Problem (1) (a) Let $P = \{I \subsetneq R \mid I \text{ is an ideal of } R\}$ be the partially ordered set of proper ideals of R. Then $\mathfrak{m} \in P$ is called a maximal ideal if it is a maximal element of this partially ordered set.

Equivalently, this means that ${}^{R}/_{\mathfrak{m}}$ is a field.

(b) Let $a \notin \mathfrak{m}$. Then $\langle a \rangle + \mathfrak{m} \supsetneq \mathfrak{m}$, so $\langle a \rangle + \mathfrak{m} = R$. Hence there is $b \in R$ and $c \in \mathfrak{m}$ with $ab + c = 1$, so that $ab = 1 - c$. But now $1 - c \in 1 + \mathfrak{m}$, which by assumption only consists of units. Hence ab is a unit. But then both factors must be units, so that a must be a unit.

We have shown that $R \setminus \mathfrak{m} \subseteq R^\times$, i.e. $\mathfrak{m} \supseteq R \setminus R^\times$. But any proper ideal only consists of non-units, so that $\mathfrak{m} \subseteq R \setminus R^\times$. Hence $\mathfrak{m} = R \setminus R^\times$, which means exactly that R is local with unique maximal ideal $\mathfrak{m}$.

(c) Let $\overline{\mathfrak{m}} \subseteq {}^{\mathbb{Q}[x,y]}/_{\langle x^{20}, y^{20} \rangle}$ be a maximal ideal. This corresponds to a maximal ideal $\mathfrak{m} \subseteq \mathbb{Q}[x,y]$ with $\langle x^{20}, y^{20} \rangle \subseteq \mathfrak{m}$. Hence $x^{20}, y^{20} \in \mathfrak{m}$. But since $\mathfrak{m}$ is a maximal ideal, it's also a prime ideal. Hence $x, y \in \mathfrak{m}$, hence $\langle x, y \rangle \subseteq \mathfrak{m}$. Since $\langle x, y \rangle \subseteq \mathbb{Q}[x,y]$ is a maximal ideal, we have $\langle x, y \rangle = \mathfrak{m}$. Hence ${}^{\mathbb{Q}[x,y]}/_{\langle x^{20}, y^{20} \rangle}$ has a unique maximal ideal, i.e. it's a local ring.

(d)

Problem (2) (a)

(b)

(c)

(d)

(e)

Problem (3) (a)

(b)

(c)

(d)

(e)