1 Proof

Define the Lagrangian function

$$
\begin{aligned}
\mathcal{L} = \frac{1}{2}\|w\|^2 + C\sum_i(\xi_i+\xi_i^*) &+ \sum_i \alpha_i(y_i - w\phi_i - b - \varepsilon - \xi_i) \\
&+ \sum_i \alpha_i(-y_i + w\phi_i - b - \varepsilon - \xi_i^*) \\
&- \sum_i \mu_i\xi_i \\
&- \sum_i \mu_i^*\xi_i^*
\end{aligned} \tag{1}
$$

$$
\textit{where} \quad \forall_i\ \xi_i\xi_i^* \geq 0
$$

Variables of the two definition of the problem:

$$
\begin{array}{ll}
\textit{Primal problem} & w,\ b,\ \xi_i,\ \xi_i^* \\
\textit{Dual Problem} & \alpha_i,\ \alpha_i^*,\ \mu_i,\ \mu_i^*
\end{array}
$$

Next step is try to simplify the definition of the Lagrangian wrt the problem that needs to be solved. Since the objective is to find the *minimum* the developments proceeds imposing this condition.

$$
\frac{\partial\mathcal{L}}{\partial w} = 0 \qquad \longrightarrow \qquad w + \sum_i \alpha_i(-\phi_i) + \sum_i \alpha_i^*\phi_i = 0 \tag{2a}
$$

$$
\frac{\partial\mathcal{L}}{\partial b} = 0 \qquad \longrightarrow \qquad \sum_i -\alpha_i + \sum_i \alpha_i^* = 0 \tag{2b}
$$

$$
\frac{\partial\mathcal{L}}{\partial \xi_i} = 0 \qquad \longrightarrow \qquad C - \alpha_i - \mu_i = 0 \tag{2c}
$$

$$
\frac{\partial\mathcal{L}}{\partial \xi_i^*} = 0 \qquad \longrightarrow \qquad C - \alpha_i^* - \mu_i^* = 0 \tag{2d}
$$

From (2a) the definition of w can be derived

$$
w = \sum_i(\alpha_i - \alpha_i^*)\phi_i \tag{3}
$$

From (2b) the first constraint on the Lagrangian variables is obtained

$$
\sum_i(\alpha_i^* - \alpha_i) = 0 \tag{4}
$$

While from (2c)/(2d) with some further development the second constraint on the Lagrangian variables can be defined

$$
\begin{aligned}
&\alpha_i,\ \alpha_i^*,\ \mu_i,\ \mu_i^* \ \geq\ 0 \quad \forall_i \\
&C = \alpha_i + \mu_i \quad \longrightarrow \quad \alpha_i = C - \mu_i \\
&\Longrightarrow \quad \alpha_i\ \in\ [0,\ C] \\
&\textit{and equivalently} \quad \alpha_i^*\ \in\ [0,\ C]
\end{aligned} \tag{5}
$$

Simplify (1) using the substitution (3)

$$
\begin{aligned}
\mathcal{L} \ &= \frac{1}{2}\sum_i \sum_j (\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\phi_i\phi_j \\
&- \sum_i \sum_j (\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\phi_i\phi_j \\
&+ \sum_i (\alpha_i - \alpha_i^*)y_i + \sum_i (\alpha_i - \alpha_i^*)b - \sum_i (\alpha_i + \alpha_i^*)\varepsilon \\
&+ \sum_i \alpha_i(-\xi_i) + \sum_i \alpha_i^*(-\xi_i^*) \\
&- \sum_i \mu_i\xi_i - \sum_i \mu_i^*\xi_i^* \\
&+ C\sum_i \xi_i + \xi_i^*
\end{aligned} \tag{6}
$$

Apply condition (4) and (2c) to simplify some terms and obtain the final formulation

$$
\begin{aligned}
\mathcal{L}(\alpha, \alpha^*) \ &= -\frac{1}{2}\sum_i \sum_j (\alpha_i - \alpha_i^*)(\alpha_j - \alpha_j^*)\phi_i\phi_j \\
&+ \sum_i (\alpha_i - \alpha_i^*)y_i \\
&- \sum_i (\alpha_i + \alpha_i^*)\varepsilon \\
&\quad With\ the\ constraints \qquad \begin{cases} \sum_i (\alpha_i^* - \alpha_i) = 0 \\ \alpha_i \in [0,\ C] \\ \alpha_i^* \in [0,\ C] \end{cases}
\end{aligned} \tag{7}
$$