Interest Rate Curve Interpolations and their Assessment via Hedge Simulations

in a post-financial crisis market environment.

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Abstract

In this paper we discuss interest rate curve interpolation methods and their properties in the context of financial applications. We review the modern (multi-curve) theory of interest rate curve modeling, taking into account collateralization. Building on this solid foundation we re-consider several curve interpolation methods and derive some modifications thereof. We revise well-known criteria for the goodness of interpolation and introduce the hedge error, arising from dynamic delta-hedging over a certain time frame, as enhanced evaluation parameter. This new criterion represents both a practical and meaningful choice since hedging is the essential instrument applied by banks to mitigate interest rate risks. The hedge error particularly indicates the amount of money at risk. Numerical results in a multi-curve financial framework conclude this work. They are derived by means of a Java-based implementation performed on actual market data.

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1 Introduction

The standard approach to value so called "linear" interest rate products, is to construct curves, effectively representing risk neutral expectations of specific reference indices or products. After the financial crisis, funding and collateralization have resulted in a re-thinking of this approach, in particular the distinction between discount curves and forward curves as well as between forward curves with different maturities. This is also called "multi-curve modeling".

A subtle part of curve construction deals with the choice of an interpolation method, which effectively constitutes a model for the underlying indices. Hagen and West examined a wide range of interpolation methods in 2006 [10]. We supplement this work by an examination of a hedge simulation, to provide a more application focused criterion for the quality of a given interpolation method: from the perspective of a bank, the hedge error reflects a crucial key figure and thus justifies as an appropriate quality criterion for the choice of interpolation.

Interest rate curve interpolation methods and some continuous modifications thereof are reviewed in section 2. The multi-curve approach is introduced in part 3. Section 4 comprises the hedge algorithms. Two continuous modifications are presented in part 5. Numerical results are presented in section 6.

2 Curve interpolation methods and their assessment

Starting with a brief summary of the relevant literature, we discuss a revised criteria for the assessment of a curve interpolation method based on dynamic delta hedging.

2.1 Some previous results from the literature

Hagen and West have examined a large number of interpolations for the forward curve construction in [10]. The analysis was done in 2006, i.e. before the beginning of the credit crunch. The following criteria were considered:

(a) Is the algorithm to find the best fit curve sufficiently rapid and the error of the created curve sufficiently small?

(b) Is the forward curve positive and continuous?[1]

(c) Does a local change of the curve have only an impact nearby? Or can the changes elsewhere be significant?

[1] Positivity of the forwards is required to avoid arbitrage, continuity is required for the pricing of interest sensitive instruments ([10], [12]).

(d) Is the forward curve stable? The degree of stability is here defined as the maximum change of the forward curve if one of the input parameters is modified.

(e) How local are hedges? Does most of the delta risk get assigned to the hedge products that have close maturities to the maturity of the derivative?

Recall that in the pre-crisis framework the forwards are defined in terms of zero-coupon bonds

$$F(T,S;t) = \frac{1}{S-T}\left(\frac{P(T;t)}{P(S;t)} - 1\right). \tag{1}$$

It is common market practice to derive the forward curve by constructing the discount curve. In order to simplify notation we set the discount factor $D(T) := P(T;t)$. Note that we also have the following relation

$$D(T) = e^{-R(T;t)(T-t)},$$

i.e. the continuously compounded spot rate represents the *yield* of the discount factor.

Hagan and West studied interpolations on the discount factors, on the logarithm of discount factors, on the yield and directly on the forwards. They observed the following results.

The linear interpolation on the log of discount factors fulfills all criteria but is not smooth. Linear interpolation on the discount factor or the yield violates (b) and (e). They examined a curve where all cubic splines lead to negative forwards, inter alia the standard cubic spline, a Bessel cubic spline and the monotone preserve cubic spline.

Furthermore, they introduced two new interpolation methods (monotone convex spline and minimal cubic spline) which overcome most of the shortfalls of the other interpolations.

However, Le Floc'h revealed some issues of the new methods in [7]. He enhanced the list of criteria by another stability criterion comparing the delta between sequential and parallel shifts. In his analyses he favours a harmonic spline interpolation as it is a fast and intuitive algorithm.

Ametrano and Bianchetti showed in [2] that a modified Bessel spline on the logarithm of the discount factors is also a good alternative.

A spreadsheet containing multi-curve calibration and interpolation and its related theory [8] has inspired an interesting discussion about interpolation in financial markets in the forum of Wilmott[2] in 2013. Thereupon, mainly Le Floc'h and Duffy argue about several interpolation methods. The former recommends to apply the harmonic spline on the log of discount factors and on the forwards once more [7], whereas the latter brings in the Akima interpolation as a meaningful alternative [5]. It fits a smooth and low oscillating curve to the grid points.

2.2 Dynamic delta-hedging as revised interpolation criteria

If we have a closer look at the quality criteria (a) – (e) listed by Hagan and West in 2006, we observe that nowadays some of the factors are obsolete. In times of powerful multi-core supercomputers, criterion (a) does not seem to be a worrying issue any more. In the past years, we have also seen that zero and even negative forwards are a possible scenario[3] – mitigating the importance of (b).

