1 Demonstration

Define the Lagrangian function

$$\mathcal{L} = \frac{1}{2}\|w\|^2 + C\sum_i(\xi_i + \xi_i^*) + \sum_i \alpha_i(y_i - w\phi_i - b - \varepsilon - \xi_i)$$
$$+ \sum_i \alpha_i(-y_i + w\phi_i - b - \varepsilon - \xi_i^*)$$
$$- \sum_i \mu_i\xi_i \tag{1}$$
$$- \sum_i \mu_i^*\xi_i^*$$

$$where \quad \forall_i \, \xi_i\xi_i^* \geq 0$$

Variables of the two definition of the problem:

$$Primal \; problem \qquad w, \; b, \; \xi_i, \; \xi_i^*$$
$$Dual \; Problem \qquad \alpha_i, \; \alpha_i^*, \; \mu_i, \; \mu_i^*$$

Next step is try to simplify the definition of the Lagrangian wrt the problem that needs to be solved. Since the objective is to find the *minimum* the developments proceeds imposing this condition.

$$\frac{\partial\mathcal{L}}{\partial w} = 0 \qquad \longrightarrow \qquad w + \sum_i \alpha_i(-\phi_i) + \sum_i \alpha_i^*\phi_i = 0 \tag{2a}$$

$$\frac{\partial\mathcal{L}}{\partial b} = 0 \qquad \longrightarrow \qquad \sum_i -\alpha_i + \sum_i \alpha_i^* = 0 \tag{2b}$$

$$\frac{\partial\mathcal{L}}{\partial \xi_i} = 0 \qquad \longrightarrow \qquad C - \alpha_i - \mu_i = 0 \tag{2c}$$

$$\frac{\partial\mathcal{L}}{\partial \xi_i^*} = 0 \qquad \longrightarrow \qquad C - \alpha_i^* - \mu_i^* = 0 \tag{2d}$$

From (2a) the definition of w can be derived

$$w = \sum_i(\alpha_i - \alpha_i^*)\phi_i \tag{3}$$

From (2b) the first constraint on the Lagrangian variables is obtained

$$\sum_i(\alpha_i^* - \alpha_i) = 0 \tag{4}$$

While from (2c)/(2d) with some further development the second constraint on the Lagrangian variables can be defined

$$\alpha_i, \; \alpha_i^*, \; \mu_i, \; \mu_i^* \; \geq \; 0 \quad \forall_i$$
$$C = \alpha_i + \mu_i \quad \longrightarrow \quad \alpha_i = C - \mu_i$$
$$\implies \quad \alpha_i \; \in \; [0, \; C] \tag{5}$$
$$and \; equivalently \quad \alpha_i^* \; \in \; [0, \; C]$$

Simplify (1) using the substitution (3)

$$
\begin{aligned}
\mathcal{L} \; = \; & \frac{1}{2}\sum_i\sum_j(\alpha_i-\alpha_i^*)(\alpha_j-\alpha_j^*)\phi_i\phi_j \\
& -\sum_i\sum_j(\alpha_i-\alpha_i^*)(\alpha_j-\alpha_j^*)\phi_i\phi_j \\
& +\sum_i(\alpha_i-\alpha_i^*)y_i+\sum_i(\alpha_i-\alpha_i^*)b-\sum_i(\alpha_i+\alpha_i^*)\varepsilon \\
& +\sum_i\alpha_i(-\xi_i)+\sum_i\alpha_i^*(-\xi_i^*) \\
& -\sum_i\mu_i\xi_i-\sum_i\mu_i^*\xi_i^* \\
& +C\sum_i\xi_i+\xi_i^*
\end{aligned}
\tag{6}
$$

Apply condition (4) and (2c) to simplify some terms and obtain the final formulation

$$
\begin{aligned}
\mathcal{L}(\alpha,\alpha^*) \; = \; & -\frac{1}{2}\sum_i\sum_j(\alpha_i-\alpha_i^*)(\alpha_j-\alpha_j^*)\phi_i\phi_j \\
& +\sum_i(\alpha_i-\alpha_i^*)y_i \\
& -\sum_i(\alpha_i+\alpha_i^*)\varepsilon
\end{aligned}
\tag{7}
$$

$$
With\ the\ constraints \qquad
\begin{cases}
\sum_i(\alpha_i^*-\alpha_i)=0 \\
\alpha_i\in[0,\;C] \\
\alpha_i^*\in[0,\;C]
\end{cases}
$$