

Problem (1) (a) Let  $P = \{I \subsetneq R \mid I \text{ is an ideal of } R\}$  be the partially ordered set of proper ideals of  $R$ . Then  $\mathfrak{m} \in P$  is called a maximal ideal if it is a maximal element of this partially ordered set.

Equivalently, this means that  $R/\mathfrak{m}$  is a field.

(b) Let  $a \notin \mathfrak{m}$ . Then  $\langle a \rangle + \mathfrak{m} \supsetneq \mathfrak{m}$ , so  $\langle a \rangle + \mathfrak{m} = R$ . Hence there is  $b \in R$  and  $c \in \mathfrak{m}$  with  $ab + c = 1$ , so that  $ab = 1 - c$ . But now  $1 - c \in 1 + \mathfrak{m}$ , which by assumption only consists of units. Hence  $ab$  is a unit. But then both factors must be units, so that  $a$  must be a unit.

We have shown that  $R \setminus \mathfrak{m} \subseteq R^\times$ , i.e.  $\mathfrak{m} \supseteq R \setminus R^\times$ . But any proper ideal only consists of non-units, so that  $\mathfrak{m} \subseteq R \setminus R^\times$ . Hence  $\mathfrak{m} = R \setminus R^\times$ , which means exactly that  $R$  is local with unique maximal ideal  $\mathfrak{m}$ .

(c) Let  $\overline{\mathfrak{m}} \subseteq \mathbb{Q}[x, y]/\langle x^{20}, y^{20} \rangle$  be a maximal ideal. This corresponds to a maximal ideal  $\mathfrak{m} \subseteq \mathbb{Q}[x, y]$  with  $\langle x^{20}, y^{20} \rangle \subseteq \mathfrak{m}$ . Hence  $x^{20}, y^{20} \in \mathfrak{m}$ . But since  $\mathfrak{m}$  is a maximal ideal, it's also a prime ideal. Hence  $x, y \in \mathfrak{m}$ , hence  $\langle x, y \rangle \subseteq \mathfrak{m}$ . Since  $\langle x, y \rangle \subseteq \mathbb{Q}[x, y]$  is a maximal ideal, we have  $\langle x, y \rangle = \mathfrak{m}$ . Hence  $\mathbb{Q}[x, y]/\langle x^{20}, y^{20} \rangle$  has a unique maximal ideal, i.e. it's a local ring.

(d) One can see that the map

$$\begin{aligned} \mathbb{C}[x, y]/\langle x^3 - y^5 \rangle &\rightarrow \mathbb{C}[t^3, t^5], \\ [x] &\mapsto t^5, \\ [y] &\mapsto t^3 \end{aligned}$$

is an isomorphism.

Clearly,  $\mathbb{C}[t^3, t^5]$  is an integral domain but not a field. Hence  $\langle x^3 - y^5 \rangle \subseteq \mathbb{C}[x, y]$  is a prime ideal but not a maximal ideal.

Problem (2) (a) Let  $M$  and  $N$  be  $R$ -modules. Then the tensor product  $M \otimes_R N$  is an  $R$ -module equipped with a bilinear map  $\alpha : M \times N \rightarrow M \otimes_R N$  such that for any  $R$ -module  $P$  and any bilinear map  $\beta : M \times N \rightarrow P$  there is a unique linear map  $\tilde{\beta} : M \otimes_R N \rightarrow P$  with  $\beta = \tilde{\beta} \circ \alpha$ .

$$\begin{array}{ccc} M \times N & \xrightarrow{\alpha} & M \otimes_R N \\ & \searrow \beta & \downarrow \tilde{\beta} \\ & & P \end{array}$$

(b) Take any element of  $U^{-1}R \otimes_R M$ . Such an element can be written as  $\sum_{i=1}^n \frac{a_i}{b_i} c_i$  with  $a_i \in R$ ,  $b_i \in U$  and  $c_i \in M$ . But now

$$\sum_{i=1}^n \frac{a_i}{b_i} c_i = \sum_{i=1}^n \frac{(\prod_{j \neq i} b_j) a_i}{\prod_j b_j} c_i = \frac{1}{\prod_j b_j} \sum_{i=1}^n \left( \prod_{j \neq i} b_j \right) a_i c_i$$

with  $\prod_j b_j \in U$  and  $\sum_{i=1}^n \left( \prod_{j \neq i} b_j \right) a_i c_i \in M$ .

