Assignment 2 - SYSID

A2-Group 46
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1 Sample mean and Maximum Likelihood

By using the data samples given in the task, which are shown in equation (1). The white Gaussian noise is w[k], where w[k] $\sim$ N (0,σ^2).

$$x[k] = A + w[k] \quad \text{with} \quad k = 1, 2, 3, \ldots, N-1 \tag{1}$$

There is a possibility to estimate the parameter A as the sample mean in the expression below.

$$\hat{A} = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \tag{2}$$

The sample mean should be shown to be a Maximum Likelihood estimator of parameter A. The noise is having a normal distribution, containing a mean and a σ which is the variable for its standard deviation. Calculations of the maximum Likelihood estimator of A is shown in the following steps down below. The final result can be seen in equation 4

$$L(A) = P[x_k = A + w_k | A] = P[w_k = x_k - A | A] \tag{3}$$

$$L(A) = \prod_{k=0}^{N-1} \tfrac{1}{\sqrt{\sigma^2 \pi 2}} e^{-\left(x_k - A\right)^2 / 2\sigma^2}$$
$$\implies L(A) = \tfrac{1}{\sqrt{\sigma^2 \pi 2}} e^{-\sum_{k=0}^{N-1} (x_k - A)^2 / 2\sigma^2}$$
$$\implies \ln L(A) = \ln\left(\tfrac{1}{\sqrt{\sigma^2 \pi 2}}\right) \tfrac{1}{2\sigma^2} \sum_{k=0}^{N-1} (x_k - A)^2$$
$$\implies \tfrac{\partial}{\partial A} \ln L(A) = \ln\left(\tfrac{1}{\sqrt{\sigma^2 \pi 2}}\right) \tfrac{1}{2\sigma^2} \sum_{k=0}^{N-1} 2(x_k - \hat{A}) = 0$$
$$\implies \sum_{k=0}^{N-1} 2(x_k - \hat{A}) = 0$$
$$\implies 2 \sum_{k=0}^{N-1} \hat{A} = 2 \sum_{k=0}^{N-1} x_k$$

$$\implies \hat{A} = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \tag{4}$$

2 Linear least squares

The linear system below should be taken in consideration, where θ is the parameter used to estimate and e[k]$\sim$ N (1,σ^2)
y[k]=θ u[k]+e[k], k=0,...,N-1

(a)

In sub question 2a the task is to investigate what happens to the estimatio of θ when $u_k = 0, \forall k$. The system will by its equation cancel out the estimate and thereby have no impact on the system at any sample k. This will not lead to any valid results.

(b)

To be able to come to any valid conclusions, the least squares estimate of θ is calculated by using equation 5. The calculations are shown down below.

$$\hat{\theta} = e^T e = (y - U\theta)^T (y - U\theta) \tag{5}$$

$$\implies e^T e = y^T y - y\theta^T \theta - y^T U\theta - \theta^T U^T U\theta$$
$$\implies \frac{\partial \delta_r}{\partial \delta} = -U^T y - y^T U + 2U^T U\hat{\theta} = 0$$
$$\implies -2U^T y + 2U^T U\hat{\theta} = 0$$

$$\implies U^T U\hat{\theta} = U^T y$$

$$\implies \hat{\theta} = (U^T U)^{-1} U^T y \tag{6}$$

The result shows that the estimates of θ are shown in equation 6.

(c)

Now that we have the estimate of θ, we will compute the bias of the result in 2b.

$$E[\hat{\theta}] = E[(U^T U)^{-1} U^T y] \tag{7}$$

$$\implies E[\hat{\theta}] = E[(U^T U)^{-1} U^T (U\theta + e_k)]$$
$$\implies E[\hat{\theta}] = E[\theta(U^T U)(U^T U)^{-1} + e_k U^T (U^T U)^{-1}]$$
$$\implies E[\hat{\theta}] = E[\theta I + e_k U^T (U^T U)^{-1}]$$

$$\implies E[\hat{\theta}] = \theta + E[e_k] E[U^T (U^T U)^{-1}] \tag{8}$$

The results shows us that it is biased since the error is not centered around zero which is visual in equation 8.