Regarding the stability criteria (c) – (e) we will focus on the local delta-hedge criterion as the delta is typically calculated by means of

1. bumping the rate of a stripping instrument,
2. re-constructing the underlying interest rate curve,
3. re-valuing the instrument by using the new curve.

Note that bumping one of the instruments will potentially have an impact on other products as well.[4] That means the hedging criteria (e) incorporates (c) and (d) to a certain extent as a change in one of the input parameters may result in further shifts of the interest rate curve.

In the context of financial application criterion (e) also seems to be quite appropriate due to the fact that hedging is a crucial instrument in financial markets to mitigate interest rate risks.

From the perspective of a bank the locality of the delta-hedge at some time point represents a poor indication only. It is particularly not clear if the chosen weights reflect the best choice of hedge products as an actual hedging process has not been considered yet.

Typically, hedging is a dynamic long-term strategy where (available) traded products can be added to the hedge portfolio at any time. The *hedge error* reflects the money which is currently at risk. On the other hand, the hedge error depends directly on the chosen interpolation method since interest rate products

[2] See http://www.wilmott.com/messageview.cfm?catid=11&threadid=94185&forumid=1.

[3] See [3] for instance.

[4] E.g. bumping a three-year swap will also influence the payment at year three of a four-year swap, and, as swaps are traded at par, the four-year curve value will hence be shifted as well.

are valued by means of their underlying curves.

So, we will enhance criterion (e) and take the error of a dynamic delta-hedge simulation over a certain time period to assess the goodness of interpolation. In order to analyse distributions of the hedge error, we will also go one step further and study a whole bunch of time frames. This kind of examination will be called *moving window analysis*.

3 From single-curve modeling to multiple curves

Traditionally, practitioners have applied LIBOR and LIBOR-swap rates as proxies for risk-free rates making use of classical interest rate theory.[5] This practice has been questioned by the credit crisis that have started in 2007 and led to a re-thinking of funding and collateralization. Nowadays, collateralized portfolios are discounted by overnight index rates obtained from overnight index swaps (OIS) [11].

Due to significant spreads between OIS and LIBOR rates, we have to differentiate between discount and forward curves as well as to distinguish forward curves with different maturities [4]. This is called multi-curve modeling.

In the following we adapt the formal approach from [8], also defining a generalized swap structure which allows to represent OIS, LIBOR and basis swaps by means of one definition only.

3.1 Re-definition of discount and forward curves

In order to define the zero-coupon bond, we consider a constant cash flow X, paid in T and secured by the collateral account N^C. The value at time t is

$$V(t) = N^C(t)\,\mathrm{E}^{\mathbb{Q}^{N^C}}\left[\frac{X}{N^C(T)}\,\middle|\,\mathcal{F}_t\right] = X \cdot N^C(t)\,\mathrm{E}^{\mathbb{Q}^{N^C}}\left[\frac{1}{N^C(T)}\,\middle|\,\mathcal{F}_t\right],$$

where $\mathbb{Q}^{N^C}$ denotes the risk neutral measure with respect to the numéraire $N = N^C$, given in terms of the collateral account.[6] We define the theoretical zero-coupon bond by

$$P^C(T;t) := N^C(t)\ \mathrm{E}^{\mathbb{Q}^{N^C}}\left[\frac{1}{N^C(T)}\,\middle|\,\mathcal{F}_t\right],$$

in order to obtain

$$V(t) = X \cdot P^C(T;t).$$

[5] The classical interest rate theory is introduced in the textbook of Filipovic [6] for instance.
[6] See [9] for more details.

Definition 1 (Discount Curve): ⌝
Let $P^C(T;t)$ denote the time t value of a payment of 1 currency unit (e.g. dollar, euro) in T collateralized by the collateral C. We call the map

$$T \mapsto P^C(T;t)$$

the *discount curve* for cash-flows (secured by the collateral C). ⌟

As usual, we consider three time points for the forward curve: valuation, fixing and payment date, $t \leq T \leq S$. Let I be the stochastic process of an index fixed in T and paid in S. The value in t is

$$V(t) = N^C(t)\,\mathrm{E}^{\mathbb{Q}^{N^C}}\left[\frac{I(T)}{N^C(S)}\,\middle|\,\mathcal{F}_t\right].$$

Recall the valuation of a constant payment Y at time t

$$Y \cdot P^C(S;t).$$

The two are equal if

$$Y \cdot P^C(S;t) = N^C(t)\,\mathrm{E}^{\mathbb{Q}^{N^C}}\left[\frac{I(T)}{N^C(S)}\,\middle|\,\mathcal{F}_t\right],$$

which is equivalent to

$$\mathrm{E}^{\mathbb{Q}^{N^C}}\left[\frac{I(T)-Y}{N^C(S)}\,\middle|\,\mathcal{F}_t\right] = 0.$$

This means, Y is the fair rate such that the value in t of this “forward contract” is zero.[7] We denote the fair forward rate by $F_I^C(T,S;t) := Y$. So we have

$$V(t) = F_I^C(T,S;t) \cdot P^C(T;t),$$

where the forward rate is given by

$$F_I^C(T,S;t) = N^C(t)\,\mathrm{E}^{\mathbb{Q}^{N^C}}\left[\frac{I(T)}{N^C(S)}\,\middle|\,\mathcal{F}_t\right] \cdot \left(P^C(S;t)\right)^{-1}.$$

We are now able to define the forward curve.

