Sets of Numbers
- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$
- Integer Numbers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$
- Rational Numbers: $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\}$
- Real Numbers: $\mathbb{R}$ = numbers with decimal representation, finite or infinite
- Complex Numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$

Set Notation
- $\emptyset$ = Empty Set
- $\in$ = Element of: $2 \in \{1, 2\}$
- $\subseteq$ = Subset; $\{2\} \subseteq \{1, 2\}$, $\{1, 2\} \subseteq \{1, 2\}$
- $\subset$ = Strict subset; $\{2\} \subset \{1, 2\}$

Properties of Addition & Multiplication
- Commutativity: Order of operands doesn't matter.
  - Add: $a + b = b + a$
  - Mult: $ab = ba$
- Associativity: Order of association doesn't matter.
  - Add: $(a+b)+c = a+(b+c)$
  - Mult: $(ab)c = a(bc)$
- Identity: Operand that does nothing
  - Add: $a + 0 = a$
  - Mult: $a \cdot 1 = a$
- Inverse: Operand that yields identity when applied.
  - Add: $\forall a \in \mathbb{R}, \exists -a \in \mathbb{R}$ st $a + (-a) = 0$
  - Mult: $\forall a \in \mathbb{R} \wedge a \neq 0, \exists 1/a \in \mathbb{R}$ st. $a \cdot 1/a = 1$

Euclidean Spaces

In this class, we'll always put vectors vertical.

- 2-Space: $\mathbb{R}^2 = \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R}\right\}$
- 3-Space: $\mathbb{R}^3 = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R}\right\}$
- n-Space ($n \in \mathbb{N}$): $\mathbb{R}^n = \left\{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \ \forall i \in 1, .., n\right\}$

Vector Operation

<u>Def</u>: Vector Addition

Let $\bar{u}, \bar{v} \in \mathbb{R}^n$

$$\bar{u} + \bar{v} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

Def: Scalar Multiplication
Let $\bar{u} \in \mathbb{R}^n$, $a \in \mathbb{R}$

$$a\bar{u} = a\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} au_1 \\ \vdots \\ au_n \end{bmatrix}$$

Vector Spaces (VS)

Def: Vector Space
A set V over a field F w/ addition $\oplus$ & scalar multiplication $\odot$ satisfying the following axioms for $\bar{u}, \bar{v}, \bar{w} \in V$ & $a, b \in F$.

*Note: A# = addition, M# = multiplication, D# = distributivity

A0. Closure under addition: $\bar{u} \oplus \bar{v} \in V$
A1. Addition is commutative: $\bar{u} \oplus \bar{v} = \bar{v} \oplus \bar{u}$
A2. Addition is associative: $(\bar{u} \oplus \bar{v}) \oplus \bar{w} = \bar{u} \oplus (\bar{v} \oplus \bar{w})$
A3. Additive Identity: $\exists \bar{0} \in V$ st. $\forall \bar{u} \in V$: $\bar{u} \oplus \bar{0} = \bar{u}$
A4. Additive Inverse: $\forall \bar{u} \in V$, $\exists -\bar{u} \in V$ st. $\bar{u} \oplus (-\bar{u}) = \bar{0}$

M0. Closure under scalar multiplication: $a \odot \bar{u} \in V$
M1. Multiplicative Identity: $\exists 1 \in F$ st. $\forall \bar{u} \in V$: $1 \odot \bar{u} = \bar{u}$

D1. $a \odot (\bar{u} \oplus \bar{v}) = a \odot \bar{u} \oplus a \odot \bar{v}$ ← Not D0 to match her :\^
D2. $(a+b) \odot \bar{u} = (a \odot \bar{u}) \oplus (b \odot \bar{u})$
D3. $(ab) \odot \bar{u} = a \odot (b \odot \bar{u})$ ← You can pick off bits

Associativity & commutativity fall out of the other axioms

Examples of Vector Spaces (VS):
- $(\mathbb{R}^n, +, \cdot)$ over $F = \mathbb{R}$
- $(\mathbb{C}^n, +, \cdot)$ over $F = \mathbb{C}$

Proving $\mathbb{R}^2$ is a Vector Space (VS): (Using operations defined earlier)

A0. Let $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$. $u_1, u_2, v_1, v_2 \in F$

$$\bar{u} \oplus \bar{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \in \mathbb{R}^2 \text{ since } u_1 + v_1,\ u_2 + v_2 \in \mathbb{R} \quad \square$$

A1. Let $\bar{u}, \bar{v} \in \mathbb{R}^2$

$$\bar{u} \oplus \bar{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \qquad \bar{v} \oplus \bar{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \end{bmatrix}$$

$u_1 + v_1 = v_1 + u_1$ & $u_2 + v_2 = v_2 + u_2$

$\therefore \bar{u} \oplus \bar{v} = \bar{v} \oplus \bar{u} \quad \square$

And so forth for all axioms. We say these properties are <u>inherited</u> from their field (in this case $\mathbb{R}$).

Other Vector Spaces:

- Functions ($\mathbb{R} \to \mathbb{R}$)
- $m \times n$ Matrices (Can translate to vectors) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
- Solutions to differential equations
- Polynomials w/ real coefficients.
- For $n \in \mathbb{N}$, $\mathcal{P}_n = \{ p : p(x) = a_n x^n + \dots + a_0,\ a_i \in \mathbb{R},\ i = \overline{1,n} \}$

Notation
$\overline{1,n} = 1, \dots, n$

Surprising Failed Vector Spaces:

- V all polynomials of degree 2 w/ real coeff

  Ex. x^2, $\frac{1}{3}x^2 - \sqrt{2}x + 15 \in V$

  Ex. 1, $x+5 \notin V$

  Fails A0, closure under addition.

  $x^2 + (-x^2) = 0$

  $x^2, -x^2 \in V$

  $0 \notin V$

  Fails A3, no identity b/c 0 would be identity but $0 \notin V$.

- $V = \mathbb{R}, \oplus, \odot$

  $a \oplus b = 2a + 2b$

  $k \odot a = ka$

  Fails A2, not associative

  $(a \oplus b) \oplus c$ $\quad$ $a \oplus (b \oplus c)$

  $2(2a+2b) + 2c$ $\quad$ $2a + 2(2b+2c)$

  $4a + 4b + 2c$ $\quad$ $2a + 4b + 4c$

  Suppose $a = 1$, $b = 0$, $c = 0$. $4 \neq 2$

Suprising Successful Vector Spaces:

- $V = \mathbb{R}^2$

  $\bar{u} \oplus \bar{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + 1 \\ u_2 + v_2 + 1 \end{bmatrix}$

  $a \odot \bar{u} = a \odot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} au_1 + a - 1 \\ au_2 + a - 1 \end{bmatrix}$

Prove for yourself!

Theorem: The additive Identity is unique
By A2, we have at least one zero vector.

Suppose we have 2 zero vectors both satisfying A2, $\bar{0}$ & $\bar{0}'$

We show $\bar{0} = \bar{0}'$.

$$\underset{A3}{\bar{0}' = \bar{0}' + \bar{0}} \underset{A1}{=} \bar{0} + \bar{0}' \underset{A3}{=} \bar{0} \quad \square$$

Theorem: The additive inverse is unique.
Let $\bar{u} \in V$ & assume $\bar{v}_1, \bar{v}_2 \in V$ that are both additive inverses of $\bar{u}$. (A4)
We show $\bar{v}_1 = \bar{v}_2$

$$\bar{v}_1 = \bar{v}_1 + \bar{0} = \bar{v}_1 + (\bar{u} + \bar{v}_2) = (\bar{v}_1 + \bar{u}) + \bar{v}_2 = \bar{0} + \bar{v}_2 = \bar{v}_2 \quad \square$$

Theorem: For every $\bar{u} \in V$, $0 \cdot \bar{u} = \bar{0}$

Thm: For every $a \in F$, $a \cdot \bar{0} = \bar{0}$

Subspace

How do we prove something is a vector space w/o enumerating the axioms? Previously we only got a sense of what worked w/o any formal proof.

Def: Subspace
A non-empty subset U of a vector space V is called a subspace of V if U itself is a vector space wrt inherited operations of vector addition & scalar multiplication of V.

Examples:

$V = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$ subspace of $\mathbb{R}^3$

$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z, a, b, c \in \mathbb{R},\ ax + by + cz = 0 \right\}$ is a subspace of $\mathbb{R}^3$

$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R},\ 2x + y - z = -4 \right\}$ is not a subspace of $\mathbb{R}^3$ b/c $\bar{0} \notin W$

*Note: Essentially a subset of a vector space & a vector space itself.

Thm: Minimum Conditions for a Subspace

Let U be a non-empty subset of v.s. V.

U is a subspace of V iff ($\Rightarrow$ & $\Leftarrow$)
1) U is closed under addition.
2) U is closed under scalar multiplication.

Proof of $\Rightarrow$: Trivial as U satisfies A0 & M0.

Proof of $\Leftarrow$:

Suppose U is closed under $+$ & $\cdot$. We prove the rest of the axioms are satisfied.

The trace on an $n \times n$ matrix $A = (a_{ij})$ is $trace(A) = \sum_{i=1}^{n} a_{ij}$

A0 & M0 are given.

A1. Let $\bar{u}_1, \bar{u}_2 \in U$.
$\therefore \bar{u}_1, \bar{u}_2 \in V$ since $U \subseteq V$
A1 is satisfied in V.
$\therefore \bar{u}_1 + \bar{u}_2 = \bar{u}_2 + \bar{u}_1$ satisfying A1.

Zero vector is a good litmus test for a v.s. but its existence is not a proof.

Similarly A2, M1, D1-D3 are satisfied

A3 & A4.
Let $\bar{u} \in U \subseteq V$.
By A3 is V, $\exists -\bar{u} \in V$ s.t. $\bar{u} + (-\bar{u}) = \bar{0}$.
Recall $-\bar{u} = (-1) \cdot \bar{u}$. By M0, $(-1) \cdot \bar{u} = -\bar{u} \in U$.
By A0, $\bar{u} + (-\bar{u}) = \bar{0} \in U$.
$\therefore$ A4 is satisfied.
Since $\bar{0} \in U$, A3 is satisfied.

On homework, assume
- $\mathbb{R}^n$
- polynomials
- functions

are v.s.'s

* Note: You can disprove any v.s. by any of the axioms.

Example:

Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}; a + d = 0 \right\} \subseteq M_{2\times 2}(\mathbb{R})$

Is V a v.s.? * Note $d = -a$

Subspaces are just a proof shortcut (more or less).

Let $\bar{u}_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix} \in V$

A0) $\bar{u}_1 + \bar{u}_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & -(a_1 + a_2) \end{bmatrix} \in V$

M0) $k \cdot \bar{u} = \begin{bmatrix} ka & kb \\ kc & -(ka) \end{bmatrix} \in V$

$\therefore V$ is a vector space.

Linear Span:

Def: Linear Combination of Vectors (l.c.)
Let $S = \{\bar{v}_1, \dots, \bar{v}_m\}$ from a set of vectors from v.s. $(V, +, \cdot)$.
Let $a_1, a_2, \dots, a_m$ be scalars in F ($\mathbb{R}$ or $\mathbb{C}$ here).

A linear combination of vectors from S is a vector $\bar{v} \in S$ s.t.
$\bar{v} = a_1\bar{v}_1 + a_2\bar{v}_2 + \dots + a_m\bar{v}_m = \sum_{i=1}^{m} a_i\bar{v}_i$
We say $\bar{v}$ is a l.c. of $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_m$

many

Examples: Linear Combinations
- $S = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} \subseteq \mathbb{R}^3$ ($\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$)

$\bar{v} = \begin{bmatrix}1\\3\\5\end{bmatrix}$ is a l.c. of $\bar{e}_1, \bar{e}_2, \bar{e}_3$: $\bar{v} = 1\cdot\bar{e}_1 + 3\cdot\bar{e}_2 + 5\cdot\bar{e}_3$

* Note: All vectors $\bar{u} \in \mathbb{R}^3$ can be written as l.c. of $\bar{e}_1, \bar{e}_2, \bar{e}_3$.

$\bar{u} = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = u_1\bar{e}_1 + u_2\bar{e}_2 + u_3\bar{e}_3$

- $V = \mathcal{P}_2$ (all polynomials degree 2 or less)

$f_1 = 2$; $f_2 = 1 - x$; $f_3 = 1 + x^2$

Is $g = 2x^2 - 3x + 3$ a l.c. of f_1, f_2, f_3?

Yes. $g = -1\cdot f_1 + 3\cdot f_2 + 2\cdot f_3$

We can also use other f_1, f_2, f_3 which work!

- Is $B = \begin{bmatrix}2 & -1\\3 & 8\end{bmatrix}$ a l.c. of $A_1 = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$, $A_2 = \begin{bmatrix}0 & -5\\0 & 0\end{bmatrix}$, $A_3 = \begin{bmatrix}2 & 4\\5 & 0\end{bmatrix}$? Adding an A_4 makes this work!

No. For all linear combinations of A_1, A_2, A_3 the $(2,2)$-entry is 0.

- Is $\bar{v} = \begin{bmatrix}2\\4\\6\end{bmatrix}$ a l.c. of $\bar{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$?

Yes where $\bar{v} = -2\bar{v}_1 + 6\bar{v}_2 - 2\bar{v}_3$.

Further, we can represent $\bar{u} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$ as $\bar{u} = (a_3 - a_2)\bar{v}_1 + a_3\bar{v}_2 + (a_1 - a_3 - a_2)\bar{v}_3$

Def: Linear Span Space
Let $(V, +, \cdot)$ be a v.s. & $S = \{\bar{v}_1, \dots, \bar{v}_m\}$ be a set of vectors in V.

The linear span of S or the linear space of $\bar{v}_1, \dots, \bar{v}_m$ is the set of all l.c. of S.

$$\text{span}(\bar{v}_1, \dots, \bar{v}_m) = \text{span}(S) = \{\bar{v} \in V \mid \exists a_1, \dots, a_m \in \mathbb{R} \text{ s.t. } \bar{v} = a_1\bar{v}_1 + a_2\bar{v}_2 + \dots + a_m\bar{v}_m\}$$

Examples:

Examples: Linear Span

- $\text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \left\{\begin{bmatrix}a\\b\\0\end{bmatrix} \middle| a, b \in \mathbb{R}\right\}$ xy-plane
- $\text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right) = \left\{\begin{bmatrix}a+2b\\2a\\3a+b\end{bmatrix} \middle| a, b \in \mathbb{R}\right\}$

Terms:

Given $W = \text{span}(\bar{v}_1, .., \bar{v}_2)$

- $\bar{v}_1, .., \bar{v}_m$ span W
- $\{\bar{v}_1, .., \bar{v}_m\}$ is a spanning set for W
- W is the linear span of $\bar{v}_1, .., \bar{v}_m$
- If $\bar{w} \in W$ then $\bar{w}$ is a linear combination (l.c.) of $\bar{v}_1, .., \bar{v}_m$

Thm: All Linear Spans are Subspaces

Let $(V, +, \cdot)$ be a vector space

& $\bar{v}_1, .., \bar{v}_m \in V$

We show that $\text{span}(\bar{v}_1, .., \bar{v}_m)$ is closed under addition & scalar multiplication.

Let $c_1\bar{v}_1 + .. + c_m\bar{v}_m,\ d_1\bar{v}_1 + .. + d_m\bar{v}_m \in \text{span}(v_1, .., v_m)$. & $a \in \mathbb{R}$.

AO $(c_1\bar{v}_1 + .. + c_m\bar{v}_m) + (d_1\bar{v}_1 + .. + d_m\bar{v}_m) = (c_1 + d_1)\bar{v}_1 + .. + (c_m + d_m)\bar{v}_m$

MO $a(c_1\bar{v}_1 + .. + c_m\bar{v}_m) = (ac_1)\bar{v}_1 + .. + (ac_m)\bar{v}_m$

We have thus shown closure under addition (AO) & scalar multiplication (MO). It is thus a vector space. $\square$

Thm: $\text{span}(\bar{v}_1, .., \bar{v}_m)$ is the smallest subspace containing all $\bar{v}_1, .., \bar{v}_m$

Let U be a subspace containing $\bar{v}_1, .., \bar{v}_m$. & $U \subseteq \text{span}(\bar{v}_1, .., \bar{v}_m)$

We show $\text{span}(\bar{v}_1, .., \bar{v}_m) \subseteq U$ (i.e. $U = \text{span}(\bar{v}_1, .., \bar{v}_m)$)

Let $c_1\bar{v}_1 + .. c_m\bar{v}_m \in \text{span}(\bar{v}_1, .., \bar{v}_m)$

$\bar{v}_1, .., \bar{v}_m \in U$

$\Rightarrow c_1\bar{v}_1, .., c_m\bar{v}_m \in U$ (by MO)

$\Rightarrow c_1\bar{v}_1 + .. + c_m\bar{v}_m \in U$ (by AO).

We have shown $x \in U \Rightarrow x \in \text{span}(\bar{v}_1, .., \bar{v}_m)$.

Thus $\text{span}(\bar{v}_1, .., \bar{v}_m) \subseteq U \subseteq \text{span}(\bar{v}_1, .., \bar{v}_m)$

$\therefore U = \text{span}(\bar{v}_1, .., \bar{v}_m)$ $\square$

* Note: A vector space has infinitely many spanning sets.

Def: Finite Dimensional Vector Space
When a vector space has a finite spanning set.

Examples:
Finite Dimensional v.s.: $\mathcal{P}_2, \mathbb{R}^3, \mathcal{M}_{2\times 3}$
Infinite Dimensional v.s.: $\mathcal{P}$, all sequences, $f: \mathbb{R} \to \mathbb{R}$

II Linear Independence

Def: Linearly Independent (l.i.)
A set of vectors $\{\bar{v}_1, \ldots, \bar{v}_m\}$ from a v.s. V is linearly independent if the only solution to $a_1\bar{v}_1 + \ldots + a_m\bar{v}_m = \bar{0}$ is $a_1 = a_2 = 0$

This essentially means that all vectors bring something new that no other vector can cancel.

Def: Linearly Dependent (l.d.)
Not linearly independent. In other words, there are non-trivial solutions to the above equation.

Example: Easy
$\{1, 1+x, x^2\}$ is l.i. b/c $0 + 0x + 0x^2 = 0(1) + 0(1+x) + 0(x^2)$

$\{1, x, x^2, 2x^2 - x\}$ is l.d. b/c $0 + 0x + 0x^2 = 0(1) + 1(x) - 2(x^2) + 1(2x^2 - x)$

Example: More Difficult
Are $\left\{ \underset{\bar{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}}, \underset{\bar{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}}, \underset{\bar{v}_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}} \right\}$ l.i. in $\mathbb{R}^4$?

To start this, we multiply the vectors by scalars & sum them. We then try to find if we can find non-trivial solutions.

$a\bar{v}_1 + b\bar{v}_2 + c\bar{v}_3$

$\Rightarrow a + c = 0$
$b + c = 0$ $\Rightarrow a = 0$ $\Rightarrow c = 0$, $\Rightarrow b = 0$
$a + b + c = 0$
$2a + 2b + 3c = 0$

Since $a = b = c = 0$, they are l.i.

Example: Generate your own

Set of l.i. vectors in $\mathcal{M}_{2\times 2}$: $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -1 \\ +1 & -1 \end{bmatrix} \right\}$

Set of l.d. vectors in $\mathcal{M}_{2\times 2}$: $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \right\}$

Thm: Spanning Sets & Linear Independence
Let $(V, +, \cdot)$ be a v.s. & $S = \{\bar{v}_1, \ldots, \bar{v}_m\} \subseteq V$.

S is l.d. iff there exists one vector $\bar{v}_j \in S$ s.t. $\bar{v}_j$ is a l.c. of the other vectors.

In other words, you have an unnecessary vector b/c
$(a_1\bar{v}_1 + \ldots a_{m-1}\bar{v}_{m-1}) + 1 \cdot \bar{v}_j = \bar{0}$ is a non-trivial solution.

Proof:

Assume S is l.d.
Then $a_1\bar{v}_1 + \ldots + a_m\bar{v}_m = \bar{0}$ for some non-trivial $a_1, \ldots, a_m$

Let $a_j \neq 0$ be the non-trivial scalar.

$a_j\bar{v}_j = -a_1\bar{v}_1 - \ldots - a_{j-1}\bar{v}_{j-1} - a_{j+1}\bar{v}_{j+1} - \ldots - a_m\bar{v}_m$

$\bar{v}_j = -\frac{a_1}{a_j}\bar{v}_1 - \ldots - \frac{a_{j-1}}{a_j}\bar{v}_{j-1} - \frac{a_{j+1}}{a_j} - \ldots - \frac{a_m}{a_j}\bar{v}_m$

$\therefore \bar{v}_j$ is a l.c. of $S \setminus \{\bar{v}_j\}$. $\longleftarrow$ $\text{span}(\{\bar{v}_1, \ldots, \bar{v}_m\}) = \text{span}(\{\bar{v}_1, \ldots, \bar{v}_{j-1}, \bar{v}_{j+1}, \ldots, \bar{v}_m\})$

Example: Creating l.d. set of vectors
Given a set of vectors $\{2+x, 7x-x^2, 3x^3+1, 4x^5-3\}$, give a linearly dependent vector

$$(2+x) + (7x - x^2) + (3x^3 + 1) + (4x^5 - 3)$$

Thm: Subsets + Linear (In)Dependence
Given $(V, +, \cdot)$ & two sets of vectors $S \subseteq T$.

If T is l.i., S is also l.i.
If S is l.d., T is also l.d.

Example: Tough l.d.
Let $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\bar{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Are they l.d.?

You can brute force this & do $\bar{v}_1 - \bar{v}_2 + \bar{v}_3 - 2\bar{v}_4 = \bar{0}$, but this doesn't scale.

The most general way is to go w/ the definition.

$a\bar{v}_1 + b\bar{v}_2 + c\bar{v}_3 + d\bar{v}_4 = 0$

$$\Rightarrow \begin{aligned} a \quad\; + c + d &= 0 \quad \Rightarrow d = 2b \\ b + c \quad\; &= 0 \quad \Rightarrow c = -b \\ a + b \quad\quad\; &= 0 \quad \Rightarrow a = -b \end{aligned}$$

Suppose $b = 1$. This gives us a non-trivial solution.

4 variables & 3 equations. B/c of this, there are infinitely many solutions.

$\therefore$ There exists a non-trivial solution
$\therefore$ $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ are l.d.

In this case, all the vectors are the "bad guy". Meaning removing any one of them would create linear independence.

Example: Take it further w/ Span
What is span $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$?

It is $\mathbb{R}^3$ b/c We can create $e_1, e_2,$ & e_3 from $\bar{v}_1, \bar{v}_2, \bar{v}_3$. The span of $\{e_1, e_2, e_3\}$ is trivially $\mathbb{R}^3$.

