

Problem (1) (a) Let $P = \{I \subsetneq R \mid I \text{ is an ideal of } R\}$ be the partially ordered set of proper ideals of R . Then $\mathfrak{m} \in P$ is called a maximal ideal if it is a maximal element of this partially ordered set.

Equivalently, this means that R/\mathfrak{m} is a field.

- (b) Let $a \notin \mathfrak{m}$. Then $\langle a \rangle + \mathfrak{m} \supsetneq \mathfrak{m}$, so $\langle a \rangle + \mathfrak{m} = R$. Hence there is $b \in R$ and $c \in \mathfrak{m}$ with $ab + c = 1$, so that $ab = 1 - c$. But now $1 - c \in 1 + \mathfrak{m}$, which by assumption only consists of units. Hence ab is a unit. But then both factors must be units, so that a must be a unit.

We have shown that $R \setminus \mathfrak{m} \subseteq R^\times$, i.e. $\mathfrak{m} \supseteq R \setminus R^\times$. But any proper ideal only consists of non-units, so that $\mathfrak{m} \subseteq R \setminus R^\times$. Hence $\mathfrak{m} = R \setminus R^\times$, which means exactly that R is local with unique maximal ideal \mathfrak{m} .

- (c) Let $\overline{\mathfrak{m}} \subseteq \mathbb{Q}[x, y]/\langle x^{20}, y^{20} \rangle$ be a maximal ideal. This corresponds to a maximal ideal $\mathfrak{m} \subseteq \mathbb{Q}[x, y]$ with $\langle x^{20}, y^{20} \rangle \subseteq \mathfrak{m}$. Hence $x^{20}, y^{20} \in \mathfrak{m}$. But since \mathfrak{m} is a maximal ideal, it's also a prime ideal. Hence $x, y \in \mathfrak{m}$, hence $\langle x, y \rangle \subseteq \mathfrak{m}$. Since $\langle x, y \rangle \subseteq \mathbb{Q}[x, y]$ is a maximal ideal, we have $\langle x, y \rangle = \mathfrak{m}$. Hence $\mathbb{Q}[x, y]/\langle x^{20}, y^{20} \rangle$ has a unique maximal ideal, i.e. it's a local ring.
- (d) One can see that the map

$$\begin{aligned} \mathbb{C}[x, y]/\langle x^3 - y^5 \rangle &\rightarrow \mathbb{C}[t^3, t^5], \\ [x] &\mapsto t^5, \\ [y] &\mapsto t^3 \end{aligned}$$

is an isomorphism.

Clearly, $\mathbb{C}[t^3, t^5]$ is an integral domain but not a field. Hence $\langle x^3 - y^5 \rangle \subseteq \mathbb{C}[x, y]$ is a prime ideal but not a maximal ideal.

Problem (2) (a) Let M and N be R -modules. Then the tensor product $M \otimes_R N$ is an R -module equipped with a bilinear map $\alpha : M \times N \rightarrow M \otimes_R N$ such that for any R -module P and any bilinear map $\beta : M \times N \rightarrow P$ there is a unique linear map $\tilde{\beta} : M \otimes_R N \rightarrow P$ with $\beta = \tilde{\beta} \circ \alpha$.

$$\begin{array}{ccc} M \times N & \xrightarrow{\alpha} & M \otimes_R N \\ & \searrow \beta & \downarrow \tilde{\beta} \\ & & P \end{array}$$

- (b) Take any element of $U^{-1}R \otimes_R M$. Such an element can be written as $\sum_{i=1}^n \frac{a_i}{b_i} c_i$ with $a_i \in R$, $b_i \in U$ and $c_i \in M$. But now

$$\sum_{i=1}^n \frac{a_i}{b_i} c_i = \sum_{i=1}^n \frac{\left(\prod_{j \neq i} b_j\right) a_i}{\prod_j b_j} c_i = \frac{1}{\prod_j b_j} \sum_{i=1}^n \left(\prod_{j \neq i} b_j\right) a_i c_i$$

with $\prod_j b_j \in U$ and $\sum_{i=1}^n \left(\prod_{j \neq i} b_j\right) a_i c_i \in M$.

- (c) We have

$$\begin{aligned} &\mathbb{Q} \otimes_{\mathbb{Z}} \left(\mathbb{Z} \oplus \mathbb{Z}/\langle 42 \rangle\right) \\ &\cong (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus \left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/\langle 42 \rangle\right) \\ &\cong \mathbb{Q} \oplus 0 \\ &\cong \mathbb{Q} \end{aligned}$$

- (d) It is a well-known fact from linear algebra that every module over a field is free.

On the other hand, assume that every R -module is free. Now take $a \in R$ with $a \neq 0$. Then, by assumption, the R -module $R/\langle a \rangle$ is free. If it is free of nonzero rank, multiplication by a must be a nonzero endomorphism of it. However, multiplication by a is just the zero endomorphism. Hence it must be free of rank zero, i.e. $R/\langle a \rangle \cong 0$, so that $\langle a \rangle = R$, i.e. a is a unit. Hence every nonzero element of R is invertible, so that R is a field.

- (e)

Problem (3) (a)

- (b)

- (c)

- (d)

- (e)