly continuous.

Uniformly continuity is a stronger condition than continuity b/c it requires one value of δ to work for many pairs of points.

Def:
Let X & Y be metric spaces w/ $f: X \to Y$.

Iff there exists a $k \in \mathbb{R}^+$ such that $d_Y(f(a), f(b)) \leq k\, d_X(a,b) \quad \forall a, b \in X$ then f is said to satisfy a Lipschitz condition w/ constant k & we say f is a Lipschitz function.

Thm:
If $f: X \to Y$ is a Lipschitz function, then f is uniformly continuous.

Limits, Continuity, & Order

Thm:
Let X be a metric space w/ $f: X \to \mathbb{R}$ & a limit point $a \in X$.
Assume that $\lim_{x \to a} f(x)$ exists. Fix $t \in \mathbb{R}$

Suppose $\exists\, r > 0$ st $f(x) \geq t \quad \forall x \in B_r(a) \setminus \{a\}$. Then $\lim_{x \to a} f(x) \geq t$.
Suppose $\exists\, r > 0$ st $f(x) \leq t \quad \forall x \in B_r(a) \setminus \{a\}$. Then $\lim_{x \to a} f(x) \leq t$.

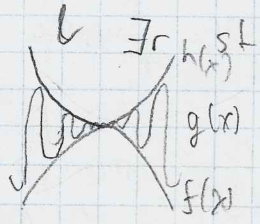
Thm:
Let X be a metric space w/ $a \in X$ is a limit point of X & $f: X \to \mathbb{R}$ & $g: X \to \mathbb{R}$ where $\lim_{x \to a} f(x)$ & $\lim_{x \to a} g(x)$ both exist.

If $\exists\, r > 0$ such that $f(x) \leq g(x) \quad \forall x \in B_r(a) \setminus \{a\}$, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

Thm: Sandwich Theorem
Let X be a metric space w/ a limit point $a \in X$. & $f, g, h: X \to \mathbb{R}$.

Suppose $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ & $\exists\, r$ st. $f(x) \leq g(x) \leq h(x) \quad \forall x \in B_r(a) \setminus \{a\}$.

Then $\lim_{x \to a} g(x) = L$ as well.



Thm:

Let X be a metric space w/ limit point $a \in X$ & $f: X \to \mathbb{R}$.
Suppose $\lim_{x \to a} f(x) = L$.

Then for all $K > |L|$ $\exists \delta > 0$ st if $x \in B_\delta(a) \setminus \{a\}$ then $|f(x)| < K$.

Similar Theorems

Thm:

Let X be a metric space w/ limit point $a \in X$ & $f: X \to \mathbb{R}$.
Suppose $\lim_{x \to a} f(x) = L \neq 0$.

Then for all $K \in \mathbb{R}^+$ where $0 < K < |L|$, $\exists \delta > 0$ st $x \in B_\delta(a) \setminus \{a\} \Rightarrow |f(x)| > K$.

Coro:

Let X, a, f, & L be defined as above.

If $\exists r > 0$ st $f(x) \geq 0$ $\forall x \in B_r(a) \setminus \{a\}$ then $\lim_{x \to a} f(x) = L \geq 0$.

Coro:

Let X, a, f, & L be defined as above

$\forall \varepsilon > 0$ $\exists \delta > 0$ st $0 < d(x,a) < \delta \Rightarrow |f(x)| < |L| + \varepsilon$

Coro:

Let X be a metric space w/ $a \in X$ & $f: X \to \mathbb{R}$. Suppose f is continuous at a (i.e. $\lim_{x \to a} f(x) = f(a)$). Fix some $t \in \mathbb{R}$.

If $\exists r > 0$ st $f(x) \geq t$ $\forall x \in B_r(a) \setminus \{a\}$, then $f(a) \geq t$.
If $\exists r > 0$ st $f(x) \leq t$ $\forall x \in B_r(a) \setminus \{a\}$, then $f(a) \leq t$.

Coro:

Let X be a metric space w/ $a \in X$ & $f: X \to \mathbb{R}$, $g: X \to \mathbb{R}$.
Suppose f & g are continuous at a.

If $\exists r > 0$ st $f(x) \leq g(x)$ $\forall x \in B_r(a) \setminus \{a\}$, then $f(a) \leq g(a)$.

Coro: Sandwich Theorem Redux

Let X be a metric space w/ limit point $a \in X$.
Let $f, g, h \in X \to \mathbb{R}$. Suppose
- f & h are continuous at a
- $f(a) \leq h(a)$
- $\exists r > 0$ st $f(x) \leq g(x) \leq h(x)$ $\forall x \in B_r(a) \setminus \{a\}$

Then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x)$.

One-Sided Limits

Def:

Let $K \subseteq \mathbb{R}$ & X any metric space. Fix $c \in \mathbb{R}$ & $\eta > 0$ st $(c-\eta, c) \subseteq K$.
We say the limit of f as x approaches c from the left is L
iff $\forall \varepsilon > 0$ $\exists \delta \leq \eta$ st $x \in (c-\delta, c) \Rightarrow d(f(x), L) < \varepsilon$. We write
$\lim_{x \to c^-} f(x) = L$

We can define the limit on the right similarly & write
$\lim_{x \to c^+} f(x) = L$.