$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\bar{v}_1 - \frac{1}{2}\bar{v}_2 + \frac{1}{2}\bar{v}_3$ The

$\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \bar{v}_3 - \bar{e}_1$

$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \bar{v}_2 - \bar{e}_2$

Alternatively, what kind of vectors $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ from $\mathbb{R}^3$ can be expressed as l.c. of $\bar{v}_1, \bar{v}_2, \bar{v}_3$

$$a_1\bar{v}_1 + a_2\bar{v}_2 + a_3\bar{v}_3 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \Rightarrow \begin{cases} a_1 + a_3 = \alpha \\ a_2 + a_3 = \beta \\ a_1 + a_2 = \gamma \end{cases} \Rightarrow 2a_1 + 2a_2 + 2a_3 = \alpha + \beta + \gamma$$

Y.000: Lecture Notes

Systematically Finding Span (& Solving Linear Equations)

We've seen earlier that determining the span decays into solving systems of linear equations. How do we do that efficiently & methodically.

<u>Def:</u> System of Linear Equations

A system of linear equations is a finite set of equations w/ shared variables. We want to solve them all by assigning the variables to exact values.

They have the form of the following where our variables are $a_1, \ldots, a_n$ & we have m equations.

$$\left.\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}\right\} \text{List of Equations}$$

We can write the coefficients in a $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = A \text{ is the coefficient matrix}$$

We can "augment" this by adding the constants on the righ separated by a bar.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] = \bar{A} \text{ is the augmented matrix.}$$

Gaussian Elimination

Gaussian elimination is systematic elimination.

<u>Example:</u>

$$\begin{cases} x_1 + x_2 - 3x_3 = 3 \\ -2x_1 - x_2 = -4 \\ 4x_1 + 2x_2 + 3x_3 = 7 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ -2 & 1 & 0 & -4 \\ 4 & 2 & 3 & 7 \end{array}\right]$$

We replace row 2 (R_2) w/ $R_2 + 2R_1$
& row 3 (R_3) w/ $R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -6 & 2 \\ 0 & -2 & 15 & -5 \end{array}\right]$$

We have now "eliminated" x_1 from R_2 & R_3 by combining them w/ R_1.
We now 'eliminate x_2' from R_3 by combining it w/ R_2.

We replace R_3 w/ $R_3 + 2R_2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -6 & 2 \\ 0 & 0 & 3 & -1 \end{array}\right]$$ ← This is called <u>row echelon form</u>.

Now we have isolated x_3 in R_3. We solve for x_3

$3x_3 = -1$ (R_3)
$\underline{x_3 = -1/3}$

Now that we've solved for x_3 in the bottom form, we cascade up to find the rest.

$x_2 - 6x_3 = 2$ (R_2)
$x_2 + 2 = 2$
$\underline{x_2 = 0}$

$x_1 + x_2 - 3x_3 = 3$
$x_1 + 0 + 1 = 3$
$\underline{x_1 = 2}$

Reduced Row Echelon Form: The coefficient matrix is the identity matrix; i.e. everything but the leading 1s is zero.

We have thus solve it w/ $x_1 = 2$, $x_2 = 0$, $x_3 = -1/3$.

Now we've seen an example; let's formalize it.

<u>Def</u>: Row Operations
Applying the following row operations on an augmented matrix does not change the system.

i) Replace row w/ multiple of itself $R_i := \alpha R_i$
ii) Switch two rows $R_i \leftrightarrow R_j$
iii) Replace row w/ sum of itself & some multiple of another $R_i := R_i + \alpha R_j$

Could be $R_i = \alpha_0 R_0 + \dots + \alpha_n R_n$ to be most general

<u>Def</u>: Row Echelon Form (REF)
Each non-zero row starts w/ a 1. All entries below leading 1 is 0.
Every zero row is below all non-zero rows. All leading 1s are left of leading 1 below

<u>Def</u>: Method of Gaussian Elimination
Applying row operations on an augmented matrix to reach row echelon form.
Once you reach row echelon form, you then solve the bottom & cascade up.

Zero rows are "unnecessary eqns" or are linearly dependent w/ others

Example:

$$\begin{cases} -3x_2 + 4x_3 = -5 \\ 3x_1 - 7x_2 + 8x_3 = 9 \\ 3x_1 - 9x_2 + 6x_3 = 15 \end{cases}$$

> You want to get the zeros, then get the one. The one makes it easier to move forward w/ making more zeros.

$$\left[\begin{array}{ccc|c} 0 & 3 & 4 & -5 \\ 3 & -7 & 8 & 9 \\ 3 & -9 & 6 & 15 \end{array}\right] \quad R_1 \leftrightarrow \tfrac{1}{3}R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 3 & -7 & 8 & 9 \\ 0 & 3 & 4 & -5 \end{array}\right] \quad R_2 := R_2 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 2 & 2 & -6 \\ 0 & 3 & 4 & -5 \end{array}\right] \quad R_2 := \tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 4 & -5 \end{array}\right] \quad R_3 := R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

R_3: $\underline{x_3 = 4}$

R_2: $x_2 + x_3 = -3$

$x_2 + 4 = -3$

$\underline{x_2 = -7}$

R_1: $x_1 - 3x_2 + 2x_3 = 5$

$x_1 + 21 + 8 = 5$

$x_1 = -3 - 21$

$\underline{x_1 = -24}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -24 \\ -7 \\ 4 \end{bmatrix}$$

Example: Gaussian Elimination & Span

Is $3+5x+9x^2 \in span\{1+2x+3x^2, 1+x+2x^2, 1+3x+4x^2\}$.

In other words is a l.c. that gives us $3+5x+9x^2$.

$3+5x+9x^2 = c_1(1+2x+3x^2)+c_2(1+x+2x^2)+c_3(1+3x+4x^2)$

This gives us

$$\begin{cases} c_1+c_2+c_3=3 \\ 2c_1+c_2+3c_3=5 \\ 3c_1+2c_2+4c_3=9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1&1&1&3 \\ 2&1&3&5 \\ 3&2&4&9 \end{array}\right] \begin{array}{l} R_2 := R_2-2R_1 \\ R_3 := R_3-3R_1 \end{array} \left[\begin{array}{ccc|c} 1&1&1&3 \\ 0&-1&1&-1 \\ 0&-1&1&0 \end{array}\right] R_2 := -R_2 \left[\begin{array}{ccc|c} 1&1&1&3 \\ 0&1&-1&1 \\ 0&-1&1&0 \end{array}\right] R_3 := R_3+R_2 \left[\begin{array}{ccc|c} 1&1&1&3 \\ 0&1&-1&1 \\ 0&0&0&1 \end{array}\right]$$

$0 \neq 1$

$\therefore 3+5x+9x^2 \notin span\{\ldots\}$

You also could have done

$$\left[\begin{array}{ccc|c} 1&1&1&3 \\ 2&1&3&5 \\ 3&2&4&9 \end{array}\right] R_3 := R_3-R_2 \left[\begin{array}{ccc|c} 1&1&1&3 \\ 2&1&3&5 \\ 1&1&1&4 \end{array}\right]$$

$3 = x_1+x_2+x_3 = 4$
$\therefore$ not in span

OR

$$\left[\begin{array}{ccc|c} 1&1&1&3 \\ 0&-1&1&-1 \\ 0&-1&1&0 \end{array}\right]$$

$-1 = -x_2+x_3 = 0$
$\therefore$ not in span

Systems w/ no solutions are call <u>inconsistent</u> systems.

Example:

$$\begin{cases} x-y+z=5 \\ x-y=2 \\ 2x-2y+z=7 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1&-1&1&5 \\ 1&-1&0&2 \\ 2&-2&1&7 \end{array}\right] \begin{array}{l} R_2 := R_2-R_1 \\ R_3 := R_3-2R_2 \end{array} \left[\begin{array}{ccc|c} 1&-1&1&5 \\ 0&0&-1&-3 \\ 0&0&-1&-3 \end{array}\right] \begin{array}{l} R_2 := -R_2 \\ R_3 := R_3-R_2 \end{array} \left[\begin{array}{ccc|c} 1&-1&1&5 \\ 0&0&1&3 \\ 0&0&0&0 \end{array}\right] \text{REF}$$

$R_2: \underline{z=3}$ $\quad R_1: x-y+z=5$
$x-y+3=5$
$\underline{x-y=2}$

x & y are dependent on each other.

Since y doesn't have a leading zero, we pick that as our free variable.

Our solution is

$$\begin{cases} x=s+2 \\ y=s \\ z=3 \end{cases} \quad s \in \mathbb{R} \text{ parameter}$$

We chose s as the parameter rather than y to differentiate the solution

Info from Row Echelon Form

Examples

3 vars, 3 eqn

i) $\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 0 & 8 \end{array}\right]$ Inconsistent System ($R_3: 0 \neq 8$)
No solution.

4 vars, 3 eqn

ii) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 8 & 1 \\ 0 & 0 & 1 & -6 & -1 \end{array}\right]$ Infinitely many solutions w/ x_4 being free (no leading 1)

$x_1 = 2$ $\quad x_3 = -1 + 6s$ $\quad \bar{x} = \begin{bmatrix} 2 \\ 1-8s \\ -1+6s \\ s \end{bmatrix} \forall s \in \mathbb{R}$

$x_2 = 1 - 8s$ $\quad x_4 = s \in \mathbb{R}$

5 vars, 3 eqn

iii) $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -6 \end{array}\right]$ Infinitely many solutions w/ x_3 & x_5 being free (no leading 1)

$x_1 = -2 - s - t$ $\quad x_3 = s \in \mathbb{R}$ $\quad x_5 = t \in \mathbb{R}$ $\quad \bar{x} = \begin{bmatrix} -2-s-t \\ 1+2s+t \\ s \\ -6-2t \\ t \end{bmatrix} \forall s,t \in \mathbb{R}$

$x_2 = 1 + 2s + t$ $\quad x_4 = -6 - 2t$

5 vars, 3 eqn

iv) $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$ Infinitely many solutions w/ x_3 & x_5 being free (no leading 1)

$x_1 = -s - t$ $\quad x_3 = s \in \mathbb{R}$ $\quad x_5 = t \in \mathbb{R}$

$x_2 = 2s + t$ $\quad x_4 = -2t$

$$x = \begin{bmatrix} -s-t \\ 2s+t \\ s \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} t \quad \forall s,t \in \mathbb{R}$$

Solution Set: $\text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

* Note: In general the minimum number of parameters / free variables is # variables − # equantions = cols − rows

Rank

Def: Rank

Number of leading 1's in the REF (row echelon form) of a matrix. (or the number of non-zero rows)

Examples:

We use the above matrixes, naming them A_{-}

$\text{rank}(\bar{A}_i) = 3$ ← third row 1/8 has leading 1!

$\text{rank}(A_i) = 2$ ← coefficient matrix

$\text{rank}(\bar{A}_{ii}) = 3$

$\text{rank}(\bar{A}_{iii}) = 3$

$\text{rank}(\bar{A}_{iv}) = 3$

Now that we have the concept of rank, we can properly define more properties.

Thm: Consistency & Solutions w/ Rank
Consider linear system w/ coefficient matrix A & augmented matrix $\bar{A}$.

i) The system is consistent iff $\text{rank}(A) = \text{rank}(\bar{A})$.
In particular the system has
unique solution iff $\text{rank}(A) = $ # variables
infinitely many iff $\text{rank}(A) <$ # variables

i.e. you have some number of parameters / free variables.

ii) Inconsistent system iff $\text{rank}(A) < \text{rank}(\bar{A})$

It's impossible for $\text{rank}(A) > \text{rank}(\bar{A})$

Def: Homogeneity of Linear Systems
A linear system w/ all constant coefficients being zero is called homogeneous.
In other words, the matrixes are of the form

$\bar{A} = [A \mid \bar{0}]$ ← prevents rows like $[0\ 0 \ldots 0 \mid \neq 0]$, which cause inconsistency.

All homogeneous systems are consistent. The trivial solution $0, \ldots, 0$ always exists.

We can still have infinitely many solutions when $\text{rank}(A) <$ # variables.
That is, if # rows < # columns of A or you have sufficiently many l.d. rows.

Example:
Given a homogeneous system w/ 4 equations & 6 variables, answer the following questions.

1) Can the system have a unique solution? No
2) How many parameters are there? At least 2.
3) How many parameters are there if one row is a multiple of another? At least 3.
4) How many parameters are there if $\text{rank}(A) = 4$? 2.

Dimension & Basis of Vector Spaces

Example:
Are the vectors $\bar{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix}4\\5\\8\end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix}0\\6\\7\end{bmatrix}$, $\bar{v}_4 = \begin{bmatrix}-3\\2\\8\end{bmatrix}$ l.i. or l.d.?

We rephrase this to try to find a non-trivial l.c. for $\bar{0}$.

$a\bar{v}_1 + b\bar{v}_2 + c\bar{v}_3 + d\bar{v}_4 = \bar{0} \Rightarrow$ homogeneous system of 3 eqn. & 4 var.

The REF has at most rank 3 w/ 4 variables, meaning there is at least 1 free variable & thus infinitely many non-trivial solutions.

Thm: Linear Dependence & Hardcoded Dimensions
Any set of more than n vectors in $\mathbb{R}^n$ are l.d.
Any set of more than $n+1$ vectors in P_n are l.d.
Any set of more than mn vectors in $M_{m \times n}$ are l.d.

Thm: Linear Dependence & Generalized Dimensions
Let $(V, +, \cdot)$ be a vector space. Then
(# vectors in l.i. set) $\leq$ (dimension of V)

Now let's go the other direction, talking about span & dimension.

Example:
Do $\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -5 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 3 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 9 \\ 6 \\ 7 \\ -1 \end{bmatrix}$ span $\mathbb{R}^4$?

In a sense, you cannot go in the 4th direction

We rephrase this to find any vector $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ as a l.c. of those vectors.

$$a\bar{v}_1 + b\bar{v}_2 + c\bar{v}_3 = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow \text{system of 4 eqn. \& 3 var.}$$

The REF can have at most 3 leading ones (1 for each var).
The bottom row will thus be of the form

$$[0\ 0\ 0\ 0 \mid \text{l.c. of } x, y, z, w]$$

This effectively "traps" the values x, y, z, w can be, for example $2x - y + 3z + 8w = 0$. This means not all $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ will lead to solution for $\mathbb{R}^4$, meaning $\bar{v}_1, \bar{v}_2, \bar{v}_3$ do not span $\mathbb{R}^4$.

Thm: Spanning Sets & Hardcoded Dimensions
Any set of less than n vectors in $\mathbb{R}^n$ cannot be a spanning set.
Any set of less than $n+1$ vectors in P_n cannot be a spanning set.
Any set of less than $m \cdot n$ vectors in $M_{m \times n}$ cannot be a spanning set.

Thm: Spanning Sets & Generalized Dimensions
Let $(V, +, \cdot)$ be a vector space. Then
(# vectors in spanning set) $\geq$ (dimension of V)

Def: Bases. (Basises) (bæsīz)
Let $(V, +, \cdot)$ be a vector space.
A set $S \neq \emptyset$ of vectors from V is called a <u>basis for V</u> iff

S spans V &
S is l.i.

Example: Different Bases for $\mathbb{R}^3$

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$ (The Standard Basis)

$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is NOT a basis b/c it doesn't span $\mathbb{R}^3$

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is NOT a basis b/c it is l.d.

$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}2\\2\\2\end{bmatrix},\begin{bmatrix}3\\3\\3\end{bmatrix}\right\}$ is NOT a basis b/c it is l.d. & not a spanning set

Example: Infinite Bases for Infinite Dimension V.S.

$\{x^n : n \geq 0, n \in \mathbb{Z}\}$ is a basis for all polynomials $\mathcal{P}$

Thm: The Earlier Theorems Wrapped w/ Bases

Let $(V, +, \cdot)$ be a v.s. w/ basis S w/ n vectors.

If T is a set of m vectors from V then the following statements hold:

i) If $m < n$ then T is not a spanning set.

ii) If $m > n$ then T is l.d.

iii) If $m = n$ then if T is either a basis or neither l.i. nor a spanning set.

strong statement, will prove later.

Def: Dimension of v.s.

If a v.s. V has a basis w/ n vectors, then any other basis for V has n vectors, & n is the dimension of V $(n = \dim(V))$.

If the basis is is finite, then the v.s. is called finite dimensional.

Example: Dimensions

$\dim(\mathbb{R}^4) = 4$

$\dim(\mathcal{P}_5) = 6$

$\dim(M_{2\times3}) = 2 \cdot 3 = 6$

$\dim(\{ax^2 + bx : a, b \in \mathbb{R}\}) = 2$ (x^2, x are basis)

$\dim\left(\underbrace{\left\{\begin{bmatrix}2a+b\\-2a+3b\\b\end{bmatrix} : a, b \in \mathbb{R}\right\}}\right) = 2$ $\left(\begin{bmatrix}2\\-2\\0\end{bmatrix},\begin{bmatrix}1\\3\\1\end{bmatrix}\right.$ i.e basis$\left.\right)$

because it is $\text{span}\left\{\begin{bmatrix}2\\-2\\0\end{bmatrix},\begin{bmatrix}1\\3\\1\end{bmatrix}\right\}$ & $\left\{\begin{bmatrix}2\\-2\\0\end{bmatrix},\begin{bmatrix}1\\3\\1\end{bmatrix}\right\}$ is l.i.

Thm: Linear Combinations & Bases

Let $(V, +, \cdot)$ be a v.s. & $S = \{\bar{v}_1, \dots, \bar{v}_n\} \subseteq V$.

S is a basis for V

iff

every vector $\bar{v} \in V$ can be written uniquely as a linear combination of $\bar{v}_1, \dots, \bar{v}_n$. That is $\exists!\, a_1, \dots, a_n \in \mathbb{R}$ s.t. $\bar{v} = a_1\bar{v}_1 + \dots + a_n\bar{v}_n$.

Proof: $\Rightarrow$

Assume S is a basis for V.

Let $\bar{v} \in V$ be some arbitrary vector.

Since S is a basis, $\exists a_1, \dots, a_n \in \mathbb{R}$ s.t. $\bar{v} = a_1\bar{v}_1 + \dots + a_n\bar{v}_n$ ← uses span

& $\exists b_1, \dots, b_n \in \mathbb{R}$ s.t. $\bar{v} = b_1\bar{v}_1 + \dots + b_n\bar{v}_n$.

We show $a_1, \dots, a_n = b_1, \dots, b_n$ (using linear independence).

To do this, we find $\bar{v} - \bar{v} = \bar{0}$ & manipulate it.

$\bar{v} - \bar{v} = \bar{0}$

$a_1\bar{v}_1 + \dots + a_n\bar{v}_n - (b_1\bar{v}_1 + \dots + b_n\bar{v}_n) = \bar{0}$

$(a_1 - b_1)\bar{v}_1 + \dots + (a_n - b_n)\bar{v}_n = \bar{0}$.

Since S is linearly independent, $a_1 - b_1, \dots, a_n - b_n = 0, \dots, 0$.

Thus $a_1, \dots, a_n = b_1, \dots, b_n$.

Therefore there exists only one unique linear combination of S for $\bar{v}$. □

Proof: $\Leftarrow$

Assume every vector $\bar{v} \in V$ can be written uniquely as a linear combination of $S = \{\bar{v}_1, \dots, \bar{v}_n\}$. We show S is a basis for V.

Trivially, $\exists a_1, \dots, a_n$ s.t. $\bar{v} = a_1\bar{v}_1 + \dots + a_n\bar{v}_n$. Therefore $\text{span}(S) = V$. (i)

Obviously, $0\bar{v}_1 + \dots + 0\bar{v}_n = \bar{0}$. Since from assumption of uniqueness, this is the only unique solution. Thus there are no non-trivial solutions to $a_1\bar{v}_1 + \dots + a_n\bar{v}_n = \bar{0}$, so S is linearly independent. (ii)

By (i) & (ii), S is a basis for V. □

Reducing l.d. Spanning Sets to Bases

If we have a l.d. spanning set, we know that we can remove one or more vectors to make it l.i. & also still span the v.s. (i.e. it is a basis). How do we know which ones to remove?

First we'll do an example & then a vector.

<u>Example:</u>

Give the l.d. spanning set $S=\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}-1\\1\\-2\end{bmatrix}\right\}$.

(l.d. b/c $|S|=5>3=\dim(\mathbb{R}^3)$.)

We solve $c_1\bar{v}_1+c_2\bar{v}_2+c_3\bar{v}_3+c_4\bar{v}_4+c_5\bar{v}_5=\bar{0}$ to find the problem vectors.

$$\left[\begin{array}{ccccc|c}1&0&1&1&-1&0\\0&1&1&2&1&0\\1&1&2&1&-2&0\end{array}\right] R_3-R_1 \quad \left[\begin{array}{ccccc|c}1&0&1&1&-1&0\\0&1&1&2&1&0\\0&1&1&0&-1&0\end{array}\right] R_3-R_2 \quad \left[\begin{array}{ccccc|c}1&0&1&1&-1&0\\0&1&1&2&1&0\\0&0&0&-2&-2&0\end{array}\right] R_3\cdot -\tfrac{1}{2}$$

$$\left[\begin{array}{ccccc|c}1&0&1&1&-1&0\\0&1&1&2&1&0\\0&0&0&1&1&0\end{array}\right].$$

As you can see, we have 2 parameters $c_3=s$ & $c_5=t$. This gives us infinitely many solutions.

To get only one solution, we eliminate the original vectors that led to the free variables. Thus acheiving l.i. (You can just cross out their columns.)

Therefore $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}\subseteq S$ is l.i.

How do we know it still has the same span? Since it is l.d. we can write the eliminated vectors as a l.c. of the others. Subbing those l.c.'s in for the eliminated vector, we get the same expression only in terms of the final vectors.

<u>Thm:</u>

Every spanning set of a vector space V can be reduced to a basis.

Given a set of vectors $\bar{v}_1,\dots,\bar{v}_k$, you reduce them to a basis by finding the REF of the system of equations produced by $a_1\bar{v}_1+\dots+a_k\bar{v}_k=\bar{0}$ & eliminating the vectors associated w/ the columns w/ no leading one.

<u>Thm:</u> Lazy Proving Bases

Let V be a v.s. w/ $\dim(V)=n$.

Let $S\subseteq V$.

If S is l.i. & $|S|=n$, then S is a basis.

If S is a spanning set & $|S|=n$, then S is a basis.

H Extending l.i. Set of Vectors to Basis

Sometimes we have some vectors that we really want as a basis, but it doesn't span, how do we expand it?

Thm

Every l.i. set of vectors in a finite dimensional v.s. V can be extended to a basis for V.

To do this, you add any known basis (e.g. the standard basis) to the given l.i. set of vectors & then reduce the spanning set to a basis.

Example:

Given $\begin{bmatrix}2\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}$ in $\mathbb{R}^3$, generate a basis.

These are not guaranteed to be unique by any means.

We add the standard basis, getting

$$\begin{bmatrix}2\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

& solve $a_1\bar{v}_1 + \dots + a_5\bar{v}_5 = \bar{0}$ to reduce the l.d. spanning set.

$$\left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Reduced row-echelon form (0s above & below leading 1s) is unique, but non-reduced is not.

