The Econometrics of the Hodrick-Prescott filter

Robert M. de Jong[*] Neslihan Sakarya[†]

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Abstract

The Hodrick-Prescott (HP) filter is a commonly used tool in macroeconomics, and is used to extract a trend component from a time series. In this paper, we derive a new representation of the transformation of the data that is implied by the HP filter. This representation highlights that the HP filter is a symmetric weighted average plus a number of adjustments that are important near the beginning and end of the sample. The representation also allows us to carry out a rigorous analysis of properties of the HP filter without using the ARMA based approximation that has been used in the previous literature. Using this new representation, we characterize the large T behavior of the HP filter and find conditions under which it is asymptotically equivalent to a symmetric weighted average with weights independent of sample size. We also find that the cyclical component of the HP filter possesses weak dependence properties when the HP filter is applied to a stationary mixing process, a linear deterministic trend process and/or a process with a unit root. This provides the first formal justification of the use of the HP filter as a tool to achieve weak dependence in a time series. In addition, a large smoothing parameter approximation to the HP filter is derived, and using this approximation we find an alternative justification for the procedure given in Ravn and Uhlig (2002) for adjusting the smoothing parameter for the data frequency.

[*]Department of Economics, Ohio State University, 444 Arps Hall, Columbus, OH 43210, USA, Email `de-jong.8@osu.edu`.

[†]Department of Economics, Ohio State University, 469 Arps Hall, Columbus, OH 43210, USA, Email `sakarya.2@osu.edu`.

1 Introduction

The Hodrick-Prescott (HP) filter is the standard technique in macroeconomics for separating the long run trend in a data series from short run fluctuations. While it seems intuitively clear that no smoothing technique should be equally well applicable to all types of trended macroeconomic data, the HP filter is universally used in macroeconomics, and while different types of criticism can be found in the literature on the HP filter, as Ravn and Uhlig (2002) state, "the HP-filter has withstood the test of time and the fire of discussion remarkably well." Two well-cited papers in which the HP filter is applied are Kydland and Prescott (1990) and Backus and Kehoe (1992); however, the HP filter is used widely in the literature and is a commonplace tool in macroeconomics.

The HP filter smoothed series $\hat{\tau}_T = (\hat{\tau}_{T1}, \hat{\tau}_{T2}, \ldots, \hat{\tau}_{TT})'$ as introduced into Economics in Hodrick and Prescott (1980,1997) results from minimizing, over all $\tau \in \mathbb{R}^T$,

$$\sum_{t=1}^{T}(y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1}(\tau_{t+1} - 2\tau_t + \tau_{t-1})^2, \tag{1}$$

where T denotes sample size, λ is a (nonnegative) smoothing parameter that for quarterly data is often chosen to equal 1600, and $y = (y_1, \ldots, y_T)'$ is the data series to be smoothed. It has been pointed out that in Whittaker (1923) a similar filtering technique was introduced. The $\hat{\tau}_{Tt}$ are typically referred to as the "trend component," while $\hat{c}_{Tt} = y_t - \hat{\tau}_{Tt}$ is referred to as the "cyclical component." It can be shown that there exists a unique minimizer to the minimization problem of Equation (1), and that for a known positive definite $(T \times T)$ matrix F_T, letting I_T denote the $(T \times T)$ identity matrix,

$$y = (\lambda F_T + I_T)\hat{\tau}_T \text{ and } \hat{\tau}_T = (\lambda F_T + I_T)^{-1} y; \tag{2}$$

see for example Kim (2004). Therefore the trend component $\hat{\tau}_{Tt}$ and the cyclical component $\hat{c}_{Tt}$ are both weighted averages of the y_t, and in the sequel of this paper, we write $\hat{\tau}_{Tt} = \sum_{s=1}^{T} w_{Tts} y_s$; note that the dependence of w_{Tts} and $\hat{\tau}_{Tt}$ on λ is suppressed for notational convenience. However, the inability to find a simple expression for the elements of $(\lambda F_T + I_T)^{-1}$ has prevented researchers from finding a simple expression for the weights that are implicit in the HP filter. This paper will set out to find a new representation for the w_{Tts} and investigates some of the consequences of this representation.

Other properties of the HP filter that follow from the definition are the following. First, $\hat{\tau}_{Tt}(y_1 + 1, y_2 + 1, \ldots, y_T + 1) = \hat{\tau}_{Tt}(y_1, y_2, \ldots, y_T) + 1$, and therefore, $\sum_{s=1}^{T} w_{Tts} = 1$ for $t \in \{1, 2, \ldots, T\}$. Secondly, $\hat{\tau}_{Tt}(1, 2, \ldots, T) = t$, implying that $\sum_{s=1}^{T} w_{Tts} s = t$ for $t \in \{1, 2, \ldots, T\}$. Note that a quadratic trend is not absorbed into $\hat{\tau}_{Tt}$ in this way.

Much of the previous literature on the HP filter is based on the observation that the first order condition for $\hat{\tau}_{Tt}$, $t \in \{3, \dots, T-2\}$ is

$$-2(y_t-\hat{\tau}_{Tt})-4\lambda(\hat{\tau}_{T,t+1}-2\hat{\tau}_{Tt}+\hat{\tau}_{T,t-1})+2\lambda(\hat{\tau}_{Tt}-2\hat{\tau}_{T,t-1}+\hat{\tau}_{T,t-2})+2\lambda(\hat{\tau}_{T,t+2}-2\hat{\tau}_{T,t+1}+\hat{\tau}_{Tt}) = 0 \tag{3}$$

which, letting $\bar{B}$ denote the forward operator and B the backward operator, simplifies to

$$y_t = (\lambda\bar{B}^2 - 4\lambda\bar{B} + (1+6\lambda) - 4\lambda B + \lambda B^2)\hat{\tau}_{Tt}, \tag{4}$$

which can be written as

$$y_t = (\lambda|1-B|^4 + 1)\hat{\tau}_{Tt}. \tag{5}$$

Papers that analyze the HP filter based on the above first order condition are for example King and Rebelo (1993), Cogley and Nason (1995), Phillips and Jin (2002), and McElroy (2008). However, the above first order condition fails to hold for $t = 1, 2$ and for $t = T-1, T$. Therefore, by effectively taking the above first order condition as the definition of the HP filter, these papers study an approximate procedure that intuitively makes sense in large samples for values of t away from the begin and end points of the sample. Hence, such an approach gives an informal large T approximation to the HP filter without a formal large T justification. Given the informal nature of such an analysis, no rigorous large T results for the properties of the HP filter could be derived in the previous literature, and this paper seeks to fill that void.

The organization of this paper is as follows. In section 2, we derive the weights of the HP filter for finite sample size. This result should be of independent interest, as it provides insight into the structure of the filter and allows an easy analytical analysis of the filter. Section 3 gives a formal large T approximation to the HP filter by considering the large T behavior of the weights that were derived in Section 2. In Section 4, we show a weak dependence property for the cyclical component of the HP filter when applied to a series with a deterministic time trend, a unit root and/or a stationary mixing process. A large smoothing parameter analysis is provided in Section 5, where a large T, large smoothing parameter approximation to the HP filter is derived. Using the approximation of Section 5, Section 6 shows the equivalence of the trends of HP filtered quarterly data and HP filtered annual data if the smoothing parameter is adjusted in the same way as suggested in Ravn and Uhlig (2002).

2 The weights of the HP filter

In this section, we derive the exact weights w_{Tts} implied by the HP filter. Our analysis starts by noting that we can also equivalently minimize over $(\theta_1, \dots, \theta_T)$, for a basis of functions $p_j(\cdot) : [0,1] \to \mathbb{R}$ for $T, j \in \mathbb{N}^+$,

$$\sum_{t=1}^{T}(y_t - \sum_{j=1}^{T}\theta_j p_j(t/T))^2 + \lambda \sum_{t=2}^{T-1}(\sum_{j=1}^{T}\theta_j p_j((t+1)/T) - 2\sum_{j=1}^{T}\theta_j p_j(t/T) + \sum_{j=1}^{T}\theta_j p_j((t-1)/T))^2$$

$$= \sum_{t=1}^{T}(y_t - \theta' p_{Tt})^2 + \lambda \sum_{t=2}^{T-1}(\theta' \Delta^2 p_{T,t+1})^2 \tag{6}$$

where $\theta' = (\theta_1, \dots, \theta_T)$ and $p'_{Tt} = (p_1(t/T), p_2(t/T), \dots, p_T(t/T))$. Differentiation with respect to θ now yields for the minimizer $\hat{\theta}$

$$0 = -2\sum_{t=1}^{T} y_t p_{Tt} + 2\sum_{t=1}^{T} p_{Tt} p'_{Tt} \hat{\theta} + 2\lambda \sum_{t=2}^{T-1}(\Delta^2 p_{T,t+1})(\Delta^2 p_{T,t+1})' \hat{\theta}, \tag{7}$$

which, if the inverse exists, is solved for

$$\hat{\theta} = (T^{-1}\sum_{t=1}^{T} p_{Tt} p'_{Tt} + \lambda T^{-1} \sum_{t=2}^{T-1}(\Delta^2 p_{T,t+1})(\Delta^2 p_{T,t+1})')^{-1} T^{-1} \sum_{t=1}^{T} y_t p_{Tt}. \tag{8}$$

Now we choose $p_1(t/T) = 1$ and $p_j(t/T) = \sqrt{2}\cos(\pi(j-1)(t-1/2)/T)$, $j = 2, \dots, T$. Note that this is a minor abuse of notation, since formally, $p_j(\cdot)$ depends on T. We can then derive the following result. Below, I_T denotes the identity matrix of dimension $(T \times T)$.

Lemma 1. *Let $p_{Tt} = (p_1(t/T), \dots, p_T(t/T))'$, where $p_1(t/T) = 1$ and $p_j(t/T) = \sqrt{2}\cos(\pi(j-1)(t-1/2)/T)$, $j = 2, \dots, T$. Then*

$$T^{-1}\sum_{t=1}^{T} p_{Tt} p'_{Tt} = I_T \tag{9}$$

and

$$T^{-1}\sum_{t=2}^{T-1}(\Delta^2 p_{T,t+1})(\Delta^2 p_{T,t+1})'$$

$$= D_T - 32T^{-1} q_{T1} q_{T1}' - 32T^{-1} q_{T2} q_{T2}', \tag{10}$$

where $D_T = \mathrm{diag}(\{16\sin(\pi(j-1)/(2T))^4, j = 1, \ldots, T\})$ *and* $q_{T1} = (q_{T11}, q_{T12}, \ldots, q_{T1T})'$, $q_{T2} = (q_{T21}, q_{T22}, \ldots, q_{T2T})'$, *where for* $j \in \{1, ..., T\}$, $q_{T1j} = \sin(\pi(j-1)/(2T))^2 \cos(\pi(j-1)/(2T))$ *and* $q_{T2j} = \sin(\pi(j-1)/(2T))^2 \cos(\pi(j-1)(T-1/2)/T)$.

The proof of this result and all proofs for this paper can be found in the Mathematical Appendix.

The importance of the above result is that the matrix to be inverted is now "close" to an easily invertible diagonal matrix (in the sense that two matrices of rank 1 have been added to the diagonal matrix), and therefore allows for a more tractable expression. When minimizing the criterion of Equation (1) over τ, no such structure occurs. It is well-known that explicit formulas can be obtained for the inverse of the sum of a matrix plus another matrix of rank 1, and such results can be adapted to deal with the inverse of a matrix plus a matrix of rank 2 as well. Our strategy will be to use such a result, as obtained in Miller (1981), in order to obtain a tractable expression for $\hat{\tau}_{Tt}$.

For the theorem below in which we derive the exact weights of the HP filter, we need to define, for $m \in \mathbb{Z}$, $\lambda \in [0, \infty)$, and $T \geq 1$,

$$\begin{aligned} f_{T\lambda}(m) &= 1/(2T) + (-1)^m (2T)^{-1} (1+16\lambda)^{-1} \\ &\quad + T^{-1} \sum_{j=2}^{T} \cos(\pi(j-1)m/T)(1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1} \end{aligned} \tag{11}$$

and for $m \geq 1$, $\lambda \in [0, \infty)$, and $T \geq 1$,

$$\begin{aligned} g_{T\lambda}(m) &= T^{-1} p_{Tm}' (I_T + \lambda D_T)^{-1} q_{T1} \\ &= T^{-1} \sum_{j=1}^{T} \sqrt{2} \cos(\pi(j-1)(m-1/2)/T) q_{T1j} (1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1}. \end{aligned} \tag{12}$$

In addition, define the sequences

$$\delta_{T\lambda} = T^{-1} q_{T1}' (I_T + \lambda D_T)^{-1} q_{T1}, \tag{13}$$

$$\eta_{T\lambda} = T^{-1} q_{T1}' (I_T + \lambda D_T)^{-1} q_{T2}, \tag{14}$$

$$\xi_{T\lambda} = 32\lambda(1-64\lambda\delta_{T\lambda})(1-64\lambda\delta_{T\lambda}+32^2\lambda^2(\delta_{T\lambda}^2-\eta_{T\lambda}^2))^{-1}$$
$$+32^2\lambda^2(1-64\lambda\delta_{T\lambda}+32^2\lambda^2(\delta_{T\lambda}^2-\eta_{T\lambda}^2))^{-1}\delta_{T\lambda}, \tag{15}$$

and

$$\phi_{T\lambda} = 32^2\lambda^2(1-64\lambda\delta_{T\lambda}+32^2\lambda^2(\delta_{T\lambda}^2-\eta_{T\lambda}^2))^{-1}\eta_{T\lambda}. \tag{16}$$

Note that $f_{T\lambda}(m) = f_{T\lambda}(-m)$ for all $m \in \mathbb{Z}$. With these definitions in place, we can now obtain the following result.