Thm:

Let $K \subseteq \mathbb{R}$ & X any metric space w/ $f: K \to X$.
Take a $c \in \mathbb{R}$ & suppose $\exists \eta > 0$ st $(c-\eta, c) \cup (c, c+\eta) \subseteq K$

Then $\lim_{x \to c} f(x)$ exists iff $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$ & we have

$$\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x).$$

Limits & Arithmetic

Thm:

Let X be a metric space w/ $f, g: X \to \mathbb{R}$. Suppose $\lim_{x \to a} f(x) = L$ & $\lim_{x \to a} g(x) = M$.
Then we know

1) $\lim_{x \to a} (f(x) + g(x)) = L + M$
2) $\lim_{x \to a} (f(x) - g(x)) = L - M$
3) $\lim_{x \to a} [f(x) g(x)] = LM$
4) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ (if $g(x)$ on some interval containing a & $M \neq 0$)

Coro:

Let X be a metric space w/ continuous functions $f: X \to \mathbb{R}$ & $g: X \to \mathbb{R}$.
Then the following functions are continuous.

1) $f + g$
2) $f - g$
3) fg
4) $\frac{f}{g}$ (if $0 \notin \text{range}(g)$)

Chapter 7 & 8: Compactness & Connectedness

Def:

Let X be a metric space w/ $S \subseteq X$. If $\{U_\alpha\}_{\alpha \in \Lambda}$ is a collection of open subsets of X & $S \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$ then

Then $\{U_\alpha\}_{\alpha \in \Lambda}$ is an <u>open cover</u> for S.

Any subcollection of $\{U_\alpha\}_{\alpha \in \Lambda}$ whose union still contains S is a <u>subcover</u> of S.

Def:

Let X be a metric space w/ $S \subseteq X$.
We say S is <u>compact</u> iff for all open covers $\{U_\alpha\}_{\alpha \in \Lambda}$ of S there exists a finite subcover for S.

Formally, $\exists B \subseteq \Lambda$ where B is finite & $S \subseteq \{U_\beta\}_{\beta \in B}$.

Examples:

- All finite sets are compact. Just put a ball around every point.
- Not all sets are compact.
  The interval $(0,1)$ is not compact.

  Consider the open cover $\{U_n\}$ for S where $U_n = (1/n, 1)$..

  $\{U_n\}_{n \in \mathbb{N}}$ covers S but no finite subcollection covers S b/c it always misses something near 0.
- Some infinite sets are compact.
  Let (a_n) be a sequence of distinct terms in metric space X converging to a. Th
  The set $S = \{a, a_1, a_2, a_3, \dots\}$ is compact b/c any ball containing S contains infinitely many points of S, leaving a finite number of other balls.

Thm:

Let $a, b \in \mathbb{R}$ where $a < b$. Then $[a, b]$ is compact.

Thm:

Every compact subset of a metric space is bounded. Further every compact metric space is bounded.

Thm:
Every compact subset of a metric space is closed.

Pf:
Let S be a compact subset of a metric space X. We show S^c is open.

Let $x \in S^c$. By problem 3.1 #4, we know that for any $y \in S$ there exist disjoint open sets U_y containing x & V_y containing y.

The set $\{V_y\}_{y \in S}$ clearly covers all of S so since S is compact there exists some finite subcover $\{V_{y_1}, \dots, V_{y_n}\}$.

The set $U = U_{y_1} \cap \dots \cap U_{y_n}$ is open as the finite intersection of open sets. Further $x \in U$ & U does not intersect w/ $V_{y_1} \cup \dots \cup V_{y_n}$ so it doesn't intersect w/ S either.

Thus S^c is open so S is closed.

Thm:
Every closed subset of a compact set is compact.

Coro: Heine-Borel Theorem
A subset $C \subset \mathbb{R}^n$ is compact iff it is closed & bounded.

Note: This isn't true for general metric spaces.

Thm:
Let X be a metric space w/ $S \subseteq X$. Then the following are equivalent

i) S is compact
ii) Every sequence in S has a subsequence converging to a point of S
iii) Every infinite subset of S contains a limit point in S.

Thm:
Every compact metric space is complete.

Thm:
The continuous image of a compact set is compact.
More formally consider metric spaces X & Y w/ $S \subseteq X$.
If $f: X \to Y$ is a continuous function then $f(S) \subseteq Y$ is compact.

Thm: Max-Min Theorem
Let X be a non-empty compact metric space w/ continuous function $f: X \to \mathbb{R}$.
Then f acheives a minimum & maximum value.
That is $\exists x, y \in X$ st $f(x) \leq f(z) \leq f(y)$ $\forall z \in X$.

Thm:
Let X be a compact metric space & Y any metric space.
Then if $f: X \to Y$ is a continuous function, then f is uniformly continuous.

Thm: Heine-Borel Theorem
Any closed & bounded subset of $\mathbb{R}^n$ is closed.

Intuitively, this means we can cover $\mathbb{R}^n$ w/ countably many balls.

Coro:

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Intermediate Value Theorem (IVT)

Def:

A real number z is <u>between</u> real numbers x & y iff
$x \le z \le y$ or $y \le z \le x$.

Thm: Intermediate Value Theorem
Let $a, b \in \mathbb{R}$ w/ $f: [a,b] \to \mathbb{R}$ be a continuous function.