This gives us the spanning set

$$\left\{\begin{bmatrix}2\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}.$$

Example:

Find a basis for $V = \{p(x) \mid p(0) = 0, p \in \mathcal{P}_2\}$.

We rewrite V as $V = \{ax^2 + bx \mid a, b \in \mathbb{R}\}$. (since $p(0) = 0 \Rightarrow c = 0$)

It is clear to see that $\{x^2, x\}$ is the basis b/c $\operatorname{span}\{x^2, x\} = V$ & $\{x^2, x\}$ is l.i.

Example:

Find a basis for $V = \{p(x) \mid p(2) = 0, p \in \mathcal{P}_2\}$.

We rewrite V as

$V = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}, 4a + 2b + c = 0\}$.

We can see $c = -4a - 2b$. Therefore

$V = \{ax^2 + bx - 4a - 2b \mid a, b \in \mathbb{R}\}$

is an equivalent construction.

This gives us
$V = \{ a(x^2-4) + b(x-2) \mid a, b \in \mathbb{R} \}$.
It is now clear to see $\{x^2-4, x-2\}$ is a basis for V b/c
$\text{span}\{x^2-4, x-2\} = V$ &
$\{x^2-4, x-2\}$ is linearly independent.

Matrixes

Def: Matrix
An $m \times n$ matrix $(m, n \in \mathbb{N})$ is an array of objects arranged in <u>m rows</u> & <u>n cols</u>.

Things in the matrix are called <u>entries</u>.

Notation: Matrix
Matrixes are denoted w/ uppercase letters (e.g. A, B, M, Q).

To get entry at row $1 \le i \le m$ & col $1 \le j \le n$, you notate it w/ a_{ij}
(where the letter is the lowercase of the matrix).

Def: Matrix & Vectors
The rows can be broken into row vectors & columns into column vectors.
We say $\mathbb{R}^n$ is a $n \times 1$ matrix.

Def: Matrix Addition
Let $A, B, C \in M_{m \times n}$.
$C = A + B$
iff
$c_{ij} = a_{ij} + b_{ij}$ for $1 \le i \le m$ & $1 \le j \le n$.

Def: Scalar Multiplication
Let $A, B \in M_{m \times n}$ & $\alpha \in \mathbb{R}$.
$B = \alpha A$
iff
$b_{ij} = \alpha a_{ij}$ for $1 \le i \le m$ & $1 \le j \le n$.

Def: Matrix Equality
Let $M_1 = M_{m_1 \times n_1}$ & $M_2 = M_{m_2 \times n_2}$.
$M_1 = M_2$
iff
$m_1 = m_2$,
$n_1 = n_2$, &
$a_{ij} = b_{ij}$ for $1 \le i \le m$ & $1 \le j \le n$.

Matrix multiplication is a generalization of the dot product.

Def: Matrix Multiplication.
Let $A \in M_{m \times n}$ & $B \in M_{n \times p}$ & $C \in M_{m \times p}$.
$C = A \cdot B$
iff
vector dot product.
$c_{ij} = \underbrace{\text{row}(i, A) \cdot \text{col}(j, B)}_{\text{These must be the same length, which is why they share } n.}$, where $1 \le i \le m$ & $1 \le j \le p$.

We normally elide the dot.

Example: Matrix Multiplication

Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \end{bmatrix} \in M_{2\times 3}$$

$$B = \begin{bmatrix} -1 & 2 & 4 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & 0 & 1 & 3 \end{bmatrix} \in M_{3\times 4}.$$

$$A \cdot B = \begin{bmatrix} -1 & 4 & 2 & 1 \\ 3 & 5 & 11 & -17 \end{bmatrix}.$$

$B \cdot A$ is undefined.

Example: More Matrix Multiplication

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in M_{2\times 2}$$

$$B = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} \in M_{2\times 2}.$$

$$A \cdot B = \begin{bmatrix} 6 & 3 \\ 12 & 5 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} -3 & -4 \\ 9 & 14 \end{bmatrix}$$

Not commutative! (Should also know from size constraint)

Def: Properties

Let A, B, & C be matrices w/ appropriate sizes.

- $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
- $\alpha(A \cdot B) = \alpha A \cdot B = A \cdot \alpha B$
- $A \cdot (B + C) = A \cdot B + A \cdot C$
- $(A + B) \cdot C = A \cdot C + B \cdot C$
- $A \cdot B \neq B \cdot A$ ← not guaranteed to be equal

- $A + B = B + A$

Example: Who does Commute?

Find all 2×2 matrices that commuted w/ $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

In other words, find

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$AB = BA.$$

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

$$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & b \\ c+d & d \end{bmatrix}$$

Since we want $AB = BA$, we get

$a = a+b \Rightarrow \underline{b=0}$

$b = b$

$c+d = a+c \Rightarrow \underline{a=d}$

$b+d = d \Rightarrow$

This means this set commutes w/ A

$$\left\{ \begin{bmatrix} a & 0 \\ b & a \end{bmatrix} \ \forall a, c \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

__Def:__ Transpose of Matrix

Let $A \in M_{m \times n}$ & $B \in M_{n \times m}$.

$B = A^T$

iff

$b_{ji} = a_{ij}$ for $1 \le i \le m$ & $1 \le j \le n$.

Basically, flip the colums & rows

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}.$$

__Example:__ Matrices w/ Linear Equations

$$\begin{cases} x + 2y + 3z = 8 \\ -x \quad\quad + 4z = 12 \\ 2x - y + z = 0 \end{cases} \Rightarrow \begin{matrix} (-1, 2, 3) \cdot (x, y, z) = 8 \\ (-1, 0, 4) \cdot (x, y, z) = 12 \\ (2, -1, 1) \cdot (x, y, z) = 0 \end{matrix}$$

Notice that we can rewrite this using matrix multiplication.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 2 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\bar{v}} = \underbrace{\begin{bmatrix} 8 \\ 12 \\ 0 \end{bmatrix}}_{\bar{b}} \Rightarrow A \cdot \bar{v} = \bar{b}$$

We call this the matrix equation of the linear system.

Now that we have this, we can start defining an algebra for solving linear systems (linear algebra!) w/ matrices.

__Def:__ Identity matrix

The $n \times n$ identity matrix I_n is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>Thm:</u> Inverse & Products
Let $A_1, \dots, A_k$ be invertible matrices.

$$(A_1 \cdot \dots \cdot A_k)^{-1} = A_k^{-1} \cdot \dots \cdot A_k^{-1}.$$

Note: If A & B are not invertible, $A+B$ may or may not be invertible.

<u>Thm:</u> Solutions to Linear Systems & Invertibility
Let $A \in M_{n \times n}$ be an invertible matrix.
For all vectors $\bar{b} \in \mathbb{R}^n$, $A\bar{x} = \bar{b}$ has a unique solution where: $\bar{x} = A^{-1}\bar{b}$

Further, if A is <u>not</u> an invertible matrix, $[A|\bar{b}]$ either has infinitely many or no solutions.

<u>Example:</u> Solving Systems w/ Inverse of Matrix
Solve
$$\begin{cases} x_1 + x_2 - 2x_3 = 5 \\ -x_1 + 2x_2 = 3 \\ -x_2 + x_3 = -2 \end{cases}.$$

$$\underbrace{\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}}_{\bar{b}}$$

We find A^{-1} (if it exists) to find $\bar{x}$ by $\bar{x} = A^{-1}\bar{b}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R_2 + R_1 \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R_2 + 2R_3 \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_1 - R_2 \\ \\ R_3 + R_1 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] R_1 + 2R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$
$$\qquad\qquad I \qquad\qquad A^{-1}$$

Now we isolate $\bar{x}$.

$A\bar{x} = \bar{b}$
$A^{-1}A\bar{x} = A^{-1}\bar{b}$
$\bar{x} = A^{-1}\bar{b}$

$$\bar{x} = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

Example: Span w/ Inverse of Matrix

Is $\text{span}\left\{\begin{bmatrix}1\\3\\3\end{bmatrix}, \begin{bmatrix}-1\\-3\\-3\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}\right\} = \mathbb{R}^3$?

Only square matrixes are invertible, so whenever we talk about inverse existing, we're talking about square matrixes.

That is, $\forall \vec{v} \in \mathbb{R}^3$, is there a l.c. of $\vec{v}$?

That is, can $\underbrace{\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be solved?

In other words if ...

A is invertible: Yes, any system can be solved, $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$.

A is not invertible: No

Remark: Let $A \in M_{n\times n}$

A is not invertible $\overset{\Leftrightarrow}{\text{iff}}$ rows are l.d. $\overset{\Leftrightarrow}{\text{iff}}$ cols are l.d.

A is invertible iff rows/cols are l.i.

Thm:

Let $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ & $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$. $A \in M_{n\times n}$

$\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ $\overset{\Leftrightarrow}{\text{iff}}$ A is invertible $\overset{\Leftrightarrow}{\text{iff}}$ the rows form a basis, iff the cols form a basis

Thm: Inverse of Inverse

If $A \in M_{n\times n}$ is invertible, A^{-1} is also invertible, where

$(A^{-1})^{-1} = A$

Helps us show only square matrixes are invertible

Example:

Find a basis for P_2. Using

Using columns of earlier

$\{x^2 - x, x^2 + 2x - 1, -2x^2 + 1\}$.

Find basis for $\left\{\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R}\right\} = S$.

Using columns of earlier example's inverse

$\left\{\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}\right\}$

Find basis for $\left\{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R}\right\}$

Using rows from earlier example

$\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$

Notice that $\underset{m \times n}{A} \times I_n = A$ & $I_n \times \underset{n \times p}{B} = B$. This is b/c the zeros "kill" all the entries aren't the one you want.

Def: Matrix Equation of Linear System

If a linear system can be written as

$[\bar{A} \,\vdots\, \bar{b}]$

then you can write it as a matrix product

$\bar{A} \cdot \bar{x} = \bar{b}$

where

$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ variables.

Why do we care about the matrix equation? It can make solving linear systems way easier. For example

$\bar{A} \cdot \bar{x} = \bar{b}$

$\bar{A}^{-1} \cdot (\bar{A} \cdot \bar{x}) = \bar{A}^{-1} \cdot \bar{b}$

$(\bar{A}^{-1} \bar{A}) \cdot \bar{x} = \bar{A}^{-1} \cdot \bar{b}$

$\bar{I} \cdot \bar{x} = \bar{A}^{-1} \cdot \bar{b}$

$\bar{x} = \bar{A}^{-1} \cdot \bar{b}$.

How do we find $\bar{A}^{-1}$ such that $\bar{A}^{-1} \cdot \bar{A} = \bar{I}$

Def: Invertible Matrixes

Let $A \in M_{n \times n}$.

Iff $\exists B \in M_{n \times n}$ s.t. $A \cdot B = B \cdot A = I_n$, we say A is invertible & $B = A^{-1}$ (& $A = B^{-1}$ similarly).

Thm: Uniqueness of Inverse

Let $A \in M_{n \times n}$. If there exists an inverse, it is <u>unique</u>.

Proof:

Assume $A \in M_{n \times n}$ has 2 inverses, B & C. Then

$AB = BA = I$ & $AC = CA = I$

$B = BI = B(AC) = (BA)C = IC = C$ $\square$

How do we find the inverse? We basically just solve

$A \cdot A^{-1} = I$.

Suppose $A, A^{-1} \in \mu_{3\times3}$ where $A^{-1} = \begin{bmatrix} b_{11} & \cdots \\ b_{12} & \cdots \\ b_{13} & \cdots \end{bmatrix}$

To find $A \cdot A^{-1} = I$, we do

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} b_{11} & \cdots \\ b_{12} & \cdots \\ b_{13} & \cdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which breaks down to

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \end{array}\right] \xrightarrow{REF} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix}$$

$$\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \end{array}\right] \xrightarrow{REF} \begin{bmatrix} b_{21} \\ b_{22} \\ b_{23} \end{bmatrix}$$

$$\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \end{array}\right] \xrightarrow{REF} \begin{bmatrix} b_{31} \\ b_{32} \\ b_{33} \end{bmatrix}$$

As you can see, we repeat a lot of work, so we can do it simultaneously.

Example:

Let $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Find A^{-1}. We do $[A|I]$ & get reduced REF on the right hand side.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_2 + R_1 \\ R_3 \cdot -1 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \end{array}\right] R_2 \Leftrightarrow R_3 \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 3 & -2 & 1 & 1 & 0 \end{array}\right] R_3 - 3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \begin{matrix} R_1 + 2R_3 \\ R_2 + R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 2 & 6 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] R_1 - R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$

You should check

Now, if you look only at the 1st column on the right, that is the first column of A^{-1}. Likewise for the other columns.

In other words, we have transformed $[A|I] \to [I|A^{-1}]$. This means A is invertible.

Similarly, if $[A|I] \Rightarrow [\neq I|\ldots]$, then A is not invertible.

Questions:

Let A, B be invertible. Is $A+B$ always invertible? !

No. Suppose $A = -B$, then you get $\mathbf{0}$, which is not invertible.

Let A, B be invertible. Is $A \cdot B$ always invertible?

Yes; let $(AB)^{-1} = B^{-1}A^{-1}$

$(AB)(AB)^{-1} = (AB)(B^{-1}A^{-1}) = A(B^{-1}B)A^{-1} = A(IA^{-1}) = AA^{-1} = I$. $\square$

Def: Column & Null Space

Let $A \in M_{m \times n}$.

The null space, denoted by null(A),
$\text{null}(A) = \{ \bar{x} \in \mathbb{R}^n \mid A\bar{x} = \bar{0} \}$.

> Matrixes are basically transforms from rows to columns.

also output space or codomain/range of $A: \mathbb{R}^n \to \mathbb{R}^m$

The column space of A, denoted by col(A), is
$\text{col}(A) = \text{span}\{\text{column vectors of } A\}$.

> Null space is either the single $\bar{0}$ vector or a line, plane, etc. intersecting $\bar{0}$

Thm:

Let $A \in M_{m \times n}$.

null(A) is a subspace of $\mathbb{R}^n$.

col(A) is a subspace of $\mathbb{R}^m$ (as it is a linear span of m-length vectors).

Proof: For null(A)

Let $\bar{x}, \bar{y} \in \text{null}(A)$ (i.e. $A\bar{x} = \bar{0}$ & $A\bar{y} = \bar{0}$).
$A(\bar{x} + \bar{y}) = A\bar{x} + B\bar{y} = \bar{0} + \bar{0} = \bar{0}$.

Let $\alpha \in R$
$A(\alpha \bar{x}) = \alpha(A\bar{x}) = \alpha \bar{0} = \bar{0}$.

> Proof for col(A) is trivial b/c it is the span of vectors

Thm:

$A\bar{x} = \bar{b}$ is consistent iff $\bar{b} \in \text{col}(A)$.

Proof:

Let $A \in M_{m \times n}$ where $A = \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix}$.

Let $\bar{x} \in M_{n \times 1} = \mathbb{R}^n$ where $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

By the definition of matrix multiplication

$$A\bar{x} = \begin{bmatrix} x_1 v_{11} + \cdots + x_n v_{n1} \\ x_1 v_{12} + \cdots + x_n v_{n2} \\ \vdots \quad \vdots \quad \vdots \\ x_1 v_{1m} + \cdots + x_n v_{nm} \end{bmatrix} = x_1 \bar{v}_1 + \cdots + x_n \bar{v}_n.$$

Recall $col(A) = span\{\bar{v}_1, \ldots, \bar{v}_n\} = a_1\bar{v}_1 + \ldots + a_n\bar{v}_n \quad a_1, \ldots, a_n \in \mathbb{R}$.

Since $A\bar{x} = x_1\bar{v}_1 + \ldots + x_n\bar{v}_n$ & $x_1, \ldots, x_n \in \mathbb{R}$, $A\bar{x} \in col(A)$.

If $A\bar{x} = \bar{b}$ is consistent (i.e. correct), then $A\bar{x} = \bar{b} \in col(A)$. $\square$

For the reverse direction, trivially if $\bar{b} \in col(A)$, then $A\bar{x} = \bar{b}$ is consistent where $\bar{x} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. $\square$

Extra Section for My Understanding

<u>Thm:</u> No Non-square Inverses
Let $A \in M_{m \times n}$ where $m \neq n$. A is not invertible.

<u>Proof:</u>
To prove this, we split the problem into 2 cases.

Case $m < n$:
Interpret A as a linear transformation $A: \mathbb{R}^m \to \mathbb{R}^n$.

Since the dimension of the domain ($\dim(\mathbb{R}^m) = m$) is less than the dimension of the codomain ($\dim(\mathbb{R}^n) = n$), A is not surjective (onto).

Since A must be surjective (onto) & injective (one-to-one) (i.e. bijective), for its inverse A^{-1} to exist, <u>A is not invertible.</u>

Case $m > n$:
Suppose for contradiction that $A \in M_{m \times n}$ exists. Therefore $A^{-1} \in M_{n \times m}$ exists.

By our earlier theorem, all inverses A^{-1} are invertible.
However, we have already shown that matrixes $B \in M_{m' \times n'}$ where $m' < n'$ are not invertible.

We have thus reached a contradiction, therefore <u>A is not invertible.</u>

Why is A^{-1} a $n \times m$ matrix? If you interpret A as a linear transformation, the $A: \mathbb{R}^m \to \mathbb{R}^n$. Therefore its inverses have the domain & codomain swapped, so $A^{-1}: \mathbb{R}^n \to \mathbb{R}^m$. Reinterpreting this as a matrix gives you $A^{-1} \in M_{n \times m}$.

To find a basis for null(A), you find the solution set to $A\bar{x} = \bar{0}$ & create a basis for it. This should be easy b/c getting a solution set gets you an REF w/ free variables, meaning you'll have something like

$$\bar{x} = \begin{bmatrix} -s \\ 2/35 \\ s \\ 0 \end{bmatrix} \text{ for } s \in \mathbb{R}.$$

To find a basis for col(A), you just reduce the column vectors to a basis.

Def: Nullity

Let $A \in M_{n \times m}$. We call the dimension of the null space the nullity of A.

Def: Rank w/ Column Space

Let $A \in M_{n \times m}$. We call the dimension of the column space the rank of A.

Properties:

Let $A \in M_{m \times n}$.

$$\left.\begin{array}{l} n = \text{rank}(A) + \text{nullity}(A) \\ \text{rank}(A) \le n \\ \text{nullity}(A) \le n \\ \text{rank}(A) < n \Rightarrow \text{nullity}(A) > 0 \\ \text{nullity}(A) < n \Rightarrow \text{rank}(A) > 0 \end{array}\right\} \text{All kind of saying the same thing.}$$

Thm: Rank Theorem ← Wrapping up the above

Let $A \in M_{m \times n}$.

i) $\text{rank}(A) = \dim(\text{col}(A)) = \#$ leading ones in REF

ii) $\dim(\text{col}(A)) + \dim(\text{null}(A)) = n$

or

$\text{rank}(A) + \text{nullity}(A) = n$

Remarks:

Let $A \in M_{m \times n}$.

i) $\text{null}(A) \subseteq \mathbb{R}^n \Rightarrow \text{nullity}(A) \le n$

ii) $\text{col}(A) \subseteq \mathbb{R}^m \Rightarrow \text{rank}(A) \le m$

By the rank theorem, $\text{rank}(A) \le n$, therefore $\text{rank}(A) \le \min(m, n)$.

Linear Transformations

Def: Linear Transformation

A linear map or linear transformation (l.t.) from a v.s. V to v.s. W is a function
$$T: V \to W$$
that preserves the linearity of V & W.

Linearity of a transformation means

V & W must have same field of scalars.

i) $\forall \bar{u}, \bar{v} \in V$, $\quad T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$ ← Additivity

ii) $\forall \bar{u} \in V, \lambda \in \mathbb{R}$ $\quad T(\lambda \bar{u}) = \lambda T(\bar{u})$ ← Homogeneity

Really V's scalars $\subseteq$ W's scalars

Def: Linear Operation

If $T: V \to V$ is a linear transformation, we call it a linear operator.

Example:

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by
$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}.$$
Show T is a linear operator.

Image = output of function

We show T preserves additivity.

Let $\bar{u}_1, \bar{u}_2 \in dom(T) = \mathbb{R}^2$ be $\bar{u}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ & $\bar{u}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$,

$$T(\bar{u}_1 + \bar{u}_2) = T\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} (x_1+x_2) + (y_1+y_2) \\ (x_1+x_2) - (y_1+y_2) \end{bmatrix}$$

$$T(\bar{u}_1) + T(\bar{u}_2) = \begin{bmatrix} x_1 + y_1 \\ x_1 - y_1 \end{bmatrix} + \begin{bmatrix} x_2 + y_2 \\ x_2 - y_2 \end{bmatrix} = \begin{bmatrix} (x_1+x_2) + (y_1+y_2) \\ (x_1+x_2) - (y_1-y_2) \end{bmatrix}$$

Since $T(\bar{u}_1 + \bar{u}_2) = T(\bar{u}_1) + T(\bar{u}_2)$, addivity is preserved.

We show T preserves homogeneity

Let $\bar{u} \in dom(T) = \mathbb{R}^2$ & $\lambda \in \mathbb{R}$ where

$$T(\lambda \bar{u}) = T\begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} = \begin{bmatrix} \lambda x + \lambda y \\ \lambda x - \lambda y \end{bmatrix}$$

$$\lambda T(\bar{u}) = \lambda \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} \lambda x + \lambda y \\ \lambda x - \lambda y \end{bmatrix}$$

Since $T(\lambda \bar{u}) = \lambda T(\bar{u})$, homogeneity is preserved.

Since $dom(T) = cod(T)$ & homogeneity & additivity is preserved, T is a linear operator.

Here is a note about linear transformations. All vectors $\bar{u} \in V$ can be written as a l.c. of a basis (i.e. $\bar{u} = a_1\bar{e}_1 + \dots + a_n\bar{e}_n$). Thus, we can describe the linear transformation of a vector as how the basis is transformed.

Let V be a v.s., $\bar{u} \in V$, & $\bar{e}_1, \dots, \bar{e}_n$ be a basis for V.
We know $\bar{u} = a_1\bar{e}_1 + \dots + a_n\bar{e}_n$ for $a_1, \dots, a_n \in \mathbb{R}$.
Let $T: V \to W$ be a linear transformation.
$T(\bar{u}) = T(a_1\bar{e}_1 + \dots + a_n\bar{e}_n) = a_1 T(\bar{e}_1) + \dots + a_n T(\bar{e}_n)$.

Example:

Let $T: V \to W$ $T(\bar{v}) = \bar{0}$.
T is a (trivial) linear transformation.

Example:

Let $T: \mathcal{P}_3 \to \mathcal{P}_2$ be $T(f) = f'$.

The sum of linear transformations is a linear transformation.