Theorem 1. *For any $\lambda \in [0,\infty)$, $\hat{\tau}_{Tt} = \sum_{s=1}^{T} w_{Tts}y_s$, where*

$$\begin{aligned}
w_{Tts} &= f_{T\lambda}(t-s) + f_{T\lambda}(T)I(t+s-1=T) \\
&+ f_{T\lambda}(t+s-1)I(t+s-1<T) + f_{T\lambda}(2T-t-s+1)I(t+s-1>T) \\
&+ \xi_{T\lambda}g_{T\lambda}(t)g_{T\lambda}(s) + \phi_{T\lambda}g_{T\lambda}(T-t+1)g_{T\lambda}(s) \\
&+ \phi_{T\lambda}g_{T\lambda}(t)g_{T\lambda}(T-s+1) + \xi_{T\lambda}g_{T\lambda}(T-t+1)g_{T\lambda}(T-s+1) \\
&= \sum_{j=1}^{8} w_{Tts}^j
\end{aligned} \tag{17}$$

where $|f_{T\lambda}(0)| \leq 1$ and for $m \in \{1,2,\ldots,T\}$, $|f_{T\lambda}(m)| \leq Cm^{-3}$ and $|g_{T\lambda}(m)| \leq Cm^{-3}$ for some constant C not depending on T.

A simple computer program to verify the correctness of the formula of Equation (17), as well as other material related to this paper, can be found at `http://dl.dropbox.com/u/2159931/hp.html`.

The above formula highlights that a key feature of the HP filter is the creation of a symmetric weighted average of the data with w_{Tts}^1. From the results of the next paragraph, it will follow that $\phi_{T\lambda}$ and $\xi_{T\lambda}$ are uniformly bounded for $T \geq 1$. This then has the following implications. w_{Tts}^2 is of little consequence since $f_{T\lambda}(T)$ is bounded by CT^{-3} by the bound on $f_{T\lambda}(\cdot)$. w_{Tts}^3 is small as long as t is away from the begin point of the sample, since it is bounded by $C(t+s-1)^{-3}$. Similarly, w_{Tts}^4 is small as long as t is away from T. By the bound on $g_{T\lambda}(\cdot)$, w_{Tts}^5 and w_{Tts}^7 are also small for t away from 1, and w_{Tts}^6 and w_{Tts}^8 are small for t away from T. This can be formalized by noting that for any $r \in (0,1)$,

$$\sup_{T\geq 1}\sup_{1\leq s\leq T}|T^3\sum_{j=2}^{8} w_{T,[rT],s}^j| < \infty. \tag{18}$$

Here and below, $[x]$ denotes the largest integer that is smaller than x. In conclusion, from the above theorem it follows that the HP filter behaves like a symmetric weighted average with several correction terms that are important near the beginning and the end of the sample.

Note that since $\cos(\pi(j-1)(2T-t-s+1)/T) = \cos(\pi(j-1)(t+s-1)/T)$, it follows that $f_{T\lambda}(t+s-1) = f_{T\lambda}(2T-t-s+1)$, and therefore,

$$f_{T\lambda}(s+t-1) = f_{T\lambda}(2T-t-s+1) = f_{T\lambda}(T)I(t+s-1=T)$$
$$+f_{T\lambda}(t+s-1)I(t+s-1<T) + f_{T\lambda}(2T-t-s+1)I(t+s-1>T); \qquad (19)$$

however, the formula of Theorem 1 brings out the fact that the HP filter is invariant to reversing the time order, i.e., makes it clear that

$$\hat{\tau}_{Tt}(y_1, y_2, \ldots, y_{T-1}, y_T) = \hat{\tau}_{T,T-t+1}(y_T, y_{T-1}, \ldots, y_2, y_1),$$

and is more useful when considering large T and large λ approximations to the HP filter. It can also be noted that $\lim_{\lambda\to\infty} f_{T\lambda}(m) = (2T)^{-1}$ and that $\lim_{\lambda\to\infty} g_{T\lambda}(m) = 0$, which corresponds to the result $\lim_{\lambda\to\infty} \hat{\tau}_{Tt} = T^{-1}\sum_{t=1}^{T} y_t$ that can be observed directly from the definition of the HP filter.

In Tables 1 and 2 in Appendix 1, values for $\xi_{T\lambda}$ and $\phi_{T\lambda}$ for various values of λ and T are listed. In Section 3, it is proven that limit values for these sequences are reached as $T \to \infty$, and from Tables 1 and 2 it can be seen that values near the limit are typically reached for sample sizes as small as $T = 50$. Appendix 1 also contains graphs (Figure 1 and Figure 2) of the $f_{T\lambda}(m)$ and $g_{T\lambda}(m)$ functions. It is notable that for $\lambda = 1600$, we found that

$$\max_{m=-50,\ldots,50} |f_{50,1600}(m) - f_{1000,1600}(m)| = 1.033 \cdot 10^{-4}$$

and

$$\max_{m=1,\ldots,50} |g_{50,1600}(m) - g_{1000,1600}(m)| = 2.428 \cdot 10^{-6},$$

and therefore we only provided one graph for $\lambda = 1600$ and $T \geq 50$. The Matlab programs used for arriving at this conclusion and used for creating Tables 1 and 2 and Figure 1 and Figure 2 can be found at `http://dl.dropbox.com/u/2159931/hp.html`.

The importance of the representation for the HP filter given in Theorem 1 is that it clarifies the nature of the mapping of the data that is implied by the use of the HP filter. It does this by providing an explicit finite sample formula for the weights that are implicit in the use of the HP filter. In the sections below, we will investigate some of the consequences of Theorem 1. However, since it provides additional insight into the mapping of the data that is used in procedures that make use of the HP filter, it is very well possible that there are many more possible applications of Theorem 1 than those provided in the next sections only. For example, it is now straightforward to calculate the properties of the correlogram and periodogram when calculated based on the cyclical component of a stationary time series process.

3 Large T results

In this section, we gather some large T results which will be used to show that it is possible to formulate weighted average procedures with weights not dependent on T that are asymptotically equivalent to the HP filter. First, we prove that the constants present in Theorem 1 achieve limit values:

Theorem 2. *For all $\lambda \geq 0$, $\lim_{T\to\infty} \eta_{T\lambda} = 0$, $\lim_{T\to\infty} \phi_{T\lambda} = 0$,*

$$\lim_{T\to\infty} \delta_{T\lambda} = \int_0^1 \sin(\pi r/2)^4 \cos(\pi r/2)^2 (1 + 16\lambda \sin(\pi r/2)^4)^{-1} dr = \delta_\lambda, \tag{20}$$

and

$$\lim_{T\to\infty} \xi_{T\lambda} = \frac{32\lambda}{1 - 32\lambda\delta_\lambda} = \xi_\lambda. \tag{21}$$

Part of the assertion of Theorem 2 is that $1 - 32\lambda\delta_\lambda \neq 0$. Together with Theorem 1, the above result implies

$$\sup_{T\geq 1} \sup_{1\leq t\leq T} \sum_{s=1}^{T} |w_{Tts}| < \infty. \tag{22}$$

The following result shows that the functions $f_{T\lambda}(\cdot)$ and $g_{T\lambda}(\cdot)$ converge pointwise to limit functions.

Theorem 3. *Pointwise in (λ, m), for all $\lambda \geq 0$ and $m \in \mathbb{Z}$,*

$$\lim_{T\to\infty} f_{T,\lambda}(m) = f_\lambda(m) = \int_0^1 \cos(\pi r m)(1 + 16\lambda \sin(\pi r/2)^4)^{-1} dr$$

$$= \frac{2q|r_1|^{m+2} \sin(\theta)(|r_1|^2 \sin((m-1)\theta) - \sin((m+1)\theta))}{(1 - 2\cos(2\theta)|r_1|^2 + |r_1|^4)(|r_1|^2 - 1)(1 - \cos(2\theta))}$$

where

$$r_1 = \frac{(2i - \sqrt{q}) + \sqrt{q - 4i\sqrt{q}}}{2i}$$

and

$$\theta = \tan^{-1}\left(\frac{\sqrt{q - 4i\sqrt{q}} + \sqrt{q + 4i\sqrt{q}} - 2\sqrt{q}}{i(4i + \sqrt{q - 4i\sqrt{q}} - \sqrt{q + 4i\sqrt{q}})}\right).$$

and for all $\lambda \geq 0$ *and* $m \geq 1$,

$$\lim_{T\to\infty} g_{T,\lambda}(m) = g_\lambda(m) = \sqrt{2}\int_0^1 \cos(\pi r(m-1/2))\sin(\pi r/2)^2\cos(\pi r/2)(1+16\lambda\sin(\pi r/2)^4)^{-1}dr$$

$$= (\sqrt{2}/8)(-f_\lambda(m-2) + f_\lambda(m-1) + f_\lambda(m) - f_\lambda(m+1)).$$

By working with the approximate result of Equation 5, McElroy (2008) obtained a formula that appears to be identical to our $f_\lambda(m)$. Given these results, we can now define the weighted average

$$\tilde{\tau}_{Tt} = \sum_{s=1}^{T} y_s f_\lambda(t-s)$$

and formally show that it is asymptotically equivalent to $\hat{\tau}_{Tt}$, in a sense to be made precise below. In order to be able to formulate our result, we use the index r for t/T in the theorem below. Also, we used a scaling factor of $E|y_t|$ to account for the possibly increasing nature of y_t and $\hat{\tau}_{Tt}$.

Theorem 4. *Assume that for any* $r \in (0,1)$,

$$\limsup_{T\to\infty} \sup_{1\leq s\leq T} E|y_s|(E|y_{[rT]}|)^{-1} < \infty.$$

Then

$$\limsup_{T\to\infty}(E|y_{[rT]}|)^{-1}E|\hat{\tau}_{T,[rT]} - \tilde{\tau}_{T,[rT]}| = 0. \tag{23}$$

Noting that $f_\lambda(m) = f_\lambda(-m)$, the above theorem shows that the HP filter is asymptotically equivalent to the symmetric weighted average

$$\sum_{s=1}^{T} y_s f_\lambda(t-s)$$

as long as the regularity conditions hold and as long as we stay away from the beginning and end of the sample. These regularity conditions holds for a wide class of processes. For a series $y_t = \alpha_1 + \alpha_2 t + \alpha_3 z_t + u_t$ for which $\alpha_2 \neq 0$, $\sup_{t\geq 1} E|u_t| < \infty$, and $E|z_t| = o(t)$ (such as in the case where z_t is a unit root process that satisfies mild regularity conditions), we have

$$\alpha_2 r T (E|y_{[rT]}|)^{-1} \to 1 \quad \text{and} \quad E|y_T|(E|y_{[rT]}|)^{-1} \to r^{-1},$$

implying that the conditions of the theorem hold. In the case where $\alpha_2 = 0$ and z_t is a unit root process such that $t^{-1/2}E|z_t| \to c$ for some constant $c > 0$, we have

$$\alpha_3 c T^{1/2} r^{1/2} (E|y_{[rT]}|)^{-1} \to 1 \quad \text{and} \quad E|y_T|(E|y_{[rT]}|)^{-1} \to r^{-1/2},$$

again implying that the conditions of the theorem hold. Similar results can be established if the regressors are integrated of finite order and/or polynomials of the time trend.

In conclusion, it appears that away from the beginning and end of the sample for a wide range of data-generating processes, the HP filter is effectively a symmetric weighted average with weights $f_\lambda(t-s)$. It should be noted that $f_\lambda(\cdot)$ takes on both positive and negative values, and therefore some observations are given negative weights, even asymptotically.

4 Weak dependence of the cyclical component

Given the fact that the HP filter is a weighted average of T observations, it follows that the cyclical component is a triangular array. Also, for every value of t, the HP filter necessarily weighs $t-1$ earlier observations and $T-t$ later observations; and therefore, it is not possible for the cyclical component of the HP filter to be strictly stationary or weakly stationary. However, it is possible to show that the cyclical component of the HP filter has weak dependence properties when the HP filter is applied to the sum of a stationary mixing process, a linear trend process, and a unit root process. Such a property then ensures that laws of large numbers and central limit theorems can hold for functions of the cyclical component. This provides a justification for the use of the HP filter, as it implies that inference based on the cyclical component can be asymptotically correct. Therefore, the results below illustrate that the finding in empirical macro that time series tend to possess time trends and/or unit roots, and the practice of using inference procedures based on the cyclical component of the HP filter, are not contradictory.

In addition, the weak dependence of the cyclical component indicates that the unit root process has essentially been absorbed into the trend component. This can be taken to imply that the HP filter is capable of removing unit roots from a data-generating process.

In this section, we consider weak dependence properties of the cyclical component when the HP filter has been applied to the sum of a stationary mixing process, a linear trend process, and a unit root process; that is, we assume that y_t satisfies the following assumption:

Assumption 1.

$$y_t = \alpha_1 + \alpha_2 t + \alpha_3 z_t + u_t,$$

where $z_t = \sum_{j=1}^{t} \varepsilon_j$, $\sup_{s\geq 1} \parallel \varepsilon_s \parallel_p < \infty$ *and* $\sup_{s\geq 1} \parallel u_s \parallel_p < \infty$.