If γ is b/w $f(a)$ & $f(b)$ then $\exists c \in [a,b]$ st
$f(c) = \gamma$.

Thm:

Let X be a metric space w/ $S \subseteq X$.
Suppose A & B are non-empty, disjoint subsets of S such that
$S = A \cup B$. Then the following are equivalent
1) No point of A is a limit point of B & vice versa
2) A is both a closed & open subset of S
3) B is both a closed & open subset of S.

Def:

Sets that are both open & closed are <u>clopen sets</u>.

Def:

A metric space is <u>disconnected</u> if it can be written as the disjoint union of two non-empty clopen sets.
A set is <u>connected</u> if it is not disconnected.

~~Coro~~:

Let X be a metric space. The following are equivalent.
1) X is connected
2) The only subsets of X that are clopen are X & $\emptyset$.
3) There do not exist non-empty, disjoint open sets A & B such that $X = A \cup B$
4) Same as 3 but w/ open

Thm:

Let X & Y be metric spaces w/ continuous function $f: X \to Y$.
If X is connected so is $f(X)$.

Def:

Let $A \subseteq K \subseteq \mathbb{R}$ w/ $f: K \to \mathbb{R}$. We say f

We say f is increasing on A iff $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ $\forall x_1, x_2 \in A$.
We say strictly increasing iff $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

We similarly define decreasing & strictly decreasing.

We say f is monotonic on A iff f is increasing or decreasing on A.
Similarly for strictly monotonic.

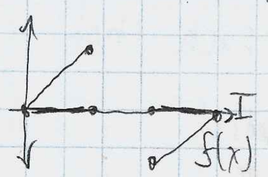
Thm:

Let $K \subseteq \mathbb{R}$ be a D-domain w/ function $f: K \to \mathbb{R}$ & interval $I \subseteq K$. Suppose f is differentiable on I.

Then f is increasing on I iff $f'(x) \geq 0$ $\forall x \in I$.
Likewise f is decreasing on I iff $f'(x) \leq 0$ $\forall x \in I$.

Similarly strictly increasing $\Leftrightarrow f'(x) > 0$ & strictly decreasing $\Leftrightarrow f'(x) < 0$.

We restrict ourselves to intervals b/c it's not clear what we expect from



Thm:

Let $K \subseteq \mathbb{R}$ be a D-domain w/ $f: K \to \mathbb{R}$. Suppose f is differentiable on $[a,b] \subseteq K$.
If for all $x \in (a,b)$ $f'(x) \neq 0$ then f is one-to-one on $[a,b]$.

Chapter 9: Differentiation

There's no standard for a differentiable domain, so we come up w/ a sufficient one.

Def:

A subset $K \subseteq \mathbb{R}$ is called a D-domain if K is all of $\mathbb{R}$ or a union of countably many non-degenerate disjoint intervals.

Let K be a D-domain w/ $f: K \to \mathbb{R}$ & $x \in K$. To see if the derivative of f exists at x, we need to inspect the difference quotient of f at x DQ_x

$$DQ_x(y) = \frac{f(y) - f(x)}{y - x} \quad \text{where } DQ_x : K \setminus \{x\} \to \mathbb{R}$$

To find $f'(x)$ we take the limit of DQ_x at x.

Def:

Let K be a D-domain w/ $f: K \to \mathbb{R}$.
We say f is differentiable at $x \in \mathbb{R}$ iff

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} \text{ exists.}$$

If f is differentiable at every point in $S \subseteq K$, then we say f is differentiable on S & define the derivative $f': S \to \mathbb{R}$

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

Iff $S = K$, we say f is differentiable.

There is another way to look at derivatives tho. Rather than seeing it as a limit of secant lines, imagine zooming in infinitely at a point. The derivative is the slope of that straight line.

Thm:

Let K be a D-domain w/ $f: K \to \mathbb{R}$.
f is differentiable at $x \in K$ iff there exists real number $f'(x)$ & function $r: K \to \mathbb{R}$ which satisfy

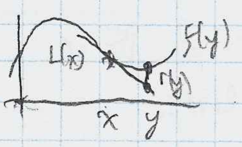
$$f(y) = f'(x)(y - x) + f(x) + r(y) \iff r(y) = f(y) - \underbrace{\left(f'(x)(y-x) + f(x)\right)}_{\text{tangent line at } x}$$

where (see next page)

$r(y)$ is residual.

where
$$\lim_{y \to x} \frac{r(y)}{y-x} = 0.$$

Geometrically, $r(y)$ is the difference b/w the original curve & the tangent line at x.



where $L(x) = f'(x)(y-x) + f(x)$

Lemma:
Let K be a D-domain w/ $r: K \to \mathbb{R}$. Then
$$\lim_{y \to x} \frac{r(y)}{y-x} \Rightarrow \lim_{y \to x} r(y) = 0.$$

Coro:
Let K be a D-domain w/ $f: K \to \mathbb{R}$.
If f is differentiable at $x \in K$ then f is continuous at x.