T is a linear transformation b/c

$T(f+g) = (f+g)' = f' + g'$
$T(f) + T(g) = f' + g'$

$T(af) = (af)' = af'$
$aT(f) = af'$

T is a linear transformation.

Example:

Let $T: \mathcal{P} \to \mathcal{P}$ be $T(f) = \int f$. (Indefinite integral)

T is not a linear transformation b/c it is not a function.
Given a single input, there is an infinite family of outputs.

We can "fix it" by bounding one side $\int_0^x f(t)\,dt$ to do the definite integral. That way we don't get $+C$.

Example:

i) Is $T: \mathbb{R}^2 \to \mathbb{R}$ $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + y^2$ a linear transformation?
No. $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 4$ & $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2$

ii) Is $T: \mathcal{P}_n \to \mathcal{P}_n$ $T(p(x)) = p(x) + x$ a l.t.?
No. $2T(0) = 2x$. $T(2(0)) = T(0) = x$.

iii) Is $T: \mathcal{P}_3 \to \mathcal{P}_3$ $T(p(x)) = 2p''(x) - 3p'(x) + p(x)$

This is called a differential operator b/c it's a function that returns diff eqs.

Yes.
$T(ap(x)) = 2ap''(x) - 3ap'(x) + p(x) = aT(p(x))$
$T(p(x) + g(x)) = 2p''(x) + 2g''(x) - 3p'(x) - 3g'(x) + p(x) + g(x) =$
$= T(p(x)) + T(g(x))$

Not super clean. On test be more explicit w/ $(p(x) + g(x))' = p'(x) + g'(x)$

iv) Is $T: \mathbb{R}^3 \to \mathbb{R}^3$ $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \cos(x) \\ \sin(y) \\ \sin(x)+\sin(z) \end{bmatrix}$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos(0) \\ \sin(0) \\ \sin(0)+\sin(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

v) Is $T: \mathbb{R}^3 \to \mathbb{R}^2$ $T(ax^3+bx^2+cx+d) = \begin{bmatrix} -a-b+1 \\ c+d \end{bmatrix}$

No.

$$2T(0) = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
$$T(2(0)) = T(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$\neq$

Example:

Create a function $T: \mathcal{P}_2 \to M_{2\times 2}$ s.t. T is a l.t. & another where T is not a l.t.

i) T is l.t. $T(ax^2+bx+c) = \begin{bmatrix} a+c & b \\ b & a+c \end{bmatrix}$

ii) T is not l.t. $T(ax^2+bx+c) = \begin{bmatrix} a+c & b^2 \\ b^2 & a+c \end{bmatrix}$

A cooler linear transform is
$T(p(x)) = \begin{bmatrix} p(0) & p(1) \\ p(2) & p(3) \end{bmatrix}$

A cooler not linear transform is
$T(ax^2+bx+c) = \begin{bmatrix} e^a & \ln(b) \\ c^2 & 1 \end{bmatrix}$

Let V & W be v.s.
Let $S, T \in \mathcal{L}(V, W)$ & λ be a scalar
$(S+T)(\bar{v}) = S(\bar{v}) + T(\bar{v})$
$(\lambda T)(\bar{v}) = \lambda T(\bar{v})$

I won't exhaustively prove this b/c I intuitively see it & it's a lot of writing I don't have room for.

Since linear transformations are closed under addition & scalar multiplication, & a subset of the vector space of all functions they are vector spaces!

Thm: Linear Transformations are Vector Spaces
Let V & W be vector spaces.
The set of all linear transformations from V to W is a vector space (& subspace of functions from V to W).
We notate this set w/
$\mathcal{L}(V, W) = \{T \mid T: V \to W;\ T \text{ is linear transform}\}$.

Property:
Let $T: V \to W$ be a l.t. $T(\bar{0}) = \bar{0}$.

This is a quick, easy check for linear transformations.

Proof:
$T(\bar{0}) = T(\bar{0}+\bar{0}) = T(\bar{0}) + T(\bar{0})$
$\therefore T(\bar{0}) = \bar{0}$.

It further follows that $0\,T(\bar{x}) = \bar{0}$ b/c

$0\,T(\bar{x}) = T(0 \cdot \bar{x}) = T(\bar{0}) = \bar{0}$

Thm: Vector Spaces of Same Dimension are Connected

Let V, W be v.s.

Let $\{\bar{v}_1, \ldots, \bar{v}_n\}$ be a basis for V.

Let $\{\bar{w}_1, \ldots, \bar{w}_n\} \subseteq W$. ← doesn't have to be a basis!

There exists a <u>unique linear map</u> $T: V \to W$ such that $T(\bar{v}_i) = \bar{w}_i$ for $i = 1, \ldots n$

This is going to be a long proof, so let's break it up into segments w/ examples.

Proof:

Part 1: Existence of T

If $\bar{v} \in V$, how do we define $T(\bar{v})$, given $T(v_i) = w_i$?

Example:

$$T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\-3\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix} + (-3)\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + (-3)T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = x\,T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + y\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

Notice that

$\bar{v} = c_1\bar{v}_1 + \ldots + c_n\bar{v}_n$ for $c_i \in \mathbb{R}$

This also preserves the property $T(\bar{v}_i) = \bar{w}_i$.

We thus define

$T(\bar{v}) = c_1 T(\bar{v}_1) + \ldots + c_n T(\bar{v}_n) = c_1\bar{w}_1 + \ldots + c_n\bar{w}_n$ where $\bar{v} = c_1\bar{v}_1 + \ldots + c_n\bar{v}_n$.

Part 2: Linearity of T & Homogeneity of T

Let $\bar{v}, \bar{u} \in V$.

$\bar{v} + \bar{u}$

$= c_1\bar{v}_1 + \ldots + c_n\bar{v}_n + d_1\bar{v}_1 + \ldots + d_n\bar{v}_n$

$= (c_1 + d_1)\bar{v}_1 + \ldots + (c_n + d_n)\bar{v}_n$

$T(\bar{v} + \bar{u}) = (c_1 + d_1)\bar{w}_1 + \ldots + (c_n + d_n)\bar{w}_n = c_1\bar{w}_1 + \ldots + c_n\bar{w}_n +$

$= c_1\bar{w}_1 + \ldots + c_n\bar{w}_n + d_1\bar{w}_1 + \ldots + d_n\bar{w}_n = T(\bar{v}) + T(\bar{u})$.

$\therefore$ T preserves linearity.

Likewise for homogeneity

Part 3: Uniqueness

We suppose there two such linear transformations T & $\tilde{T}$.

Let $\bar{v} \in V$. We show $T(\bar{v}) = \tilde{T}(\bar{v})$.

$T(\bar{v}) = T(c_1\bar{v}_1 + \dots + c_n\bar{v}_n) = c_1\bar{w}_1 + \dots + c_n\bar{w}_n = \tilde{T}(c_1\bar{v}_1 + \dots + c_n\bar{v}_n) = \tilde{T}(\bar{v})$.

This works b/c $T(\bar{v}_i) = \bar{w}_i$ by the earlier criteria.

Example: $T: V \to W$, where $\{w_1, \dots, w_n\}$ is not a basis

$T: P_2 \to P_2$. $T(1) = 1$, $T(x) = 2$, $T(x^2) = 3$.

Example: Deriving T

Given $T: P_2 \to P_2$ where $T(1) = 1+x$, $T(x) = 2+x^2$, $T(x^2) = x - 3x^2$

$T(-3 + x - x^2) = -3T(1) + T(x) - T(x^2) = -3 - 3x + 2 + x^2 - x + 3x^2 = -1 - 4x + 4x^2$

$T(a + bx + cx^2) = a + ax + 2b + bx^2 + cx - 3cx^2 = (a+2b) + (a+c)x + (b-3c)x^2$

Example:

Given $T: M_{2\times2} \to M_{2\times2}$ where

$$T\begin{bmatrix}1&0\\0&0\end{bmatrix} = \begin{bmatrix}0&1\\0&0\end{bmatrix} \quad T\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}1&0\\0&-1\end{bmatrix} \quad T\begin{bmatrix}0&0\\1&0\end{bmatrix} = \begin{bmatrix}1&1\\0&0\end{bmatrix} \quad T\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\2&0\end{bmatrix}$$

$$T\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}0&a\\0&0\end{bmatrix} + \begin{bmatrix}b&0\\0&-b\end{bmatrix} + \begin{bmatrix}c&c\\0&0\end{bmatrix} + \begin{bmatrix}0&0\\2d&0\end{bmatrix}$$

$$= \begin{bmatrix}b+c & a+c\\ 2d & -b\end{bmatrix}$$

Example:

Given $T: \mathbb{R}^3 \to \mathbb{R}^2$ where $T(\bar{e}_1) = \begin{bmatrix}1\\1\end{bmatrix}$, $T(\bar{e}_2) = \begin{bmatrix}-1\\2\end{bmatrix}$, & $T(\bar{e}_3) = \begin{bmatrix}0\\1\end{bmatrix}$

$$T\begin{bmatrix}a\\b\\c\end{bmatrix} = aT(\bar{e}_1) + bT(\bar{e}_2) + cT(\bar{e}_3) = \begin{bmatrix}a-b\\a+2b+c\end{bmatrix}$$

Notice that we can represent T using matrix multiplication. <u>This is universally true for linear transformations.</u>

We can write T as

$$T(\bar{a}) = \begin{bmatrix}1&-1&0\\1&2&1\end{bmatrix}\bar{a}$$

$$\begin{array}{ccc} T(\bar{e}_1) & T(\bar{e}_2) & T(\bar{e}_3) \end{array}$$

Thm: Matrixes $\Rightarrow$ Linear Maps

Let $A \in M_{m\times n}$ matrix. Then $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\bar{v}) = A\bar{v}$ is a linear map.

Proof:

Let $\bar{u}, \bar{v} \in \mathbb{R}^n$ & λ scalar.

$T(\bar{u}+\bar{v}) = A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v} = T(\bar{u}) + T(\bar{v})$

$T(\lambda\bar{u}) = A(\lambda\bar{u}) = \lambda(A\bar{u}) = \lambda T(\bar{u})$

<u>Thm: Linear Maps $\Rightarrow$ Matrixes</u>

For every linear map $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ there exists a unique $m \times n$ matrix A s.t. $T(\bar{v}) = A\bar{v}$.

We won't prove uniqueness, but to create matrix A you make the column vectors of A be $T(e_i)$ where i is the index of the column vector.

<u>Example: Linear Map $\Rightarrow$ Matrix</u>

Given $T: \mathbb{R}^3 \to \mathbb{R}^4$ where $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4x - 3y + z \\ 2x - y + z \\ x - z \\ 2x + 5z \end{bmatrix}$, find a matrix A s.t. $T(\bar{v}) = A\bar{v}$

Let $A = \begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) & T(\bar{e}_3) \\ | & | & | \\ | & | & | \end{bmatrix}$. We just plug in to find the column vectors.

$$T(\bar{v}) = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & 0 & 5 \end{bmatrix} \bar{v}$$

We will now show that linear maps under composition. (We also call this the product of linear maps to hint that this is the same as taking the product of the linear maps' corresponding matrixes.)

<u>Def: Product of Linear Maps</u>

Let $T \in \mathcal{L}(U, V)$ & $S \in \mathcal{L}(V, W)$.

The product $ST: U \to W$ defined by
$$(ST)(\bar{u}) = S(T(\bar{u})) = (S \circ T)(\bar{u})$$

<u>Thm: Linear Maps are Closed under Product</u>

$ST \in \mathcal{L}(U, W)$. (ST is a linear map.)

<u>Proof:</u>

$$(ST)(\bar{u} + \bar{v}) = S(T(\bar{u} + \bar{v})) = S(T(\bar{u}) + T(\bar{v})) = S(T(\bar{u})) + S(T(\bar{v})) = ST(\bar{u}) + ST(\bar{v})$$

Similarly $(ST)(\lambda \bar{v}) = \lambda (ST)(\bar{v})$.

Properties of Product:

Associative: $(S_1 S_2) S_3 = S_1 (S_2 S_3)$

Identity: $S \cdot I = I \cdot S = S$

Distributivity: $(S_1 + S_2) T = S_1 T + S_2 T$
$S(T_1 + T_2) = ST_1 + ST_2$

Not Commutative: $S_1 S_2 = S_2 S_1$ not always true

Example: Not Commutative
$T: \mathcal{P} \to \mathcal{P}$, $D: \mathcal{P} \to \mathcal{P}$ where $T(f) = x^2 f$ & $D(f) = f'$
Show $TD \neq DT$

Suppose $f = x^2 - 3x + 2$.

$TD(f) = x^2(2x - 3) = 2x^3 - 3x^2$
$DT(f) = (x^4 - 3x^3 + 2x^2)' = 4x^3 - 9x^2 + 4x$

Example:
Suppose $T \in \mathcal{L}(V, W)$ & $\bar{v}_1, \ldots, \bar{v}_n \in V$ s.t. $T(\bar{v}_1), \ldots, T(\bar{v}_n)$ are l.i. Then $\bar{v}_1, \ldots, \bar{v}_n$ are l.i. Prove!

Suppose for contradiction $\bar{v}_1, \ldots, \bar{v}_n$ are l.d. Then
$a_1 \bar{v}_1 + \ldots + a_n \bar{v}_n = \bar{0}$
for some non-trivial $a_1, \ldots, a_n$.

Apply T to both side
$T(a_1 \bar{v}_1 + \ldots + a_n \bar{v}_n) = T(\bar{0})$ by earlier theorem
$a T(\bar{v}_1) + \ldots + a_n T(\bar{v}_n) = \bar{0}$.

I prefer the construction

Could have done a construction by showing $a_1 = \ldots = a_n = 0$. It goes the exact same way.

This implies $T(\bar{v}_1), \ldots, T(\bar{v}_n)$ are l.d. giving us a contradiction. Thus $\bar{v}_1, \ldots, \bar{v}_n$ are l.i.

Thm: Product of Linear Maps is Product of Matrixes
Let $S \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ & $T \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$.
Then there exists some $A \in M_{m \times n}$ & $B \in M_{n \times p}$ s.t. $S(\bar{u}) = A\bar{u}$ & $T(\bar{u}) = B\bar{v}$.

The product $ST \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^m)$ is defined by a matric $C \in M_{m \times p}$ s.t. $ST(\bar{v}) = C\bar{v}$.
We show $C = AB$

$(ST)(\bar{v}) = S(T(\bar{v})) = S(B\bar{v}) = A(B\bar{v}) = \underbrace{(AB)}_{C}\bar{v}$

We can have linear maps b/w linear maps b/c linear maps are vector spaces!

Example:
Let $B \in M_{n \times n}$.
Let $T: M_{n \times n} \to M_{n \times n}$ where $T(A) = AB - BA$.
Show T is a linear map.

$T(A_1 + A_2) = (A_1 + A_2)B - B(A_1 + A_2) = A_1 B + A_2 B - BA_1 - BA_2$
$= A_1 B - BA_1 + A_2 B - BA_2 = T(A_1) + T(A_2)$

$T(\lambda A) = (\lambda A)B - B(\lambda A) = \lambda(AB) - \lambda(BA) = \lambda(AB - BA) = \lambda T(A)$

Def: Null Space & Range

Consider $T \in \mathcal{L}(V, W)$ where V & W are v.s.

The <u>null space of T</u>, denoted by null(T), is a subset of V defined by

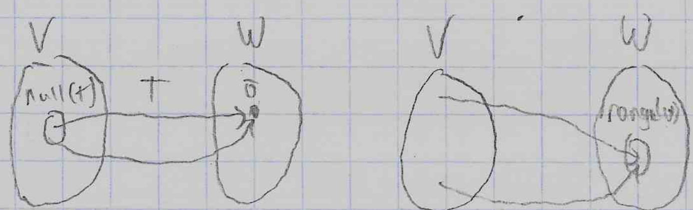
$$\text{null}(T) = \{\bar{v} \in V \mid T(\bar{v}) = \bar{0}\}$$

The <u>range of T</u> is a subset of W defined by

$$\text{range}(T) = \{T(\bar{v}) \mid \bar{v} \in V\}$$

In other words, the null space is the set of input vectors that map to zero in W.

The range is the set of all output vectors.



<u>Example:</u> Trivial

$T: V \to W$, $T(\bar{v}) = \bar{0}$, $\forall \bar{v} \in V$

null$(T) = V$

range$(T) = \{\bar{0}\}$

<u>Example:</u> Plane & Common

$T: \mathbb{R}^3 \to \mathbb{R}$ $\quad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + 2y + 3z$

null$(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + 3z = 0 \right\}$ ← This is a plane!

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \text{null}(T)$; $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in \text{null}(T)$

Also, null$(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x = -2y - 3z \right\} = \left\{ \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} \mid y, z \in \mathbb{R} \right\}$

$= \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

range$(T) = W$ $\quad \left(\forall \bar{w} \in \mathbb{R},\ T\begin{bmatrix} \bar{w} \\ 0 \\ 0 \end{bmatrix} = \bar{w} \in \text{range}(T) \right)$

Example: Differentiation

$D: \mathcal{P} \to \mathcal{P} \quad D(p) = p'$

$\text{null}(D) = \mathcal{P}_0 = \{a \mid a \in \mathbb{R}\} = \{p \in \mathcal{P} \mid \deg(p) = 0\} = \mathbb{R}$

$\text{range}(D) = \mathcal{P}$ ← there is no largest. you can always go up a level

Thm: Null Space & Range are Vector Space

Let V, W be v.s.

Let $T \in \mathcal{L}(V, W)$.

$\text{null}(T)$ is a subspace of V.

$\text{range}(T)$ is a subspace of W.

Proof: Range is a Subspace

We have already show $\text{range}(T) \subseteq W$ by definition, so we must only show closure

Closure under addition:

We claim that $\forall \bar{w}_1, \bar{w}_2 \in \text{range}(T)$, $\bar{w}_1 + \bar{w}_2 \in \text{range}(T)$. In other words $\exists \bar{v}_1, \bar{v}_2 \in V$ s.t. $\bar{w}_1 = T(\bar{v}_1)$ & $\bar{w}_2 = T(\bar{v}_2)$ & $T(\bar{v}_1 + \bar{v}_2) = \bar{w}_1 + \bar{w}_2$.

$\bar{w}_1 + \bar{w}_2 = T(\bar{v}_1) + T(\bar{v}_2) = T(\bar{v}_1 + \bar{v}_2) = T(\bar{v}) \in \text{range}(T)$

Closure under scalar multiplication

Let $\bar{w} \in \text{range}(T)$ & c be a scalar. Is $c\bar{w} \in \text{range}(T)$?

$\bar{w} \in \text{range}(T) \Rightarrow \bar{w} = T(\bar{v})$ for some $\bar{v} \in V$.

We multiply both sides by c

$c\bar{w} = cT(\bar{v}) = T(c\bar{v}) \in \text{range}(T)$

Similarly for $\text{null}(T)$.

Example:

$T: \mathbb{R}^4 \to \mathbb{R}^3 \quad T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix}$

I realized this by noticing "solving" for null(A) resulted in the system of equations

$$\begin{cases} a + b = 0 \\ b - c = 0 \\ a + d = 0 \end{cases}$$

Find a basis for $\text{null}(T)$ & $\dim(\text{null}(T))$.

Describe $\text{range}(T)$

Find a basis for $\text{range}(T)$ & $\dim(\text{range}(T))$.

We first find A s.t. $T(\bar{v}) = A\bar{v}$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} T(e_1) & T(e_2) & T(e_3) & T(e_4) \end{array} \end{array}$$

1) To find $\text{null}(T)$, we solve $A\bar{v} = \bar{0}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \cdot -1]{R_2 + R_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[R_2 \cdot -1]{R_1 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$
$$\begin{array}{cccc} & & & s \end{array}$$

$$\bar{u} = \begin{bmatrix} -s \\ s \\ s \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This is a big jump I took

$\boxed{\text{Thus } null(T) = span\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\} \text{ \& } \dim(null(T)) = 1}$

3) The range(T) is the same as the column space of A (col(A)). To find a basis for col(A), we reduce them to a basis. Relying on our earlier work

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Column 4 is redundant. Thus, the column space of A is

$$col(A) = span\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right\}$$

Therefore,

$\boxed{range(T) = col(A) = span\{\dots\} = \mathbb{R}^3 \text{ \& } \dim(range(T)) = 3}$

2) To go slower, we know

$$range(T) = \left\{\begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix} \middle| \, a,b,c,d \in \mathbb{R}\right\} = \left\{a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$= span\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Good description of range(T)

We know our span above is l.d. (too many vectors for $\mathbb{R}^3$). We now reduce the basis, relying on our earlier work.

Recall, if we know $T(\bar{v}_i)$ for all $\bar{v}_i$ that form a basis for V then $T(\bar{v})$ is completely determined for every $\bar{v} \in V$.

<u>Thm:</u>

Let V, W be v.s & $B = \{\bar{v}_1, \dots, \bar{v}_n\}$ be a basis for V. If $T \in \mathcal{L}(V, W)$, then

$$range(T) = span\{T(\bar{v}_1), \dots, T(\bar{v}_n)\}$$

makes sense b/c T is defined by its operation on $\bar{v}_i \in B$

Example:

$T: \mathbb{R}^3 \to \mathbb{R}^3$, $B = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ basis.

Suppose
$$T(\bar{v}_1) = \begin{bmatrix}1\\1\\0\end{bmatrix},\quad T(\bar{v}_2) = \begin{bmatrix}1\\0\\-1\end{bmatrix},\quad T(\bar{v}_3) = \begin{bmatrix}2\\1\\-1\end{bmatrix}$$

1) Is $\begin{bmatrix}1\\2\\1\end{bmatrix} \in \text{range}(T)$?

2) Basis for range(T)

3) Basis for null(T)

First note $\text{range}(T) = \text{span}\{T(\bar{v}_1), T(\bar{v}_2), T(\bar{v}_3)\} = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}\right\}$

1) We try to find $\begin{bmatrix}1\\2\\1\end{bmatrix} = c_1 T(\bar{v}_1) + c_2 T(\bar{v}_2) + c_3 T(\bar{v}_3)$

$$\left[\begin{array}{ccc|c}1&1&2&1\\1&0&1&2\\0&-1&-1&1\end{array}\right] \xrightarrow{\text{Skip steps}} \left[\begin{array}{ccc|c}1&1&2&1\\0&1&1&-1\\0&0&0&0\end{array}\right]$$

Since the system is consistent, $\begin{bmatrix}1\\2\\1\end{bmatrix} \in \text{range}(T)$ is true.

2) To find basis for range(T), we reuse earlier work to reduce $\text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}\right\}$ to a basis.

$$\begin{bmatrix}1&1&2\\0&1&1\\0&0&0\end{bmatrix}$$

Thus a basis for range(T) is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$ & $\dim(\text{range}(T)) = 2$.

