Hand in 1

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Introduction

In this Assignment a system with a helicopter hovering a mass was analyzed with the help of Lagrangian Modelling.

Question 1a)

Using the generalized coordinates of the form

$$q = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ \theta \\ \phi \end{bmatrix} \tag{1}$$

The position of the helicopter is simply described by

$$p_1 = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} \tag{2}$$

The hovering mass will depend on the position of the helicopter, as well ass the angles θ and ϕ. The position p_2 can then be described as

$$p_2(q) = \begin{bmatrix} p_{11} + L\,\cos(\theta)\,\sin(\phi) \\ p_{12} + L\,\sin(\phi)\,\sin(\theta) \\ p_{13} - L\,\cos(\phi) \end{bmatrix} \tag{3}$$

Using given equation for kinetic and potential energy the Lagrange function was derived. Using gradient notation, the Euler-Lagrange was then calculated according to

$$\frac{d}{dt}\nabla_{\dot{q}}L - \nabla_q L = F \tag{4}$$

where F is the external force vector only acting on mass 1

$$F = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

Calculating $\nabla_q L$ and $\nabla_{\dot{q}} L$ straightforward. To calculate $\frac{d}{dt}\nabla_{\dot{q}} L$ we use that

$$W(q)\dot{q} = \nabla_{\dot{q}} L \quad \Longrightarrow \quad \frac{d}{dt}\nabla_{\dot{q}} L = \frac{d}{dt}W(q)\dot{q} = \frac{\partial}{\partial q}(W(q)\dot{q})\dot{q} + W(q)\ddot{q} \tag{6}$$

Which gives us the Euler-Lagrange equation (4) as

$$W(q)\ddot{q} + \frac{\partial}{\partial q}(W(q)\dot{q}) - \nabla_q L = F \tag{7}$$

Equation 6 can be rewritten as

$$W(q)\ddot{q} = F + \nabla_q L - \frac{\partial}{\partial q}(W(q)\dot{q}) \tag{8}$$

Using the notation $W(q) = M(q)$ and $\ddot{q} = \dot{v}$ and referingen to the right hand side of equation (8) as $b(q, \dot{q}, u)$, this can be written as

$$M\dot{v} = b(q, \dot{q}, u) \tag{9}$$

Using symbolic toolbox in Matlab, $M\dot{v}$ was calculated to

$$M\dot{v} = \begin{bmatrix} L\, m_2\, \cos(\theta)\, \sin(\phi)\, \dot{\phi}^2 + 2\, L\, m_2\, \cos(\phi)\, \sin(\theta)\, \dot{\phi}\, \dot{\theta} + L\, m_2\, \cos(\theta)\, \sin(\phi)\, \dot{\theta}^2 + u_1 \\ L\, m_2\, \sin(\phi)\, \sin(\theta)\, \dot{\phi}^2 - 2\, L\, m_2\, \cos(\phi)\, \cos(\theta)\, \dot{\phi}\, \dot{\theta} + L\, m_2\, \sin(\phi)\, \sin(\theta)\, \dot{\theta}^2 + u_2 \\ -L\, m_2\, \cos(\phi)\, \dot{\phi}^2 + u_3 - g\, m_1 - g\, m_2 \\ -L^2\, m_2\, \dot{\phi}\, \dot{\theta}\, \sin(2\,\phi) \\ -L\, m_2\, \left(g\, \sin(\phi) - \frac{L\, \dot{\theta}^2\, \sin(2\,\phi)}{2}\right) \end{bmatrix} \tag{10}$$

Question 1b)

Using constrained Lagrange approach q was rewritten as:

$$q = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{21} \\ p_{22} \\ p_{23} \end{bmatrix} \tag{11}$$

As in question 1a, the kinetic and potential energy were found and the Lagrange function was derived. Using gradient notation, the Euler-Lagrange was then calculated according equation (4). The external force vector F, is the same as in question 1a. As in question 1a the Euler-Lagrange equation could be rewritten as in equation (8). However since new generalized coordinates, q where used a constraint C was calculated as following:

$$C(q) = \frac{(p_{11} - p_{21})^2}{2} - \frac{L^2}{2} + \frac{(p_{12} - p_{22})^2}{2} + \frac{(p_{13} - p_{23})^2}{2} \tag{12}$$

