- Problem (1) (a) Let  $P = \{I \subsetneq R \mid I \text{ is an ideal of } R\}$  be the partially ordered set of proper ideals of R. Then  $\mathfrak{m} \in P$  is called a maximal ideal if it is a maximal element of this partially ordered set. Equivalently, this means that  $R/\mathfrak{m}$  is a field.
  - (b) Let  $a \notin \mathfrak{m}$ . Then  $\langle a \rangle + \mathfrak{m} \supsetneq \mathfrak{m}$ , so  $\langle a \rangle + \mathfrak{m} = R$ . Hence there is  $b \in R$  and  $c \in \mathfrak{m}$  with ab + c = 1, so that ab = 1 c. But now  $1 c \in 1 + \mathfrak{m}$ , which by assumption only consists of units. Hence ab is a unit. But then both factors must be units, so that a must be a unit.

We have shown that  $R \setminus \mathfrak{m} \subseteq R^{\times}$ , i.e.  $\mathfrak{m} \supseteq R \setminus R^{\times}$ . But any proper ideal only consists of non-units, so that  $\mathfrak{m} \subseteq R \setminus R^{\times}$ . Hence  $\mathfrak{m} = R \setminus R^{\times}$ , which means exactly that R is local with unique maximal ideal  $\mathfrak{m}$ .

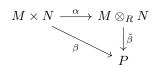
- (c) Let  $\overline{\mathfrak{m}}\subseteq \mathbb{Q}[x,y]/_{\left\langle x^{20},\,y^{20}\right\rangle}$  be a maximal ideal. This corresponds to a maximal ideal  $\mathfrak{m}\subseteq \mathbb{Q}[x,y]$  with  $\langle x^{20},y^{20}\rangle\subseteq \mathfrak{m}$ . Hence  $x^{20},y^{20}\in \mathfrak{m}$ . But since  $\mathfrak{m}$  is a maximal ideal, it's also a prime ideal. Hence  $x,y\in \mathfrak{m}$ , hence  $\langle x,y\rangle\subseteq \mathfrak{m}$ . Since  $\langle x,y\rangle\subseteq \mathbb{Q}[x,y]$  is a maximal ideal, we have  $\langle x,y\rangle=\mathfrak{m}$ . Hence  $\mathbb{Q}[x,y]/_{\left\langle x^{20},\,y^{20}\right\rangle}$  has a unique maximal ideal, i.e. it's a local ring.
- (d) One can see that the map

$$\begin{array}{c} \mathbb{C}[x,y]/\langle x^3-y^5\rangle \to \mathbb{C}[t^3,t^5], \\ [x] \mapsto t^5, \\ [y] \mapsto t^3 \end{array}$$

is an isomorphism.

Clearly,  $\mathbb{C}[t^3, t^5]$  is an integral domain but not a field. Hence  $\langle x^3 - y^5 \rangle \subseteq \mathbb{C}[x, y]$  is a prime ideal but not a maximal ideal.

Problem (2) (a) Let M and N be R-modules. Then the tensor product  $M \otimes_R N$  is an R-module equipped with a bilinear map  $\alpha: M \times N \to M \otimes_R N$  such that for any R-module P and any bilinear map  $\beta: M \times N \to P$  there is a unique linear map  $\tilde{\beta}: M \otimes_R N \to P$  with  $\beta = \tilde{\beta} \circ \alpha$ .



(b) Take any element of  $U^{-1}R \otimes_R M$ . Such an element can be written as  $\sum_{i=1}^n \frac{a_i}{b_i} c_i$  with  $a_i \in R$ ,  $b_i \in U$  and  $c_i \in M$ . But now

$$\sum_{i=1}^{n} \frac{a_i}{b_i} c_i = \sum_{i=1}^{n} \frac{\left(\prod_{j \neq i} b_j\right) a_i}{\prod_j b_j} c_i = \frac{1}{\prod_j b_j} \sum_{i=1}^{n} \left(\prod_{j \neq i} b_j\right) a_i c_i$$

with  $\prod_j b_j \in U$  and  $\sum_{i=1}^n \left(\prod_{j \neq i} b_j\right) a_i c_i \in M$ .

- (c)
- (d)
- (e)
- Problem (3) (a
  - (b)
  - (c)
  - (d)
  - (e)