3) We search for c_1, c_2, c_3 s.t. $c_1 T(\bar{v}_1) + c_2 T(\bar{v}_2) + c_3 T(\bar{v}_3) = 0$. In other words, we solve
$$\begin{bmatrix}1&1&2\\1&0&1\\0&-1&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \bar{0}.$$

Using our earlier work
$$\left[\begin{array}{ccc|c}1&1&2&0\\0&1&1&0\\0&0&0&0\end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&0&0&0\end{array}\right]$$
s

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}-s\\-s\\s\end{bmatrix}$$

Thus null(

$\text{null}(T) = \{\bar{v} \mid v = (-s)\bar{v}_1 + (-s)\bar{v}_2 + (s)\bar{v}_3,\ s \in \mathbb{R}\} = \{-s(\bar{v}_1 + \bar{v}_2 - \bar{v}_3) \mid s \in \mathbb{R}\}$

$= \text{span}\{\bar{v}_1 + \bar{v}_2 - \bar{v}_3\}$

&

$\dim(\text{null}(T)) = 1$

Note: For $T: V \to W$, $\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T))$.

Thm: Fundamental Theorem of Linear Maps
Suppose V & W are v.s. & V is finite dimensional.
For any $T \in \mathcal{L}(V, W)$,
The range(T) is finite dimensional (can't get more than you start!)
&
$$\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T))$$

Proof:
Recall null(T) is a subspace of V, so null(T) has basis $\{\bar{u}_1, \ldots, \bar{u}_m\}$.

The l.i. set $\{\bar{u}_1, \ldots, \bar{u}_m\}$ can be extended to be a basis for V:
$\{\bar{u}_1, \ldots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_n\}$.

We already know $\dim(\text{null}(T)) = m$.
We show $\dim(\text{range}(T)) = n$.

We claim $T(\bar{v}_1), \ldots, T(\bar{v}_n)$ is a basis for range(T). We show $T(\bar{v}_1), \ldots, T(\bar{v}_n)$ spans range(T).

We already know that b/c $\{\bar{u}_1, \ldots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_n\}$ spans V, so $\{T(\bar{u}_1), \ldots, T(\bar{u}_m), T(\bar{v}_1), \ldots, T(\bar{v}_n)\}$ spans for range(T). Since $\{\bar{u}_1, \ldots, \bar{u}_m\}$ is a basis for null(T), we know
$$\{T(\bar{u}_1), \ldots, T(\bar{u}_m), T(\bar{v}_1), \ldots, T(\bar{v}_n)\}$$
$$= \{\bar{0}, \ldots, \bar{0}, T(\bar{v}_1), \ldots, T(\bar{v}_n)\}.$$
We reduce this to
$$\{T(\bar{v}_1), \ldots, T(\bar{v}_n)\}.$$

We now show $\{T(\bar{v}_1), \ldots, T(\bar{v}_n)\}$ is l.i. To do this, we need to show $c_1 T(\bar{v}_1) + \ldots + c_n T(\bar{v}_n) = \bar{0}$ has no trivial solutions.
$$c_1 T(\bar{v}_1) + \ldots + c_n T(\bar{v}_n) = \bar{0}$$
$$T(c_1 \bar{v}_1 + \ldots + c_n \bar{v}_n) = \bar{0}$$
$$\Rightarrow c_1 \bar{v}_1 + \ldots + c_n \bar{v}_n \in \text{null}(T) = \text{span}\{\bar{u}_1, \ldots, \bar{u}_n\}$$
This means
$$c_1 \bar{v}_1 + \ldots + c_n \bar{v}_n = d_1 \bar{u}_1 + \ldots + d_m \bar{u}_m \quad \text{for some } d_i$$
$$c_1 \bar{v}_1 + \ldots + c_n \bar{v}_n - d_1 \bar{u}_1 - \ldots - d_m \bar{u}_m = 0$$
Since $\{\bar{v}_1, \ldots, \bar{v}_n, \bar{u}_1, \ldots, \bar{u}_m\}$ is a basis for V, they are l.i.,
so $c_1 = \ldots = c_n = d_1 = \ldots = d_m = 0$.
We has thus shown $c_1 = \ldots = c_n = 0$ must be true, so $c_1 T(\bar{v}_1) + \ldots + c_n T(\bar{v}_n) = \bar{0}$ has no trivial solutions, so $\{T(\bar{v}_1), \ldots, T(\bar{v}_n)\}$ is l.i.

$\therefore \{T(\bar{v}_1), \ldots, T(\bar{v}_n)\}$ is a basis for range(T)
& $\dim(\text{range}(T)) = n$.

Thus, $\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T))$.

Linear Transformations to Connect Different Vector Spaces

We've had a sense that P_2 & $\mathbb{R}^3$ are equivalent as are $M_{2\times2}$ & $\mathbb{R}^4$.
Let's formalize that.

We formalize this using bijections (onto/surjective & one-to-one/injective functions).

<u>Def:</u> Injective
A function $T: V \to W$ is called injective (or one-to-one) iff
$\bar{u} \neq \bar{v} \Rightarrow T(\bar{u}) \neq T(\bar{v})$
or equivalently
$T(\bar{u}) = T(\bar{v}) \Rightarrow \bar{u} = \bar{v}$. ← more useful for proofs b/c inequality is hard.

<u>Def:</u> Surjective
A function $T: V \to W$ is called surjective (or onto) iff
range$(T) = W$
or equivalently
$\forall \bar{w} \in W, \exists \bar{v} \in V$ s.t. $T(\bar{v}) = \bar{w}$. ← easier for proofs (really showing codom$(T) \subseteq$ range(T) b/c range$(T) \subseteq$ codom(T) by def)

<u>Example:</u>
$T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\bar{v}) = A\bar{v}$ where $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. Is T injective? surjective?

Let's rewrite T to help our proof
$\bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $T(\bar{v}) = \begin{bmatrix} v_1 + v_2 \\ -v_1 \end{bmatrix}$

We show T is injective
Let $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ & $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ s.t. $T(\bar{u}) = T(\bar{v})$. We show $\bar{u} = \bar{v}$
$T(\bar{u}) = T(\bar{v}) \Rightarrow \begin{bmatrix} u_1 + u_2 \\ -u_1 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ -v_1 \end{bmatrix}$
This gives us <u>$u_1 = v_1$</u>. Plugging in to the top eqn gives us
$u_1 + u_2 = v_1 + v_2$
$u_2 = v_2 + (v_1 - u_1)$
$u_2 = v_2$
Thus $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, so <u>$T$ is injective.</u>

We show T is surjective.
Let $\bar{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}_2$. We show $\exists \bar{v} \in \mathbb{R}^2$ s.t. $T(\bar{v}) = \bar{w}$.
$T(\bar{v}) = \bar{w}$
$\begin{bmatrix} v_1 + v_2 \\ -v_1 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$
$v_1 = -w_2$
$v_2 = w_1 + w_2$
$T(\bar{v}) = \bar{w}$ when $\bar{v} = \begin{bmatrix} w_1 + w_2 \\ -w_2 \end{bmatrix}$, so <u>$T$ is surjective.</u>

Note:
Asking if $T: V \to W$, $T(\bar{v}) = A\bar{v}$ is surjective is the same as asking if
$col(A) = codom(T)$
b/c $range(T) = col(A)$.

Thm:
$T: V \to W$ is injective iff $null(T) = \{\bar{0}\}$.

Proof: $\Rightarrow$
Assume T is injective. ∵ $T(\bar{0}) = \bar{0}$ & no distinct inputs map to the same output, it follows that only $\bar{0} \in V$ will map to $\bar{0} \in W$, so $null(T) = \{\bar{0}\}$.

Proof: $\Leftarrow$
Assume $null(T) = \{\bar{0}\}$.
Let $\bar{u}, \bar{v} \in V$ s.t. $T(\bar{u}) = T(\bar{v})$. We show $\bar{u} = \bar{v}$.
$T(\bar{u}) = T(\bar{v})$
$T(\bar{u}) - T(\bar{v}) = 0$
$T(\bar{u} - \bar{v}) = 0$
Thus $\bar{u} - \bar{v} \in null(T)$, so $\bar{u} - \bar{v} = \bar{0}$, meaning $\bar{u} = \bar{v}$.

This theorem makes proving injectivity much easier in general.

Example:
$T: \mathbb{R}^2 \to \mathbb{R}^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ 5x + 2y \end{bmatrix}$

Is T injective? In other words, does $null(T) = \{\bar{0}\}$?

We want $\begin{bmatrix} x \\ y \end{bmatrix}$ s.t. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 0$

$$\begin{cases} 2x - 3y = 0 \\ 5x + 2y = 0 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} 2 & -3 \\ 5 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We could have redefined T in terms of A directly & then have found A's null space.

We find $null(A) = null(T)$

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 5 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \cdot 1/2} \left[\begin{array}{cc|c} 1 & -3/2 & 0 \\ 5 & 2 & 0 \end{array}\right] \xrightarrow{R_2 - 5R_1} \left[\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 10/2 & 0 \end{array}\right]$$

15 + 4

$null(A) = \{\bar{0}\}$ ← solve for $A\bar{v} = \bar{0}$.

T is injective b/c $null(T) = null(A) = \{\bar{0}\}$.

T is surjective b/c $rank(A) = 2$ & $A\bar{v} = \bar{w}$ consistent.

Remark:

Let $T: V \to W$, $T(\bar{v}) = A\bar{v}$. $\quad null(T) = null(A)$ always

Yes b/c

$null(T) = \{\bar{v} \in V \mid T(\bar{v}) = A\bar{v} = \bar{0}\}$

&

$null(A) = \{\bar{v} \in V \mid A\bar{v} = T(\bar{v}) = \bar{0}\}$.

Thm:

Let $T: V \to W$, where V & W are finite dimensional vector spaces.

could do one where one is finite & other is infinite

If $\dim(V) > \dim(W)$, T is not injective b/c V is too large

Proof:

Recall $\dim(V) = \dim(null(T)) + \dim(range(T)) \Rightarrow \dim(null(T)) = \dim(V) - \dim(range(T))$

$\dim(range(T)) \leq \dim(W)$

$\Rightarrow \dim(null(T)) \geq \dim(V) - \dim(W) > 0$

$\dim(V) > \dim(W)$

$\Rightarrow \underline{\dim(null(T)) > 0}$

If $\dim(V) < \dim(W)$, T is not surjective b/c V is not large enough.

Proof

Recall $\dim(V) = \dim(null(T)) + \dim(range(T)) \Rightarrow \dim(range(T)) = \dim(V) - \dim(null(T))$

$\dim(V) < \dim(W)$

$\Rightarrow \dim(range(T)) < \dim(V) - \dim(null(T))$

$\Rightarrow \dim(range(T)) < \dim(V)$

Example:

$T: \mathcal{P}_2 \to \mathcal{P}_2$, $T(p) = p' - p$. Is T inj? sur?

Let $p = ax^2 + bx + c$, $T(p) = 2ax + b + ax^2 + bx + c = (a)x^2 + (2a+b)x + (b+c)$.

We show T is injective:

We show $null(T) = \{\bar{0}\}$ by solving $T(ax^2+bx+c) = 0$ for a, b, c.

$(a)x^2 + (2a+b)x + (b+c) = 0$

$\Rightarrow a = 0$

$2a + b = 0 \Rightarrow b = 0$

$b + c = 0 \Rightarrow c = 0$

Could have used matrixes!

Since $a = b = c = 0$ is the only solution to $T(ax^2+bx+c) = 0$, $null(T) = \{\bar{0}\}$ & $\underline{T \text{ is injective.}}$

We show T is surjective:

Suppose $p = ax^2 + bx + c \in \mathcal{P}_2$. We find $p' = a'x^2 + b'x + c'$ s.t. $T(p') = p$.

$(a')x^2 + (2a' + b')x + (b' + c') = ax^2 + bx + c$

$\Rightarrow a' = a$

$2a' + b' = b \Rightarrow b' = b - 2a$

$b' + c' = c \Rightarrow c' = c - b + 2a$

Since $\exists p' \in \mathcal{P}$ s.t. $T(p') = p$, $\underline{T \text{ is surjective}}$

Easier: Since $\dim(null(T)) = 0$,

$\dim(V) = \dim(range(T)) = 3$

Since $\dim(range(T)) = \dim(V)$, T is surjective.

Example:

$T: \mathcal{P}_2 \to \mathcal{P}_3$ $T(p) = xp$. Is T inj? T sur?

We show T is injective

We find null(T) by solving $T(p) = xp \Rightarrow p = 0$

$T(p) = x(ax^2 + bx + c) = 0$

only when $p = ax^2 + bx + c = 0$.

Thus null(T) $= \{\bar{0}\}$ & T is injective.

T is not surjective.

Let $p = ax^2 + bx + c$.

$T(p) = ax^3 + bx^2 + cx$.

There is no p s.t. $T(p) = 1$.

Also $\dim(\mathcal{P}_2) < \dim(\mathcal{P}_3)$, so T cannot be surjective.

Remark:

Let $T: V \to W$ where V & W are v.s. s.t. $\dim(V) = \dim(W)$.

T is injective $\Leftrightarrow$ T is surjective

Example: $\mathcal{P}_2 \to \mathbb{R}^3$

$T: \mathcal{P}_2 \to \mathbb{R}^3$, $T(ax^2 + bx + c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

T is both injective & surjective (bijective).

$S: \mathcal{P}_2 \to \mathbb{R}^3$, $S(ax^2 + bx + c) = \begin{bmatrix} a \\ 2a+b \\ b+c \end{bmatrix}$

S is both injective & surjective (bijective).

This is true in general as long as you transform the basis in $\mathcal{P}_2$ to a basis in $\mathbb{R}_3$. That is $\{f_1, f_2, f_3\}$ basis for $\mathcal{P}_2$

$\forall f \in \mathcal{P}_2$, $f = a_1 f_1 + a_2 f_2 + a_3 f_3$, $T(f) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

Thm: Isomorphisms b/w $\mathbb{R}^n$ & V

If V is a v.s. w/ $\dim(V)=n$, then $\exists T \in \mathcal{L}(V, \mathbb{R}^n)$ that is bijective (injective & surjective). That is, they are isomorphic.

Def: Isomorphism

Let V & W be v.s.
A linear map $T: V \to W$ that is bijective (injective & surjective) is called an isomorphism. ← b/c it preserves the structures properties/shape!
We write $V \cong W$.

Example:

Define $T: M_{2\times2} \to P_3$ s.t. T is an isomorphism

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ax^3 + bx^2 + cx + d$$

We could also do

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)x^3 + bx^2 + (c+d)x + d$$

Remark:

Let $T: V \to W$ where V, W are v.s. s.t. $\dim(V) \neq \dim(W)$.

T is not an isomorphism b/c it cannot be both injective & surjective

Thm:

Let $A \in M_{n\times n}$ & $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\bar{v}) = A\bar{v}$.

T is an isomorphism iff A is invertible. (aka $\text{null}(A) = \text{null}(T) = \{\bar{0}\}$ or column vectors of A are l.i.)

Proof: Isomorphisms b/w $\mathbb{R}^n$ & V

Since $\dim(V) = n$, $\exists B = \{\bar{v}_1, \dots, \bar{v}_n\}$ basis for V.

We define $T(\bar{v})$ for are arbitrary $\bar{v} \in V$. Recall we can rewrite $\bar{v}$ as a l.c. of B
$\bar{v} = a_1\bar{v}_1 + \dots + a_n\bar{v}_n$ for some unique $a_1, \dots, a_n \in \mathbb{R}$.

We thus define
$$T(\bar{v}) = T(a_1\bar{v}_1 + \dots + a_n\bar{v}_n) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

We'd now have to check vector addition & scalar multiplication, to verify that T is a linear map, but this should be easy to see.

We now verify that T is injective.
To do this, we show $\text{null}(T) = \bar{0}$.

$\text{null}(T) = \{\bar{v} \in V \mid T(\bar{v}) = \bar{0}\}$ where $\bar{v} = a_1\bar{v}_1 + \dots + a_n\bar{v}_n$.

$$T(\bar{v}) = \bar{0} \Rightarrow \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \bar{v} = \bar{0}.$$

Thus $null(T) = \bar{0}$.

We now verify that T is surjective.
To do this, we show that $\forall \bar{v}' \in \mathbb{R}^n \Rightarrow \exists \bar{v} \in V$ s.t. $T(\bar{v}) = \bar{v}'$

Let $\bar{v}' = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$. Suppose $\bar{v} = a_1\bar{v}_1 + \dots + a_n\bar{v}_n \in V$.

$T(\bar{v}) = \bar{v}'$ by definition.

Therefore T is surjective

Easier using fundamental theorem of linear maps

$\dim(\mathbb{R}^n) = \dim(null(T)) + \dim(range(T))$
$\dim(\mathbb{R}^n) = \dim(range(T))$

Since range(T) subspace $\mathbb{R}^n$ & $\dim(range(T)) = \dim(\mathbb{R}^n)$,
range$(T) = \mathbb{R}^n$

Def: Coordinate vectors
Let V be a v.s. w/ basis $B = \{\bar{v}_1, \dots, \bar{v}_n\}$ ordered.

Then, any vector $\bar{v} \in V$ can be written uniquely as l.c. of vectors from B: $\bar{v} = a_1\bar{v}_n + \dots + a_n\bar{v}_n$

In this case, we write
$$[\bar{v}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$$
& call this the coordinate vector of $\bar{v}$ w/ respect to basis B.

Recall: Invertibility
Function $f: A \to B$ is invertible
iff
$\exists g: B \to A$ s.t. $(f(a) = b \Leftrightarrow g(b) = a)$ or $(f \circ g = id_B$ & $g \circ f = id_A)$
iff
f is bijective.

Claim:
Let V, W be v.s. & $T: V \to W$ be an isomorphism.
We claim $T^{-1}: W \to V$ is also an isomorphism.

That is, $T(\vec{v}) = A\vec{v}$ where A is invertible (required b/c T is an isomorphism),
then $T^{-1}(\vec{w}) = A^{-1}\vec{w}$. ← This is true

Thm: Same-Dimensional Vector Spaces are Isomorphic
If V, W are n-dimensional v.s. the V & W are isomorphic ($V \cong W$).

Pf:

$\dim(V) = n \Rightarrow \exists\, T: V \to \mathbb{R}^n$ isomorphism
$\dim(W) = n \Rightarrow \exists\, S: W \to \mathbb{R}^n$ isomorphism

S isomorphism $\Rightarrow \exists\, S^{-1}: \mathbb{R}^n \to W$ isomorphism.

Unproven here, but not hard to agree w/

Since the composition of isomorphisms is an isomorphism
$(S^{-1} \circ T): V \to W$ is an isomorphism.

$$\underbrace{V \xrightarrow{T} \mathbb{R}^n \xrightarrow{S^{-1}} W}_{(S^{-1} \circ T)}$$

$\therefore V \cong W$

Example:
Define an isomorphism T from P_2 into the v.s. of symmetric matrices.

$T(ax^2 + bx + c) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is an isomorphism ← More fancily $T(ax^2 + bx + c) = \begin{bmatrix} b-c & a+b \\ a+b & c \end{bmatrix}$

Example
Define an isomorphism T from $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, x + 2y - z = 0 \right\}$ & $\mathbb{R}^2$.

We rewrite V by solving the eqn for z & substituting
$V = \left\{ \begin{bmatrix} x \\ y \\ x+2y \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

We define $T: V \to \mathbb{R}^2$
$T\left(\begin{bmatrix} x \\ y \\ x+2y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}$ is an isomorphism ← More fancily $T\left(\begin{bmatrix} x \\ y \\ x+2y \end{bmatrix} \right) = \begin{bmatrix} x+y \\ y \end{bmatrix}$

All Linear Transformations are Matrices

Let V, W be v.s. where $\dim(V) = n$ & $\dim(W) = m$. We know $\mathcal{L}(V, W)$ is a vector space. We show $\mathcal{L}(V, W)$ has an isomorphic matrix $A \in M_{m \times n}$.

Let

$\mathcal{B}_V = \{\bar{v}_1, \ldots, \bar{v}_n\}$ be an ordered basis for V &
$\mathcal{B}_W = \{\bar{w}_1, \ldots, \bar{w}_n\}$ be an ordered basis for W.

We (abstractly) define linear transformation $T: V \to W$.

$$T(\bar{v}_1) = a_{11}\bar{w}_1 + a_{21}\bar{w}_2 + \cdots + a_{m1}\bar{w}_m$$
$$T(\bar{v}_2) = a_{12}\bar{w}_1 + a_{22}\bar{w}_2 + \cdots + a_{m2}\bar{w}_m$$
$$\vdots$$
$$T(\bar{v}_n) = a_{1n}\bar{w}_1 + a_{2n}\bar{w}_2 + \cdots + a_{mn}\bar{w}_m$$

As hinted at by the form of the definition this corresponds to a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$[T(\bar{v}_1)]_{\mathcal{B}_W}$ $\quad [T(\bar{v}_2)]_{\mathcal{B}_W}$ $\quad [T(\bar{v}_n)]_{\mathcal{B}_W}$

Thm:

Let V, W be v.s. where $\dim(V) = n$ & $\dim(W) = m$.

The vector space of all linear maps from V into W is isomorphic to the v.s. of all $m \times n$ matrices.

$$\mathcal{L}(V, W) \cong M_{m \times n} \quad \leftarrow \text{ the isomorphism b/w them is an element of } \mathcal{L}(\mathcal{L}(V, W), M_{m \times n})!$$

Remark:

This means there is a "size" to how many linear transformations can exist b/w two v.s. It's really the dimensions

$$\dim(\mathcal{L}(V, W)) = \dim(V) \cdot \dim(W)$$

Example:

Let

$$V = \left\{ \begin{bmatrix} a+b \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}, \quad B_V = \left\{ \underset{\bar{v}_1 \,\downarrow a}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\bar{v}_2 \,\downarrow b}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \right\},$$

$$W = P_2, \quad B_W = \{1, x, x^2\}, \ \&$$

$$T : V \to W, \ T \begin{bmatrix} a+b \\ b \\ a \end{bmatrix} = ax^2 + 2ax + 5b.$$

We define $A \in M_{3\times 2}$ that corresponds w/ T.

$T(\bar{v}_1) = 0 \cdot 1 + 2 \cdot x + 1 \cdot x^2 \leftarrow$ 1st column

$T(\bar{v}_2) = 5 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \leftarrow$ 2nd column

Thus the corresponding matrix is

$$A = \begin{bmatrix} 0 & 5 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

Recall our earlier expression on an abstract linear map $T : V \to W$ where V has basis $B_V = \{\bar{v}_1, \dots, \bar{v}_n\}$ & W has basis $B_W = \{\bar{w}_1, \dots, \bar{w}_m\}$.