Definition 2 (Forward Curve): ⌝
Let $t \mapsto I(t)$ be an index, i.e. an adapted stochastic real-valued process. By $V_I^C(T,T+d;t)$ we denote the time t-value of a payment of $I(T)$ paid in $T+d$ and secured by the collateral account C, where $d \geq 0$. The *forward* $F_I^{d,C}(T;t)$

[7] Note that this definition is similar to the definition of a par swap rate.

of a payment of the index I paid in $T+d$ is defined by

$$F_I^{d,C}(T;t) := \frac{V_I^C(T, T+d; t)}{P^C(T+d;t)}.$$

The *forward curve* is the map

$$T \mapsto F_I^{d,C}(T;t),$$

where d is a fixed period. ⌟

3.2 Definition of a generalized swap

We follow the definition of a generalized swap given in [8].

Definition 3 (Swap): ⌝
A *swap* exchanges the payments of the receiver and the payer leg. The two legs can have different payment times, indices and fixed payments. They are collateralized with regard to the same account C. The value of the swap in $t \leq T_0$ is given by

$$\begin{aligned} V^C_{\text{Swap}}(t) =& V^C_{\text{SwapLeg}}(\alpha_1 I_1, X_1, \{T_i^1\}_{i=0}^{n_1}; t) - V^C_{\text{SwapLeg}}(\alpha_2 I_2, X_2, \{T_i^2\}_{i=0}^{n_2}; t) \\ = & \sum_{i=0}^{n_1-1} \left(\left(\alpha_1 F_1^C(T_i^1) + X_1 \right) \cdot P^C(T_{i+1}^1) \right) \\ & - \sum_{i=0}^{n_2-1} \left(\left(\alpha_2 F_2^C(T_i^2) + X_2 \right) \cdot P^C(T_{i+1}^2) \right), \end{aligned}$$

where F_k^C defines the forward based on some index I_k, and α_k and X_k are constants. ⌟

This definition allows us to define overnight index swaps (OIS), basis swaps exchanging two forward curves with different maturities and also classical interest rate swaps exchanging fix versus floating rates.[8] These types of swaps will be used to construct discount and forward curves.

4 Assessment of interpolation methods via hedge simulations

Having introduced the revised interpolation criteria as well as the bridge from single- to multi-curves, we will now define the financial market and, building

[8] For more details see [8]. By slight adjustments of the generalized definition we can also incorporate swaps comprising notional exchange and cross currency swaps.

on that, the hedge simulations. The latter comprises two different strategies: a short-term buy-and-sell and a long-term buy-and-hold hedge. A comparison of the two algorithms completes this section.

4.1 Financial market assumptions

We focus on a financial market comprising of interest rate swaps only. We fix a number of trading dates $t_0, ..., t_n$, where t_0 serves as reference date. This means that the money at any other time is discounted with respect to t_0. For any t_i we hold a model of curves calibrated from traded market swaps. V denotes the derivative which will be hedged. At every time point t_i a subset of the traded market swaps $V^{t_i}_{\text{swap,1}}, \dots, V^{t_i}_{\text{swap,m}}$ can be purchased and added to the hedge portfolio.

Remark 4 (No arbitrage): Absence of arbitrage is assured by the fact that swaps at par can only be purchased.[9] In particular, the fair hedge products are traded without any knowledge of their further development at any time. The future performance is stochastic.

We assume that all products are collateralized by the same account C representing the risk neutral asset.[10] This particularly means that all cash flows of the derivative and of the hedge are totally booked in the collateral account. The replicating portfolio will be of the form

$$V(t_k) \approx C(t_k) + \Pi(t_k).$$

The collateral account combined with the delta-hedge of market swaps approximately equals the value of the derivative. However, if we speak about the hedge we mean the process Π only, although the collateral is part of the replicating portfolio.

The strategy will be based on delta-hedging implying

$$\Delta V(t_k) \approx \Delta C(t_k) + \Delta\Pi(t_k).$$

We define the *hedge error* by

$$\epsilon(t_k) := V(t_k) - \big(C(t_k) + \Pi(t_k)\big),$$

i.e. the difference between the derivative and its replicating portfolio.

Remark 5 (Self-financing): The *self-financing* property is ensured by two properties. First, only fair swaps are added to the hedge portfolio. Second, as any product is collateralized by the same account C, all cash flows of the

[9] The vale of a swap at par equals zero.
[10] The collateral account will be subject to (risk-free) overnight interest rates.

derivative and the hedge are directly booked in the collateral. This guarantees that the money remains in the replicating portfolio.

4.2 Rebalancing and ageing hedge

We distinguish two different hedge algorithms: a short-term buy-and-sell strategy (the so-called *rebalancing hedge*) and a long-term buy-and-hold scheme (*ageing hedge*). The two are based on delta-hedging theory.

