Problem (1) (a) Let $P = \{I \subsetneq R \mid I \text{ is an ideal of } R\}$ be the partially ordered set of proper ideals of R. Then $\mathfrak{m} \in P$ is called a maximal ideal if it is a maximal element of this partially ordered set.

Equivalently, this means that ${}^{R}\!/_{\mathfrak{m}}$ is a field.

(b) Let $a \notin \mathfrak{m}$. Then $\langle a \rangle + \mathfrak{m} \supsetneq \mathfrak{m}$, so $\langle a \rangle + \mathfrak{m} = R$. Hence there is $b \in R$ and $c \in \mathfrak{m}$ with $ab + c = 1$, so that $ab = 1 - c$. But now $1 - c \in 1 + \mathfrak{m}$, which by assumption only consists of units. Hence ab is a unit. But then both factors must be units, so that a must be a unit.

We have shown that $R \setminus \mathfrak{m} \subseteq R^\times$, i.e. $\mathfrak{m} \supseteq R \setminus R^\times$. But any proper ideal only consists of non-units, so that $\mathfrak{m} \subseteq R \setminus R^\times$. Hence $\mathfrak{m} = R \setminus R^\times$, which means exactly that R is local with unique maximal ideal $\mathfrak{m}$.

(c) Let $\overline{\mathfrak{m}} \subseteq {}^{\mathbb{Q}[x,y]}\!/_{\langle x^{20}, y^{20} \rangle}$ be a maximal ideal. This corresponds to a maximal ideal $\mathfrak{m} \subseteq \mathbb{Q}[x,y]$ with $\langle x^{20}, y^{20} \rangle \subseteq \mathfrak{m}$. Hence $x^{20}, y^{20} \in \mathfrak{m}$. But since $\mathfrak{m}$ is a maximal ideal, it's also a prime ideal. Hence $x, y \in \mathfrak{m}$, hence $\langle x, y \rangle \subseteq \mathfrak{m}$. Since $\langle x, y \rangle \subseteq \mathbb{Q}[x,y]$ is a maximal ideal, we have $\langle x, y \rangle = \mathfrak{m}$. Hence ${}^{\mathbb{Q}[x,y]}\!/_{\langle x^{20}, y^{20} \rangle}$ has a unique maximal ideal, i.e. it's a local ring.

(d) One can see that the map

$$\begin{aligned} {}^{\mathbb{C}[x,y]}\!/_{\langle x^3 - y^5 \rangle} &\to \mathbb{C}[t^3, t^5], \\ [x] &\mapsto t^5, \\ [y] &\mapsto t^3 \end{aligned}$$

is an isomorphism.

Clearly, $\mathbb{C}[t^3, t^5]$ is an integral domain but not a field. Hence $\langle x^3 - y^5 \rangle \subseteq \mathbb{C}[x,y]$ is a prime ideal but not a maximal ideal.

Problem (2) (a) Let M and N be R-modules. Then the tensor product $M \otimes_R N$ is an R-module equipped with a bilinear map $\alpha : M \times N \to M \otimes_R N$ such that for any R-module P and any bilinear map $\beta : M \times N \to P$ there is a unique linear map $\tilde{\beta} : M \otimes_R N \to P$ with $\beta = \tilde{\beta} \circ \alpha$.

$$\begin{tikzcd} M \times N \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & M \otimes_R N \arrow[d, "\tilde{\beta}"] \\ & P \end{tikzcd}$$

(b) Take any element of $U^{-1}R \otimes_R M$. Such an element can be written as $\sum_{i=1}^n \frac{a_i}{b_i} c_i$ with $a_i \in R$, $b_i \in U$ and $c_i \in M$. But now

$$\sum_{i=1}^n \frac{a_i}{b_i} c_i = \sum_{i=1}^n \frac{\left(\prod_{j \neq i} b_j\right) a_i}{\prod_j b_j} c_i = \frac{1}{\prod_j b_j} \sum_{i=1}^n \left(\prod_{j \neq i} b_j \right) a_i c_i$$

with $\prod_j b_j \in U$ and $\sum_{i=1}^n \left(\prod_{j \neq i} b_j\right) a_i c_i \in M$.

(c) We have

$$\begin{aligned} & \mathbb{Q} \otimes_{\mathbb{Z}} \left(\mathbb{Z} \oplus {}^{\mathbb{Z}}\!/_{\langle 42 \rangle} \right) \\ \cong \; & (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus \left(\mathbb{Q} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}\!/_{\langle 42 \rangle} \right) \\ \cong \; & \mathbb{Q} \oplus 0 \\ \cong \; & \mathbb{Q} \end{aligned}$$

(d) It is a well-known fact from linear algebra that every module over a field is free.

On the other hand, assume that every R-module is free. Now take $a \in R$ with $a \neq 0$. Then, by assumption, the R-module ${}^{R}\!/_{\langle a \rangle}$ is free. If it is free of nonzero rank, multiplication by a must be a nonzero endomorphism of it. However, multiplication by a is just the zero endomorphism. Hence it must be free of rank zero, i.e. ${}^{R}\!/_{\langle a \rangle} \cong 0$, so that $\langle a \rangle = R$, i.e. a is a unit. Hence every nonzero element of R is invertible, so that R is a field.

(e) Assume that R is an integral domain.

If R is even a field, then every R-module is free and hence flat.

On the other hand, assume that every R-module is flat. Now take $a \in R$ with $a \neq 0$. Now since R is an integral domain, the map $R \xrightarrow{\cdot a} R$ is an injective endomorphism of R. Since every R-module is flat, ${}^{R}\!/_{\langle a \rangle}$ is flat, so that ${}^{R}\!/_{\langle a \rangle} \xrightarrow{\cdot a} {}^{R}\!/_{\langle a \rangle}$ must also be injective. However, multiplication by a is just the zero endomorphism on ${}^{R}\!/_{\langle a \rangle}$. Hence we must have ${}^{R}\!/_{\langle a \rangle} \cong 0$, so that $\langle a \rangle = R$. Hence a is a unit. We have shown that every nonzero element of R is invertible, i.e. that R is a field.

Problem (3) (a) A ring R is called Artinian if it satisfies the descending chain condition:

Every infinite descending chain of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

eventually stabilizes, i.e. there is $n \in \mathbb{N}$ with $\forall m \geq n : I_m = I_n$.

(b) First of all, we will show that every Artinian domain is a field.

Let R be an Artinian domain. Take $a \in R$ with $a \neq 0$. Then we have a descending chain

$$R \supseteq \langle a \rangle \supseteq \langle a^2 \rangle \supseteq \dots$$

This chain must eventually stabilize. Hence there is $n \in \mathbb{N}$ with $\langle a^n \rangle = \langle a^{n+1} \rangle$. In particular, we have $a^n \in \langle a^{n+1} \rangle$. This means that there is $b \in R$ with $a^n = ba^{n+1}$, hence $0 = ba^{n+1} - a^n = a^n\,(ba - 1)$. Since $a \neq 0$ and R is a domain, we must in fact have $0 = ba - 1$, hence $1 = ba$. Hence a is a unit. Since every nonzero unit of R is invertible, R is a field.

Now we will show that in an Artinian ring every prime ideal is maximal.

Let R be Artinian and $\mathfrak{p} \subseteq R$ be a prime ideal. Then ${}^{R}/_{\mathfrak{p}}$ is an Artinian domain. By the first part, ${}^{R}/_{\mathfrak{p}}$ is in fact a field. Hence $\mathfrak{p} \subseteq R$ is in fact a maximal ideal.

(c)

(d)

(e)