Док:

$$\mathrm{I} \Rightarrow \sum_j \frac{\mathbb{E}((aX+bZ)\mathbb{1}_{A_j})}{\mathbb{P}(A_j)}\mathbb{1}_{A_j} = \sum_j \frac{a\mathbb{E}X\mathbb{1}_{A_j} + b\mathbb{E}Z\mathbb{1}_{A_j}}{\mathbb{P}(A_j)}\mathbb{1}_{A_j} = a\sum_j \frac{\mathbb{E}X\mathbb{1}_{A_j}}{\mathbb{P}(A_j)}\mathbb{1}_{A_j} + b\sum_j \frac{\mathbb{E}Z\mathbb{1}_{A_j}}{\mathbb{P}(A_j)}\mathbb{1}_{A_j} =$$

$$= a\mathbb{E}[X|Y] + b\mathbb{E}[Z|Y]$$

X дискр

$$\mathrm{II} \Rightarrow \mathbb{E}[X|Y] = \sum_j \mathbb{E}[X|Y=y_j]\mathbb{1}_{A_j} \qquad \mathbb{E}[X|Y=y_j] = \sum_i x_i \mathbb{P}(X=x_i|Y=y_j) =$$

$$= \sum_i x_i \frac{\mathbb{P}(X=x_i)\cancel{\mathbb{P}(Y=y_j)}}{\cancel{\mathbb{P}(Y=y_j)}} = \mathbb{E}X \qquad = \sum_j \mathbb{E}X\mathbb{1}_{A_j} = \mathbb{E}X$$

$$g(Y)\mathbb{1}_{A_j} = g(Y)\cdot\mathbb{1}\{Y=y_j\} = g(y_j)\mathbb{1}\{Y=y_j\}$$

$$\mathrm{III} \Rightarrow \mathbb{E}[X|Y] = \sum_j \frac{\mathbb{E}g(Y)\mathbb{1}_{A_j}}{\mathbb{P}(A_j)}\mathbb{1}_{A_j} = \sum_j \frac{\mathbb{E}[g(y_j)\mathbb{1}_{A_j}]}{\mathbb{P}(A_j)}\mathbb{1}_{A_j} = \sum_j g(y_j)\frac{\cancel{\mathbb{P}(A_j)}}{\cancel{\mathbb{P}(A_j)}}\mathbb{1}_{A_j} =$$

$$\cancel{= \mathbb{E}g(Y) = \mathbb{E}X} \qquad = g(Y) = X$$

$$\mathbb{P}(A_j) = \mathbb{P}(Y=y_j)$$

$$\sum_j \mathbb{1}_{A_j} = 1$$

$$\mathrm{IV} \Rightarrow \mathbb{E}[X|Y] = \sum_j \mathbb{E}[X|Y=y_j]\mathbb{1}_{A_j}$$

$$\mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}\left[\sum_j \mathbb{E}[X|Y=y_j]\mathbb{1}_{A_j}\right] = \sum_j \mathbb{E}[X|Y=y_j]\,\mathbb{E}\mathbb{1}_{A_j} = \sum_j \mathbb{E}(X|Y=y_j)\mathbb{P}(Y=y_j) =$$

$$= \sum_j \frac{\mathbb{E}X\mathbb{1}_{A_j}}{\mathbb{P}(A_j)}\mathbb{P}(A_j) = \mathbb{E}\sum_j X\mathbb{1}_{A_j} = \mathbb{E}X$$