$T(\bar{v}_1) = c_{11} \cdot \bar{w}_1 + \dots + c_{m1} \bar{w}_m$

$\dots$

$T(\bar{v}_n) = c_{1n} \bar{w}_n + \dots + c_{mn} \bar{w}_m$

This means $T(\bar{v})$ is defined as follows for $\bar{v} = a_1 \bar{v}_1 + \dots + a_n \bar{v}_n$

$T(\bar{v}) = (c_{11} a_1 + \dots + c_{1n} a_n) \bar{w}_1 + \dots + (c_{m1} a_1 + \dots + c_{mn} a_n) \bar{w}_n$

This means we define the coordinate vector for $[T(\bar{v})]_{B_W}$

$$[T(\bar{v})]_{B_W} = \begin{bmatrix} c_{11} a_1 + \dots + c_{1n} a_n \\ \dots \\ c_{m1} a_1 + \dots + c_{mn} a_n \end{bmatrix} = \underbrace{\begin{bmatrix} c_{11} & \dots & c_{1n} \\ \dots & & \\ c_{m1} & \dots & c_{mn} \end{bmatrix}}_{\text{matrix of } T} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{[\bar{v}]_{B_V}}$$

(finite-dimensional)

This means its any transformation b/w arbitrary vector spaces can be expressed as a linear map b/w Euclidean spaces!

Thm:

Let $T : V \to W$ be a linear map & B_V & B_W be ordered bases for v.s. V & W respectively (resp.) where

$B_V = \{\bar{v}_1, \dots, \bar{v}_n\}$ &

$B_W = \{\bar{w}_1, \dots, \bar{w}_m\}$.

The matrix of T relative to $B_V \to B_W$ is defined as

$$[T]_{B_V \to B_W} = \begin{bmatrix} [T(\bar{v}_1)]_{B_W} & [T(\bar{v}_2)]_{B_W} & \dots & [T(\bar{v}_n)]_{B_W} \end{bmatrix} \in M_{m\times n}$$

(goes from n into m)

Then, for all $\bar{v} \in V$

$$[T(\bar{v})]_{B_W} = [T]_{B_V \to B_W} [\bar{v}]_{B_V}$$

Duca writes $B = B_V$ & $B' = B_W$

The previous theorem means all linear maps b/w finite dimensional v.s. can be expressed as matrix multiplication. ← easy to code!

Example: Standard Bases

$T: \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -y \\ z \end{bmatrix}$

$B = B' = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$.

In this case,

$$[T]_{B \to B'} = \begin{array}{ccc} \\ \left[\begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \\ T(e_1) \; T(e_2) \; T(e_3) \end{array}$$

← Should show more work to derive

We use this to compute $T\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$[T(\bar{v})]_{B'} = [T]_{B \to B'} [\bar{v}]_B$ b/c $B = B'$ is standard basis

$\bar{v} = [\bar{v}]_B = [\bar{v}]_{B'}$

$A = [T]_{B \to B'}$

$$T\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Example: Different Dimensions

$T: \mathbb{R}^2 \to \mathbb{R}^3$, $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$

Let B & B' be the standard bases for their resp. v.s.

$$[T]_{B \to B'} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

T is one-to-one (injective) b/c $\text{null}(T) = \{\bar{0}\}$.

T is not surjective b/c $\dim(V) = 2 < 3 = \dim(W)$

$$T\begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ -1 \end{bmatrix}$$

Example: Non-standard Bases

Do the previous example but where

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \quad \& \quad B' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

We define $[T]_{B \to B'}$ by finding $[T(\bar{v})]_{B'}$ for every $\bar{v} \in B$.

$T(\bar{v}_1) = T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

(i.e. $[T(\bar{v})]_{B'}$)

We write $T(\bar{v}_1)$ as a l.c. of B' by solving $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \bar{u} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$u_3 = -1$

$u_2 = 3 - u_3 = 4$

$u_1 = 2 - u_1 - u_2 = -1$

Thus

$$[T(\bar{v}_1)]_{B'} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$$

$T(\bar{v}_2) = T\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

We write $[T(\bar{v}_2)]_{B'}$ by solving $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \bar{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$u_3 = 2$

$u_2 = 4 - u_3 = 2$

$u_1 = 1 - u_1 - u_2 = -3$

Thus

$$[T(\bar{v}_2)]_{B'} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$$

$[T]_{B \to B'}$'s column vectors are $[T(v_1)]_{B'}$ & $[T(\bar{v}_2)]_{B'}$ resp. So

$$[T]_{B \to B'} = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$$

We use $[T]_{B \to B'}$ to compute $\left[T\begin{bmatrix} -3 \\ -2 \end{bmatrix}\right]_{B'}$

$[T(\bar{v})]_{B'} = [T]_{B \to B'} [\bar{v}]_B$ where $\bar{v} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

$[\bar{v}]_B$ is found by writing it as l.c. of B

$$\left[\begin{array}{cc|c} 1 & 3 & -3 \\ 2 & 1 & -2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & -3 \\ 0 & -5 & 4 \end{array}\right]$$

$u_2 = -4/5$

$u_1 = -3 + 12/5 = -3/5$

could have pulled out $1/5$ (or $-1/5$!) to make math simpler

$$[T(\bar{v})]_{B'} = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -3/5 \\ -4/5 \end{bmatrix} = \begin{bmatrix} 3/5 + 12/5 \\ -12/5 - 8/5 \\ 3/5 - 8/5 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$$

If we have $T: V \to W$ & B, B' ordered basis for V, W resp. can we discover if T is one-to-one and/or onto using only $[T]_{B \to B'}$?
Yes!

T is one-to-one $\Leftrightarrow$ $null(T) = \{\bar{0}\} \Leftrightarrow null([T]_{B \to B'}) = \{\bar{0}\}$. This relies on the fact that $T(\bar{v}) = \bar{0} \Leftrightarrow [T(\bar{v})]_{B'} = \bar{0} \Leftrightarrow [T]_{B \to B'}[\bar{v}]_B = \bar{0}$

T is onto $\Leftrightarrow$ $range(T) = W \Leftrightarrow col([T]_{B \to B'}) = \mathbb{R}^m$ where $m = dim(W)$.

T is isomorphism $\Leftrightarrow$ $[T]_{B \to B'}$ is invertible $\Leftrightarrow$ T is one-to-one & onto.

<u>Example: Polynomial's + Matrixes? (standard Basis)</u>

$T: P_2 \to P_3$, $T(f) = x^2 f'' - 2f' + xf$.

$B = \{1, x, x^2\}$ & $B' = \{1, x, x^2, x^3\}$.

I first rewrite T as

$$T(a + bx + cx^2) = x^2(2c) - 2(b + 2cx) + x(a + bx + cx^2)$$
$$= (-2b) + (-4c + a)x + (2c + b)x^2 + (c)x^3$$

I now write $[T]_{B \to B'}$ since we're using the standard bases

$$[T]_{B \to B'} = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Showing more work (& possibly making it easier)

$T(1) = x$

$$[x]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$T(x) = -2 + x^2$

$$[-2 + x^2]_{B'} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$T(x^2) = 2x^2 - 4x + x^3$

$$[T(x^2)]_{B'} = [-4x + 2x^2 + x^3] = \begin{bmatrix} 0 \\ -4 \\ 2 \\ 1 \end{bmatrix}$$

Now we use $[T(1)]_{B'}$, $[T(x)]_{B'}$, $[T(x^2)]_{B'}$ as the column vectors of $[T]_{B\to B'}$

$$[T]_{B\to B'} = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thm:
If $T: V \to W$ is a linear transformation where $\dim(V) = \dim(W)$, then T injective $\Leftrightarrow$ T surjective $\Leftrightarrow$ T isomorphism.

We use $[T]_{B\to B'}$ to find $T(x^2-3x+1)$

$$[T(x^2-3x+1)]_{B'} = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -1 \\ 1 \end{bmatrix}$$

Thus $T(x^2-3x+1) = 6 - 3x - x^2 + x^3$

Is T one-to-one? To answer this, we find $\text{null}([T]_{B\to B'})$ by solving $[T]_{B\to B'}\bar{v} = \bar{0}$ for $\bar{v}$.

$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} R_4 + 2R_2 - 2R_3 \quad \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have shown $\text{null}([T]_{B\to B'}) = \{\bar{0}\}$, so T is one-to-one.

T is not one-to-one b/c $\dim(\text{dom}(T)) = 3 < 4 = \dim(\text{codom}(T))$.

T is not an isomorphism b/c it is not one-to-one.

Thm:
Let V, W be v.s. w/ ordered bases B & B' resp.
Let $S, T \in \mathcal{L}(V, W)$ & k be scalar.

$$[S+T]_{B\to B'} = [S]_{B\to B'} + [T]_{B\to B'}$$

$$[kS]_{B\to B'} = k[S]_{B\to B'}$$

Let $T \in \mathcal{L}(U, V)$, $S \in \mathcal{L}(V, W)$.
Let U, V, W have ordered bases B, B', B''

$$\underset{B}{U} \longrightarrow \underset{B'}{V} \longrightarrow \underset{B''}{W}$$

By definition of composition

$$[(S\circ T)(\bar{v})]_{B''} = [S(T(\bar{v}))] = [S]_{B'\to B''}[T(\bar{v})]_{B'} = [S]_{B'\to B''}[T]_{B\to B'}[\bar{v}]_B$$

$$[(S\circ T)(\bar{u})]_{B''} = [S\circ T]_{B\to B''}[\bar{u}]_B$$ ← By definition of application

Synthesizing these 2 results/rules w/ pattern matching, we get

$$[S\circ T]_{B\to B''} = [S]_{B'\to B''}[T]_{B\to B'}$$

Composition of functions is the same as matrix multiplication!

Eigenvalues & Eigenvectors

Def: Eigenvalue & Eigenvector

Let $A \in M_{n \times n}$. A number λ is called an eigenvalue for A if there exists a non-zero vector $\bar{v} \in \mathbb{R}^n$ s.t.

$$A\bar{v} = \lambda \bar{v}.$$

In this case, $\bar{v}$ is called an eigenvector for A corresponding to λ.

We call $(\lambda, \bar{v})$ an eigenpair.

Why do we care about this? Some simulations (e.g. Markov chains) require repeated matrix multiplication. W/ eigenvalues/vectors, this is much easier. (If we change basis this can be even easier!)

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\bar{v}$ is an eigenvector for A. What is $\bar{v}$'s corresponding eigenvalue?

$$A\bar{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \bar{v}$$

$\bar{v}$'s corresponding eigenvalue is 1.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ Find eigenvalues & eigenvectors for A.

We solve $A\bar{v} = \lambda\bar{v}$ for $\bar{v}$ & λ. Let $\bar{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$A\bar{v} = \lambda\bar{v}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

$$\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

$$\begin{cases} -\lambda x + y = 0 \\ x - \lambda y = 0 \end{cases}$$

We convert this system of equations into a matrix equation

$$\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\bar{v} = \bar{0} \quad \text{for some non-zero } \bar{v}$$

$$\left[\begin{array}{cc|c} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \end{array}\right] \to \begin{bmatrix} 1 & -\lambda \\ -\lambda & 1 \end{bmatrix} \xrightarrow{R_2 + \lambda R_1} \left[\begin{array}{cc|c} 1 & -\lambda & 0 \\ 0 & 1-\lambda^2 & 0 \end{array}\right]$$

To get a non-zero solution $\bar{v}$, we need $1-\lambda^2=0$.

$1-\lambda^2=0$

$\lambda=\pm 1$

We thus have two eigenvalues $\lambda_1=-1$, $\lambda_2=1$. We plug back into the matrix equation to find the eigenvectors.

First we do $\lambda_1=-1$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

s

$$\bar{v}_1=\begin{bmatrix} -s \\ s \end{bmatrix}=\begin{bmatrix} -1 \\ 1 \end{bmatrix}s \quad \forall s \in \mathbb{R}$$

We have infinitely many eigenvectors corresponding to λ_1.

Now we do $\lambda_1=1$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

s

$$\bar{v}_2=\begin{bmatrix} s \\ s \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}s \quad \forall s \in \mathbb{R}$$

We have infinitely many eigenvectors corresponding to λ_2.

Let's plug $\bar{v}=5\bar{v}_1=\begin{bmatrix} -5 \\ 5 \end{bmatrix}$ into $A\bar{v}=\lambda_1\bar{v}$ to ensure it works

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ -5 \end{bmatrix}=-1\cdot\begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} -5 \\ 5 \end{bmatrix}=\begin{bmatrix} -5 \\ 5 \end{bmatrix} \checkmark$$

It works!

The important part of solving for the eigenvectors was

i) Forming a system of equations involving λ. ← the system is always homogeneous!

ii) Finding bounds on λ.

Note that our bounds on the matrix eqn. involving λ boiled down to the determinant of the matrix being zero. This should make sense b/c we want non-trivial solutions & if the determinant was zero, $\bar{0}$ would be the only solution b/c the system is homogeneous.

<u>Def:</u> Determinant

Let $A \in M_{n \times n}$ of real numbers.

$\det(A)$ is a real number formally defined as the sum of all possible products in distinct rows & columns. (doing some sign handling

Since that's a lot, we use an inductive definition b/c it's easier to do.

If $A \in M_{1 \times n}$ $A=[a_{11}]$, $\det(A)=a_{11}$

On next page...

$\therefore$ Else $A \in M_{n\times n}$ $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ These blocks are called cofactors!

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det\left(\begin{bmatrix} a_{21} & \cdots & a_{2i-1} & a_{2i+1} & \cdots & a_{2n} \\ & & \vdots & & & \\ a_{n1} & \cdots & a_{ni-1} & a_{2i+1} & \cdots & a_{nn} \end{bmatrix} \right)$$

Concretely, $\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = a_{11}a_{22} - a_{12}a_{21}$

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Note: We also write $\det(A)$ as $|A|$.

Question: Properties as Determinant

Does $\det(A) = \det(A^T)$?

Can we switch around rows?

Combining the above, can we switch around columns?

This will be answered later!

Def: Determinant w/ Cofactors ← This makes determinant easier!

Let $A \in M_{n\times n}$.

For each $i, j \in \{1, \ldots, n\}$, define the (i, j)-cofactor as

$$\boxed{A_{ij} = \underset{+/-}{(-1)^{i+j}} \cdot \det\left(\begin{bmatrix} (n-1)\times(n-1) \text{ matrix obtained by removing} \\ \text{row } i \text{ \& column } j \end{bmatrix} \right)}$$

Using this,

$$\boxed{\det(A) = a_{11} \cdot A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}}$$

We call this definition cofactor expansion across/along row 1 of A.

Thm:

Let $A \in M_{n\times n}$. Then $\det(A)$ can be calculated using the cofactor expansion across any row/column of A.

row i of $A \Rightarrow \det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$

column j of $A \Rightarrow \det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$

We generally pick the row/column w/ the most zeros.

Example: Determinant by Cofactors

Let $A = \begin{bmatrix} 2 & -3 & 0 \\ 3 & 1 & 5 \\ -2 & 7 & 0 \end{bmatrix}$

We find the determinant using column 3 to maximize the number of zeros

$$\begin{vmatrix} 2 & -3 & 0 \\ 3 & 1 & 5 \\ -2 & 7 & 0 \end{vmatrix} = (-1)^{2+3} \cdot 5 \begin{vmatrix} 2 & -3 \\ -2 & 7 \end{vmatrix} = -5 \cdot ((2)(7) - (-3)(-2)) = -5 \cdot (14-6) = -40$$

Properties of Determinants:

- $\det(A) = \det(A^T)$
- If A has an entire zero row/column, then $\det(A) = 0$.
- The determinant of an (upper/lower) triangular matrix, $\det(A)$ is the product of the main diagonal.

$$\begin{vmatrix} 2 & -5 & 3 & 1 \\ 0 & 3 & 7 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 2 \cdot (-1)^{1+1} \begin{vmatrix} 3 & 7 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 2\left(3 \cdot (-1)^{1+1} \begin{vmatrix} -1 & 5 \\ 0 & 2 \end{vmatrix}\right) = 2 \cdot 3 \left(-1 \cdot (-1)^{1+1} |2|\right) = 2 \cdot 3 \cdot -1 \cdot 2$$

- If a row/column of A is multiplied by a scalar $k \neq 0$ to obtain B,

  $\det(B) = k\det(A) \leftarrow$ b/c k appears in every cofactor expansion.

$$\begin{vmatrix} 3 & -1 & 2 \\ 33 & -77 & 121 \\ 15 & 1 & 3 \end{vmatrix} = 11 \begin{vmatrix} 3 & -1 & 2 \\ 3 & -7 & 11 \\ 15 & 1 & 3 \end{vmatrix} = 11 \cdot 3 \begin{vmatrix} 1 & -1 & 2 \\ 1 & -7 & 11 \\ 5 & 1 & 3 \end{vmatrix}$$

- If a matrix B is obtained by swapping 2 rows/columns from A, then

  $\det(B) = -\det(A) \leftarrow$ b/c of $(-1)^{i+j}$ in cofactor expansion   must both be rows or both columns

- If we add a multiple of a row/column to another row/column of A to obtain B, then

  $\det(B) = \det(A) \leftarrow$ proof is tough. basically split cofactor expansion to sum of 2 cofactor expansions, one of which is zero & the other is original.

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \xrightarrow{R_2 + 2R_1} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = -5 \cdot (-1)^{2+3} \begin{vmatrix} 1 & -4 \\ -1 & 7 \end{vmatrix} = 5((1)(7) - (-1)(-4)) = 5(7-4) = 15$$

add rows/columns together

- If A has 2 linearly dependent (or identical) rows/columns, then $\det(A) = 0$.
- If A is linearly dependent / not invertible, then $\det(A) = 0$
- $\det(A \cdot B) = \det(A) \cdot \det(B)$
- $\det(A+B) \neq \det(A) + \det(B)$ in general

- $\det(kA) = k^n \det(A)$ ← recall rule about multiplying rows by scalars
- $\det(A^n) = \det(A)^n$ ← recall multiplication rule
- $\det(A) = 1/\det(A^{-1})$. Proof $\det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1}) = \det(I) = 1$.

  unless $\det(A) \neq 0$, but if $\det(A) = 0$ then A is not invertible.

<u>Thm:</u> Determinants & Everything Else
Let $A \in M_{n \times n}$.

A invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow A\bar{x} = \bar{0}$ has unique solution
$\Leftrightarrow \det(A) \neq 0$
$\Leftrightarrow A\bar{x} = \bar{0}$ has unique solution ← important for eigenvalues
$\Leftrightarrow A\bar{x} = \bar{b}$ has unique solution
$\Leftrightarrow$ columns of A are l.i.
$\Leftrightarrow$ rows of A are l.i.
$\Leftrightarrow$ columns of A span $\mathbb{R}^n$
$\Leftrightarrow$ columns of A form basis for $\mathbb{R}^n$
$\Leftrightarrow$ rows of A span $\mathbb{R}^n$
$\Leftrightarrow$ rows of A form basis for $\mathbb{R}^n$
$\Leftrightarrow$ iff linear map $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\bar{v}) = A\bar{v}$ is an isomorphism (injective, surjective, bijective)

rows & columns are interchangable b/c $\det(A) = \det(A^T)$

<u>Def:</u> Eigenvectors for Linear Operators
Let $T: V \to V$ be a linear operator (map).
A scalar λ is called an eigenvalue of T iff
$\exists \bar{v} \neq \bar{0}$, $v \in V$ s.t.
$T(\bar{v}) = \lambda \bar{v}$.
In this case $\bar{v}$ is an eigenvector corresponding to λ & $(\lambda, \bar{v})$ is an eigenpair.

<u>Example:</u>
$T: P_2 \to P_2$, $T(ax^2 + bx + c) = (-a + b + c)x^2 + (-b - 2c)x + (-2b - c)$

Is $f = -x^2 + x + 1$ an eigenvector of T?

$T(f) = 3x^2 + (-3)x + (-3) = -3 \cdot f \Rightarrow$ yes where $\lambda = -3$.

Our eigenvectors for $M_{n \times n}$ & $T: V \to V$ are equivalent by the transformation matrix $[T]_{B \to B}$ given an ordered bases B. That is
$(\lambda, \bar{v})$ eigenpair for $T \Leftrightarrow T(\bar{v}) = \lambda \bar{v} \Leftrightarrow [T[\bar{v}]]_B = [\lambda \bar{v}]_B$
$\Leftrightarrow [T(\bar{v})]_B = \lambda [\bar{v}]_B \Leftrightarrow [T]_{B \to B} [\bar{v}]_B = \lambda [\bar{v}]_B \Leftrightarrow (\lambda, [\bar{v}]_B)$ eigenpair for $[T]_{B \to B}$

Since eigenvalues & eigenvectors are equivalent for $M_{n\times n}$ & $T: V \to V$, we'll only deal w/ matrices for now b/c it is simpler.

Let's get more systematic about finding the eigenvalues for $A \in M_{n\times n}$ (where $\bar{v} \neq \bar{0}$).

$A\bar{v} = \lambda\bar{v}$
$A\bar{v} - \lambda\bar{v} = \bar{0}$
$\boxed{(A - \lambda I)\bar{v} = \bar{0}}$ ← homogeneous system w/ non-trivial solution

We now solve for λ that makes $(A - \lambda I)$ a non-invertible matrix so that we get $\bar{v} \neq \bar{0}$ (a non-trivial solution). This means $\det(A - \lambda I) = 0$ must be true to get a homogeneous system w/ non-trivial solutions.

Thm: Eigenvalues & Determinants
A scalar λ is an eigenvalue of $A \in M_{n\times n}$ iff
$\det(A - \lambda I) = 0$.
That is eigenvalues solve the above.

Finding $\det(A - \lambda I)$ gives us a polynomial (multiply λ's on diagonal) is called the characteristic polynomial & we write

$p(\lambda) = \det(A - \lambda I)$.

Here $\deg(p(\lambda)) = n$ b/c you're multiplying the n λ's on the eigenvalue.

By the fundamental theorem of algebra, this means $p(\lambda) = \det(A - \lambda I)$ has n solutions, meaning A has n eigenvalues.

We call $\det(A - \lambda I)$ the characteristic equation.

Given an eigenvalue λ of A, we find the eigenvectors by solving $(A - \lambda I)\bar{v} = \bar{0}$.

Def: Eigenspace
We call the set of eigenvectors $\bar{v}$ corresponding to eigenvalue λ the eigenspace
$V_\lambda = \text{null}(A - \lambda I)$. (We include the $\bar{0}$ to make it a vector space.)

Example:
Let $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ w/ eigenvalue $\lambda_1 = -1$. Find the eigenspace V_{λ_1}.

$(A - \lambda_1 I)\bar{v} = \bar{0}$
$\begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix} \begin{matrix} R_1 \cdot 1/3 \\ R_2 - 1/3 R_1 \end{matrix} \quad \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$ s
$\bar{v} = \begin{bmatrix} 4s \\ s \end{bmatrix}$
$V_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$ ← excluding $\bar{0}$

Example:

Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$. Find the values & eigenvectors

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 5 & -10 \\ 1 & 0 & 2-\lambda & 0 \\ 1 & 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-1)^{1+1} \begin{vmatrix} 1-\lambda & 5 & -10 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$ ← triangular matrix!

$$= (1-\lambda)(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$\lambda_1 = 1 \quad (2)$

$\lambda_2 = 2$

$\lambda_3 = 3$

We find eigenspace for λ_1

$(A - \lambda_1 I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \xrightarrow[R_3 - R_2]{R_2 \cdot 1/5} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} & s & & t \end{array}$$

$$\vec{v} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix}$$

this isn't always the case (see below)

↓

$$V_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$ ← Eigenspace has dimension 2, which makes sense b/c it was a double root to the characteristic equation

Example: Double Root only 1D Eigenspace

Let $A = \begin{bmatrix} 0 & 5 \\ -5 & 10 \end{bmatrix}$

Solve characteristic eqn.