Rebalancing hedge

We assume that all hedge products acquired at time t_k are immediately sold at the following time t_{k+1}. So, we only hold a small number of hedge products over the entire trading time. This is a benefit with respect to the computation time as just a few products have to be valued at any time. On the other hand, there is no guarantee that every hedge product can be sold immediately on the next trading date.

The shares of the hedge products are determined according to the sensitivity of the derivative relating to this swap. I.e. the hedge weight $\alpha_j^{t_k}$ of the j-th market swap in t_k is approximated by the finite difference

$$\alpha_j^{t_k} = \frac{\partial V(t_k)}{\partial V_{\text{swap},j}^{t_k}(t_k)} \approx \frac{V(t_k) - \tilde{V}^j(t_k)}{V_{\text{swap},j}^{t_k}(t_k) - \tilde{V}_{\text{swap},j}^{t_k}(t_k)} = \frac{V(t_k) - \tilde{V}^j(t_k)}{-\tilde{V}_{\text{swap},j}^{t_k}(t_k)}.$$

Here, $V(t_k)$ and $V_{\text{swap},j}^{t_k}(t_k)$ are the values of the derivative and of the j-th swap at time t_k, both valued under the calibrated curve model in t_k. The latter is indeed zero as all available market swaps are at par. $\tilde{V}^j(t_k)$ and $\tilde{V}_{\text{swap},j}^{t_k}(t_k)$ denote the valuation of the derivative and of the j-th market swap under the *shifted* curve model.[11] In particular, $\tilde{V}_{\text{swap},j}^{t_k}(t_k)$ will be different from zero.

The hedge in t_k is defined by

$$\Pi(t_k) := \sum_{j=1}^{m} \alpha_j^{t_k} V_{\text{swap},j}^{t_k}(t_k) = 0,$$

and equals zero, as only new hedge swaps at par are incorporated. The value of the aged hedge is booked in the collateral.

The mechanism of the hedge algorithm is described in detail in [14]. It is shown that the hedge error in t_k is given by

$$\epsilon(t_k) = \epsilon(t_{k-1}) + \delta(t_k),$$

[11] The shifted model is constructed by "bumping" the quoted market spread of the respective swap.

and therefore composed of the propagated error $\epsilon(t_{k-1})$ and the error of the delta-hedge valuation change from t_{k-1} to t_k,

$$\Delta V(t_{k-1}) = \Delta C(t_{k-1}) + \Delta\Pi(t_{k-1}) + \delta(t_k).$$

Ageing hedge

Now we consider the ageing hedge, where acquired shares of hedge products are hold until maturity. As we can buy new market swaps at any trading time, the number of hedge products is consistently growing over time, at least until early products are expired. This requires a high computational effort but reflects a more realistic market situation.

The idea is to calculate not only the sensitivity of the derivative V, but also taking the sensitivity of the existing hedge portfolio into account. We define the *portfolio process*

$$W(t_0) := V(t_0), \quad W(t_k) := V(t_k) - \Pi_{t_{k-1}}^{t_k}(t_k),\ k > 0,$$

where we denote by $\Pi_{t_{k-1}}^{t_k}$ the total hedge portfolio held in t_{k-1} and evaluated with the curves in t_k. The hedge share $\alpha_j^{t_k}$ of the market swap with number j is approximated by the finite difference

$$\alpha_j^{t_k} = \frac{\partial W(t_k)}{\partial V_{\text{swap},j}^{t_k}(t_k)} \approx \frac{W(t_k) - \tilde{W}^j(t_k)}{V_{\text{swap},j}^{t_k}(t_k) - \tilde{V}_{\text{swap},j}^{t_k}(t_k)} = \frac{W(t_k) - \tilde{W}^j(t_k)}{-\tilde{V}_{\text{swap},j}^{t_k}(t_k)}.$$

Here, $W(t_k)$ and $V_{\text{swap},j}^{t_k}(t_k)$ are the values of the portfolio process and of the j-th swap at time t_k, both derived by the calibrated curves at t_k. Note that the value of the latter is indeed zero as all available market swaps are at par. $\tilde{W}^j(t_k)$ and $\tilde{V}_{\text{swap},j}^{t_k}(t_k)$ denote the corresponding values computed under the curve models *shifted* with respect to the j-th market swap. In particular, the value of the swap is different from zero under the shifted model.

The hedge in t_k is composed of the existing hedge products and the new ones,

$$\Pi(t_k) = \Pi_{t_{k-1}}^{t_k}(t_k) + \sum_{j=1}^{m} \alpha_j^{t_k} V_{\text{swap},j}^{t_k}(t_k).$$

Note that we have

$$W(t_k) - \left(C(t_k) + \sum_{j=1}^{m} \alpha_j^{t_k} V_{\text{swap},j}^{t_k}(t_k) \right) = V(t_k) - \big(C(t_k) + \Pi(t_k) \big) = \epsilon(t_k).$$

Therefore, we can interpret the ageing hedge in an alternative way. The new hedge products combined with the collateral are a replication of the portfolio process W.

The hedge algorithm is described in detail in [14]. Similarly to the rebalancing strategy we obtain the hedge error in t_k by

$$\epsilon(t_k) = \epsilon(t_{k-1}) + \delta(t_k).$$

4.3 The hedge algorithms in comparison

For a comparison of the two hedging strategies we exemplarily consider an OIS with maturity of 1 year and monthly payments (the derivative), which is hedged by a certain number of OIS swaps. The charts are plotted in figure 1.