We explicitly allow for the cases $u_t = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$ and/or $z_t = 0$. Remembering that the cyclical component of y_t is denoted as $\hat{c}_{Tt}$, i.e. $\hat{c}_{Tt} = y_t - \hat{\tau}_{Tt} = y_t - \sum_{s=1}^{T} w_{Tts} y_s$, define

$$\hat{c}_{Tt}^{m} = u_t - \sum_{s=1}^{T} w_{Tts} u_s I(|s-t| \leq m) - \sum_{j=1}^{T} \varepsilon_j (\sum_{s=j}^{T} w_{Tts} - I(j \leq t)) I(|j-t| \leq m).$$

Note that for $m \geq 0$, $\hat{c}_{Tt}^{m}$ is an approximation to $\hat{c}_{Tt}$ that uses only $(v_{\max(1,t-m)}, \ldots, v_{\min(T,t+m)})$, where $v_t = (u_t, \varepsilon_t)'$. Therefore, $\hat{c}_{Tt}^{m}$ is an approximation to $\hat{c}_{Tt}$ that uses only information that is at most m time periods away from the current time period. Concepts of closeness to such an approximation have a long history in statistics and econometrics; see for example Ibragimov (1962), Billingsley (1968), Gallant and White (1988), Wooldridge and White (1988), Andrews (1988), and Pötscher and Prucha (1997), among others. Near epoch dependence is a simpler condition than mixing to verify, but the "base" (here, v_t) needs to satisfy a mixing condition in order for the condition to be useable for showing laws of large numbers and/or central limit theorems. Our result is the following:

Theorem 5. *Assume that y_t satisfies Assumption 1. Then for any $\gamma \in (0, 1/2)$,*

$$\sup_{T\geq 1} \sup_{t\in[\gamma T,(1-\gamma)T]} \parallel \hat{c}_{Tt} - \hat{c}_{Tt}^{m} \parallel_p = O(m^{-1})$$

and

$$\sup_{T\geq 1} \sup_{t\in[\gamma T,(1-\gamma)T]} \parallel \hat{c}_{Tt} \parallel_p < \infty. \tag{24}$$

Note that if $z_t = 0$, we can improve the $O(m^{-1})$ to $O(m^{-2})$ in the above theorem.

In Gallant and White (1988) and Pötscher and Prucha (1997), the authors demonstrate how a complete theory of inference can be based on the near epoch dependence property as demonstrated above.

In the above theorem, we had to limit ourselves to sample values away from the beginning and end of the sample. However, it is also possible to choose the γ arbitrarily small, and use an asymptotic negligibility argument for the beginning and the end of the sample, and along those lines, a complete theory of inference based on the near epoch dependence property using the entire sample is feasible also. As an example, we prove the following weak law of large numbers for functions of the cyclical component:

Theorem 6. *Assume that y_t satisfies Assumption 1 for $p = 2$, and assume that $(u_t, \varepsilon_t)'$ is strong mixing. Let $f(\cdot)$ be a function that is bounded and continuous on $\mathbb{R}$. Then*

$$T^{-1} \sum_{t=1}^{T} (f(\hat{c}_{Tt}) - Ef(\hat{c}_{Tt})) \xrightarrow{p} 0.$$

The implication of the results in the section is that the HP filter removes a unit root from a time series, in the sense that the resulting cyclical component is weakly dependent. Therefore, statistical inference based on the cyclical components is not necessarily illegitimate.

5 Results for large T and large λ

While in Section 3 we found that the HP filter for large T is equivalent to a weighted average, the weight function depended on λ. When considering large values of λ, it is also possible to find an approximation to the HP filter using a weighted average that only uses λ as a bandwidth type parameter. This result is both of interest in its own right and provides a route towards a formal result on adjusting the HP filter for the frequency of the observations. Our result is based on the following theorem:

Theorem 7. *For $|m| \geq 1$, $f_\lambda(m) \leq C\lambda^{1/4} m^{-2}$ for some constant $C > 0$ not depending on λ or m. Also, pointwise in m for all $m \in \mathbb{Z}$,*

$$\lim_{\lambda \to \infty} \lambda^{1/4} f_\lambda(\lambda^{1/4} m) = f(m) = 2^{-3/2} \exp(-2^{-1/2}|m|)(\sin(2^{-1/2}|m|) + \cos(2^{-1/2}|m|))$$

and for pointwise in m for all $m \geq 1$,

$$\lim_{\lambda \to \infty} \lambda^{3/4} g_\lambda(\lambda^{1/4} m) = g(m) = 2^{-3} \exp(-2^{-1/2} m)(\cos(2^{-1/2} m) - \sin(2^{-1/2} m)).$$

Furthermore, for all $K > 0$,

$$\lim_{\lambda\to\infty} \sup_{|m|\le K} |\lambda^{1/4} f_\lambda(\lambda^{1/4} m) - f(m)| = 0.$$

Note that in the above result, $f(\cdot)$ and $g(\cdot)$ are not dependent on λ. Furthermore, let

$$\bar{\tau}_{Tt} = \sum_{s=1}^{T} y_s \lambda^{-1/4} f(\lambda^{-1/4}(t-s)),$$

and note that $\int_{-\infty}^{\infty} f(m)dm = 1$ and $\int_0^{\infty} g(m)dm = 0$. Using Theorem 7 as a basis, we can now establish the following result.

Theorem 8. *Assume that for any $r \in (0,1)$,*

$$\limsup_{T\to\infty} \sup_{1\le s\le T} E|y_s|(E|y_{[rT]}|)^{-1} < \infty.$$

Then

$$\limsup_{\lambda\to\infty} \limsup_{T\to\infty} (E|y_{[rT]}|)^{-1} E|\hat{\tau}_{T,[Tr]} - \bar{\tau}_{T,[Tr]}| = 0. \tag{25}$$

Theorem 8 suggests that for large T and λ, away from the beginning and end of our sample,

$$\hat{\tau}_{Tt} \approx \sum_{s=1}^{T} y_s \lambda^{-1/4} f(\lambda^{-1/4}(t-s)),$$

which for $\lambda = 1600$ simplifies to

$$\hat{\tau}_{Tt} \approx 0.1581 \sum_{s=1}^{T} y_s \ f(0.1581(t-s)).$$

The above theorems also imply that the presence of both positive and negative weights remain a feature of the HP filter even when both T and λ are assumed large.

6 Adjusting the HP filter for the frequency of the observations

Hodrick and Prescott's suggestion to use a quarterly smoothing parameter of 1600 raises the question as to what smoothing parameter is appropriate for annual or monthly data. For annual data, Backus and Kehoe (1992) suggested a value of 100, and Correia et al. (1992) and Cogley and Ohanian (1991) suggested a value of 400, while Backus and Kehoe (1992) suggested a value of 10. Ravn and Uhlig (2002) suggested that a value of 6.25 for the annual smoothing parameter corresponds to a quarterly smoothing parameter of 1600. They obtain this value as $1600 \times 4^{-4} = 6.25$. In this paragraph, we will give a different justification for rescaling the smoothing parameter using a fourth power when adjusting for the data frequency.

Assume for simplicity that we have $4T$ observations of quarterly data and apply the HP filter using a smoothing value λ_Q, implying that we have observations for T years. Then, letting $w_{Tts}(\lambda_Q)$ denote the weights corresponding to λ_Q and defining the notation $\hat{\tau}_{Tt}(\lambda, A)$ as the trend component of the set of observations A using smoothing parameter λ, the HP filter trend in the tth year of data in quarter q, $t = 1, ..., T$, $q = 1, 2, 3, 4$ would equal

$$\hat{\tau}_{4T,4t-4+q}(\lambda_Q, \{y_s, s = 1, \ldots, 4T\}) = \sum_{s=1}^{4T} y_s w_{4T,4t-4+q,s}(\lambda_Q),$$

while using annual data $(1/4)\sum_{q=1}^{4} y_{4t-4+q}$ and smoothing parameter λ_A, we would use

$$\hat{\tau}_{T,t}(\lambda_A, \{(1/4)\sum_{q=1}^{4} y_{4i-4+q} : i = 1, \ldots, T\}) = \sum_{i=1}^{T}((1/4)\sum_{q=1}^{4} y_{4i-4+q})w_{Tti}(\lambda_A).$$

Note that this assumes that the quarterly data are measured on a per year basis. The asymptotic equivalence of both procedures for

$$\lambda_A = 4^{-4}\lambda_Q$$

is then shown in the following theorem:

Theorem 9. *Assume that for any* $r \in (0, 1)$, $\limsup_{T\to\infty} \max_{1\le s\le 4T} E|y_s|(E|y_{[4rT]}|)^{-1} < \infty$ *and*

$$\limsup_{T\to\infty} \max_{1\le i\le T} E|(1/4)\sum_{q=1}^{4} y_{4i-4+q}|(E|(1/4)\sum_{q=1}^{4} y_{4[rT]-4+q}|)^{-1} < \infty.$$

Then for $q = 1, 2, 3,$ *or* 4,

$$\lim_{\lambda\to\infty} \limsup_{T\to\infty} (E|y_{[rT]}|)^{-1} E|\hat{\tau}_{4T,4[rT]-4+q}(\lambda, \{y_s, s = 1, \ldots, 4T\})$$

$$-\hat{\tau}_{T,[rT]}(4^{-4}\lambda, \{(1/4)\sum_{q=1}^{4} y_{4i-4+q} : i = 1, \ldots, T\})| = 0. \tag{26}$$

It should be noted that our asymptotic equivalence argument using $\lambda_A = 4^{-4}\lambda_Q$ only follows if we use quarterly data y_i and annual data $(1/4)\sum_{q=1}^{4} y_{4i-4+q}$. This implies that quarterly and annual data must be measured both on a quarterly or an annual basis for the equivalence to hold. Therefore, if we have quarterly data on domestic car sales, then the asymptotic equivalence between the HP filter for quarterly data using $\lambda_Q = 1600$ and the HP filter for annual data using $\lambda_A = 6.25$ only holds if we measure domestic car sales on a per year or a per quarter basis for both applications of the HP filter.

Using a similar argument, going from quarterly to monthly data, we find that the smoothing parameter λ_M asymptotically corresponds to $(1/3)^{-4}\lambda_Q = 81\lambda_Q$, and for $\lambda_Q = 1600$, this then suggests $\lambda_M = 129600$. This also corresponds to the suggestion of Ravn and Uhlig (2002). We omit a formal theorem to this effect, as it will be analogous to Theorem 9. Again, for the asymptotic equivalence to hold, we need to measure both monthly and annual data on either a monthly or an annual basis. Similar statements can of course be made for adjustments to the smoothing parameter when considering different frequency adjustments.

7 Conclusion

This paper provides a mathematically rigorous discussion of the properties of the HP filter by using a calculation of the explicit weights of the HP filter. It is shown that the HP filter is a symmetric weighted average plus some adjustment terms that are important in the beginning and end of the sample. When sample size is large, we show that the HP filter asymptotically approaches to a symmetric weighted average under some regularity conditions. Consistent with the use of the HP filter by macroeconomists, it is shown that the cyclical component satisfies weak dependence properties for a wide class of stochastic processes. This conclusion allows us to derive a weak law of large numbers result for functions of the cyclical component. Lastly, when the sample size and the smoothing parameter are large, we provide an alternative justification for a procedure for adjusting the smoothing parameter for the data frequency given in Ravn and Uhlig (2002).

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Appendix 1: the constants of Section 2

Table 1: Values of $\xi_{T\lambda}$ for various values of λ and T. The Matlab program used to calculate these values (and values for the $\delta_{T\lambda}$ and $\eta_{T\lambda}$ sequences) can be found at `https://dl.dropbox.com/u/2159931/hp.html`.

	$T = 25$	$T = 50$	$T = 100$	$T = 1000$
$\lambda = 100$	14491.2	14490.8	14490.8	14490.8
$\lambda = 400$	81643.7	81461.9	81461.8	81461.8
$\lambda = 1600$	461398	459391	459380	459380

Table 2: Values of $\phi_{T\lambda}$ for various values of λ and T. The Matlab program used to calculate these values (and values for the $\delta_{T\lambda}$ and $\eta_{T\lambda}$ sequences) can be found at `https://dl.dropbox.com/u/2159931/hp.html`.

	$T = 25$	$T = 50$	$T = 100$	$T = 1000$
$\lambda = 100$	-8.52750	0.33904	6.37970×10^{-6}	-5.86219×10^{-11}
$\lambda = 400$	4421.81	-57.7142	0.02290	-4.69986×10^{-10}
$\lambda = 1600$	33352.1	-403.167	10.3726	-3.73888×10^{-9}

Figure 1: The function $f_{T,\lambda}(m)$ for $T \geq 50$ and $\lambda = 1600$.

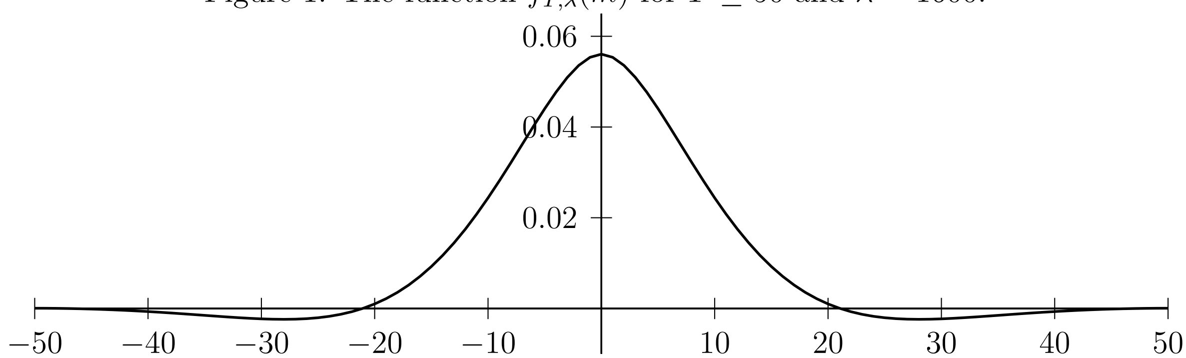
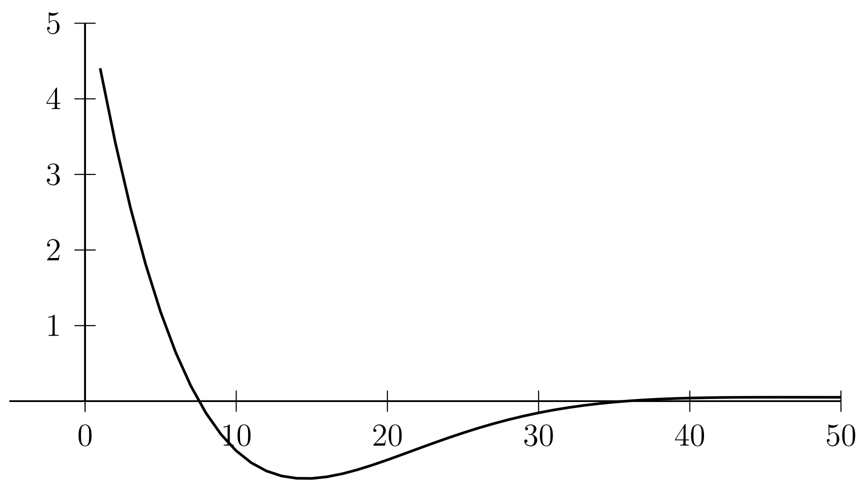


Figure 2: The function $g_{T,\lambda}(m)$ for $T \geq 50$ and $\lambda = 1600$; y axis needs to be multiplied by 10^{-4}.