Using generalized coordinates and constraint C, equation (9) was derived as following:

$$M\dot{v} = \begin{bmatrix} u_1 - z\, (p_{11} - p_{21}) \\ u_2 - z\, (p_{12} - p_{22}) \\ u_3 - g\, m_1 - z\, (p_{13} - p_{23}) \\ z\, (p_{11} - p_{21}) \\ z\, (p_{12} - p_{22}) \\ z\, (p_{13} - p_{23}) - g\, m_2 \end{bmatrix} \tag{13}$$

As we can see the equation (9) is less complex for question 1b compared to question 1a.

Question 2a)

With $a(q) = \nabla_q c$ and $c(q, \dot{q}, u)$ as

$$c(q, \dot{q}, u) = \begin{bmatrix} Q - \frac{\partial}{\partial q}(W(q)\dot{q})\dot{q} + \nabla_q T - \nabla_q V \\ -\frac{\partial}{\partial q}(\frac{\partial}{\partial q}\dot{q})\dot{q} \end{bmatrix} \tag{14}$$

the system from question 1b can be put into the following form:

$$\begin{bmatrix} M & a(q) \\ a(q)^T & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ z \end{bmatrix} = c(q, \dot{q}, u) \tag{15}$$

Since in our case $Q = F$ we get that $c(q, \dot{q}, u)$ for system in question 1b is:

$$c = \begin{bmatrix} u_1 \\ u_2 \\ u_3 - g\,m_1 \\ 0 \\ 0 \\ -g\,m_2 \\ -\mathrm{dp}_{11}{}^2 + 2\,\mathrm{dp}_{11}\,\mathrm{dp}_{21} - \mathrm{dp}_{12}{}^2 + 2\,\mathrm{dp}_{12}\,\mathrm{dp}_{22} - \mathrm{dp}_{13}{}^2 + 2\,\mathrm{dp}_{13}\,\mathrm{dp}_{23} - \mathrm{dp}_{21}{}^2 - \mathrm{dp}_{22}{}^2 - \mathrm{dp}_{23}{}^2 \end{bmatrix} \tag{16}$$

With

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 & p_{11} - p_{21} \\ 0 & m_1 & 0 & 0 & 0 & 0 & p_{12} - p_{22} \\ 0 & 0 & m_1 & 0 & 0 & 0 & p_{13} - p_{23} \\ 0 & 0 & 0 & m_2 & 0 & 0 & p_{21} - p_{11} \\ 0 & 0 & 0 & 0 & m_2 & 0 & p_{22} - p_{12} \\ 0 & 0 & 0 & 0 & 0 & m_2 & p_{23} - p_{13} \\ p_{11} - p_{21} & p_{12} - p_{22} & p_{13} - p_{23} & p_{21} - p_{11} & p_{22} - p_{12} & p_{23} - p_{13} & 0 \end{bmatrix} \tag{17}$$

from question 1b and c given equation (16) we can calculate $a(q)$ as:

$$a(q) = \begin{bmatrix} p_{11} - p_{21} \\ p_{12} - p_{22} \\ p_{13} - p_{23} \\ p_{21} - p_{11} \\ p_{22} - p_{12} \\ p_{23} - p_{13} \end{bmatrix} \tag{18}$$

Quesiton 2b)

By inverting matrix:

$$\begin{bmatrix} M & a(q) \\ a(q)^T & 0 \end{bmatrix} \tag{19}$$

and multipliying it with $c(q, \dot{q}, u)$ we get the system in explicit form. By looking at the expression, it's clear that the rather simple system became much more complex. The matrix in equation (19) seems quite simple, but since it depends on q it quickly becomes very complex by inverting it. With the system in explicit form the matrix $[\ddot{q} \quad z]^T$ becomes much more complex.