The hedge portfolio of the rebalancing hedge equals zero over the entire investing period due to the strategy of daily liquidation. Only new products are held in the hedge portfolio at any time. The cash flows of the sold aged hedge products are totally booked in the collateral. As they are determined via the delta, the collateral account moves very close to the derivative. Note that V is not visible in the graphic as the hedge error is very small.

The value changes of V are therefore compensated by the movements of the collateral, which come from the sold hedge products. Cash flows of the derivative are directly booked in the collateral.

The second graphic shows the weight of the purchased hedge products. We recognize that the hedge OIS reach their maximum shares if their respective maturity coincides with the remaining term of the derivative. Only the OIS with weekly payments is regularly acquired to a small amount.

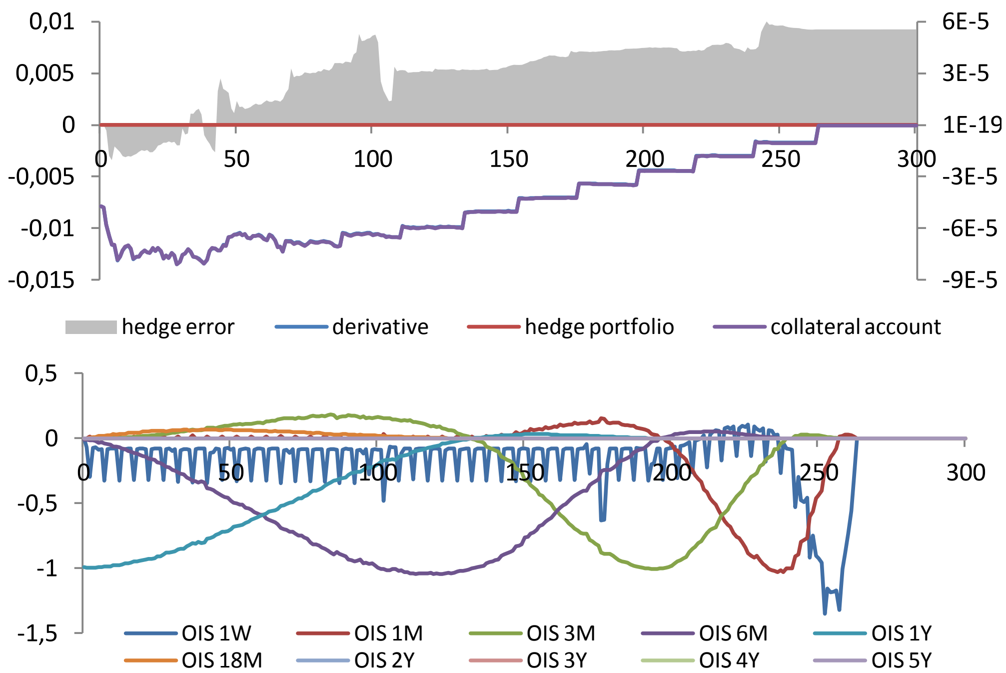
The mechanism is slightly different for the ageing hedge. We see that the hedge portfolio moves in the opposite direction as the derivative. Thus, the hedge portfolio compensates the risk evoked by value changes of the derivative. V combined with the hedge[12] – the green line – is not visible as it shows nearly the same performance as the collateral account. So, the hedge error is very small. The collateral is adjusted for monthly cash flows of the derivative.

The last chart shows the weights of the hedge products. They are mainly dominated by the 1Y swap acquired in t_0 due to the fact that this OIS has the same maturity as the derivative.[13] If we blank out this product we can see that the other hedge products are acquired as well. However, their weights are all below 1%.

[12] V and Π correspond to the portfolio process W.

[13] The standard OIS 1Y has only one cash flow at maturity whereas the considered derivative provides monthly payments.

