Problem (1)

(a) Let $P = \{I \subsetneq R \mid I \text{ is an ideal of } R\}$ be the partially ordered set of proper ideals of R. Then $\mathfrak{m} \in P$ is called a maximal ideal if it is a maximal element of this partially ordered set.

Equivalently, this means that $R/\mathfrak{m}$ is a field.

(b) Let $a \notin \mathfrak{m}$. Then $\langle a \rangle + \mathfrak{m} \supsetneq \mathfrak{m}$, so $\langle a \rangle + \mathfrak{m} = R$. Hence there is $b \in R$ and $c \in \mathfrak{m}$ with $ab + c = 1$, so that $ab = 1 - c$. But now $1 - c \in 1 + \mathfrak{m}$, which by assumption only consists of units. Hence ab is a unit. But then both factors must be units, so that a must be a unit.

We have shown that $R \setminus \mathfrak{m} \subseteq R^\times$, i.e. $\mathfrak{m} \supseteq R \setminus R^\times$. But any proper ideal only consists of non-units, so that $\mathfrak{m} \subseteq R \setminus R^\times$. Hence $\mathfrak{m} = R \setminus R^\times$, which means exactly that R is local with unique maximal ideal $\mathfrak{m}$.

(c) Let $\overline{\mathfrak{m}} \subseteq \mathbb{Q}[x,y]/\langle x^{20}, y^{20} \rangle$ be a maximal ideal. This corresponds to a maximal ideal $\mathfrak{m} \subseteq \mathbb{Q}[x,y]$ with $\langle x^{20}, y^{20} \rangle \subseteq \mathfrak{m}$. Hence $x^{20}, y^{20} \in \mathfrak{m}$. But since $\mathfrak{m}$ is a maximal ideal, it's also a prime ideal. Hence $x, y \in \mathfrak{m}$, hence $\langle x, y \rangle \subseteq \mathfrak{m}$. Since $\langle x, y \rangle \subseteq \mathbb{Q}[x,y]$ is a maximal ideal, we have $\langle x, y \rangle = \mathfrak{m}$. Hence $\mathbb{Q}[x,y]/\langle x^{20}, y^{20} \rangle$ has a unique maximal ideal, i.e. it's a local ring.

(d) One can see that the map

$$\begin{aligned} \mathbb{C}[x,y]/\langle x^3 - y^5 \rangle &\to \mathbb{C}[t^3, t^5], \\ [x] &\mapsto t^5, \\ [y] &\mapsto t^3 \end{aligned}$$

is an isomorphism.

Clearly, $\mathbb{C}[t^3, t^5]$ is an integral domain but not a field. Hence $\langle x^3 - y^5 \rangle \subseteq \mathbb{C}[x,y]$ is a prime ideal but not a maximal ideal.

Problem (2)

(a) Let M and N be R-modules. Then the tensor product $M \otimes_R N$ is an R-module equipped with a bilinear map $\alpha : M \times N \to M \otimes_R N$ such that for any R-module P and any bilinear map $\beta : M \times N \to P$ there is a unique linear map $\tilde{\beta} : M \otimes_R N \to P$ with $\beta = \tilde{\beta} \circ \alpha$.

$$\begin{tikzcd} M \times N \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & M \otimes_R N \arrow[d, "\tilde{\beta}"] \\ & P \end{tikzcd}$$

(b) Take any element of $U^{-1}R \otimes_R M$. Such an element can be written as $\sum_{i=1}^n \frac{a_i}{b_i} c_i$ with $a_i \in R$, $b_i \in U$ and $c_i \in M$. But now

$$\sum_{i=1}^n \frac{a_i}{b_i} c_i = \sum_{i=1}^n \frac{\left(\prod_{j \neq i} b_j\right) a_i}{\prod_j b_j} c_i = \frac{1}{\prod_j b_j} \sum_{i=1}^n \left(\prod_{j \neq i} b_j\right) a_i c_i$$

with $\prod_j b_j \in U$ and $\sum_{i=1}^n \left(\prod_{j \neq i} b_j\right) a_i c_i \in M$.

(c) We have

$$\begin{aligned} & \mathbb{Q} \otimes_{\mathbb{Z}} \left(\mathbb{Z} \oplus \mathbb{Z}/\langle 42 \rangle\right) \\ \cong \; & (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus \left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/\langle 42 \rangle\right) \\ \cong \; & \mathbb{Q} \oplus 0 \\ \cong \; & \mathbb{Q} \end{aligned}$$

(d)

(e)

Problem (3)

(a)

(b)

(c)

(d)

(e)