Thm:
Let K be a D-domain w/ $f: K \to \mathbb{R}$ & $g: K \to \mathbb{R}$. Suppose f & g are differentiable at $x \in K$. Then the following holds

- Constant Multiple Rule: $(kf)'(x) = k f'(x) \ \forall k \in \mathbb{R}$.
- Sum & Difference Rule: $(f \pm g)'(x) = f'(x) \pm g'(x)$.
- Product Rule: $(fg)'(x) = f'(x) g(x) + f(x) g'(x)$.

Thm: Chain Rule
Let K & K' be D-domains w/ $f: K \to \mathbb{R}$ & $g: K' \to \mathbb{R}$ where $f(K) \subseteq K'$.

Suppose f is differentiable at $x \in K$ & g is differentiable at $f(x)$.
Then $g \circ f$ is differentiable at x &
$$(g \circ f)'(x) = g'(f(x)) f'(x).$$

Thm: Mean Value Theorem
Let $K \subseteq \mathbb{R}$ be a D-domain w/ $f: K \to \mathbb{R}$ & $[a,b] \subseteq \mathbb{R}$. Suppose f is continuous on $[a,b]$ & differentiable on (a,b).

Then there exists some c such that
$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Informally, the average slope has to be hit at some point along the curve.

<u>Def:</u>

Let $f: X \to \mathbb{R}$ be a function w/ $z \in X$.

$f(z)$ is said to be a <u>(global) maximum</u> iff $f(z) \geq f(x)\ \forall x \in X$.

$f(z)$ is said to be a <u>local maximum</u> iff $\exists r > 0$ such that $f(z) \geq f(x)\ \forall x \in B_r(z) \subseteq X$.

Doesn't mention derivative!

<u>Global & local minimums</u> are defined similarly.

We say $f(z)$ is a <u>(global) extreme value/extremum</u> & say z is a <u>(global) extreme point</u> iff f attains a minimum or maximum at z.

Similarly we define <u>local extreme value/point</u>.

<u>Thm:</u> Local Extremum Theorem
Let $K \subseteq \mathbb{R}$ be a D-domain w/ $f: K \to \mathbb{R}$. Suppose f has local extremum at c.

If f is differentiable at c & $c \in \text{interior}(K)$, then $f'(c) = 0$.

<u>Thm:</u> Rolle's Theorem ← Special case of mean value theorem
Let $K \subseteq \mathbb{R}$ be a D-domain w/ $f: K \to \mathbb{R}$ & $[a,b] \subseteq \mathbb{R}$.
Suppose the following holds
i) f is continuous on $[a,b]$,
ii) f is differentiable on (a,b), &
iii) $f(a) = f(b)$.

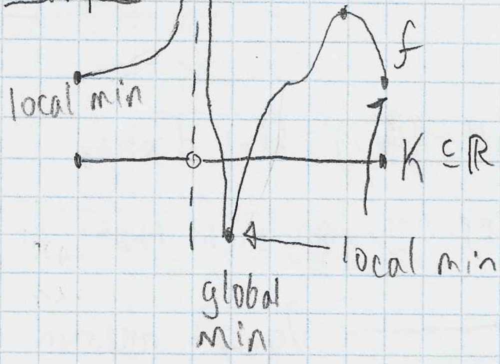
Then $\exists c \in (a,b)$ st $f'(c) = 0$.

<u>Exercise:</u>
We can more clearly connect Rolle's theorem & the mean value theorem.

Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function & consider some $a, b \in \mathbb{R}$.
Draw a straight line function g b/w $(a, f(a))$ & $(b, f(b))$.
Define a new function $h: \mathbb{R} \to \mathbb{R}$ by
$$h = f - g \iff h(x) = f(x) - g(x)$$

Here we have defined an function h where Rolle's theorem applies even tho only the mean value theorem necessarily applies to f.

Example:



There is no global max!

The endpoints are included b/c every point in a (small) open ball is no smaller than them.

Here are some corollaries to the mean value theorem.

Coro:

Let $K \subseteq \mathbb{R}$ be a D-domain w/ function $f: K \to \mathbb{R}$ & interval $I \subseteq K$.

f is constant on I iff $f'(x) = 0 \ \forall x \in I$.

Coro:

Let $K \subseteq \mathbb{R}$ be a D-domain w/ $f: K \to \mathbb{R}$ & $g: K \to \mathbb{R}$ & interval $I \subseteq K$.

Then $\exists C \in \mathbb{R}$ st $f(x) = g(x) + C \ \forall x \in I$ iff $f'(x) = g'(x) \ \forall x \in I$.

Def:

Let $K \subseteq \mathbb{R}$ be a D-domain. We say $F: K \to \mathbb{R}$ is the antiderivative of $f: K \to \mathbb{R}$ iff
$$F'(x) = f(x) \quad \forall x \in K.$$

Sometimes we restrict the domains to some interval $I \subseteq K$.

Monotonicity & MVT

Def:

Let $A \subseteq K \subseteq \mathbb{R}$ w/ $f: K \to \mathbb{R}$.

We say f is increasing on A iff $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \ \forall x_1, x_2 \in A$.
We say strictly increasing iff $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

We similarly define decreasing & strictly decreasing.

We say f is monotonic on A iff f is increasing or decreasing on A.
Similarly for strictly monotonic.