Rebalancing hedge



Ageing hedge

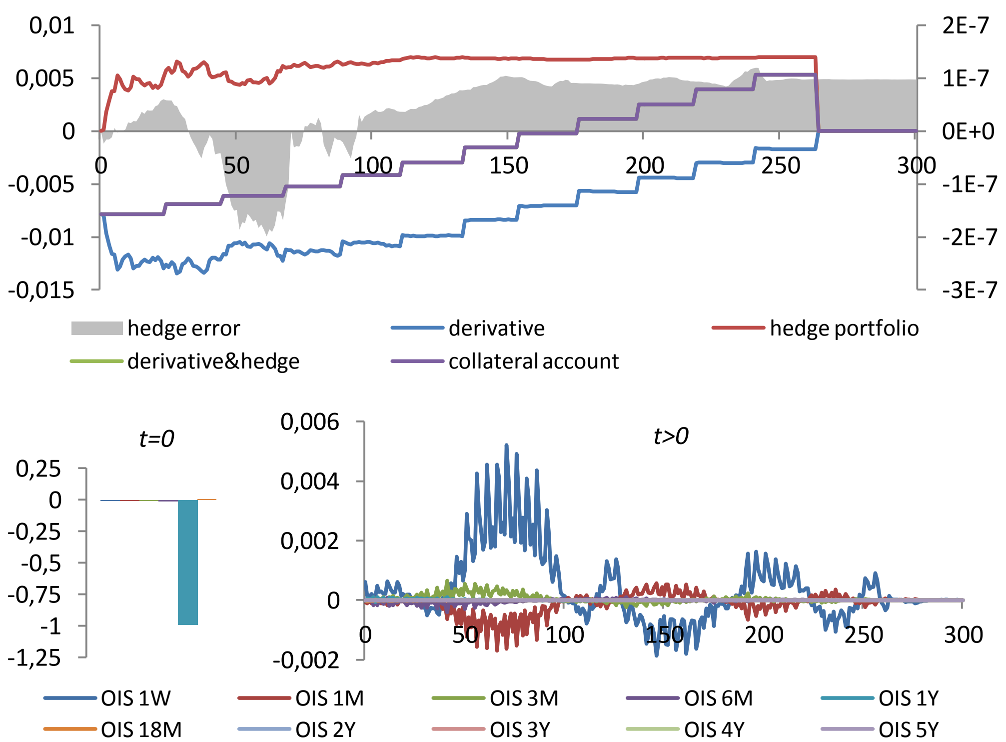


Figure 1: Rebalancing and ageing hedge in comparison

5 Some modifications of interpolation methods related to the revised assessment criterion

For the local hedge criterion (e) introduced by Hagan and West, discontinuous boundary points are negligible as they do not impact the distribution of the delta weights. Having extended this approach to our hedge simulation criteria the valuation of aged products will also require a smooth curve at boundary points to guarantee a stable hedging. In this context we present continuous modifications of two interpolation methods proposed in the literature: Akima favoured by Duffy [5] and a harmonic spline examined by Le'Floch [7].

5.1 Akima

The Akima method belongs to the class of cubic spline interpolation and aims to compute an interpolation function which looks very similar to a manually drawn one.[14] This is achieved by choosing the derivative in a certain grid point in dependence on its two adjacent left and right grid points.

In order to explain the concept we consider a grid with just five points $(t_0, f_0),\ldots,(t_4, f_4)$. We denote by m_i the *slope* between two adjacent points. It is computed by

$$m_i = \frac{f_{i+1} - f_i}{t_{i+1} - t_i}.$$

The derivative in t_2 is defined by

$$h_2 := \frac{|m_3 - m_2|\, m_1 + |m_1 - m_0|\, m_2}{|m_3 - m_2| + |m_1 - m_0|}.$$

The standard convention for the case $m_0 = m_1$ and $m_2 = m_3$ is defined by

$$h_2 := \frac{1}{2}\,(m_1 + m_2).$$

This clearly induces a discontinuity as $m_2 = m_3$ implies

$$\lim_{m_1 \to m_0} \frac{|m_1 - m_0|\, m_2}{|m_1 - m_0|} = m_2 \neq h_2.$$

We add some small constant $\delta > 0$ to the absolute differences $|m_3 - m_2|$ and $|m_1 - m_0|$. This modified Akima interpolation is continuous for $m_2 = m_3$ as

[14] The Akima interpolation was first introduced in [1].

we obtain

$$\begin{aligned}
&\lim_{m_1 \to m_0} \frac{(|m_3 - m_2| + \delta)\, m_1 + (|m_1 - m_0| + \delta)\, m_2}{(|m_3 - m_2| + \delta) + (|m_1 - m_0| + \delta)} \\
&= \lim_{m_1 \to m_0} \frac{\delta(m_1 + m_2) + |m_1 - m_0|\, m_2}{2\delta + |m_1 - m_0|} \\
&= \frac{\delta(m_1 + m_2)}{2\delta} = \frac{1}{2}(m_1 + m_2) = h_2.
\end{aligned}$$

The derivatives of the boundary points h_1 and h_0 resp. h_3 and h_4 are defined slightly different (see [1]).

5.2 Harmonic spline

The harmonic spline is a cubic spline interpolation as well, where the first derivative h_i is defined by a weighted harmonic mean

$$\frac{1}{h_i} = \frac{(t_i - t_{i-1}) + 2(t_{i+1} - t_i)}{3(t_{i+1} - t_{i-1})} \frac{1}{m_{i-1}} + \frac{2(t_i - t_{i-1}) + (t_{i+1} - t_i)}{3(t_{i+1} - t_{i-1})} \frac{1}{m_i},$$

when $m_{i-1} m_i > 0$. We set $h_i = 0$, if $m_{i-1} m_i \leq 0$.[15] This condition is continuous as we have (without loss of generality assume $m_{i-1}, m_i > 0$), that

$$\lim_{m_i \searrow 0} \frac{3(t_{i+1} - t_{i-1})\, m_{i-1}\, m_i}{((t_i - t_{i-1}) + 2(t_{i+1} - t_i))\,(m_{i-1} + m_i)} = 0 = h_i.$$

The derivatives at the boundaries are defined by

$$\begin{aligned}
h_0 &= \frac{2(t_1 - t_0) + (t_2 - t_1)}{t_2 - t_0}\, m_0 - \frac{t_1 - t_0}{t_2 - t_0}\, m_1, \\
h_n &= \frac{2(t_n - t_{n-1}) + (t_{n-1} - t_{n-2})}{t_n - t_{n-2}}\, m_{n-1} - \frac{t_n - t_{n-1}}{t_n - t_{n-2}}\, m_{n-2}.
\end{aligned}$$

and then filtered for monotonicity according to a distinction of three cases. These differentiations imply discontinuous derivatives at the outer points.[16] We drop the original filtering and keep the derivatives as defined above.