Appendix 2: Mathematical proofs

In this section, C and C_1, C_2,... denote arbitrary constants that can take different values in different proofs.

Lemma 2. *Let* $p_{Tt} = (p_1(t/T), \ldots, p_T(t/T))'$, *where* $p_1(t/T) = 1$ *and* $p_j(t/T) = \sqrt{2}\cos(\pi(j-1)(t-1/2)/T)$, $j = 2, \ldots, T$. *Then* $T^{-1}\sum_{t=1}^{T} p_{Tt}p_{Tt}' = I_T$.

Proof. First note that for $k \in \{1, 2, \ldots, 2T-1\}$,

$$A(k,T) = \sum_{t=1}^{T} \cos(\pi k(t-1/2)/T) = 0.$$

To see this, note that for $k \in \{1, 2, \ldots, 2T-1\}$, because $\sum_{t=1}^{T} \rho^{t-1/2} = (\rho-1)^{-1}\rho^{1/2}(\rho^T - 1)$ for $\rho \neq 1$,

$$\sum_{t=1}^{T} \exp(i\pi k(t-1/2)/T) = (\exp(i\pi k/T) - 1)^{-1} \exp((1/2)i\pi k/T)(\exp(i\pi k) - 1)$$

and

$$\sum_{t=1}^{T} \exp(-i\pi k(t-1/2)/T) = (\exp(-i\pi k/T) - 1)^{-1} \exp(-(1/2)i\pi k/T)(\exp(-i\pi k) - 1).$$

Now for k even, $\exp(-i\pi k) = \exp(i\pi k) = 1$ while for k odd, $\exp(-i\pi k) = \exp(i\pi k) = -1$. Therefore, for $k \in \{1, 2, \ldots, 2T-1\}$ even, by Euler's formula,

$$A(k,T) = (1/2)\sum_{t=1}^{T} \exp(i\pi k(t-1/2)/T) + (1/2)\sum_{t=1}^{T} \exp(-i\pi k(t-1/2)/T)$$

$$= 0 + 0 = 0,$$

while for $k \in \{1, 2, \ldots, 2T-1\}$ odd,

$$A(k,T) = (1/2)\sum_{t=1}^{T} \exp(i\pi k(t-1/2)/T) + (1/2)\sum_{t=1}^{T} \exp(-i\pi k(t-1/2)/T)$$

$$= (1/2)(\exp(i\pi k/T)-1)^{-1} \exp((1/2)i\pi k/T)(-2) + (1/2)(\exp(-i\pi k/T)-1)^{-1} \exp(-(1/2)i\pi k/T)(-2),$$

and setting $z = \exp((1/2)i\pi k/T)$, it now follows that the last expression equals 0 because

$$-(z^2-1)^{-1}z-(z^{-2}-1)^{-1}z^{-1}=0.$$

Therefore, $A(k,T)=0$ for $k\in\{1,2,\ldots,2T-1\}$. To prove the lemma, first note that trivially, $T^{-1}\sum_{t=1}^{T}p_1(t/T)p_1(t/T)=T^{-1}\sum_{t=1}^{T}1=1$. For $j\in\{2,3,\ldots,T\}$, by the earlier property of $A(k,T)$,

$$T^{-1}\sum_{t=1}^{T}p_j(t/T)p_1(t/T)=T^{-1}\sum_{t=1}^{T}\sqrt{2}\cos(\pi(j-1)(t-1/2)/T)$$

$$=\sqrt{2}T^{-1}A(j-1,T)=0,$$

and therefore, for $j\in\{2,3,\ldots,T\}$,

$$\Big[T^{-1}\sum_{t=1}^{T}p_{Tt}p_{Tt}'\Big]_{j,1}=\Big[T^{-1}\sum_{t=1}^{T}p_{Tt}p_{Tt}'\Big]_{1,j}=0.$$

Furthermore, by the trigonometric identity $\cos(\alpha)\cos(\beta)=(\cos(\alpha+\beta)+\cos(\alpha-\beta))/2$, for $j,k\in\{2,3,\ldots,T\}$,

$$\Big[T^{-1}\sum_{t=1}^{T}p_{Tt}p_{Tt}'\Big]_{j,k}=T^{-1}\sum_{t=1}^{T}p_j(t/T)p_k(t/T)$$

$$=T^{-1}\sum_{t=1}^{T}\cos(\pi(j+k-2)(t-1/2)/T)+T^{-1}\sum_{t=1}^{T}\cos(\pi(j-k)(t-1/2)/T)$$

$$=T^{-1}A(j+k-2,T)+T^{-1}A(|j-k|,T).$$

The above expression is zero for $j\neq k$ by the earlier result on $A(k,T)$, while for $j=k$, $T^{-1}A(0,T)=1$, which shows the result. □

Lemma 3. *Assume that $h:[0,1]\to\mathbb{R}$ is continuous on $[0,1]$. Then*

$$\lim_{T\to\infty}T^{-1}\sum_{j=1}^{T}h((j-1)/T)=\int_0^1 h(r)dr.$$

Proof. This follows because by continuity of $h(\cdot)$ and setting $j = rT$,

$$T^{-1}\sum_{j=1}^{T} h((j-1)/T) = T^{-1}\int_0^T h([j]/T)dj = \int_0^1 h([rT]/T)dr,$$

and because $\sup_{r\in[0,1]}|h(r)| < \infty$ by continuity and because $r - 1/T \leq [rT]/T \leq r$, the result now follows by the dominated convergence theorem. $\square$

Lemma 4. *Assume that $h : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)m/T)h((j-1)/T) \\
&= -(2T)^{-1}h(1/T) \\
&-(2T)^{-1}(-1)^m h((T-1)/T) \\
&-T^{-1}(2\sin(\pi m/(2T))^{-2}(h(2/T) - h(1/T))\cos(\pi m/T) \\
&+T^{-1}(2\sin(\pi m/(2T))^{-2}(h(1-1/T) - h(1-2/T))\cos(\pi m(1-1/T)) \\
&+T^{-1}(2\sin(\pi m/(2T)))^{-3}(h(3/T) - 2h(2/T) + h(1/T))\sin(\pi(3/2)m/T) \\
&-T^{-1}(2\sin(\pi m/(2T)))^{-3}(h(1-1/T) - 2h(1-2/T) + h(1-3/T))\sin(\pi(T-3/2)m/T) \\
&+T^{-1}(2\sin(\pi m/(2T)))^{-3}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\sin(\pi(j+1/2)m/T)
\end{aligned}$$

and

$$\begin{aligned}
&T^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)(m-1/2)/T)\cos(\pi(j-1)/(2T))h((j-1)/T) \\
&= \ldots
\end{aligned}$$

Proof. The proof of this result is somewhat tedious and can be found at `https://dl.dropbox.com/u/2159931/hp.html`. $\square$

Proof of Lemma 1: By Lemma 2, $T^{-1}\sum_{t=1}^{T} p_{Tt}p'_{Tt} = I_T$, and therefore it only remains to prove the second part of Lemma 1. To show this, note that for $j \in \{1, 2, ..., T\}$, remembering that $p_1(t/T) = 1$ for $t \in \mathbb{Z}$,

$$p_j(t/T) - p_j((t-1)/T) = \sqrt{2}\cos(\pi(j-1)(t-1/2)/T) - \sqrt{2}\cos(\pi(j-1)(t-3/2)/T),$$

and because

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin(\alpha)\sin(\beta)$$

and setting $\alpha + \beta = \pi(j-1)(t-1/2)/T$ and $\alpha - \beta = \pi(j-1)(t-3/2)/T$, we get

$$p_j(t/T) - p_j((t-1)/T) = -2\sqrt{2}\sin(\pi(j-1)(t-1)/T))\sin(\pi(j-1)/(2T)).$$

Therefore,

$$\begin{aligned}
&p_j((t+1)/T) - 2p_j(t/T) + p_j((t-1)/T) \\
&= (p_j((t+1)/T) - p_j(t/T)) - (p_j(t/T) - p_j((t-1)/T)) \\
&= 2\sqrt{2}\sin(\pi(j-1)/(2T))(\sin(\pi(j-1)(t-1)/T) - \sin(\pi(j-1)t/T)).
\end{aligned}$$

Now because

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\sin(\beta)\cos(\alpha)$$

and setting $\alpha + \beta = \pi(j-1)(t-1)/T$ and $\alpha - \beta = \pi(j-1)t/T)$, we have

$$\begin{aligned}
&\sin(\pi(j-1)(t-1)/T) - \sin(\pi(j-1)t/T) \\
&= -2\sin(\pi(j-1)/(2T))\cos(\pi(j-1)(t-1/2)/T).
\end{aligned}$$

Therefore, for $j \in \{1, 2, \ldots, T\}$ and $t \in \mathbb{Z}$,

$$p_j((t+1)/T) - 2p_j(t/T) + p_j((t-1)/T) = -4\sqrt{2}\sin(\pi(j-1)/(2T))^2\cos(\pi(j-1)(t-1/2)/T),$$

implying that, for $j, k \in \{1, 2, ..., T\}$,

$$\Big[\sum_{t=2}^{T-1}(p_{T,t+1} - 2p_{Tt} + p_{T,t-1})(p_{T,t+1} - 2p_{Tt} + p_{T,t-1})'\Big]_{jk}$$

$$= 32\sin(\pi(j-1)/(2T))^2\sin(\pi(k-1)/(2T))^2\sum_{t=2}^{T-1}\cos(\pi(j-1)(t-1/2)/T)\cos(\pi(k-1)(t-1/2)/T).$$

Also, for $j, k \in \{1, 2, \ldots, T\}$, by Lemma 2,

$$\sum_{t=2}^{T-1} \cos(\pi(j-1)(t-1/2)/T)\cos(\pi(k-1)(t-1/2)/T)$$

$$= (1/2)TI(j=k) - \cos(\pi(j-1)/(2T))\cos(\pi(k-1)/(2T))$$

$$- \cos(\pi(j-1)(T-1/2)/T)\cos(\pi(k-1)(T-1/2)/T),$$

implying that

$$T^{-1}\sum_{t=2}^{T-1}(p_{T,t+1} - 2p_{Tt} + p_{T,t-1})(p_{T,t+1} - 2p_{Tt} + p_{T,t-1})'$$

$$= \mathrm{diag}(\{16\sin(\pi(j-1)/(2T))^4, j = 1, \ldots, T\}) - 32T^{-1}q_{T1}q_{T1}' - 32T^{-1}q_{T2}q_{T2}',$$

where $q_{T1j} = \sin(\pi(j-1)/(2T))^2\cos(\pi(j-1)/(2T))$ and $q_{T2j} = \sin(\pi(j-1)/(2T))^2\cos(\pi(j-1)(T-1/2)/T)$. □

Proof of Theorem 1: It follows from Lemma 1 that

$$\hat{\theta} = \left[T^{-1}\sum_{t=1}^{T} p_{Tt}p_{Tt}' + \lambda T^{-1}\sum_{t=2}^{T-1}(p_{T,t+1} - 2p_{Tt} + p_{T,t-1})(p_{T,t+1} - 2p_{Tt} + p_{T,t-1})'\right]^{-1} T^{-1}\sum_{t=1}^{T} y_t p_{Tt}$$

$$= \left[I_T + \lambda D_T + H_T(I_T + \lambda D_T)\right]^{-1} T^{-1}\sum_{t=1}^{T} y_t p_{Tt}$$

$$= (I_T + \lambda D_T)^{-1}(I_T + H_T)^{-1}T^{-1}\sum_{t=1}^{T} y_t p_{Tt}$$

where

$$H_T = (-32\lambda T^{-1}q_{T1}q_{T1}' - 32\lambda T^{-1}q_{T2}q_{T2}')(I_T + \lambda D_T)^{-1}.$$

