Зад. $X_1, X_2 \sim Exp(\lambda)$ независими

$$f_{\frac{X_1}{X_1+X_2}}(t) = ? \qquad f_{X_1,X_2}(x_1,x_2) = \lambda^2 e^{-\lambda(x_1+x_2)}$$

$f_X \qquad f_{X^3} = ?$

$$\mathbb{P}(X^3 \le t) = \mathbb{P}(X \le \sqrt[3]{t}) = F_X(\sqrt[3]{t})$$

$$\Rightarrow f_{X^3}(t) = F_X'(\sqrt[3]{t}) \cdot (\sqrt[3]{t})' = f_X(\sqrt[3]{t})(\sqrt[3]{t})'$$

$$f_{g(X)}(t) = f_X(g^{-1}(t)) \cdot |(g^{-1}(t))'| \quad f_{X^3}(t) =$$

$$\left.\begin{aligned} Y_1 &= \frac{X_1}{X_1+X_2} \\ Y_2 &= X_1 \end{aligned}\right. \Rightarrow \left| \begin{aligned} X_1 &= Y_2 \\ X_2 &= \frac{Y_2 - Y_2 Y_1}{Y_1} = \frac{Y_2}{Y_1} - Y_2 \end{aligned}\right.$$

* Ако $x_1 = y_2$, $x_2 = \frac{y_2}{y_1} - y_2$, то $y_1 = \frac{x_1}{x_1+x_2} = \frac{y_2}{y_2/y_1} = y_1$

* $Y = X^3 \to X = \sqrt[3]{Y}$

$$|J| = \begin{vmatrix} \frac{\partial x_1(y_1,y_2)}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -\frac{1}{y_1^2} \cdot y_2 & \frac{1}{y_1} - 1 \end{vmatrix} = \frac{y_2}{y_1^2}$$

$$\Rightarrow f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(y_2, \frac{y_2}{y_1} - y_2\right) \cdot |J| = \lambda^2 e^{-\lambda\left(y_2 + \frac{y_2}{y_1} - y_2\right)} \cdot \frac{y_2}{y_1^2} =$$

$$= \lambda^2 e^{-\frac{\lambda y_2}{y_1}} \cdot \frac{y_2}{y_1^2}$$

$$f_{Y_1}(y_1) = \int_0^\infty \lambda^2 \cdot e^{-\lambda \frac{y_2}{y_1}} \cdot \frac{y_2}{y_1^2}\, dy_2 = \int_0^\infty \left(\frac{\lambda y_2}{y_1}\right) \cdot e^{-\frac{\lambda y_2}{y_1}}\, d\frac{\lambda y_2}{y_1} =$$

$$= \int_0^\infty t e^{-t} dt = 1 \qquad y_1 \in (0,1)$$

Всъщност $\int_0^\infty t e^{-t} dt = \mathbb{E}(Exp(1))$

Получихме, че ако $X_1, X_2 \sim Exp(\lambda)$, $X_1 \perp\!\!\!\perp X_2$, то $\frac{X_1}{X_1+X_2} \overset{d}{=} U(0,1)$