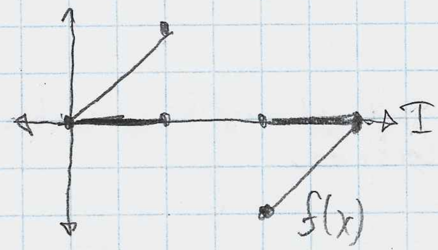
Thm:

Let $K \subseteq \mathbb{R}$ be a D-domain w/ function $f: K \to \mathbb{R}$ & interval $I \subseteq K$. Suppose f is differentiable on I.

Then f is increasing on I iff $f'(x) \geq 0 \ \forall x \in I$.
Likewise f is decreasing on I iff $f'(x) \leq 0 \ \forall x \in I$.

Similarly strictly increasing $\Leftrightarrow f'(x) > 0$ & strictly decreasing $\Leftrightarrow f'(x) < 0$.

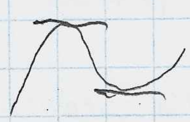
We restrict ourselves to intervals b/c it's not clear what we expect from



Thm:

Let $K \subseteq \mathbb{R}$ be a D-domain w/ $f: K \to \mathbb{R}$. Suppose f is differentiable on $[a,b] \subseteq K$.
If $f'(x) \neq 0 \ \forall x \in [a,b]$ then f is one-to-one on $[a,b]$.

Intuitively, non-one-to-one functions must have a rise & fall where their derivative is 0



Lemma:

Let $K \subseteq \mathbb{R}$ be a D-domain w/ $f: K \to \mathbb{R}$. Suppose that f is differentiable on $[a,b] \subseteq K$.
If $f'(a) = -f'(b)$ then $\exists c \in (a,b)$ st $f'(c) = 0$.

Thm: Darboux's Theorem

Let $K \subseteq \mathbb{R}$ be a D-domain w/ $[a,b] \subseteq K$ & $f: K \to \mathbb{R}$. Suppose f is differentiable on $[a,b]$. Then f' satisfies the intermediate value property, that is for any α b/w $f(a)$ & $f(b)$ there exists a $c \in [a,b]$ such that $f(c) = \alpha$.

Inverses

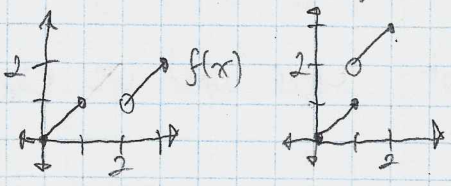
Often, speaking about inverses is useful, even if the function isn't properly invertible. We restrict the domain & codomain in these cases.

Thm:
Let $I \subseteq \mathbb{R}$ be an interval w/ function $f: I \to \mathbb{R}$ which is injective & continuous. Then $f^{-1}: f(I) \to \mathbb{R}$ is also continuous.

This theorem doesn't work for all domains in general. For example, consider $f: [0,1] \cup (2,3]$ defined by

$$f(x) = \begin{cases} x & x \in [0,1] \\ x-1 & x \in (2,3] \end{cases}$$

$f(x)$ is one-to-one & continuous but $f^{-1}(x)$ contains a jump discontinuity.



Thm:
Let $K \subseteq \mathbb{R}$ be a closed & bounded subset of $\mathbb{R}$ w/ injective, continuous function $f: K \to \mathbb{R}$. Then $f^{-1}: f(I) \to \mathbb{R}$ is also continuous.

Remark:
If $K \subseteq \mathbb{R}$ is a D-domain & $f: K \to \mathbb{R}$ is continuous, then $f(K)$ is a D-domain.

Thm:
Let $K \subseteq \mathbb{R}$ be closed & bounded w/ injective, continuous function $f: K \to \mathbb{R}$ which is differentiable at $x \in K$.

Taylor's Theorem & Polynomial Approximations

Taylor's theorem is an extension of the mean-value theorem to any infinitely-differentiable functions, describing how to (perfectly) approximate such functions w/ polynomials.

You can also think of Taylor's theorem as an extension of horizontal-line & straight line approximations to polynomial approximations

Thm: Taylor's Remainder Theorem for Linear Approximations
Let $f: [a,b] \to \mathbb{R}$ be differentiable on $[a,b] \subseteq \mathbb{R}$ & twice differentiable on (a,b). Let $t, s \in [a,b]$ arbitrary.

Then $\exists c$ b/w s & t such that

$$f(t) = f(s) + f'(s)(t-s) + \frac{f''(c)}{2!}(t-s)^2$$

Def:
The polynomial

$$P_n(x) = f(s) + f'(s)(x-s) + \frac{f''(s)}{2!}(x-s)^2 + \dots + \frac{f^{(n)}(s)}{n!}(x-s)^n$$

is called the <u>nth-degree Taylor polynomial approximation of f based at s</u> or the <u>n^{th} Taylor polynomial of f at s</u>.

When $s=0$, we call them <u>Maclaurin polynomials</u>.

<u>Thm:</u> Taylor's Theorem

Let $f:[a,b]\to\mathbb{R}$ be a function at least n-times differentiable. Further assume $f^{(n+1)}$ exists on (a,b). Let $s,t\in[a,b]$.