Now according to Miller (1981), for a matrix H_T of rank 0, 1 or 2, we have

$$(I_T + H_T)^{-1} = I_T - (a_T + b_T)^{-1}(a_T H_T - H_T^2)$$

for $a_T = 1 + \mathrm{tr}(H_T)$ and $2b_T = (\mathrm{tr}(H_T))^2 - \mathrm{tr}(H_T^2)$. Therefore,

$$
\begin{aligned}
&(I_T + \lambda D_T)^{-1}(I_T + H_T)^{-1} = (I_T + \lambda D_T)^{-1}(I_T - (a_T + b_T)^{-1}(a_T H_T - H_T^2)) \\
&= (I_T + \lambda D_T)^{-1} \\
&+32\lambda a_T(a_T + b_T)^{-1}(I_T + \lambda D_T)^{-1}(T^{-1}q_{T1}q_{T1}')(I_T + \lambda D_T)^{-1} \\
&+32\lambda a_T(a_T + b_T)^{-1}(I_T + \lambda D_T)^{-1}(T^{-1}q_{T2}q_{T2}')(I_T + \lambda D_T)^{-1} \\
&+32^2\lambda^2(a_T + b_T)^{-1}(I_T + \lambda D_T)^{-1}(T^{-1}q_{T1}q_{T1}')(I_T + \lambda D_T)^{-1}(T^{-1}q_{T1}q_{T1}')(I_T + \lambda D_T)^{-1} \\
&+32^2\lambda^2(a_T + b_T)^{-1}(I_T + \lambda D_T)^{-1}(T^{-1}q_{T1}q_{T1}')(I_T + \lambda D_T)^{-1}(T^{-1}q_{T2}q_{T2}')(I_T + \lambda D_T)^{-1} \\
&+32^2\lambda^2(a_T + b_T)^{-1}(I_T + \lambda D_T)^{-1}(T^{-1}q_{T2}q_{T2}')(I_T + \lambda D_T)^{-1}(T^{-1}q_{T1}q_{T1}')(I_T + \lambda D_T)^{-1} \\
&+32^2\lambda^2(a_T + b_T)^{-1}(I_T + \lambda D_T)^{-1}(T^{-1}q_{T2}q_{T2}')(I_T + \lambda D_T)^{-1}(T^{-1}q_{T2}q_{T2}')(I_T + \lambda D_T)^{-1} \\
&= \sum_{i=1}^{7} M_{Ti},
\end{aligned}
$$

say, and we can write

$$
\hat{\tau}_{Tt} = p_{Tt}'\hat{\theta} = p_{Tt}'\sum_{i=1}^{7} M_{Ti}T^{-1}\sum_{s=1}^{T} p_{Ts}y_s = \sum_{s=1}^{T} y_s \sum_{i=1}^{7} T^{-1}p_{Tt}'M_{Ti}p_{Ts}
$$

where

$$
T^{-1}p_{Tt}'M_{T1}p_{Ts} = T^{-1}p_{Tt}'(I_T + \lambda D_T)^{-1}p_{Ts},
$$

$$
\begin{aligned}
&T^{-1}p_{Tt}'M_{T2}p_{Ts} = 32\lambda a_T(a_T + b_T)^{-1}(T^{-1}p_{Tt}'(I_T + \lambda D_T)^{-1}q_{T1})(T^{-1}q_{T1}'(I_T + \lambda D_T)^{-1}p_{Ts}), \\
&T^{-1}p_{Tt}'M_{T3}p_{Ts} = 32\lambda a_T(a_T + b_T)^{-1}(T^{-1}p_{Tt}'(I_T + \lambda D_T)^{-1}q_{T2})(T^{-1}q_{T2}'(I_T + \lambda D_T)^{-1}p_{Ts}), \\
&T^{-1}p_{Tt}'M_{T4}p_{Ts} \\
&= 32^2\lambda^2(a_T{+}b_T)^{-1}(T^{-1}p_{Tt}(I_T{+}\lambda D_T)^{-1}q_{T1})(T^{-1}q_{T1}'(I_T{+}\lambda D_T)^{-1}q_{T1})(T^{-1}q_{T1}'(I_T{+}\lambda D_T)^{-1}p_{Ts}), \\
&T^{-1}p_{Tt}'M_{T5}p_{Ts} \\
&= 32^2\lambda^2(a_T{+}b_T)^{-1}(T^{-1}p_{Tt}(I_T{+}\lambda D_T)^{-1}q_{T1})(T^{-1}q_{T1}'(I_T{+}\lambda D_T)^{-1}q_{T2})(T^{-1}q_{T2}'(I_T{+}\lambda D_T)^{-1}p_{Ts}), \\
&T^{-1}p_{Tt}'M_{T6}p_{Ts} \\
&= 32^2\lambda^2(a_T{+}b_T)^{-1}(T^{-1}p_{Tt}(I_T{+}\lambda D_T)^{-1}q_{T2})(T^{-1}q_{T2}'(I_T{+}\lambda D_T)^{-1}q_{T1})(T^{-1}q_{T1}'(I_T{+}\lambda D_T)^{-1}p_{Ts}),
\end{aligned}
$$

and

$$T^{-1}p'_{Tt}M_{T7}p_{Ts}$$

$$= 32^2\lambda^2(a_T+b_T)^{-1}(T^{-1}p_{Tt}(I_T+\lambda D_T)^{-1}q_{T2})(T^{-1}q'_{T2}(I_T+\lambda D_T)^{-1}q_{T2})(T^{-1}q'_{T2}(I_T+\lambda D_T)^{-1}p_{Ts}).$$

Now consider the first term $T^{-1}p'_{Tt}(I_T+\lambda D_T)^{-1}p_{Ts}$. Using the identity

$$\cos(\alpha)\cos(\beta) = (1/2)\cos(\alpha+\beta) + (1/2)\cos(\alpha-\beta),$$

one can write, for $j \geq 2$,

$$p_j(t/T)p_j(s/T) = 2\cos(\pi(j-1)(t-1/2)/T)\cos(\pi(j-1)(s-1/2)/T)$$

$$= \cos(\pi(j-1)(t+s-1)/T) + \cos(\pi(j-1)(t-s)/T).$$

Using the above equation, the first term can be represented as (remembering that $p_1(t/T) = 1$ for $t \in \{1, 2, \ldots, T\}$),

$$T^{-1}p'_{Tt}(I_T+\lambda D_T)^{-1}p_{Ts}$$

$$= T^{-1} + T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)(t+s-1)/T)(1+16\lambda\sin(\pi(j-1)/(2T))^4)^{-1}$$

$$+T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)(t-s)/T)(1+16\lambda\sin(\pi(j-1)/(2T))^4)^{-1}$$

$$= T^{-1} + f_{T\lambda}(t+s-1) - (2T)^{-1} - (2T)^{-1}(-1)^{t+s-1}(1+16\lambda)^{-1}$$
$$+f_{T\lambda}(t-s) - (2T)^{-1} - (2T)^{-1}(-1)^{t-s}(1+16\lambda)^{-1}$$

$$= f_{T\lambda}(t-s) + f_{T\lambda}(t+s-1)$$

because $(-1)^{t+s-1} + (-1)^{t-s} = (-1)^t((-1)^{s-1} + (-1)^s) = 0$. By the result of Equation (19),

$$T^{-1}p'_{Tt}M_{T1}p_{Ts} = f_{T\lambda}(t-s) + f_{T\lambda}(T)I(t+s-1=T)$$

$$+f_{T\lambda}(s+t-1)I(t+s-1<T)+f_{T\lambda}(2T-t-s+1)I(t+s-1>T) = w^1_{Tts}+w^2_{Tts}+w^3_{Tts}+w^4_{Tts}.$$

Next, we consider the other terms and we will show that

$$\sum_{i=2}^{7}T^{-1}p'_{Tt}M_{Ti}p_{Ts} = w^5_{Tts}+w^6_{Tts}+w^7_{Tts}+w^8_{Tts}.$$

First note that

$$T^{-1}p'_{Tt}M_{T1}q_{T1} = g_{T\lambda}(t)$$

and note that by symmetry of D_T and I_T, $T^{-1}q'_{T1}(I_T+\lambda D_T)^{-1}q_{T2} = T^{-1}q'_{T2}(I_T+\lambda D_T)^{-1}q_{T1} = \eta_{T\lambda}$. Also, note that

$$T^{-1}p'_{Tt}M_{T1}q_{T2} = g_{T\lambda}(T-t+1).$$

To see this, first note that for $j \in \mathbb{Z}$

$$q_{T2j} = \cos(\pi(j-1))q_{T1j}.$$

This follows because, since $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$ and because $\sin(\pi(j-1)) = 0$ for $j \in \mathbb{Z}$,

$$q_{T2j} = \sin(\pi(j-1)/(2T))^2\cos(\pi(j-1)-\pi(j-1)/(2T))$$

$$= \sin(\pi(j-1)/(2T))^2\cos(\pi(j-1))\cos(\pi(j-1)/(2T)) = \cos(\pi(j-1))q_{T1j}.$$

In addition, note that

$$p_j(t/T) = p_j((T-t+1)/T)$$

because

$$p_j((T-t+1)/T) = \sqrt{2}\cos(\pi(j-1)((T-t+1/2)/T))$$

$$= \sqrt{2}\cos(\pi(j-1))\cos(\pi(j-1)(t-1/2)/(2T)) = \cos(\pi(j-1))p_j(t/T),$$

and therefore it follows that

$$T^{-1}p'_{Tt}(I_T+\lambda D_T)^{-1}q_{T2} = \sum_{j=1}^{T} T^{-1}p_j(t/T)q_{T2j}(1+\lambda d_{jj})^{-1}$$

$$= \sum_{j=1}^{T} T^{-1}p_j(t/T)\cos(\pi(j-1))q_{T1j}(1+\lambda d_{jj})^{-1}$$

$$= T^{-1}p'_{T,T-t+1}(I_T+\lambda D_T)^{-1}q_{T1} = g_{T\lambda}(T-t+1).$$

Finally, note that

$$T^{-1}q'_{T2}(I_T+\lambda D_T)^{-1}q_{T2} = T^{-1}q'_{T1}(I_T+\lambda D_T)^{-1}q_{T1} = \delta_T.$$

Therefore,

$$T^{-1}p'_{Tt}M_{T2}p_{Ts} = 32\lambda a_T(a_T+b_T)^{-1}g_{T\lambda}(t)g_{T\lambda}(s),$$

$$T^{-1}p'_{Tt}M_{T3}p_{Ts} = 32\lambda a_T(a_T+b_T)^{-1}g_{T\lambda}(T-t+1)g_{T\lambda}(T-s+1),$$
$$T^{-1}p'_{Tt}M_{T4}p_{Ts} = 32^2\lambda^2(a_T+b_T)^{-1}\delta_T g_{T\lambda}(t)g_{T\lambda}(s),$$
$$T^{-1}p'_{Tt}M_{T5}p_{Ts} = 32^2\lambda^2(a_T+b_T)^{-1}\eta_T g_{T\lambda}(t)g_{T\lambda}(T-s+1),$$
$$T^{-1}p'_{Tt}M_{T6}p_{Ts} = 32^2\lambda^2(a_T+b_T)^{-1}\eta_T g_{T\lambda}(T-t+1)g_{T\lambda}(s),$$
$$T^{-1}p'_{Tt}M_{T7}p_{Ts} = 32^2\lambda^2(a_T+b_T)^{-1}\delta_T g_{T\lambda}(T-t+1)g_{T\lambda}(T-s+1).$$

Now note that

$$a_T = 1+\mathrm{tr}(H_T) = 1+\mathrm{tr}((-32\lambda T^{-1}q_{T1}q'_{T1} - 32\lambda T^{-1}q_{T2}q'_{T2})(I_T+\lambda D_T)^{-1})$$
$$= 1 - 32\lambda T^{-1}q'_{T1}(I_T+\lambda D_T)^{-1}q_{T1} - 32\lambda T^{-1}q'_{T2}(I_T+\lambda D_T)^{-1}q_{T2}$$
$$= 1-64\lambda\delta_T,$$
$$b_T = (1/2)(\mathrm{tr}(H_T))^2 - (1/2)\mathrm{tr}(H_T^2)$$
$$= (1/2)(-64\lambda\delta_T)^2 - (1/2)32^2\lambda^2(2\delta_T^2+2\eta_T^2)$$
$$= 32^2\lambda^2(\delta_T^2-\eta_T^2),$$
$$\xi_{T\lambda} = 32\lambda a_T(a_T+b_T)^{-1} + 32^2\lambda^2(a_T+b_T)^{-1}\delta_T,$$

and

$$\phi_{T\lambda} = 32^2\lambda^2(a_T+b_T)^{-1}\eta_T.$$

Therefore, setting

$$w^5_{Tts} = \xi_{T\lambda}g_{T\lambda}(t)g_{T\lambda}(s)$$

$$w^6_{Tts} = \phi_{T\lambda}g_{T\lambda}(T-t+1)g_{T\lambda}(s)$$
$$w^7_{Tts} = \phi_{T\lambda}g_{T\lambda}(t)g_{T\lambda}(T-s+1)$$
$$w^8_{Tts} = \xi_{T\lambda}g_{T\lambda}(T-t+1)g_{T\lambda}(T-s+1)$$

it now follows that

$$\sum_{i=2}^{7} T^{-1}p'_{Tt}M_{Ti}p_{Ts} = w^5_{Tts}+w^6_{Tts}+w^7_{Tts}+w^8_{Tts},$$

which completes the proof of the representation for w_{Tts}.