(c) We have

$$\begin{aligned} &\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z}/\langle 42 \rangle) \\ &\cong (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/\langle 42 \rangle) \\ &\cong \mathbb{Q} \oplus 0 \\ &\cong \mathbb{Q} \end{aligned}$$

(d) It is a well-known fact from linear algebra that every module over a field is free.

On the other hand, assume that every  $R$ -module is free. Now take  $a \in R$  with  $a \neq 0$ . Then, by assumption, the  $R$ -module  $R/\langle a \rangle$  is free. If it is free of nonzero rank, multiplication by  $a$  must be a nonzero endomorphism of it. However, multiplication by  $a$  is just the zero endomorphism. Hence it must be free of rank zero, i.e.  $R/\langle a \rangle \cong 0$ , so that  $\langle a \rangle = R$ , i.e.  $a$  is a unit. Hence every nonzero element of  $R$  is invertible, so that  $R$  is a field.

(e) Assume that  $R$  is an integral domain.

If  $R$  is even a field, then every  $R$ -module is free and hence flat.

On the other hand, assume that every  $R$ -module is flat. Now take  $a \in R$  with  $a \neq 0$ . Now since  $R$  is an integral domain, the map  $R \xrightarrow{a} R$  is an injective endomorphism of  $R$ . Since every  $R$ -module is flat,  $R/\langle a \rangle$  is flat, so that  $R/\langle a \rangle \xrightarrow{a} R/\langle a \rangle$  must also be injective. However, multiplication by  $a$  is just the zero endomorphism on  $R/\langle a \rangle$ . Hence we must have  $R/\langle a \rangle \cong 0$ , so that  $\langle a \rangle = R$ . Hence  $a$  is a unit. We have shown that every nonzero element of  $R$  is invertible, i.e. that  $R$  is a field.

- Problem (3) (a) A ring  $R$  is called Artinian if it satisfies the descending chain condition:  
Every infinite descending chain of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

eventually stabilizes, i.e. there is  $n \in \mathbb{N}$  with  $\forall m \geq n : I_m = I_n$ .

- (b) First of all, we will show that every Artinian domain is a field.

Let  $R$  be an Artinian domain. Take  $a \in R$  with  $a \neq 0$ . Then we have a descending chain

$$R \supseteq \langle a \rangle \supseteq \langle a^2 \rangle \supseteq \dots$$

This chain must eventually stabilize. Hence there is  $n \in \mathbb{N}$  with  $\langle a^n \rangle = \langle a^{n+1} \rangle$ . In particular, we have  $a^n \in \langle a^{n+1} \rangle$ . This means that there is  $b \in R$  with  $a^n = ba^{n+1}$ , hence  $0 = ba^{n+1} - a^n = a^n(ba - 1)$ . Since  $a \neq 0$  and  $R$  is a domain, we must in fact have  $0 = ba - 1$ , hence  $1 = ba$ . Hence  $a$  is a unit. Since every nonzero unit of  $R$  is invertible,  $R$  is a field.

Now we will show that in an Artinian ring every prime ideal is maximal.

Let  $R$  be Artinian and  $\mathfrak{p} \subseteq R$  be a prime ideal. Then  $R/\mathfrak{p}$  is an Artinian domain. By the first part,  $R/\mathfrak{p}$  is in fact a field. Hence  $\mathfrak{p} \subseteq R$  is in fact a maximal ideal.

- (c) We can compute

$$\begin{aligned} & \langle xy, xz, yz \rangle \\ &= \langle x, xz, yz \rangle \cap \langle y, xz, yz \rangle \\ &= \langle x, yz \rangle \cap \langle y, xz \rangle \\ &= \langle x, y \rangle \cap \langle x, z \rangle \cap \langle y, x \rangle \cap \langle y, z \rangle \\ &= \langle x, y \rangle \cap \langle x, z \rangle \cap \langle y, z \rangle \end{aligned}$$

All of these are clearly prime. Hence we have found a primary decomposition of  $\langle xy, xz, yz \rangle$ .

- (d) No. We have an infinite strictly descending chain of ideals

$$S \supsetneq \langle [x] \rangle \supsetneq \langle [x^2] \rangle \supsetneq \dots$$

Geometrically,  $S$  is the coordinate ring of a 1-dimensional affine algebraic set.

- (e)