Then there exists $c\in\mathbb{R}$ b/w $\overset{\text{"base point"}}{s}$ & $\overset{\text{"variable"}}{t}$ such that

$$f(t) = \underbrace{f(s) + f'(s)(t-s) + \frac{f''(s)}{2!}(t-s)^2 + \dots + \frac{f^{(n)}(s)}{n!}(t-s)^n}_{\text{Taylor Polynomial}} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(t-s)^{n+1}}_{\text{Remainder}}$$

<u>Coro:</u>

Let $K\in\mathbb{R}$ w/ $f:[a,b]\to\mathbb{R}$. Assume $f',\dots,f^{(n)}$ all exist & are continuous on $[a,b]$. Suppose $f^{(n+1)}$ exists on (a,b) & that

$$|f^{(n+1)}(x)|\le K \quad \forall x\in(a,b).$$

Let $s,t\in[a,b]$. Then

$$|R_n(t)| \le \frac{K}{(n+1)!}|t-s|^{n+1}$$

where $R_n(t)$ is the remainder defined by

$$R_n(t) = f(t) - P_n(t) = \frac{f^{(n+1)}(c)}{(n+1)!}(t-s)^{n+1}.$$

Note that the nth-order Taylor polynomial shares the value of f at a & the first n derivatives at that point.

Chapter 11: The Riemann Integral

Riemann integration uses the method of exhaustion. That is you split the area into smaller partitions & keep splitting & splitting, finding the area of each split using an appropriate approximation (e.g. a Taylor polynomial).

Def:

Consider $a, b \in \mathbb{R}$ where $a<b$. Then any set
$$P = \{x_0, \ldots, x_n\} \quad \text{where} \quad a = x_0 < x_1 < \ldots < x_n = b$$
is called a partition of $[a,b]$ & the intervals $[x_{i-1}, x_i]$ are called subintervals of $[a,b]$ determined by P.

The mesh of the partition $\|P\|$ is the length of the longest subinterval. That is
$$\|P\| = \max_{1 \le i \le n} (x_i - x_{i-1}).$$

If f is a real-valued function where $[a,b] \subseteq Dom(f)$, then the Riemann sum of f corresponding to partition P is given by
$$R(f,P) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \quad \text{where} \quad x_i^* \in [x_{i-1}, x_i].$$

The points $\{x_i^*\}_{i=1}^{n}$ are called sampling points.

Def:

Let $a, b \in \mathbb{R}$ w/ $a<b$. Let f be a real-valued function where $[a,b] \subseteq Dom(f)$. We say f is Riemann integrable on $[a,b]$ if there exists a real number $I \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that
$$|R(f,P) - I| < \varepsilon \quad \text{whenever} \quad \|P\| < \delta.$$

We denote I by $\int_a^b f$ & call it the Riemann integral of f over $[a,b]$. We write

Thm:

Let $a, b \in \mathbb{R}$ w/ $a<b$. Suppose f is a real-valued function that is Riemann integrable on $[a,b]$. Then the Riemann integral is unique.

Example:

Let $f: [a,b] \to \mathbb{R}$ be a function & $\gamma \in [a,b]$. Define
$$f(x) = \begin{cases} 1 & x = \gamma \\ 0 & x \neq \gamma \end{cases}$$

Then $\int_a^b f = 0$.

Exercise:

Let
$$f(x) = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$$
Then $f(x)$ is not Riemann integrable.

Def:

Let $a,b \in \mathbb{R}$ w/ $a<b$. Suppose f is Riemann integrable on $[a,b]$. Then for any $c \in [a,b]$ we say
$\int_c^c f = 0$ & $\int_a^b f = -\int_b^a f$.

Thm:

Let $a,b \in \mathbb{R}$ w/ $a<b$. Suppose f is Riemann integrable on $[a,b]$. Then f is bounded on $[a,b]$.

Thm: Translation Invariance of Integral.

Let $a,b \in \mathbb{R}$ w/ $a<b$. Suppose f is Riemann integrable on $[a,b]$. For a fixed $h \in \mathbb{R}$ define
$g_h(x) = f(x+h)$, where $x \in Dom(g_h) \iff x+h \in Dom(f)$.

Then g_h is Riemann integrable on $[a-h, b-h]$ & further
$\int_{a-h}^{b-h} g_h = \int_a^b f$.

Examining the Definition of Riemann Integrable

Def:

Let $a,b \in \mathbb{R}$ where $a<b$. Consider partition P of $[a,b]$ defined by
$P = \{x_0, \dots, x_n\}$.
We say P is a <u>regular partition</u> iff every subinterval has the same length.

Def:

Let $f: \mathbb{R} \to \mathbb{R}$ be a function & $P = \{x_0, \dots, x_n\}$ be a regular partition of $[a,b]$.

We define the <u>right endpoint Riemann sum</u> for f corresponding to P as
$R_n(f) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$
& the <u>left endpoint Riemann sum</u> as
$L_n(f) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$.

Thm:

Let $a,b \in \mathbb{R}$ w/ $a<b$ & function $f: [a,b] \to \mathbb{R}$. Suppose either $(R_n(f))$ or $(L_n(f))$ is a convergent series. Then the other sequence also converges & both converge to the same point.

If f is Riemann integrable on $[a,b]$ then $(R_n(f))$ & $(L_n(f))$ both converge &
$\lim_{n\to\infty} R_n(f) = \lim_{n\to\infty} L_n(f) = \int_a^b f$

Def:

Fix an interval $[a,b]$. We say f is <u>right integrable</u> iff $(R_n(f))$ converges.
We denote the <u>right integral of</u> f by
$R_a^b(f) = \lim_{n\to\infty} R_n(f)$.