In order to find the upper bounds for $f_{T\lambda}(m)$ and $g_{T\lambda}(m)$, note that by the first result of Lemma 4, it follows that for $h(r) = 1/(1 + 16\lambda \sin(\pi r/2)^4)$,

$$
\begin{aligned}
& f_{T\lambda}(m) = -(2T)^{-1}(h(1/T) - h(0)) \\
& -(2T)^{-1}(-1)^m(h((T-1)/T) - h(1)) \\
& -T^{-1}(2\sin(\pi m/(2T))^{-2}(h(2/T) - h(1/T))\cos(\pi m/T) \\
& +T^{-1}(2\sin(\pi m/(2T))^{-2}(h(1-1/T) - h(1-2/T))\cos(\pi m(1-1/T)) \\
& +T^{-1}(2\sin(\pi m/(2T)))^{-3}(h(3/T) - 2h(2/T) + h(1/T))\sin(\pi(3/2)m/T) \\
& -T^{-1}(2\sin(\pi m/(2T)))^{-3}(h(1-1/T) - 2h(1-2/T) + h(1-3/T))\sin(\pi(T-3/2)m/T) \\
& +T^{-1}(2\sin(\pi m/(2T)))^{-3}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\sin(\pi(j+1/2)m/T).
\end{aligned}
$$

Noting that $h'(0) = h'(1) = 0$ and that $\sup_{x\in[0,1]}|h''(x)| < \infty$, it follows from a Taylor series expansion of order 2 that the first and second terms are bounded in absolute value by

$$(1/4)T^{-3}\sup_{x\in[0,1]}|h''(x)|.$$

Similarly, the third and fourth terms are bounded by a multiple of

$$T^{-3}(2\sin(\pi m/(2T))^{-2},$$

and the fifth, sixth and seventh terms by a multiple of

$$T^{-3}(2\sin(\pi m/(2T))^{-3}.$$

By noting that

$$\sup_{r\in[0,1]}|r/\sin(\pi r/2)| = 1$$

and that $m \leq T$, it now follows that $|f_{T\lambda}(m)| \leq Cm^{-3}$. By using the second result of Lemma 4 and setting $h(r) = \sin(\pi r/2)^2(1 + 16\lambda\sin(\pi r/2)^4)^{-1}$, the bound for $g_{T\lambda}(m)$ can be proven analogously. $\square$

Proof of Theorem 2: By Lemma 3,

$$\delta_{T\lambda} = T^{-1}q'_{T1}(I_T + \lambda D_T)^{-1}q_{T1}$$

$$= T^{-1}\sum_{j=1}^{T} \sin(\pi(j-1)/(2T))^4 \cos(\pi(j-1)/(2T))^2(1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1}$$

$$\to \int_0^1 \sin(\pi r/2)^4 \cos(\pi r/2)^2(1 + 16\lambda \sin(\pi r/2)^4)^{-1}dr = \delta_\lambda$$

and

$$\eta_{T\lambda} = T^{-1}q'_{T1}(I_T + \lambda D_T)^{-1}q_{T2}$$

$$= T^{-1}\sum_{j=1}^{T} q_{T1j}q_{T2j}(1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1},$$

and because $q_{T2j} = \cos(\pi(j-1))q_{T1j}$,

$$\eta_{T\lambda} = T^{-1}\sum_{j=1}^{T} \cos(\pi(j-1))q_{T1j}^2(1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1}$$

$$= T^{-1}\sum_{j=1, j \text{ odd}}^{T} q_{T1j}^2(1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1}$$

$$-T^{-1}\sum_{j=1, j \text{ even}}^{T} q_{T1j}^2(1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1}$$

$$= T^{-1}\sum_{j=0}^{[(T-1)/2]} \sin(\pi j/T)^4 \cos(\pi j/T)^2(1 + 16\lambda \sin(\pi j/T)^4)^{-1}$$

$$-T^{-1}\sum_{j=1}^{[T/2]} \sin(\pi(2j-1)/(2T))^4 \cos(\pi(2j-1)/(2T))^2(1 + 16\lambda \sin(\pi(2j-1)/(2T))^4)^{-1},$$

and by an argument similar to that of Lemma 3, it now follows that both terms converge to the same number, implying that $\lim_{T\to\infty} \eta_{T\lambda} = 0$. Note that since $0 \leq \sin(\pi r/2)^4\lambda(1 + 16\lambda \sin(\pi r/2)^4)^{-1} \leq 1/16$, it follows that

$$\lambda\delta_\lambda \leq (1/16)\int_0^1 \cos(\pi r/2)^2 dr = 1/32.$$

Therefore,

$$\xi_{T\lambda} = 32\lambda(1-64\lambda\delta_{T\lambda})(1-64\lambda\delta_{T\lambda}+32^2\lambda^2(\delta_{T\lambda}^2-\eta_{T\lambda}^2))^{-1}+32^2\lambda^2(1-64\lambda\delta_{T\lambda}+32^2\lambda^2(\delta_{T\lambda}^2-\eta_{T\lambda}^2))^{-1}\delta_{T\lambda}$$
$$\to 32\lambda(1-64\lambda\delta_\lambda)(1-64\lambda\delta_\lambda+32^2\lambda^2\delta_\lambda^2)^{-1}+32^2\lambda^2(1-64\lambda\delta_\lambda+32^2\lambda^2\delta_\lambda^2)^{-1}\delta_\lambda$$
$$= \frac{32\lambda(1-64\lambda\delta_\lambda)+32^2\lambda^2\delta_\lambda}{(1-64\lambda\delta_\lambda+32^2\lambda^2\delta_\lambda^2)} = \frac{32\lambda}{1-32\lambda\delta_\lambda},$$

and

$$\phi_{T\lambda} = 32^2\lambda^2(1-64\lambda\delta_{T\lambda}+32^2\lambda^2(\delta_{T\lambda}^2-\eta_{T\lambda}^2))^{-1}\eta_{T\lambda}$$
$$\to 32^2\lambda^2(1-32\lambda\delta_\lambda)^{-2}\times 0 = 0.$$

□

Proof of Theorem 3: The results $\lim_{T\to\infty} f_{T\lambda}(m) = \int_0^1 \cos(\pi rm)(1+16\lambda\sin(\pi r/2)^4)^{-1}dr$ and $\lim_{T\to\infty} g_{T\lambda}(m) = \sqrt{2}\int_0^1 \cos(\pi r(m-1/2))\sin(\pi r/2)^2\cos(\pi r/2)(1+16\lambda\sin(\pi r/2)^4)^{-1}dr$ are direct consequences of Lemma 3. To show that both representations for $f_\lambda(\cdot)$ and $g_\lambda(\cdot)$ are identical, note that one can represent $f_\lambda(m)$ as

$$f_\lambda(m) = \int_0^1 \frac{\cos(\pi rm)}{1+16\lambda\sin(\pi r/2)^4}dr = \int_0^1 \frac{\mathrm{Re}(\exp(i\pi rm))q}{q+(1-\exp(i\pi r))^2(1-\exp(-i\pi r))^2}dr$$
$$= \mathrm{Re}\left(\frac{1}{2}\int_{-1}^1 \frac{\exp(i\pi rm)q}{q+(1-\exp(i\pi r))^2(1-\exp(-i\pi r))^2}dr\right)$$

by setting $\cos(\pi rm) = \mathrm{Re}(\exp(i\pi rm))$, where $\mathrm{Re}(x)$ stands for the real part of the complex number x, $\sin(\pi r/2)^4 = (1-\exp(i\pi r))^2(1-\exp(-i\pi r))^2/16$, and $q = 1/\lambda$. Using the change of variable $z = \exp(i\pi r)$ and $dz = i\pi z dr$, we now obtain

$$f_\lambda(m) = \frac{q}{2i\pi}\oint \frac{z^{m-1}}{q+(1-z)^2(1-z^{-1})^2}dz.$$

In the above expression, the denominator is a fourth order polynomial that has roots

$$r_1 = \frac{(2i-\sqrt{q})+\sqrt{q-4i\sqrt{q}}}{2i}$$

$$r_2 = \frac{(2i-\sqrt{q})-\sqrt{q-4i\sqrt{q}}}{2i}$$

$$r_3 = \frac{(2i+\sqrt{q}) - \sqrt{q+4i\sqrt{q}}}{2i}$$

$$r_4 = \frac{(2i+\sqrt{q}) + \sqrt{q+4i\sqrt{q}}}{2i}$$

These roots have the following relations: $r_2 = r_1^{-1}$, $r_4 = r_3^{-1}$, $r_3 = \bar{r}_1$, and $r_4 = \bar{r}_2$. Note that r_1 and r_3 are inside the unit circle, whereas r_2 and r_4 are outside the unit circle. Therefore,

$$f_\lambda(m) = \frac{q}{2i\pi} \oint \frac{z^{m-1}}{(z-r_1)(z-r_1^{-1})(z-\bar{r}_1)(z-\bar{r}_1^{-1})} dz.$$

Using Cauchy's Residue Theorem, the solution to this contour integral is

$$f_\lambda(m) = qRe\left(\frac{r_1^{m+1}}{(r_1 - r_1^{-1})(r_1 - \bar{r}_1)(r_1 - \bar{r}_1^{-1})} + \frac{\bar{r}_1^{m+1}}{(\bar{r}_1 - r_1)(\bar{r}_1 - r_1^{-1})(\bar{r}_1 - \bar{r}_1^{-1})} \right).$$

By simplifying the above result, we can express $f_\lambda(m)$ as:

$$f_\lambda(m) = \frac{2q|r_1|^{m+2}\sin(\theta)(|r_1|^2\sin((m-1)\theta) - \sin((m+1)\theta))}{(1-2\cos(2\theta)|r_1|^2 + |r_1|^4)(|r_1|^2-1)(1-\cos(2\theta))}$$

where

$$\theta = \tan^{-1}\left(\frac{\sqrt{q-4i\sqrt{q}} + \sqrt{q+4i\sqrt{q}} - 2\sqrt{q}}{i(4i + \sqrt{q-4i\sqrt{q}} - \sqrt{q+4i\sqrt{q}})} \right).$$

Furthermore, one can also solve $g_\lambda(m)$ for $m \geq 1$ using the above approach, and this then gives

$$g_\lambda(m) = \sqrt{2}\int_0^1 \frac{\cos(\pi r(m-1/2))\sin(\pi r/2)^2\cos(\pi r/2)}{1+16\lambda\sin(\pi r/2)^4} dr.$$

In order to obtain our formula that relates $f_\lambda(m)$ to $g_\lambda(m)$, note that

$$\cos(\pi r(m-1/2))\sin(\pi r/2)^2\cos(\pi r/2) = \frac{1}{8}(\cos(\pi r(m-1)) - \cos(\pi r(m-2)) + \cos(\pi r m) - \cos(\pi r(m+1))),$$

which implies that $g_\lambda(m)$ can be written as

$$g_\lambda(m) = \frac{\sqrt{2}}{8}\int_0^1 \frac{\cos(\pi r(m-1)) - \cos(\pi r(m-2)) + \cos(\pi r m) - \cos(\pi r(m+1))}{1+16\lambda\sin(\pi r/2)^4} dr,$$

which in turn is equivalent to

$$g_\lambda(m) = (\sqrt{2}/8)\big(f_\lambda(m-1) - f_\lambda(m-2) + f_\lambda(m) - f_\lambda(m+1)\big)$$

for $m \geq 1$ by the definition of the function $f_\lambda(m)$. $\square$

Proof of Theorem 4: First note that

$$E|(E|y_{[rT]}|)^{-1}\sum_{s=1}^{T} y_s(w_{T,[rT],s} - f_\lambda([rT]-s))|$$

$$\leq (E|y_{[rT]}|)^{-1}\sum_{s=1}^{T} E|y_s||w_{T,[rT],s} - f_\lambda([rT]-s))|$$

$$\leq \sup_{1\leq s\leq T}(E|y_s|(E|y_{[rT]}|)^{-1})\sum_{s=1}^{T}|w_{T,[rT],s} - f_\lambda([rT]-s)|,$$

and since by assumption,

$$\limsup_{T\to\infty}\sup_{1\leq s\leq T}(E|y_s|(E|y_{[rT]}|)^{-1}) < \infty,$$

it suffices to show

$$\limsup_{T\to\infty}\sum_{s=1}^{T}|f_{T\lambda}([rT]-s) - f_\lambda([rT]-s)| = 0$$

and

$$\limsup_{T\to\infty}\sum_{s=1}^{T}\sum_{j=2}^{8}|w^j_{T,[rT],s}| = 0.$$

The second result follows from the result of Equation (18). To show the first result, note that

$$\limsup_{T\to\infty}\sum_{s=1}^{T}|f_{T\lambda}([rT]-s) - f_\lambda([rT]-s)|$$

$$= \limsup_{T\to\infty} \sum_{s=0}^{[rT]-1} |f_{T\lambda}(s) - f_\lambda(s)| + \limsup_{T\to\infty} \sum_{s=1}^{T-[rT]} |f_{T\lambda}(s) - f_\lambda(s)|$$

and both terms vanish by the dominated convergence theorem because for $s \neq t$, by Theorem 1,

$$|f_{T\lambda}(t-s) - f_\lambda(t-s)| \leq C|t-s|^{-3}.$$

This now completes the argument. $\square$

Proof of Theorem 5: Note that, by the properties $\sum_{s=1}^{T} w_{Tts} = 1$ and $\sum_{s=1}^{T} w_{Tts}s = t$ that were noted in Section 2,

$$\hat{c}_{Tt} = y_t - \sum_{s=1}^{T} w_{Tts}(\alpha_1 + \alpha_2 s + \alpha_3 z_s + u_s)$$

$$= u_t - \sum_{s=1}^{T} w_{Tts}u_s + \alpha_3(z_t - \sum_{s=1}^{T} w_{Tts}z_s)$$

$$= u_t - \sum_{s=1}^{T} w_{Tts}u_s + \alpha_3(\sum_{j=1}^{t} \varepsilon_j - \sum_{s=1}^{T}\sum_{j=1}^{s} \varepsilon_j w_{Tts})$$

$$= u_t - \sum_{s=1}^{T} w_{Tts}u_s - \alpha_3 \sum_{j=1}^{T} \varepsilon_j(\sum_{s=j}^{T} w_{Tts} - I(j \leq t)).$$

Since $\hat{c}_{Tt} = \hat{c}_{Tt}^m$ for $m \geq T-1$ and because $\hat{c}_{Tt}^{-1} = u_t$ and $\sup_{t\geq 1} \parallel u_t \parallel_p < \infty$, it suffices to show that for every $m \geq -1$,

$$\sup_{T:T\geq m+2}\ \sup_{t\in[\gamma T,(1-\gamma)T]} \parallel \hat{c}_{Tt} - \hat{c}_{Tt}^m \parallel_p \leq C_1(m+2)^{-1}$$

and therefore it also suffices to show that for every $m \geq 0$,

$$\sup_{T:T\geq m+1}\ \sup_{t\in[\gamma T,(1-\gamma)T]} \parallel \hat{c}_{Tt} - \hat{c}_{Tt}^{m-1} \parallel_p \leq C_1(m+1)^{-1}.$$