6 Numerical results

Giving a short overview of the actual market data framework and the examined interpolations, we will perform the hedge simulation primarily on three simple

[15] Note that by this condition the derivative h_i is always zero, if the adjacent slopes m_{i-1} and m_i stand contrarily to each other.

[16] See [7] for more details.

methods. Subsequently all interpolation methods will be taken into account.

6.1 Market data framework

product	EONIA OIS													EURIBOR 3M IRS									
maturity	3M	6M	1Y	18M	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	3M	6M	9M	1Y	2Y	3Y	4Y	5Y	7Y	10Y
floating leg	tenor		annual											quarterly									
fixed leg	tenor		annual											tenor			annual						
hedge product	yes								no					yes								no	

Table 1: Overview of calibration and hedge products

The simulation is applied to an interest rate market comprising the EIONIA discount curve as well as the EURIBOR 3M forward curve, constructed from overnight index swaps (OIS) or classical interest rate swaps (IRS) respectively. An overview of the applied products is displayed in figure 1. Only a part of the calibration swaps is also used for hedging.

For the derivative we exemplary consider a classical interest rate swap exchanging quarterly the EURIBOR 3M rate with a constant spread having a non-traded maturity of 3 years and 7 months.[17]

In a first step we study the hedge performance of the simulation for two time frames, each of them comprising 500 trading days. The first period takes place from 2nd August 2010 to 29th June 2012. We denote this frame by *aftermath of crisis* as it covers the time after the credit crunch of 2007-2009. The second period lasting from 1st July 2012 to 3rd June 2014 is called *low interest rates market*. This name is inspired by the development of rates market in the corresponding time frame. Both periods are highlighted in figure 2, which shows the development of several quoted EURIBOR 3M IRS spreads from 2007 to 2014.

In a second step the moving window analysis is carried out. We fix a period length of 250 trading days and compare the performance of the hedge simulation over 1000 different starting times to gain an insight into the distribution of the hedge error. We study the entire time frame from 2nd August 2010 to 3rd June 2014.

[17] Further derivatives showing similar results have been studied in [14].

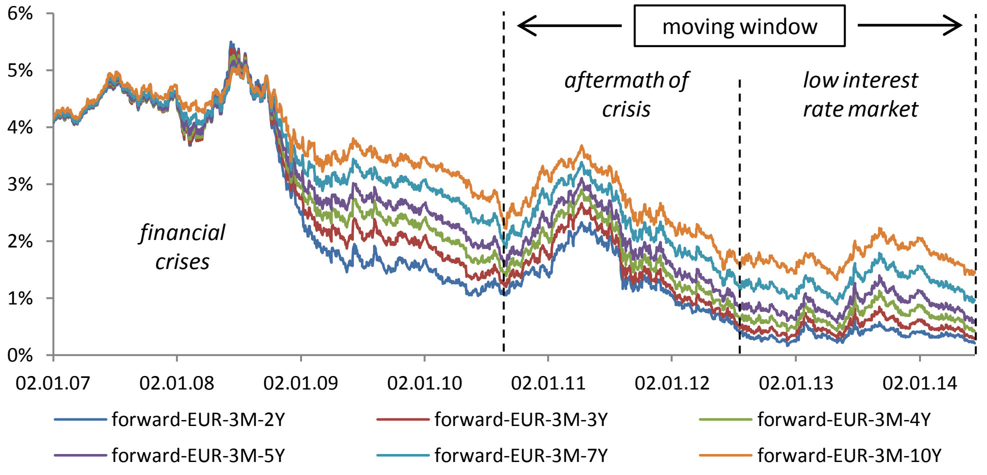


Figure 2: Quoted spread rates of EURIBOR 3M interest rate swaps

6.2 Interpolation methods and entities

Based on the reviewed scientific research and particularly the discussion in the Wilmott forum,[18] we will focus on the interpolation methods as presented in figure 3.

Interpolation methods	Interpolation entities for the	
	discount curve	forward curve
• piecewise constant • linear spline • cubic spline • Akima • harmonic spline	• discount factor (df) • log of df • yield (log of df per time)	• forward • index (forward times df) • synthetic discount factor (all discount curve entities)

Figure 3: Overview of examined interpolation methods and entities

The piecewise constant interpolation is only included to have a really poor method for comparative purposes. Linear and cubic spline represent two classical spline interpolations.[19] Akima and harmonic spline are considered as we have provided continuous modifications in view of our dynamic hedge simulation assessment criteria (see sec. 5).

[18] See section 2.1.

[19] For a detailed introduction to spline interpolation see [13] for instance.