The reason we don't define Riemann integrals like this in general is b/c it "let's thru" some weird functions & doesn't have useful & intuitive properties we'd want integrals to have. in general

Example:

The characteristic function of the rational numbers f is right integrable on $[0,1]$

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

The reason this works is b/c every normal partition of P only splits $[0,1]$ on rational numbers.

Thm:

There exists a non-negative function $f: \mathbb{R} \to \mathbb{R}$ & real numbers $a<c<b$ such that

$$R_a^b(f) \neq R_a^c(f) + R_c^b(f)$$

Thm: Integral & Arithmetic

Let $a,b \in \mathbb{R}$ w/ $a<b$. Let f,g be real-valued functions that are Riemann integrable on $[a,b]$. Let $k \in \mathbb{R}$. Then we know...

The function $f+g$ is Riemann integrable on $[a,b]$ w/
$\int_a^b f+g = \int_a^b f + \int_a^b g$.

The function $f-g$ is Riemann integrable on $[a,b]$ w/
$\int_a^b f-g = \int_a^b f - \int_a^b g$.

The function kg is Riemann integrable on $[a,b]$ w/
$\int_a^b kf = k\int_a^b f$.

Thm: Integral & Order

Let $a,b \in \mathbb{R}$ w/ $a<b$ & $m, M \in \mathbb{R}$ w/ $m \leq M$. Let f,g be real-valued functions Riemann integrable on $[a,b]$. Then

i) $f(x) \geq 0 \ \forall x \in [a,b] \Rightarrow \int_a^b f \geq 0$

ii) $f(x) \leq g(x) \ \forall x \in [a,b] \Rightarrow \int_a^b f \leq \int_a^b g$

iii) $m \leq f(x) \leq M \ \forall x \in [a,b]$
$\Rightarrow m(b-a) \leq \int_a^b f \leq M(b-a)$

Families of Riemann Sums

Given a partition P of an interval $[a,b]$ there are infinitely many ways to choose sample points which yields a family of Riemann sums.

How can we describe/bound these families.

<u>Def:</u>

Let $a,b \in \mathbb{R}$ w/ $a<b$ & $[a,b] \subseteq K \subseteq \mathbb{R}$. Let $f: K \to \mathbb{R}$ be bounded on $[a,b]$. Fix a partition $P = \{x_0, \ldots, x_n\}$.

B/C f is bounded on $[a,b]$ it is bounded on each subinterval so we know $\forall i \in [n]$
$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ & $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.

Using these M_i & m_i we can define the upper & lower sum of f on P

$$\mathcal{U}(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \quad \& \quad \mathcal{L}(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

We pick supremum b/c consider

$\mathcal{U}(f,P)$ & $\mathcal{L}(f,P)$ may or may not be Riemann sums.

<u>Lemma:</u>

Let $a,b \in \mathbb{R}$ w/ $a<b$ & $[a,b] \subseteq K \subseteq \mathbb{R}$. Let $f: K \to \mathbb{R}$ be a bounded function. Let P be an arbitrary partition of $[a,b]$ & consider some $\varepsilon > 0$.

Then there exists Riemann sums $R_1(f,P)$ & $R_2(f,P)$ such that
$\mathcal{U}(f,P) \leq R_1(f,P) + \varepsilon$ & $\mathcal{L}(f,P) \geq R_2(f,P) - \varepsilon$.

<u>Def:</u>

Let $a,b \in \mathbb{R}$ w/ $a<b$ & let P & Q be partitions of $[a,b]$. If $P \subseteq Q$ we say Q is a <u>refinement</u> of P.

That is to get Q from P we just split it up into finer parts.

If Q is a refinement of P then the family of Riemann sums $R(f,Q)$ will be "tighter" than the family of $R(f,P)$. Formally...

<u>Thm:</u>

Let $a,b \in \mathbb{R}$ w/ $a<b$ & $[a,b] \subseteq K \subseteq \mathbb{R}$. Let $f: K \to \mathbb{R}$ be bounded on $[a,b]$. If P is a partition of $[a,b]$ & Q is a refinement of P then

$$\mathcal{L}(f,P) \leq \mathcal{L}(f,Q) \leq \mathcal{U}(f,Q) \leq \mathcal{U}(f,P).$$

<u>Lemma:</u>

Let $a,b \in \mathbb{R}$ w/ $a<b$ & $[a,b] \subseteq K \subseteq \mathbb{R}$. Let $f: K \to \mathbb{R}$ be a function & P be a partition of $[a,b]$. Let $\varepsilon > 0$. Suppose whenever $R_1(f,P)$ & $R_2(f,P)$ are Riemann sums of f on P then
$|R_1(f,P) - R_2(f,P)| < \varepsilon$.

Then we know
i) f is bounded on $[a,b]$
ii) $\mathcal{U}(f,P) - \mathcal{L}(f,P) \leq \varepsilon$.
iii) If Q is a refinement of P & $R(f,P)$ & $R(f,Q)$ are Riemann sums then
$|R(f,P) - R(f,Q)| \leq \varepsilon$.