Therefore, it suffices to show that

$$\sup_{T:T\geq m+1}\ \sup_{t\in[\gamma T,(1-\gamma)T]} \parallel \sum_{s=1}^{T} w_{Tts}u_s I(|t-s| \geq m) \parallel_p \leq C_2(m+1)^{-2}$$

and

$$\sup_{T:T\geq m+1}\sup_{t\in[\gamma T,(1-\gamma)T]}\parallel\sum_{j=1}^{T}\varepsilon_j(\sum_{s=j}^{T}w_{Tts}-I(j\leq t))I(|j-t|\geq m)\parallel_p\leq C_3(m+1)^{-1}.$$

Because of the result of Equation (18), it follows that

$$\sup_{r\in[\gamma,1-\gamma]}\sup_{T:T\geq m+1}T^3\sup_{1\leq s\leq T}\sum_{k=2}^{8}|w^k_{T,[rT],s}|<\infty.$$

To see the first result, note now that for $m\geq 0$

$$\sup_{T:T\geq m+1}\sup_{t\in[\gamma T,(1-\gamma)T]}\parallel\sum_{s=1}^{T}w_{Tts}u_sI(|t-s|\geq m)\parallel_p$$

$$\leq\sup_{T:T\geq m+1}\sup_{t\in[\gamma T,(1-\gamma)T]}\sum_{k=2}^{8}\sum_{s=1}^{T}|w^k_{Tts}|\sup_{s\geq 1}\parallel u_s\parallel_p$$

$$+\sup_{T:T\geq m+1}\sup_{t\in[\gamma T,(1-\gamma)T]}\sum_{s=1}^{T}|f_{T\lambda}(|t-s|)|I(|t-s|\geq m)\sup_{s\geq 1}\parallel u_s\parallel_p$$

$$\leq(m+1)^{-2}\sup_{r\in[\gamma,1-\gamma]}\sup_{T:T\geq m+1}T^3\sup_{1\leq s\leq T}\sum_{k=2}^{8}|w^k_{T,[rT],s}|\sup_{s\geq 1}\parallel u_s\parallel_p$$

$$+(2\sum_{k=0}^{\infty}Ck^{-3}I(k\neq 0)+I(k=0))I(|k|\geq m)\sup_{s\geq 1}\parallel u_s\parallel_p$$

$$\leq C_3(m+1)^{-2}+C_4I(m>0)\sum_{k=m}^{\infty}k^{-3}+C_5I(m=0)\leq C_2(m+1)^{-2}.$$

For the second result, note that

$$z_t-\sum_{s=1}^{T}z_sw_{Tts}$$

$$=\sum_{j=1}^{t}\varepsilon_j-\sum_{s=1}^{T}\sum_{j=1}^{s}\varepsilon_jw_{Tts}=-\sum_{j=1}^{T}\varepsilon_j(\sum_{s=j}^{T}w_{Tts}-I(j\leq t)),$$

and therefore the second result follows because, using that $\sum_{s=1}^{T} w_{Tts} = 1$, defining sums over empty index sets as 0,

$$\| \sum_{j=1}^{T} \varepsilon_j (\sum_{s=j}^{T} w_{Tts} - I(j \leq t)) I(|j-t| \geq m) \|_p$$

$$\leq \sum_{j=1}^{t} I(t-j \geq m) |\sum_{s=j}^{T} w_{Tts} - 1| \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$+ \sum_{j=t+1}^{T} I(j-t \geq m) |\sum_{s=j}^{T} w_{Tts}| \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$\leq \sum_{j=2}^{t} I(t-j \geq m) \sum_{s=1}^{j-1} |f_{T\lambda}(t-s)| \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$+ \sum_{j=t+1}^{T} I(j-t \geq m) \sum_{s=j}^{T} |f_{T\lambda}(t-s)| \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$+ \sum_{j=2}^{t} I(t-j \geq m) \sum_{k=2}^{8} \sum_{s=1}^{j-1} |w_{Tts}^k| \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$+ \sum_{j=t+1}^{T} I(j-t \geq m) \sum_{k=2}^{8} \sum_{s=j}^{T} |w_{Tts}^k| \sup_{j \geq 1} \| \varepsilon_j \|_p \,.$$

The first summation, divided by $\sup_{j \geq 1} \| \varepsilon_j \|_p$, is bounded by

$$\sum_{j=2}^{t} I(t-j \geq m) \sum_{s=1}^{j-1} C_6 (t-s)^{-3} = \sum_{s=1}^{t-1} \sum_{j=2}^{t} I(t-j \geq m) I(s \leq j-1) C_6 (t-s)^{-3}$$

$$\leq \sum_{s=1}^{t-1} C_6 I(s \leq t-m-1)(t-s)^{-2} \leq C_6 \sum_{k=m+1}^{\infty} k^{-2} \leq C_7 (m+1)^{-1}.$$

For the second term, a similar argument holds. For the third term, note that

$$\sup_{T: T \geq m+1} \sup_{t \in [\gamma T, (1-\gamma) T]} \sum_{j=2}^{t} I(t-j \geq m) \sum_{k=2}^{8} \sum_{s=1}^{j-1} |w_{Tts}^k| \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$\leq (m+1)^{-1} \sup_{T:T\geq m} \sup_{r\in[\gamma,1-\gamma]} T^3 \sup_{1\leq s\leq T} \sum_{k=2}^{8} |w^k_{T,[rT],s}| \sup_{j\geq 1} \| \varepsilon_j \|_p$$

$$\leq C_8(m+1)^{-1}$$

and a similar argument holds for the fourth term. Therefore, the conclusion of the theorem now follows. □

Proof of Theorem 6: Write

$$T^{-1}\sum_{t=1}^{T}(f(\hat{c}_{Tt}) - Ef(\hat{c}_{Tt}))$$

$$= T^{-1} \sum_{t\in\{1,\ldots,T\},t\notin[\gamma T,(1-\gamma)T]} (f(\hat{c}_{Tt}) - Ef(\hat{c}_{Tt})),$$

$$+T^{-1} \sum_{t\in\{1,\ldots,T\},t\in[\gamma T,(1-\gamma)T]} (f(\hat{c}_{Tt}) - Ef(\hat{c}_{Tt}))$$

and note that the first term is bounded in absolute value by

$$4\gamma \sup_{x\in\mathbb{R}} |f(x)|,$$

and because γ can be chosen arbitrarily small, it therefore suffices to show a weak law of large numbers for the second term. To show this, first note that for all $K > 0$ and $\eta > 0$,

$$\sup_{T\geq 1,t\in[\gamma T,(1-\gamma)T]} \| f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}) \|_2$$

$$\leq \sup_{T\geq 1,t\in[\gamma T,(1-\gamma)T]} \| f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m})I(|\hat{c}_{Tt}| > K)| \|_2$$

$$+ \sup_{T\geq 1,t\in[\gamma T,(1-\gamma)T]} \| f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m})I(|\hat{c}_{Tt}| \leq K)I(|\hat{c}_{Tt} - \hat{c}^m_{Tt}| > \eta)| \|_2$$

$$+ \sup_{T\geq 1,t\in[\gamma T,(1-\gamma)T]} \| f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m})I(|\hat{c}_{Tt}| \leq K)I(|\hat{c}_{Tt} - \hat{c}^m_{Tt}| \leq \eta)| \|_2 .$$

Because $f(\cdot)$ is bounded in absolute value,

$$\sup_{T\geq 1,t\in[\gamma T,(1-\gamma)T]} \| f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m})I(|\hat{c}_{Tt}| > K)| \|_2^2$$

$$\leq \sup_{x\in\mathbb{R}} |f(x)| \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} P(|\hat{c}_{Tt}| > K)$$

$$\leq \sup_{x\in\mathbb{R}} |f(x)| K^{-2} \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} E|\hat{c}_{Tt}|^2,$$

while similarly,

$$\sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}) I(|\hat{c}_{Tt}| \leq K) I(|\hat{c}_{Tt} - \hat{c}^m_{Tt}| > \eta)| \|_2^2$$

$$\leq \sup_{x\in\mathbb{R}} |f(x)| \eta^{-2} \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} E|\hat{c}_{Tt} - \hat{c}^m_{Tt}|^2,$$

and

$$\sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}) I(|\hat{c}_{Tt}| \leq K) I(|\hat{c}_{Tt} - \hat{c}^m_{Tt}| \leq \eta)| \|_2$$

$$\leq \sup_{|x|\leq K} \sup_{x'\in\mathbb{R}:|x-x'|\leq\eta} |f(x) - f(x')|.$$

Therefore, by making first m approach infinity, the making η approach 0, and then K approach infinity, it now follows that

$$\lim_{m\to\infty} \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}) \|_2 = 0.$$

Therefore, $f(\hat{c}_{Tt})$ is near epoch dependent on v_t for $t \in [\gamma T, (1-\gamma)T]$, and it therefore is a bounded L_1-mixingale as defined in Andrews (1988). Therefore, Andrews' weak law of large numbers applies, and the proof of the theorem is complete. □

Proof of Theorem 7: Since $f_\lambda(m) = \int_0^1 \cos(\pi rm) h_\lambda(r) dr$ where $h_\lambda(r) = l(2\lambda^{1/4}\sin(\pi r/2))$ for $l(x) = (1+x^4)^{-1}$, it follows by partial integration that for any integer m

$$f_\lambda(m) = (\pi m)^{-1} \int_0^1 h_\lambda(r) d\sin(\pi rm) = -(\pi m)^{-1} \int_0^1 \sin(\pi rm) h'_\lambda(r) dr$$

$$= (\pi m)^{-2} \int_0^1 h'_\lambda(r) d\cos(\pi rm)$$

$$= \Big[(\pi m)^{-2} h'_\lambda(r)\cos(\pi rm)\Big]_0^1 - (\pi m)^{-2} \int_0^1 \cos(\pi rm) dh'_\lambda(r).$$

Now

$$h_\lambda'(r) = l'(2\lambda^{1/4}\sin(\pi r/2))2\lambda^{1/4}\cos(\pi r/2))(\pi/2),$$

and therefore

$$\Big[(\pi m)^{-2}\ h_\lambda'(r)\cos(\pi r m)\Big]_0^1 = 0,$$

implying that

$$|f_\lambda(m)| \leq (\pi m)^{-2}\int_0^1 |h_\lambda''(r)|dr.$$

Furthermore,

$$h_\lambda''(r) = l''(2\lambda^{1/4}\sin(\pi r/2))(2\lambda^{1/4}\cos(\pi r/2))(\pi/2))^2 - l'(2\lambda^{1/4}\sin(\pi r/2))2\lambda^{1/4}\sin(\pi r/2)(\pi/2)^2,$$

and

$$\int_0^1 |l'(2\lambda^{1/4}\sin(\pi r/2))2\lambda^{1/4}\sin(\pi r/2)(\pi/2)^2|dr$$

$$\leq (\pi^2/2)\lambda^{1/4}\sup_{x\geq 0}|l'(x)|,$$

and therefore it only remains to show that

$$\int_0^1 |l''(2\lambda^{1/4}\sin(\pi r/2))(2\lambda^{1/4}\cos(\pi r/2))(\pi/2))^2|dr \leq C_1\lambda^{1/4}.$$

Since

$$|l''(x)| = |4x^2(5x^4-3)(x^4+1)^{-3}| \leq (x^4+1)^{-3/2}\sup_{x\in\mathbb{R}}|4x^2(5x^4-3)(x^4+1)^{-3/2}|$$

$$\leq C_2(x^4+1)^{-3/2}$$

and because $\sin(\pi r/2) \geq r$ for $r \in [0,1]$, it now follows that

$$\int_0^1 |l''(2\lambda^{1/4}\sin(\pi r/2))(2\lambda^{1/4}\cos(\pi r/2))(\pi/2))^2|dr$$

$$\leq C_2\int_0^1 (1+(2\lambda^{1/4}\sin(\pi r/2))^4)^{-3/2}(2\lambda^{1/4}\cos(\pi r/2))(\pi/2))^2 dr$$

$$\leq C_2\lambda^{1/4}\int_0^1 (1+(2\lambda^{1/4}r))^4)^{-3/2}\lambda^{1/4}dr$$

$$= C_2\lambda^{1/4}\int_0^{\lambda^{1/4}} (1+(2s)^4)^{-3/2}ds$$

$$\leq C_2\lambda^{1/4}\int_0^{\infty} (1+(2s)^4)^{-3/2}ds,$$

and therefore,

$$|f_\lambda(m)| \leq C_3 m^{-2}\lambda^{1/4}.$$

From this it follows that

$$\lambda^{1/4}|f_\lambda(\lambda^{1/4}m)| \leq C_3\lambda^{1/4}\lambda^{1/4}(\lambda^{1/4}m)^{-2} = C_3 m^{-2}.$$