3 One step ahead predictor

In the task description a model structure is given, where it is possible to carachterize it in several ways. The model is shown in equation 9.

$$y(t) + a_1 y(t-1) + a_2 y(t-2) = b_0 U(t) + e(t) + c_1 e(t-1) \tag{9}$$

(a)

By looking at equation 11 in task 3b) the conclusion that can be made is that the model structure is a ARMAX structure.

(b)

The model structure given in the task description should be z-transformed in to equation 10 and simplified as in to equation 11, in order to get the expression for the plant models G(z) and H(z) which are visible in expression 12.

$$Y(z) = a_1 z^{-1} 1 Y(z) + a_2 z^{-2} Y(z) = b_0 U(z) + E(z) + c_1 z^{-1} E(z) \tag{10}$$

$$Y(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}} U(z) + \frac{1 + c_1 Z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} E(z) \tag{11}$$

$$G(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}} \quad \text{and} \quad H(z) = \frac{1 + c_1 Z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{12}$$

(c)

Through equation 13 it is possible to find the 1-step-ahead predictor. By inserting the values from the previous task 3b) and by some manipulations we get the result in equation 14.

$$Y(z) = \frac{G(z)U(z)}{H(z)} + (1 - \frac{1}{H(z)})Y(z) \tag{13}$$

$$\hat{Y}(z) = \frac{b_0}{1 + c_1 z^{-1}} U(z) + \frac{c_1 z^{-1} - a_1 z^{-1} - a_2 z^{-2}}{1 + c_1 z^{-1}} Y(z) \tag{14}$$

(d)

If data based samples are used and not containing previous measurements the 1-step-ahead predictor will be a linear function of the parameters. When taking equation 14 in consideration it is visible that the predictor is nonlinear since it is depending on previous measurements, where the previous measurements are based on the same parameters.

4 Prediction or simulation

In the task description of a model structure is given and its shown in equation 15.

$$y(t) + a_1 y(t-1) = b_0 U(t) + e(t) + a_1 e(t-1) \tag{15}$$

(a)

By looking at equation 16 it is clearly visible that the model structure is OE.

$$Y(z) = \frac{b_0}{1 + a_1 z^{-1}} U(z) + E(z) \tag{16}$$

(b)

Through equation 17 it is possible to find the 1-step-ahead predictor. By inserting the values from the previous task 4a) and by some manipulations we get the result in equation 18.

$$Y(z) = \frac{G(z)U(z)}{H(z)} + (1 - \frac{1}{H(z)})Y(z) \tag{17}$$

$$\hat{Y}(z) = \frac{b_0}{1 + a_1 z^{-1}} U(z) \tag{18}$$

The predictor fails to include an output expression since H(z)=1, which is causing the equation to act weird. The linear system model does not take past values into consideration, just the input.

5 Computer exercise. Identification of an ARX model

(a)

By using linear regression in Matlab we will compute the estimated values of the parameters, which are visible in figure 1 below.

Model_1	Value1
'a1'	-0.94884
'a2'	0.31921
'b0'	0.045182

Model_2	Value2
'a1'	-0.90446
'a2'	0.31526
'b0'	-0.0028883
'b1'	0.99441

Model_3	Value3
'a1'	-1
'a2'	0.6
'a3'	-0.3
'b1'	0.99974

Figure 1: The estimated values of the parameters for the different models.

(b)

In this task we compute the simulated and predicted (1-step) output in Matlab by using the validation data and the estimated models from 5a). To be able to compare them we compute the root mean square and the result is visible in figure 2. In figure 2 we clearly see that model 3 has the lowest error for both simulation and prediction, and therefore is the best in both cases.

Model	Prediction	Simulation
'Model 1'	1.0109	1.427
'Model 2'	0.30688	0.42701
'Model 3'	0.059369	0.055995

Figure 2: The root mean square error for the different estimated models.

(c)

In this task we compute the covariance for the different models in Matlab by using equation 19, because in our case we have additive white Gaussian noise.

$$\sum_{\hat{\theta}} = \sigma^2 (U^T U)^{-1} \tag{19}$$

When the covariance analysis was done it showed that model 3 is the model with the highest covariance which means it has the greatest variation out of the three models. Model 1 and model 2 has similar values and both have less variance than model 3. According to the analysis model 1 is the one with least covariance, slightly lower than model 2 on a few parameters.