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 5 \\ -5 & 10-\lambda \end{vmatrix}$$

$= -10\lambda + \lambda^2 + 25$

$= (\lambda - 5)^2 = 0$

$\lambda = 5$

$$\begin{bmatrix}-5 & 5\\ -5 & 5\end{bmatrix} \Rightarrow \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

$$\bar{v} = \begin{bmatrix} s \\ s \end{bmatrix}$$

$$V_\lambda = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

<u>Example:</u>

Find eigenvectors of $A = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}$.

We solve the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = 0$$

$$\begin{vmatrix}-\lambda & 0 & 0\\ 0 & -\lambda & -1\\ 0 & 1 & -\lambda\end{vmatrix}$$

$$= (-\lambda)(\lambda^2 + 1) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = i, \quad \lambda_3 = -i$$

Don't ignore complex eigenvalues!

Eigenspace for $\lambda_1 = 0$ is $V_{\lambda_1=0} = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

To find eigenspace for $\lambda_2 = i$, we solve $(A - \lambda I)\bar{u} = \bar{0}$ for $\bar{u}$

$$(A - iI)\bar{u} = \bar{0}$$

$$\begin{bmatrix}-i & 0 & 0\\ 0 & -i & -1\\ 0 & 1 & -i\end{bmatrix} \begin{matrix}R_1 \cdot i\\ R_2 \cdot i\\ \\ \end{matrix} \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -i\\ 0 & 1 & -i\end{bmatrix} R_3 - R_2 \quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -i\\ 0 & 0 & 0\end{bmatrix}$$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad s$

$$\bar{u} = \begin{bmatrix}0\\ is\\ s\end{bmatrix} = \begin{bmatrix}0\\ i\\ 1\end{bmatrix} s$$

$$V_{\lambda_2=i} = \text{span}\left\{\begin{bmatrix}0\\i\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} i\right\}$$

Because complement solutions to polynomials come in pairs, $V_{\lambda_2=i} = \bar{V}_{\lambda_3=-i}$. Similarly, the eigenvectors of λ_3 are in the space of λ_2. We get

$$V_{\lambda_3=-i} = \text{span}\left\{\begin{bmatrix}0\\-i\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\-1\\0\end{bmatrix} i\right\}$$

<u>Properties: Eigenvalues</u>

1) For triangular or diagonal matrixes, the entries on the main diagonal are the eigenvalues.

2) $\det(A)$ = product of eigenvalues, taking multiplicities into account.

Should make sense if you think about solving

3) $\text{trace}(A)$ = sum of eigenvalues

<u>trace</u> is sum of all entries on the main diagonal.

<u>Thm:</u> Linear Independence of Eigenvectors

Let $A \in M_{n \times n}$ & suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of A w/ corresponding eigenvectors $\bar{v}_1, \ldots, \bar{v}_m$ (note: $m \le n$).

$\bar{v}_1, \ldots, \bar{v}_m$ are l.i.

Pf:

Suppose for contradiction that $\bar{v}_1, \ldots, \bar{v}_m$ are l.d.

WLOG (w/o loss of generality), we assume $\bar{v}_1, \ldots, \bar{v}_k$ are l.i. & $\bar{v}_1, \ldots, \bar{v}_k, \bar{v}_{k+1}$ are l.d. By definition, $\exists\ c_1, \ldots, c_k$ s.t.

$$\underline{\bar{v}_{k+1} = c_1 \bar{v}_1 + \ldots + c_k \bar{v}_k} \quad (*)$$

Multiply (*) by A on left.

$$A\bar{v}_{k+1} = c_1 A\bar{v}_1 + \ldots + c_k A \bar{v}_k$$ b/c they are eigenvectors

$$\lambda_{k+1}\bar{v}_{k+1} = c_1\lambda_1\bar{v}_1 + \ldots + c_k\lambda_k\bar{v}_k \quad (i)$$

Multiply (*) by λ_{k+1}

$$\lambda_{k+1}\bar{v}_{k+1} = c_1\lambda_{k+1}\bar{v}_1 + \ldots + c_k\lambda_k\bar{v}_k \quad (ii)$$

(1) - (2) gives us

$$\bar{0} = c_1(\lambda_1 - \lambda_{k+1})\bar{v}_1 + \ldots + c_k(\lambda_k - \lambda_{k+1})$$

We know $c_1, \ldots, c_k$ are not all zero b/c then $\bar{v}_{k+1} = \bar{0}$ by definition of eigenvectors.
We know $\lambda_1, \ldots, \lambda_k \neq \lambda_{k+1}$ b/c we are dealing w/ distinct eigenvectors.

This is a contradiction, so our assumption was wrong, so the conclusion holds.

Example:

Let A be an invertible matrix.
If λ is an eigenvalue of A, find an eigenvalue for A^{-1}. A.

$1/\lambda$ is an eigenvalue for A^{-1}

$$A\bar{v} = \lambda\bar{v}$$
$$A^{-1}A\bar{v} = A^{-1}\lambda\bar{v}$$
$$\bar{v} = \lambda A^{-1}\bar{v}$$
$$\underline{1/\lambda\ \bar{v} = A^{-1}\bar{v}}$$ ← valid b/c $\lambda \neq 0$ b/c it's an eigenvalue

Example:

Prove A is not inv. iff $\lambda = 0$ is eigenvalue

A not inv
$\Leftrightarrow \det(A) = 0$
$\Leftrightarrow \prod_{i=1}^{n} \lambda_i = 0$
$\Leftrightarrow \exists$ eigenvalue λ s.t. $\lambda = 0$
$\Leftrightarrow \lambda = 0$ is eigenvalue

Example:

Suppose λ is an eigenvalue of A.
Prove λ^n is an eigenvalue for A^n. $\forall n \in \mathbb{N}$.

We build an inductive proof.

(Just doing $\det(A) = \det(A)^n$ is not sound proof.)

When $n=1$, $A^1 = A$, so λ is given as an eigenvalue

When $n>1$, assume λ^{n-1} is eigenvalue of A^{n-1}.

$A^n \vec{v} = A(A^{n-1}\vec{v}) = A(\lambda^{n-1}\vec{v}) = \lambda^{n-1}(A\vec{v}) = \lambda^{n-1}(\lambda\vec{v}) = \lambda \vec{v}$

$\Rightarrow \lambda^n$ is eigenvalue of A^n.

By the pricipal of mathematic induction (PMI), λ^n is eigenvalue for A^n w/ eigenvector λ.

Example:

Let $C = B^{-1}AB$ w/ eigenpair $(\lambda, \vec{v})$.
Find eigenpair for A

$B^{-1}AB\vec{v} = \lambda\vec{v}$
$AB\vec{v} = B(\lambda\vec{v})$
$A(B\vec{v}) = \lambda(B\vec{v})$

$\therefore (\lambda, B\vec{v})$ eigenpair for A.

Example:

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $T: M_{2\times2} \to M_{2\times2}$, $T(B) = AB - BA$

Show $\vec{e} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ eigenvector for T.

$T(\vec{e}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

$\vec{e}$ eigenvector w/ corresponding eigenvalue 2.

Thm:

$C = B^{-1}AB \Leftrightarrow BCB^{-1} = A$.

The proof is trivial by matrix multiplication.

Def: Symmetric Matrixes

Let $A, B \in M_{n\times n}$. A & B are similar matrixes iff
$A = PBP^{-1}$
for some invertible matrix P.

Properties: Symmetry of Matrixes

- Symmetry: A similar to $B \Leftrightarrow B$ similar to A.
  $A = PBP^{-1} \Leftrightarrow B = P^{-1}A(P^{-1})^{-1} = P^{-1}AP$
- Reflexive: A similar to A. (P is identity)

- Transitivity: A similar to B & B similar to C implies A similar to C.

All of these properties mean symmetry of matrixes is an <u>equivalence relation</u>. (Allows algebra & equivalence classes.)

<u>Thm:</u> Similar Matrixes & Eigenvalues
Similar matrixes have the same characteristic polynomial & thus the same eigenvalues.

<u>Pf:</u>
Let $A = PBP^{-1}$, A, B similar.

$$
\begin{aligned}
&\det(A - \lambda I) \\
&= \det(PBP^{-1} - \lambda I) \\
&= \det(PBP^{-1} - \lambda PIP^{-1}) \\
&= \det(P(B - \lambda I)P^{-1}) \quad \text{distribute determinant} \\
&= \det(P)\det(B - \lambda I)\det(P^{-1}) \\
&= \cancel{\det(P)}\det(B - \lambda I)\cancel{\det(P)^{-1}} \\
&= \det(B - \lambda I) \quad \square
\end{aligned}
$$

<u>Note:</u> In terms of linear maps $A = PBP^{-1}$ means A & B are the same linear map in terms of different bases. P transforms the bases.

<u>Def:</u> Diagonalizable
A square matrix A is <u>diagonalizable</u> if it is similar to a diagonal matrix D.

$$A = PDP^{-1} \Longleftrightarrow AP = PD$$

Why do we care? It makes finding eigenvectors way easier.

<u>Example:</u>
Check that $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ diagonalizes $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ where $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

We check that $AP = PD$.

$$AP = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -5 & -6 \end{bmatrix} \checkmark$$

$$PD = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -5 & -6 \end{bmatrix}$$

Example:

Use the same matrixes as above.

If A similar to B, then A^{-1} similar to B^{-1}!

Compute A^k $k \in \mathbb{N}$

$A^k = PDP^{-1} \cdot PDP^{-1} \cdot \ldots \cdot PDP^{-1} = PD^kP^{-1}$

We compute D^k

$D^2 = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$, $D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$

We now compute $A^k = PD^kP^{-1}$ (We don't do it here to save time).

Compute A^{-k} $k \in \mathbb{N}$

$A^{-k} = (A^{-1})^k$

$A^{-1} = (PDP^{-1})^{-1} = P^{-1} D^{-1} (P^{-1})^{-1} = P^{-1} D^{-1} P$

$A^{-k} = P^{-1} D^{-k} P = P^{-1} (D^{-1})^k P$

$D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 5^{-1} & 0 \\ 0 & 3^{-1} \end{bmatrix}$

For diagonal matrixes, the power just distributes along the main diagonal.

$A^{-k} = P^{-1} \begin{bmatrix} 5^{-k} & 0 \\ 0 & 3^{-k} \end{bmatrix} P$

We can then multiply it out as you'd expect.

Thm:

Let $A \in M_{n \times n}$. A is diagonalizable iff A has $\underbrace{n \text{ l.i. eigenvectors}}_{}$. basis!

In this case, $A = PDP^{-1}$ where columns of P are l.i. eigenvectors & the diagonal entries of D are the corresponding eigenvalues.

ensure order consistent!

Pf: $\Rightarrow$

Suppose you have $A = PDP^{-1} \Leftrightarrow AP = PD$.

$AP = A \begin{bmatrix} \bar{v}_1 & \ldots & \bar{v}_n \\ \vert & & \vert \end{bmatrix} = \begin{bmatrix} A\bar{v}_1 & \ldots & A\bar{v}_n \\ \vert & & \vert \end{bmatrix}$

Not every square matrix is diagonalizable.

$PD = \begin{bmatrix} \bar{v}_1 & \ldots & \bar{v}_n \\ \vert & & \vert \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1\bar{v}_1 & \ldots & \lambda_n\bar{v}_n \\ \vert & & \vert \end{bmatrix}$

Since $AP = PD$,

$A\bar{v}_1 = \lambda_1 \bar{v}_1, \ldots, A\bar{v}_n = \lambda_n \bar{v}_n$

$\Rightarrow (\lambda_i, \bar{v}_i)$ eigenpairs for A.

Since P must be invertible, $\bar{v}_1, \ldots, \bar{v}_n$ are invertible. $\square$

The reverse direction follows similarly.

Example:

If A diagonalizable, are P & D unique? No. Reorder the eigenpairs or scale the eigenvectors.

For each eigenvalue, there are infinitely many eigenvectors from its eigenspace.

Example:

Diagonalize $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ if possible.

We find the eigenvalues of A.

$$0 = \det(A - \lambda I)$$

$$= \begin{vmatrix} 1- & 3 & 3 \\ -3 & -5- & -3 \\ 3 & 3 & 1- \end{vmatrix} \quad R_1 + R_2 + R_3$$

$$= \begin{vmatrix} 1-\lambda & 1-\lambda & 1-\lambda \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} \quad R_2 + 3R_1$$

$$= (1-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2-\lambda & 0 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-2-\lambda)(-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-2-\lambda)(1-\lambda-3)$$

$$= (1-\lambda)(-2-\lambda)^2 = 0$$

$\lambda_1 = 1$

$\lambda_2 = -2$ (double)

$V_{\lambda_1} = \text{span}\left\{ \overset{\bar{v}_1}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}} \right\}$ Given for brevity

$V_{\lambda_2} = \text{span}\left\{ \underset{\bar{v}_2}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\bar{v}_3}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\}$

We have 3 l.i. eigenvectors, so we can diagonalize A w/

$$P = \begin{array}{c} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ \bar{v}_1 \ \ \bar{v}_2 \ \ \bar{v}_3 \end{array} \ \& \ D = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ \lambda_1 \ \ \lambda_2 \ \ \lambda_2 \end{array}.$$

Example:

Is $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ diagonalizable?

Given the characteristic polynomial
$p(\lambda) = \det(A - \lambda I) = (2-\lambda)(\lambda-1)^2$

Given
$V_{\lambda=2} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$
$V_{\lambda=1} = \text{span}\left\{\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}\right\}$

Since we only have 2 eigenvectors, A is not diagonalizable.

Thm:

Let $A \in M_{n \times n}$ w/ $\lambda_1, \ldots, \lambda_k$ $(k \leq n)$ distinct eigenvalues w/ $m_1, \ldots, m_k$ corresponding algebraic multiplicities. (multiplicity of solution to characteristic polynomial)

A is diagonalizable iff $\dim V_{\lambda_i} = m_i$ for $i = 1, \ldots, k$

geometric multiplicity | algebraic multiplicity

Example:

Find a 2x2 matrix that is

i) Invertible but not diagonalizable.
$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ inv, $\lambda_1 = 1$ $m_1 = 2$
$\dim V_{\lambda_1} = 1$

ii) Not invertible but diagonalizable
$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$
$\det A = 0 \Leftrightarrow$ not inv
$\lambda_1 = 2$, $\lambda_2 = 0$
distinct eigenvalues $\Leftrightarrow$ l.i. eigenvectors

iii) Not invertible & not diagonalizable
$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\det A = 0 \Leftrightarrow$ not inv
$\lambda_1 = 0$ $m_1 = 2$
$\dim V_{\lambda_1} = 1$

iv) Invertible & diagonalizable
$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
$\det A \neq 0 \Leftrightarrow$ inv
$\lambda_1 = 1$, $\lambda_2 = 2$
distinct eigenvalues $\Leftrightarrow$ l.i. eigenvectors

Recall:
- $\det A$ = product of eigenvalues
- A inv $\Leftrightarrow$ $\det(A) \neq 0$
- Eigenvalues of triangular matrix along diagonal
- Eigenvectors corresponding to distinct eigenvalues are l.i.

Eigenvalues & Eigenvectors with Systems of Diff. Eqs.

Example: 1 Eqn

$y(t) = ?$

$y'(t) = k\,y(t)$

k constant

$y(0) = y_0$

Solution is $y(t) = Ce^{kt} = y_0 e^{kt}$

Example: 2 Eqn Diagonal Matrix

$$\begin{cases} y_1' = -3y_1 \\ y_2' = 2y_2 \end{cases} \Rightarrow \bar{y}' = \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}}_{D} \bar{y} \Rightarrow \bar{y}' = D\bar{y}$$

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

uncoupled system!

Using earlier method (b/c it's an uncoupled system)

$$\begin{aligned} y_1 &= y_1(0)\,e^{-3t} \\ y_2 &= y_2(0)\,e^{2t} \end{aligned} \Rightarrow \bar{y}(t) = \underbrace{\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{bmatrix}}_{e^{Dt}} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} \Rightarrow \bar{y} = e^{Dt}\bar{y}_0$$

e^{Dt} ← matrix exponential of D

What if the matrix of the system is not diagonal

Recall the definition of e^a

$$e^a = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \dots$$

We extend this to a matrix A

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

The power of arbitrary matrixes is hard. Let's do where matrix is

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$e^D = I + \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix} + \dots$$

$$e^D = \begin{bmatrix} (1 + \lambda_1 + \frac{1}{2!}\lambda_1^2 + \dots) & 0 & \dots \\ 0 & (1 + \lambda_2 + \frac{1}{2!}\lambda_2^2 + \dots) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$e^{D} = \begin{bmatrix} e^{\lambda_1} & & 0 \\ 0 & \ddots & e^{\lambda_n} \end{bmatrix}$$

If some arbitrary matrix A is diagonalizable, $A = PDP^{-1}$, P inv., D diag.

$$e^{A} = I + PDP^{-1} + \frac{1}{2!}(PDP^{-1})^2 + \dots$$

$$e^{A} = I + PDP^{-1} + \frac{1}{2!} PD^2 P^{-1}$$

$$e^{A} = P(I + D + \frac{1}{2!} D^2) P^{-1}$$

$$e^{A} = P e^{D} P^{-1}$$

Thm: Solve Uncoupled system

Suppose $\bar{y}' = D\bar{y}$, $\bar{y}(0) = \bar{y}_0$ is an uncoupled system.

$\bar{y} = e^{Dt}\bar{y}_0$ is a solution.

Thm: Solve Coupled System

Suppose $\bar{y}' = A\bar{y}$, $\bar{y}(0) = \bar{y}_0$ is a coupled system & A is similar to diagonalizable matrix D ($A = PDP^{-1}$).

$$\bar{y}' = A\bar{y}$$
$$\Rightarrow \bar{y}' = PDP^{-1}\bar{y}$$
$$\Rightarrow P^{-1}\bar{y}' = DP^{-1}\bar{y}$$

We change the variables

$$\bar{w} = P^{-1}\bar{y} \quad \Leftrightarrow \quad \bar{y} = P\bar{w}$$
$$\bar{w}(0) = P^{-1}\bar{y}_0 = \bar{w}_0$$

Thus we have

$$\bar{w}' = D\bar{w}$$
$$\bar{w}(0) = \bar{w}_0$$

which gives us

$$\bar{w} = e^{Dt}\bar{w}_0$$
$$\Rightarrow P^{-1}\bar{y} = e^{Dt}P^{-1}\bar{y}_0$$
$$\Rightarrow \bar{y} = \underbrace{P e^{Dt} P^{-1}}_{e^{At}} \bar{y}_0$$

Therefore

$$\boxed{\bar{y} = e^{At}\bar{y}_0}$$

solves the system.

Going further,

$$\bar{y} = \underbrace{\begin{bmatrix} \bar{v}_1 & \dots & \bar{v}_n \\ | & & | \end{bmatrix}}_{\substack{P \\ \text{eigenvectors}}} \overbrace{\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ 0 & \ddots & e^{\lambda_n t} \end{bmatrix}}^{e^{Dt}} \cdot \overbrace{P^{-1}\bar{y}_0}^{\text{known}} = \begin{bmatrix} e^{\lambda_1 t}\bar{v}_1 & \dots & e^{\lambda_n t}\bar{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\boxed{\bar{y} = c_1 e^{\lambda_1 t}\bar{v}_1 + \dots + c_n e^{\lambda_n t}\bar{v}_n} \longleftarrow \text{where diff eq gets the equation!}$$

<u>Thm:</u> Real Symmetric
Any real symmetric $n \times n$ matrix has n l.i. eigenvectors.

A real symmetric matrix A is one w/ only real entries & one where $A = A^T$.

For example
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

How do you handle cases where there's not so nice solutions? You use generalized eigenvectors from defective matrices.

<u>Def:</u> Generalized Eigenvectors
Let $A \in M_{n \times n}$ w/ λ eigenvalue.

A vector $\bar{u} \in \mathbb{R}^n$ is called a generalized eigenvector for λ iff $(A - \lambda I)^k \bar{u} = \bar{0}$ for some $k \in \mathbb{Z}^+$.

Note: Any normal eigenvector is a generalized eigenvector (where $k=1$).

Suppose we want to solve $\bar{x}' = A\bar{x}$.

We solve the characteristic polynomial $p(\lambda) = \det(A - \lambda I) = 0$ &
eigenvalues: $\lambda_1, \ldots, \lambda_k$
w/ multiplicities: $m_1, \ldots, m_k$.

Our goal is to have m_i l.i. eigenvectors for each eigenvalue λ_i.

Suppose each fundamental solution of $\bar{x}' = A\bar{x}$ has the form
$$e^{\lambda_i t}\left[\bar{u} + t(A - \lambda_i I)\bar{u} + \frac{t^2}{2!}(A - \lambda_i I)^2 \bar{u} + \cdots\right]$$
Sum is guaranteed to reduce to finite number of terms

where $\bar{u}$ is a generalized eigenvector.

Because $\bar{u}$ is a generalized eigenvector $(A - \lambda I)^k \bar{u} = \bar{0}$ for some $k > 0$. That means we have finitely many terms.

Example: Missing 1

Let $A \in M_{3\times3}$ w/ λ triple eigenvector w/ 2 l.i. eigenvectors $\bar{v}_1, \bar{v}_2$.

Our fundamental solutions of $\bar{x}' = A\bar{x}$ are

$\bar{x}_1 = e^{\lambda t}\bar{v}_1$

$\bar{x}_2 = e^{\lambda t}\bar{v}_2$

$\bar{x}_3 = ?$

We look for a generalized eigenvector $\bar{u}$. We know $\bar{u}$ solves

$(A - \lambda I)^2 \bar{u} = \bar{0}$

$\Rightarrow (A - \lambda I)(A - \lambda I)\bar{u} = \bar{0}$

To solve this we want $(A-\lambda I)\bar{u}$ to be a vector $\bar{u}'$ s.t. $(A-\lambda I)\bar{u}' = \bar{0}$. Luckily, we know $\bar{u}'$ must be an eigenvector ($\bar{v}_1$ or $\bar{v}_2$). We therefore find $\bar{u}$ by solving

$(A - \lambda I)\bar{u} = \bar{v}_1$ (or $\bar{v}_2$)

This gives us

$$\bar{x}_3 = e^{\lambda t}\left[\bar{u} + t\overbrace{(A-\lambda I)\bar{u}}^{v_1} + \overbrace{0 + \dots}^{\text{rest zeroed out b/c } (A-\lambda I)^2\bar{u} = \bar{0}}\right]$$

$$\bar{x}_3 = e^{\lambda t}[\bar{u} + t\bar{v}_1]$$

This gives us 3 fundamental solutions.