Also, for any $m \in \mathbb{R}$,

$$\lambda^{1/4}f_\lambda(\lambda^{1/4}m) = \lambda^{1/4}\int_0^1 \cos(\pi r\lambda^{1/4}m)h_\lambda(r)dr = \int_0^{\lambda^{1/4}} \cos(\pi ym)h_\lambda(\lambda^{-1/4}y)dy$$

$$= \int_0^\infty I(y \leq \lambda^{1/4})\cos(\pi ym)l(2\lambda^{1/4}\sin(\pi y/(2\lambda^{1/4})))dy$$

and because $\sin(\pi x/2) \geq x$ for $x \in [0,1]$,

$$|I(y \leq \lambda^{1/4})\cos(\pi ym)l(2\lambda^{1/4}\sin((1/2)\pi y\lambda^{-1/4}))| \leq |l(2y)|$$

and $\int_0^\infty |l(2y)|dy < \infty$. Therefore, by the dominated convergence theorem,

$$\lim_{\lambda\to\infty} \lambda^{1/4}f_\lambda(\lambda^{1/4}m)$$

$$= \int_0^\infty \lim_{\lambda\to\infty} I(y \leq \lambda^{1/4})\cos(\pi ym)l(2\lambda^{1/4}\sin((1/2)\pi y\lambda^{-1/4}))dy$$

$$= \int_0^\infty \cos(\pi ym)l(\pi y)dy = f(m).$$

Similarly,

$$\lambda^{3/4}g_\lambda(\lambda^{1/4}m)$$

$$= \sqrt{2}\int_0^\infty I(y \leq \lambda^{1/4})\cos(\pi ym - \pi y/(2\lambda^{1/4}))\lambda^{1/2}\sin(\pi y/(2\lambda^{1/4}))^2\cos(\pi r/(2\lambda^{1/4}))l(2\lambda^{1/4}\sin(\pi y/(2\lambda^{1/4})))dy$$

and, noting that $\lambda^{1/2}\sin(\pi y/(2\lambda^{1/4}))^2 \leq (1/4)\pi^2 y^2$ for $y \in \mathbb{R}$,

$$|I(y \leq \lambda^{1/4})\cos(\pi y m - \pi y/(2\lambda^{1/4}))\lambda^{1/2}\sin(\pi y/(2\lambda^{1/4}))^2\cos(\pi r/(2\lambda^{1/4}))l(2\lambda^{1/4}\sin(\pi r/(2\lambda^{1/4})))|$$
$$\leq (1/4)\pi^2 y^2 l(2y)$$

and $\int_0^\infty (1/4)\pi^2 y^2 |l(2y)| dy < \infty$. Therefore,

$$\begin{aligned}&\lim_{\lambda\to\infty} \lambda^{3/4} g_\lambda(\lambda^{1/4} m)\\ &= \sqrt{2}\int_0^\infty \cos(\pi y m)(\lim_{\lambda\to\infty}\lambda^{1/2}\sin(\pi y/(2\lambda^{1/4}))^2) l(\pi y) dy\\ &= \sqrt{2}\int_0^\infty \cos(\pi y m)(\pi y/2)^2 l(\pi y) dy = g(m).\end{aligned}$$

For showing the uniform convergence result, it now suffices to show equicontinuity on $[0, K]$, viz.

$$\limsup_{\eta\downarrow 0}\limsup_{\lambda\to\infty}\sup_{|m|\leq K, |m-m'|\leq\eta} |\lambda^{1/4} f_\lambda(\lambda^{1/4} m) - \lambda^{1/4} f_\lambda(\lambda^{1/4} m')| = 0.$$

To show this, note that

$$\lambda^{1/4} f_\lambda(\lambda^{1/4} m) = \lambda^{1/4}\int_0^1 \cos(m r\pi\lambda^{1/4}) h_\lambda(r) dr = \int_0^{\lambda^{1/4}} \cos(m y\pi) l(2\lambda^{1/4}\sin(\pi\lambda^{-1/4} y/2)) dy,$$

so, because $l(x)$ is nonincreasing on $[0, \infty)$ and $\sin(\pi x/2) \geq x$ for $x \in [0, 1]$,

$$\begin{aligned}&\sup_{|m|\leq K, |m-m'|\leq\eta} |\lambda^{1/4} f_\lambda(\lambda^{1/4} m) - \lambda^{1/4} f_\lambda(\lambda^{1/4} m')|\\ &\leq \int_0^{\lambda^{1/4}} \sup_{m, |m-m'|\leq\eta} |\cos(m y\pi) - \cos(m' y\pi)| l(2\lambda^{1/4}\sin(\pi\lambda^{-1/4} y/2)) dy\\ &\leq \int_0^\infty \sup_{m, |m-m'|\leq\eta} |\cos(m y\pi) - \cos(m' y\pi)| l(2y) dy,\end{aligned}$$

and by the dominated convergence theorem, using the integrability of $|l(2y)|$, it now follows that

$$\limsup_{\eta\downarrow 0}\limsup_{\lambda\to\infty}\sup_{|m|\leq K, |m-m'|\leq\eta} |\lambda^{1/4} f_\lambda(\lambda^{1/4} m) - \lambda^{1/4} f_\lambda(\lambda^{1/4} m')| = 0.$$

□

Proof of Theorem 8: Because of the result of Theorem 4, it suffices to show that

$$(E|y_{[rT]}|)^{-1}E|\sum_{s=1}^{T} y_s(f_\lambda([rT]-s)-\lambda^{-1/4}f(\lambda^{-1/4}([rT]-s)))| \to 0.$$

Similarly to the proof of Theorem 4, for all $K>0$,

$$\limsup_{T\to\infty}(E|y_{[rT]}|)^{-1}E|\sum_{s=1}^{T} y_s(f_\lambda([rT]-s)-\lambda^{-1/4}f(\lambda^{-1/4}([rT]-s)))|$$

$$\leq \limsup_{T\to\infty}\max_{1\leq s\leq T} E|y_s|(E|y_{[rT]}|)^{-1}\sum_{s=-\infty}^{\infty}|f_\lambda([rT]-s)-\lambda^{-1/4}f(\lambda^{-1/4}([rT]-s))|$$

$$\leq C_1\sum_{m=0}^{\infty}|f_\lambda(m)-\lambda^{-1/4}f(\lambda^{-1/4}m)|$$

$$\leq C_1\sum_{m=0}^{[K\lambda^{1/4}]}|f_\lambda(m)-\lambda^{-1/4}f(\lambda^{-1/4}m)|+C_1\sum_{m=[K\lambda^{1/4}]+1}^{\infty}|f_\lambda(m)|+C_1\sum_{m=[K\lambda^{1/4}]+1}^{\infty}\lambda^{-1/4}|f(\lambda^{-1/4}m)|.$$

By Theorem 7,

$$\limsup_{\lambda\to\infty}\sum_{m=0}^{[K\lambda^{1/4}]}|f_\lambda(m)-\lambda^{-1/4}f(\lambda^{-1/4}m)|$$

$$\leq K\limsup_{\lambda\to\infty}\lambda^{1/4}\sup_{m=0,1,\ldots,[K\lambda^{1/4}]}|f_\lambda(m)-\lambda^{-1/4}f(\lambda^{-1/4}m)|$$

$$\leq K\limsup_{\lambda\to\infty}\sup_{0\leq y\leq K}|\lambda^{1/4}f_\lambda(\lambda^{1/4}y)-f(y)|=0.$$

Also, because $|f_\lambda(m)|\leq C_2\lambda^{1/4}m^{-2}$ for $m\geq 1$ by Theorem 7,

$$\sum_{m=[K\lambda^{1/4}]+1}^{\infty}|f_\lambda(m)|\leq C_2\lambda^{1/4}\sum_{m=[K\lambda^{1/4}]+1}^{\infty}m^{-2}\leq C_2K^{-1}$$

and because for $m\geq 1$, $|f(m)|\leq C_3m^{-2}$,

$$\sum_{m=[K\lambda^{1/4}]+1}^{\infty}\lambda^{-1/4}|f(\lambda^{-1/4}m)|\leq C_3\sum_{m=[K\lambda^{1/4}]+1}^{\infty}\lambda^{1/4}m^{-2}\leq C_4K^{-1}$$

and since K was arbitrary, the result now follows. $\square$

Proof of Theorem 9: Note that

$$\limsup_{T\to\infty}(E|y_{[4rT]}|)^{-1}E|\hat{\tau}_{4T,4[rT]-4+q}(\lambda,\{y_s,s=1,\ldots,4T\})-\hat{\tau}_{4T,[4Tr]}(\lambda,\{y_s,s=1,\ldots,4T\})|$$

$$\leq \limsup_{T\to\infty}\max_{1\leq s\leq 4T}E|y_s|(E|y_{[4rT]}|)^{-1}\sum_{s=1}^{4T}|w_{4T,4[rT]-4+q,s}-w_{4T,[4Tr],s}|$$

$$\leq C_1\limsup_{T\to\infty}\sum_{s=1}^{4T}|f_{4T,\lambda}(4[rT]-4+q-s)-f_{4T,\lambda}([4Tr]-s)|$$

$$+C_1\limsup_{T\to\infty}\sum_{s=1}^{4T}\sum_{k=2}^{8}|w^k_{4T,4[rT]-4+q,s}|+C_1\limsup_{T\to\infty}\sum_{s=1}^{4T}\sum_{k=2}^{8}|w^k_{4T,[4Tr],s}|.$$

The result of Equation (18) ensures that the third term is $O(T^{-2})$. Using the definition of the w^k_{Tts} for $k=2,..,8$, it is easy to see that the second term is also $O(T^{-2})$. Because

$$|4[rT]-4+q-[4rT]|\leq 7,$$

the first term is bounded by

$$\limsup_{T\to\infty}\sum_{m=-\infty}^{\infty}I(|m|\leq 4T-1)I(|m+l|\leq 4T-1)\max_{|l|\leq 7}|f_{4T,\lambda}(m)-f_{4T,\lambda}(m+l)|$$

and therefore it suffices to show that for all l, $|l|\leq 7$,

$$\limsup_{\lambda\to\infty}\limsup_{T\to\infty}\sum_{m=-\infty}^{\infty}I(|m|\leq 4T-1)I(|m+l|\leq 4T-1)|f_{4T,\lambda}(m)-f_{4T,\lambda}(m+l)|$$

equals 0. By the upper bound on $f_{T,\lambda}(m)$ of Theorem 1 and the dominated convergence theorem, the last expression equals

$$\limsup_{\lambda\to\infty}\sum_{m=-\infty}^{\infty}|f_\lambda(m)-f_\lambda(m+l)|$$

and is therefore bounded by

$$\limsup_{\lambda\to\infty}\sum_{m=-\infty}^{\infty}|\lambda^{-1/4}f(\lambda^{-1/4}m)-\lambda^{-1/4}f(\lambda^{-1/4}(m+l))|$$

$$+4\limsup_{\lambda\to\infty}\sum_{m=0}^{\infty}|f_\lambda(m)-\lambda^{-1/4}f(\lambda^{-1/4}m)|,$$

and the second term was shown to equal 0 in the proof of Theorem 8. For the first term, note that we can write

$$\limsup_{\lambda\to\infty}\sum_{m=-\infty}^{\infty}|\lambda^{-1/4}f(\lambda^{-1/4}m)-\lambda^{-1/4}f(\lambda^{-1/4}(m+l))|$$

$$=\limsup_{\lambda\to\infty}\int_{-\infty}^{\infty}|\lambda^{-1/4}f(\lambda^{-1/4}[m])-\lambda^{-1/4}f(\lambda^{-1/4}([m]+l))|dm$$
$$=\limsup_{\lambda\to\infty}\int_{-\infty}^{\infty}f(\lambda^{-1/4}[\lambda^{1/4}y])-f(\lambda^{-1/4}([\lambda^{1/4}y]+l))|dy$$

and the last term equals 0 by the dominated convergence theorem because $\sup_{m\in\mathbb{R}}|f(m)|(m+1)^2<\infty$ and the continuity of $f(\cdot)$.

Now by Theorem 8,

$$\limsup_{\lambda\to\infty}\limsup_{T\to\infty}(E|y_{[4rT]}|)^{-1}E|\hat{\tau}_{4T,[4rT]}(\lambda,\{y_s,s=1,\ldots,4T\})-\bar{\tau}_{4T,[4rT]}(\lambda,\{y_s,s=1,\ldots,4T\})|=0$$

and

$$\limsup_{\lambda\to\infty}\limsup_{T\to\infty}(E|y_{[rT]}|)^{-1}E|\hat{\tau}_{T,[rT]}(\lambda,\{(1/4)\sum_{q=1}^{4}y_{4i-4+q}:i=1,\ldots,T\})$$

$$-\bar{\tau}_{T,[rT]}(\lambda,\{(1/4)\sum_{q=1}^{4}y_{4i-4+q}:i=1,\ldots,T\})|=0$$

because the regularity condition of Theorem 8 was assumed in the statement of the Theorem.

Since

$$\bar{\tau}_{T,[rT]}(\lambda,\{z_s:s=1,...,T\})=\sum_{s=1}^{T}z_s\lambda^{-1/4}f(\lambda^{-1/4}(t-s))$$

it follows that

$$\bar{\tau}_{4T,[4rT]}(\lambda,\{y_s,s=1,\ldots,4T\})=\sum_{s=1}^{4T}y_s\lambda^{-1/4}f(\lambda^{-1/4}([4rT]-s))$$

and

$$\bar{\tau}_{T,[rT]}(4^{-4}\lambda, \{(1/4)\sum_{q=1}^{4} y_{4i-4+q} : i = 1, \ldots, T\})$$

$$= \sum_{i=1}^{T}(1/4)\sum_{q=1}^{4} y_{4i-4+q}(4^{-4}\lambda)^{-1/4} f((4^{-4}\lambda)^{-1/4}([rT]-i))$$

$$= \sum_{i=1}^{T}\sum_{q=1}^{4} y_{4i-4+q}\lambda^{-1/4} f(\lambda^{-1/4}4([rT]-i))$$

$$= \sum_{s=1}^{4T} y_s\lambda^{-1/4} f(\lambda^{-1/4}4([rT]-([(s-1)/4]+1))).$$

Since

$$|([4rT]-s) - 4([rT]-([(s-1)/4]+1))| \leq 7,$$

it follows that

$$(E|y_{[4rT]}|)^{-1}E|\sum_{s=1}^{4T} y_s\lambda^{-1/4} f(\lambda^{-1/4}([4rT]-s)) - \lambda^{-1/4} f(\lambda^{-1/4}4([rT]-([(s-1)/4]+1)))|$$

$$\leq C_2 \sum_{m=-\infty}^{\infty} \max_{|l|\leq 7} |\lambda^{-1/4} f(\lambda^{-1/4}m) - \lambda_Q^{-1/4} f(\lambda^{-1/4}(m+l))|,$$

and this expression was earlier shown to approach 0 as $\lambda \to \infty$. $\square$