Lemma:

Let $a, b \in \mathbb{R}$ w/ $a<b$ & $[a,b] \subseteq K \subseteq \mathbb{R}$. Let $f: K \to \mathbb{R}$ be a bounded function on $[a,b]$. Fix arbitrary $\varepsilon > 0$ & $n \in \mathbb{N}$. We know $\exists \delta > 0$ such if $P = \{x_0, \dots, x_n\}$ is a partition of $[a,b]$ & Q is a partition of $[a,b]$ w/ $\|Q\| < \delta$ then

$$0 \le \mathcal{U}(f, Q) - \mathcal{U}(f, Q^*) < \varepsilon \quad \& \quad 0 \le \mathcal{L}(f, Q^*) - \mathcal{L}(f, Q) < \varepsilon$$

where $Q^* = Q \cup P$.

Thm:

Let $a, b \in \mathbb{R}$ w/ $a<b$ & $[a,b] \subseteq K \subseteq \mathbb{R}$. Let $f: K \to \mathbb{R}$ be a function.

Then integrability of f on $[a,b]$ is equivalent to the Cauchy Criterion for the Existence of the Integral.

Cauchy Criterion for the Existence of the Integral: For every $\varepsilon > 0$ $\exists \delta > 0$ such that if P is a partition of $[a,b]$ w/ $\|P\| < \delta$ & $R_1(f, P)$ & $R_2(f, P)$ are Riemann sums of f on P then

$$|R_1(f, P) - R_2(f, P)| < \varepsilon.$$

Coro:

Let $a, b \in \mathbb{R}$ w/ $a<b$ & $[a,b] \subseteq K \subseteq \mathbb{R}$. Let $f: K \to \mathbb{R}$ be a function.

Then f is integrable on $[a,b]$ if & only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if P is a partition of $[a,b]$ w/ $\|P\| < \delta$ then

$$\mathcal{U}(f, P) - \mathcal{L}(f, P) < \varepsilon$$

Existence of Integrals

Thm:

Let $a, b \in \mathbb{R}$ w/ $a<b$ & $[a,b] \subseteq K \subseteq \mathbb{R}$. Suppose $f: K \to \mathbb{R}$ is a function.

If f is continuous on $[a,b]$ then f is Riemann integrable on $[a,b]$

If f is monotonic on $[a,b]$ then f is Riemann integrable on $[a,b]$.

Thm:

Let $a, b, c \in \mathbb{R}$. Let I be the smallest closed interval containing a, b, c. Suppose $I \subseteq K \subseteq \mathbb{R}$ w/ function $f: K \to \mathbb{R}$.

The integral $\int_a^b f$ exists iff $\underbrace{\int_a^c f \ \& \ \int_c^b f}$ exist.

In this case

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Think like bridging the gap

Thm: Composition of Integrals

Let $a,b \in \mathbb{R}$ w/ $a<b$. Let f be a real-valued function Riemann integrable on $[a,b]$. Suppose $f([a,b]) \subseteq S$ & $h: S \to \mathbb{R}$ is uniformly continuous & bounded on $f([a,b])$.

Then $h \circ f$ is Riemann integrable on $[a,b]$.

Coro: Composition Theorem for Integrals

Let $a,b \in \mathbb{R}$ w/ $a<b$. Let f be a real-valued function Riemann integrable on $[a,b]$. Suppose $f([a,b]) \subseteq S$ & that $f([a,b])$ is compact. Let $h: S \to \mathbb{R}$ be continuous on $f([a,b])$.

Then $h \circ f$ is Riemann integrable on $[a,b]$.

Thm:

Let $a,b \in \mathbb{R}$ w/ $a<b$. Let f & g be real-valued functions that are Riemann integrable on $[a,b]$.

Then fg is Riemann integrable on $[a,b]$.

Thm:

Let $a,b \in \mathbb{R}$ w/ $a<b$. Let f be a real-valued function that is Riemann integrable on $[a,b]$. Further

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Fundamental Theorem of Calculus

Def:

Let I be an interval w/ $a \in I$ & $f: I \to \mathbb{R}$. Suppose $\int_a^x f$ exists for all $x \in I$. Define the <u>area accumulation function of f on I based at a</u> as
$$F_a(x) = \int_a^x f = \int_a^x f(t)\,dt.$$

Thm: Fundamental Theorem of Calculus

Let I be an interval w/ $a \in I$ & $I \subseteq K \subseteq \mathbb{R}$. Suppose $f: K \to \mathbb{R}$ is integrable on all $[a,x]$ where $x>a$ & $x \in I$ & $[x,a]$ where $x<a$ & $x \in I$.

Let F_a be the area accumulation for f on I based at a. Then F_a is uniformly continuous on I.

If f happens to be continuous on I then F_a is differentiable on I &
$$F_a'(x) = f(x) \quad \forall x \in I.$$

Coro: Integral Evaluation Theorem

Let $a,b \in \mathbb{R}$ w/ $a<b$ & $[a,b] \subseteq K \subseteq \mathbb{R}$. Let $f: K \to \mathbb{R}$ be a continuous function. If $F: K \to \mathbb{R}$ is an antiderivative for f on $[a,b]$ then

$$\int_a^b f = F(b) - F(a).$$