Example: Missing 3

Let $A \in M_{3\times3}$ w/ λ triple eigenvector w/ 1 eigenvector $\bar{v}$

Our fundamental solutions of $\bar{x}' = A\bar{x}$ are

$\bar{x}_1 = e^{\lambda t}\bar{v}$

$\left.\begin{array}{l}\bar{x}_2 = ?\\ \bar{x}_3 = ?\end{array}\right\}$ generalized eigenvector solutions

We find the first generalized eigenvector $\bar{u}$ similar to earlier

$(A-\lambda I)^2\bar{u} = \bar{0}$

$\Rightarrow (A-\lambda I)\underbrace{(A-\lambda I)\bar{u}}_{\bar{v}} = \bar{0}$

$\Rightarrow (A-\lambda I)\bar{u} = \bar{v}$

We find the next generalized eigenvector $\bar{w}$ going to the next power

$(A-\lambda I)^3\bar{w} = \bar{0}$

$\Rightarrow (A-\lambda I)^2\underbrace{(A-\lambda I)\bar{w}}_{\bar{u}} = \bar{0}$

$\Rightarrow (A-\lambda I)\bar{w} = \bar{u}$

Thus the fundamental solutions are

$\bar{x}_1 = e^{\lambda t}\bar{v}$

$\bar{x}_2 = e^{\lambda t}[\bar{u} + \bar{v}t]$

$$\bar{x}_3 = e^{\lambda t}\left[\bar{w} + t(A-\lambda I)\bar{w} + \frac{1}{2!}t^2(A-\lambda I)^2\bar{w} + \overbrace{0 + \dots}^{\text{rest zeroed out b/c } (A-\lambda I)^3\bar{w} = \bar{0}}\right]$$

$$= e^{\lambda t}\left[\bar{w} + t\bar{u} + \frac{1}{2!}t^2\bar{v}\right]$$

The method follows for larger vectors.

Inner Product Spaces

We want to generalize taking the product of 2 vectors. (e.g. dot product).

Def:

An *inner product* on a real v.s. V is a function

$$\langle \bar{v}, \bar{u} \rangle : V \times V \to \mathbb{R}$$

such that

i) $\langle \bar{v}, \bar{v} \rangle \geq 0 \quad \forall \bar{v} \in V$	Positivity	
ii) $\langle \bar{v}, \bar{v} \rangle = 0$ iff $\bar{v} = \bar{0}$	Definite	
iii) $\langle \bar{v} + \bar{u}, \bar{w} \rangle = \langle \bar{v}, \bar{w} \rangle + \langle \bar{u}, \bar{w} \rangle$	Additivity	} apply to 2nd term by symmetry
iv) $\langle \lambda \bar{u}, \bar{v} \rangle = \lambda \langle \bar{u}, v \rangle$	Homogeneity	}
v) $\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$	Symmetry	(not commutative b/c dealing w/ $A \times A \to B$ not $A \times A \to A$)
vi) $\langle \bar{v}, \bar{0} \rangle = 0$		

Examples: Dot Product

The dot product of vectors on $\mathbb{R}^n$ is an inner product

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = x_1 y_1 + \dots + x_n y_n$$

By homogeneity $\langle \bar{v}, \bar{0} \rangle = \bar{0}$

Example: Weighted Dot Product

On $\mathbb{R}^n$

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = c_1 x_1 y_1 + \dots + c_n x_n y_n \quad \text{where } c_i > 0$$

Example: Functions (PDEs taste)

$$V = \{ f : [-1, 1] \to \mathbb{R} \mid f \text{ cont. on } [-1, 1] \}$$

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$$

Example:

On $\mathcal{P}$

$$\langle p, q \rangle = \int_0^\infty p(x) q(x) e^{-x}\, dx$$

← guarantees improper integral is defined

Def: Inner Product Space

A vector space V w/ an inner product is called an *inner product space.*

Def: Norm

The norm is a function

$$\| \bar{v} \| : V \to \mathbb{R}^+$$

where

$$\| \bar{v} \| = \sqrt{\langle \bar{v}, \bar{v} \rangle}$$

The norm is also called *length of a vector.*

Some properties of the norm are

- $||\bar{v}|| = \bar{0} \Leftrightarrow \bar{v} = \bar{0}$
- $||\bar{v}|| \geq 0$
- $||\lambda \bar{v}|| = |\lambda| \cdot ||\bar{v}||$
- $|\langle \bar{u}, \bar{v} \rangle| \leq ||\bar{u}|| \cdot ||\bar{v}||$ Cauchy-Schwartz Inequality

Triangle Inequality

- $||\bar{u} + \bar{v}|| \leq ||\bar{u}|| + ||\bar{v}||$

Equal only in degenerate triangle (i.e. two $0°$ angles & one $180°$ angle) aka line

Def: Orthogonality
Two vectors $\bar{v}, \bar{u} \in V$ are orthogonal iff $\langle \bar{v}, \bar{u} \rangle = \bar{0}$.

Example:
Which of the following functions are inner products?

i) $V = \mathbb{R}^2$; $\langle \bar{u}, \bar{v} \rangle = u_1 v_1 - 2u_1 v_2 - 2u_2 v_1 + 3u_2 v_2$
No. When $\bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\langle \bar{v}, \bar{v} \rangle = 0$.

ii) $V = \mathbb{R}^3$; $\langle \bar{u}, \bar{v} \rangle = u_1^2 v_1^2 + u_2^2 v_2^2$
No. When $\lambda = 2$ & $\bar{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\lambda \langle \bar{v}, \bar{v} \rangle \neq \langle \lambda \bar{v}, \bar{v} \rangle$ alternatively $\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle = 0$

$\langle \lambda \bar{v}, \bar{v} \rangle = \langle \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle = 4 + 4 = 8$

$\lambda \langle \bar{v}, \bar{v} \rangle = 2 \langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle = 2(1+1) = 4$

iii) $V = M_{m \times n}$; $\langle A, B \rangle = \text{trace}(B^t A)$
This is an inner product. To see this, let's do an example where $m = n = 2$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$; $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$

$\text{trace}\left(\begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = \text{trace} \begin{bmatrix} b_{11} a_{11} + b_{21} a_{21} & \text{—} \\ \text{—} & b_{12} a_{12} + b_{22} a_{22} \end{bmatrix} = b_{11} a_{11} + b_{21} a_{21} + b_{12} a_{12} + b_{22} a_{22}$

This looks like the dot product on $\mathbb{R}^4$ & $M_{2 \times 2}$ & $\mathbb{R}^4$ are isomorphic.
Although this isn't a proof, this is an inner product.

iv) $V = P_n$; $\langle p, q \rangle = \sum_{i=0}^{n} p_i q_i$ where $p = p_0 + p_1 x + \dots + p_n x^n$ & $q = q_0 + q_1 x + \dots + q_n x^n$
This is an inner product & again looks a lot like the dot product.

Example: Integer Inner Product
$V = P_2$; $\langle p, q \rangle = \int_{-1}^{1} p(x) q(x) dx$

Show that $S = \{1, x, \frac{1}{2}(3x^2 - 1)\}$ are mutually orthogonal & find their length (norm).

$\langle 1, x \rangle = \int_{-1}^{1} (1)(x) dx = \frac{1}{2} x^2 \Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$ ✓

$\langle 1, \frac{1}{2}(3x^2 - 1) \rangle = \int_{-1}^{1} (1)(\frac{1}{2}(3x^2 - 1)) dx = \frac{1}{2} \left[\frac{1}{2} x^3 - \frac{1}{2} x \right]_{-1}^{1} = \frac{1}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{1}{2}(-\frac{1}{2} + \frac{1}{2}) = 0$ ✓

$\langle x, \frac{1}{2}(3x^2 - 1) \rangle = \dots = 0$ ✓

We find their lengths (norms)

$||1|| = \sqrt{\int_{-1}^{1} (1)(1) dx} = \sqrt{2}$

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{\left[\tfrac{1}{3}x^3\right]_{-1}^{1}} = \sqrt{\tfrac{1}{3}+\tfrac{1}{3}} = \sqrt{\tfrac{2}{3}}$$

$$\|\tfrac{1}{3}(3x^2-1)\| = \ldots$$

Example:

Use the same set S but the inner product
$$\langle\langle p, q\rangle\rangle = p_0 q_0 + p_1 q_1 + p_2 q_2.$$

Are they still orthogonal? No!

Different inner product spaces have different orthogonal vectors

$$\langle\langle 1, x\rangle\rangle = 1\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0$$

$$\langle\langle 1, \tfrac{1}{2}(3x^2-1)\rangle\rangle = 1\cdot -\tfrac{1}{2} + 0\cdot 0 + \tfrac{3}{2}\cdot 0 = -\tfrac{1}{2} \neq 0$$

Def: Orthogonal & Orthonormal Sets

A set of vectors $S = \{\bar{v}_1, \ldots, \bar{v}_n\}$ in an inner product space is said to be orthogonal iff the vectors are mutually orthogonal.
That is
$$\forall i, j \in \{1, \ldots, n\}\ i \neq j,\ \langle v_i, v_j\rangle = 0$$

The zero vector is orthogonal to everything

If in addition
$$\|\bar{v}_i\| = 1 \quad \forall i \in \{1, \ldots, n\}$$
the set S is called orthonormal.

Thm: Linear Independence & Orthogonality

Let V be an inner product space & $S = \{\bar{v}_1, \ldots, \bar{v}_n\}$ be an orthogonal set of non-zero vectors.

Then S is l.i.

Pf:

Let $c_1\bar{v}_1 + \ldots + c_n\bar{v}_n = \bar{0}$.
We want to show $c_1 = \ldots = c_n = 0$ is the only solution.

Let c_i be one of the scalars.
$\langle \bar{v}_i, c_1\bar{v}_1 + \ldots + c_n\bar{v}_n\rangle = \langle \bar{v}_i, \bar{0}\rangle$) distribute & pull out scalar
$$\underbrace{c_1\langle \bar{v}_i, \bar{v}_1\rangle}_{=0} + \ldots + \underbrace{c_n\langle \bar{v}_i, \bar{v}_n\rangle}_{\langle \bar{v}_i, \bar{v}_i\rangle \neq 0} = \underbrace{0}_{=0}$$
$$c_i\langle \bar{v}_i, \bar{v}_i\rangle = 0$$

Since $\bar{v}_i \neq 0$, $\langle \bar{v}_i, \bar{v}_i\rangle \neq 0$, so $c_i = 0$

Since we chose an arbitrary c_i, $c_1 = \ldots = c_n = 0$. □

Corollary: Basis & Orthogonal Set
If V is an inner product space of dimension n, then any set of n orthogonal vectors is a basis for V.

Thm:
Let $B = \{\bar{v}_1, \ldots, \bar{v}_n\}$ be an ordered orthonormal basis for v.s. V.

Suppose $\bar{v} \in V$ can be written as $\bar{v} = c_1\bar{v}_1 + \ldots + c_n\bar{v}_n$.

Then the coordinates of $\bar{v}$ wrt B are given by
$c_i = \langle \bar{v}_i, \bar{v} \rangle \quad i \in \{1, \ldots, n\}$

sometimes really easy!

Pf:
Let $\bar{v} = c_1\bar{v}_1 + \ldots + c_n\bar{v}_n$

$\langle \bar{v}_i, \bar{v} \rangle = \langle \bar{v}_i, c_1\bar{v}_1 + \ldots + c_n\bar{v}_n \rangle$
$= c_1\langle \bar{v}_i, \bar{v}_1 \rangle + \ldots + c_n\langle \bar{v}_i, \bar{v}_n \rangle \leftarrow$ all $\langle \bar{v}_i, \bar{v}_j \rangle = 0$ when $i \neq j$ b/c B is orthogonal set
$= c_i \langle \bar{v}_i, \bar{v}_i \rangle$
$= c_i \|\bar{v}_i\| \leftarrow$ b/c orthonormal $\|\bar{v}_i\| = 1$
$= c_i$

$\therefore \langle \bar{v}_i, \bar{v} \rangle = c_i$

Corollary: Normalizing Orthogonal Sets
Let $B = \{\bar{u}_1, \ldots, \bar{u}_n\}$ be an orthogonal basis for $(V, \langle , \rangle)$ & $\bar{v} \in V$.
Then
$$\bar{v} = \frac{\langle \bar{v}, \bar{u}_1 \rangle}{\underbrace{\langle \bar{u}_1, \bar{u}_1 \rangle}_{\text{or } \|\bar{u}_1\|^2}}\bar{u}_1 + \ldots + \frac{\langle \bar{v}, \bar{u}_n \rangle}{\langle \bar{u}_n, \bar{u}_n \rangle}\bar{u}_n$$
b/c you're doing $\left\langle \frac{\bar{u}_i}{\|\bar{u}_i\|}, \bar{v} \right\rangle = \frac{1}{\|\bar{u}_i\|^2}\langle \bar{u}_i, \bar{v} \rangle$
or $\langle \bar{u}_i, \bar{u}_i \rangle$

Examples:
$V = \mathcal{P}_2$; $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$
$S = \{1, x, \frac{1}{2}(3x^2-1)\}$ orthogonal set (basis!)

Find the l.c. for $f = 1 + x - 2x^2$ wrt basis S

$\langle 1, f \rangle = \int_{-1}^{1} 1 + x - 2x^2\,dx = x + \frac{1}{2}x^2 - \frac{2}{3}x^3 \Big|_{-1}^{1} = (1 + \cancel{\tfrac{1}{2}} - \frac{2}{3}) - (-1 + \cancel{\tfrac{1}{2}} + \frac{2}{3}) = 2 - \frac{4}{3} = \underline{\frac{2}{3}}$

$\langle x, f \rangle = \int_{-1}^{1} x + x^2 - 2x^3\,dx = \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 \Big|_{-1}^{1} = (\cancel{\tfrac{1}{2}} + \frac{1}{3} - \cancel{\tfrac{1}{2}}) - (\cancel{\tfrac{1}{2}} - \frac{1}{3} - \cancel{\tfrac{1}{2}}) = \underline{\frac{2}{3}}$

$\langle \frac{1}{2}(3x^2-1), f \rangle = \ldots = -\frac{8}{15}$

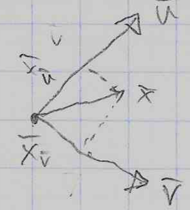
$\langle 1, 1 \rangle = \ldots = 2$
$\langle x, x \rangle = \ldots = \frac{2}{3}$
$\langle \frac{1}{2}(3x^2-1), \frac{1}{2}(3x^2-1) \rangle = \ldots = \frac{2}{5}$

$$f = \frac{2/3}{2}(1) + \frac{\cancel{2/3}}{\cancel{2/3}}(x) + \frac{-\cancel{8}^{4}/\cancel{15}_{3}}{\cancel{2}/\cancel{5}}\left(\tfrac{1}{2}(3x^2-1)\right) = \tfrac{1}{3}(1) + 1(x) - \tfrac{4}{3}\left(\tfrac{1}{2}(3x^2-1)\right)$$

$$\Rightarrow [f]_S = \begin{bmatrix} 1/3 \\ 1 \\ -4/3 \end{bmatrix} \leftarrow \text{Could have gone straight here}$$

Note: Orthogonal Projections

$\mathbb{R}^2$ w/ $\bar{u}, \bar{v}$ orthogonal basis for $\mathbb{R}^2$



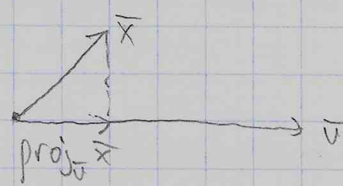
$\bar{x} = \bar{x}_{\bar{u}} + \bar{x}_{\bar{v}}$ (orthogonal projection of $\bar{x}$ onto $\bar{u}$)

$$\bar{x} = \frac{\langle \bar{x}, \bar{u} \rangle}{\langle \bar{u}, \bar{u} \rangle}\bar{u} + \frac{\langle \bar{x}, \bar{v} \rangle}{\langle \bar{v}, \bar{v} \rangle}\bar{v}$$

Def: Orthogonal Projection

Let $(V, \langle , \rangle)$ be an inner product space & $\bar{x}, \bar{u} \in V$.
The orthogonal projection of $\bar{x}$ onto $\bar{u}$ is defined to be the vector

$$\text{proj}_{\bar{u}}\, \bar{x} = \frac{\langle \bar{x}, \bar{u} \rangle}{\langle \bar{u}, \bar{u} \rangle}\bar{u}$$



Thm:
$\bar{x} - \text{proj}_{\bar{u}}\, \bar{x}$ & $\bar{u}$ are orthogonal.

Pf:

$$\langle \bar{x} - \text{proj}_{\bar{u}}\, \bar{x}, \bar{u} \rangle$$
$$= \langle \bar{x}, \bar{u} \rangle - \left\langle \left(\frac{\langle \bar{x}, \bar{u} \rangle}{\langle \bar{u}, \bar{u} \rangle}\right)\bar{u}, \bar{u} \right\rangle$$
$$= \langle x, \bar{u} \rangle - \frac{\langle \bar{x}, \bar{u} \rangle}{\cancel{\langle \bar{u}, \bar{u} \rangle}}\left(\cancel{\langle \bar{u}, \bar{u} \rangle}\right)$$
$$= 0 \quad \square$$

Example:

P_2, $\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx$

$p = x$, $q = x^2$

- $\text{proj}_q p = \frac{\langle p, q \rangle}{\langle q, q \rangle} q = \frac{\int_0^1 x^3}{\int_0^1 x^4} \cdot x^2 = \frac{\frac{1}{4}x^4 \Big|_0^1}{\frac{1}{5}x^5\Big|_0^1} \cdot x^2 = \frac{5}{4}x^2$

- Verify $p - \text{proj}_q p$ orthogonal

$$\langle p - \text{proj}_q p, q \rangle$$
$$= \langle x - \tfrac{5}{4}x^2, x^2 \rangle$$
$$= \int_0^1 x^3 - \tfrac{5}{4}x^4$$
$$= \tfrac{1}{4}x^4 - \tfrac{1}{4}x^5 \Big|_0^1$$
$$= 0 \checkmark$$

Remarks:

i) If $\bar{x}$ is orthogonal to every vector to $S = \{\bar{v}_1, \dots, \bar{v}_k\}$, then $\bar{x}$ is orthogonal to every vector in $\text{span}(S) = W$.

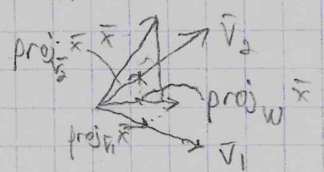
Think about a plane

We say $\bar{x}$ is orthogonal to W.

$$\langle \bar{x}, c_1\bar{v}_1 + \dots + c_k\bar{v}_k \rangle = c_1 \underbrace{\langle \bar{x}, \bar{v}_1 \rangle}_{=0} + \dots + c_k \underbrace{\langle \bar{x}, \bar{v}_k \rangle}_{=0} = 0 \checkmark$$

ii) If $\bar{v}_1, \dots, \bar{v}_k$ is an orthogonal basis for W & $\bar{x} \notin W$ then
$$\text{proj}_W \bar{x} = \text{proj}_{\bar{v}_1} \bar{x} + \dots + \text{proj}_{\bar{v}_k} \bar{x}$$

Suppose W is plane in $\mathbb{R}^3$



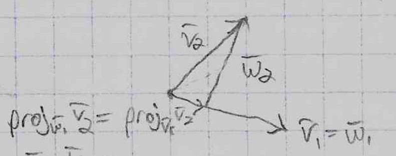
$\therefore$ get infinitely many orthogonal basis from one orthonormal

Thm: Existence of Orthonormal Basis
Any finite dimensional vector space has at least one orthonormal basis.

Pf: Gram-Schmidt Algorithm for Orthogonal Basis
Suppose $B = \{\bar{v}_1, \dots, \bar{v}_n\}$ basis for V.
We want to create an orthogonal basis $B' = \{\bar{w}_1, \dots, \bar{w}_n\}$.

Take $\bar{w}_1 = \bar{v}_1$.

Take $\bar{w}_2 = \bar{v}_2 - \text{proj}_{\bar{w}_1} \bar{v}_2$



Take $\bar{w}_3 = \bar{v}_3 - \text{proj}_{\text{span}\{\bar{w}_1, \bar{w}_2\}} \bar{v}_3 = \bar{v}_3 - \text{proj}_{\bar{w}_1}\bar{v}_3 - \text{proj}_{\bar{w}_2}\bar{v}_2$

This continues until you get B'. You can also normalize $\bar{w}_1, \dots, \bar{w}_k$ if you want. □

Here's a high-level overview.

1) Choose basis $B = \{\bar{v}_1, \dots, \bar{v}_k\}$.

2) Compute new vectors $B' = \{\bar{w}_1, \dots, \bar{w}_k\}$ orthogonal

$\bar{w}_1 = \bar{v}_1$

$$\bar{w}_n = \bar{v}_n - \left(\sum_{i=1}^{n-1} \text{proj}_{\bar{w}_i} \bar{v}_n\right) = \bar{v}_n - \left(\sum_{i=1}^{n-1} \frac{\langle \bar{v}_n, \bar{w}_i \rangle}{\langle \bar{w}_i, \bar{w}_i \rangle} \bar{w}_i\right)$$

Example:
Let $V = \text{span}\left\{\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}$. Find orthogonal basis for V. using dot product.

If you do this on a test, notate it.

We could change the order to possibly make things easier.

$\bar{w}_1 = \begin{bmatrix}-1\\1\\1\\0\end{bmatrix}$

$$\bar{w}_2 = \begin{bmatrix}-1\\0\\0\\0\end{bmatrix} - \frac{\langle \bar{v}_2, \bar{w}_1 \rangle}{\langle w_1, w_1 \rangle} \bar{w}_1 = \begin{bmatrix}-1\\0\\0\\0\end{bmatrix} - \frac{2}{3} \cdot \begin{bmatrix}-1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/3\\-2/3\\1/3\\0\end{bmatrix} \Rightarrow \begin{bmatrix}-1\\-2\\1\\0\end{bmatrix}$$

This is kind ugly. We can scale it to be integers, since we only care about direction

$$\bar{w}_3 = \bar{v}_3 - \overbrace{\frac{\langle \bar{v}_3, \bar{w}_1 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle} \bar{w}_1}^{\text{proj}_{\bar{w}_1} \bar{v}_3} - \frac{\langle \bar{v}_3, \bar{w}_2 \rangle}{\langle \bar{w}_2, \bar{w}_2 \rangle} \bar{w}_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{-1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{6} \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{3} - \frac{1}{6} \\ \frac{1}{3} - \frac{1}{3} \\ \frac{1}{3} + \frac{1}{6} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\Rightarrow \bar{w}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore B' = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Let's make an orthonormal basis B'' from B'

$$B'' = \left\{ \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \right\}$$