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Interpolation entities

Recall the re-definition of the discount and the forward curve from section 3.1. The discount rate is defined by

$$P^C(T;t) := N^C(t)\ \mathrm{E}^{\mathbb{Q}^{N^C}}\left[\frac{1}{N^C(T)}\,\middle|\,\mathcal{F}_t\right],$$

i.e. the risk-neutral conditional expectation of a payment of 1 secured by the collateral account N^C serving as numéraire.

The forward rate is re-defined in terms of an index and an associated discount curve by

$$F_I^{d,C}(T;t) := \frac{V_I^C(T,T+d;t)}{P^C(T+d;t)}. \tag{2}$$

$V_I^C(T,T+d;t)$ denotes the time t-value of a payment of the index $I(T)$ paid in $T+d$ and secured by the collateral account C, where $d \geq 0$. The discount factor $P^C(T+d;t)$ is also collateralized by C.

If we consider the forward calculated via an index based on the account N^C accruing with the overnight rate,

$$N^C(t) := \prod_k \Big(1 + r(t_k)(t_{k+1} - t_k)\Big),$$

the (pre-crisis) relation between rates and forwards (eq. 1) still holds true [8]. So we interpolate the discount curve (based on overnight rates) as usual on the entities discount factor $D(T)$, logarithm of discount factor $\log(D(T))$ and yield $\frac{\log(D(T))}{T}$.

In terms of the forward curve, eq. 2 motivates to interpolate not only the forward but also the index, which is the forward times its associated discount curve.

In the pre-crisis theory it was common market practice to construct the forwards by interpolating the discount factors. So we define *synthetic* discount factors D_F related to the forward F by assuming that

$$F(T) = \frac{1}{d}\left(\frac{D_F(T)}{D_F(T+d)} - 1\right).$$

In order to obtain F we simply interpolate D_F with one of the entities provided for discount curves.

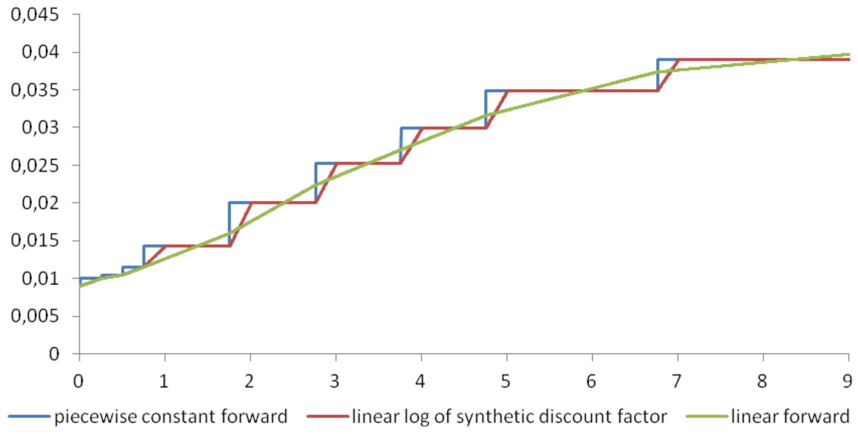


Figure 4: Sample curve interpolation for simple interpolation methods

6.3 Simple interpolation methods

At first we focus on three simple interpolation methods. We compare the piecewise constant interpolation of the forward, the linear interpolation on the logarithm of synthetic discount factor and the linear interpolation on the forward. A sample of the three curves is plotted in figure 4. The linear interpolation on the log of synthetic discount factor coincides with a linear "smoothing" of a piecewise constant interpolation[20] (red line in the chart).

Due to the shape of the curves, we expect the piecewise constant method to imply the worst performance. In particular, the discontinuity should cause jumps. The linear interpolation on the log of the synthetic discount factor should lead to a better performance as it is at least continuous. The best performance is expected for the linear interpolation since this method is the smoothest in comparison to the other two interpolations.

Aftermath of crisis and low interest rates market

At first, we have a look on the hedge errors of the each 500 trading days between 2010-12 and between 2012-14. In the upper part of figure 5 the results for the rebalancing strategy are displayed.

Both in the aftermath of crisis and in the low interest market the performance of the simple interpolations is as expected. We notice jumps in case of the piecewise constant interpolation evoked by its discontinuity. They take place when cash flows of the derivative are due (1M, 4M, 7M, etc.) and some fixings of outstanding cash flows switch from one constant forward value to the previous

[20] See appendix A.1.

one.

The results of the ageing hedge are presented in the lower part of fig. 5. We observe jumps for the piecewise constant interpolation as well. They take place at the cash flow times of the derivative and of the initial hedge products, i.e. at 1M, 3M, 4M, 6M and so on. The reason is the same as mentioned before.

If we compare the other two interpolation methods the results are different. During the aftermath of crisis the interpolation on the synthetic discount factor leads to the best performance whereas in the low interest rate market the best result is derived by the interpolation on the forward. We can see that in the first period the linear interpolation is very sensitive to changes in market. However, at the end of the period the hedge error becomes quite small.

Moving window analysis

For a deeper insight especially into the unexpected result in case of the ageing hedge we examine the distribution of the moving window. That means, we perform the hedge simulation for a fixed time frame of 500 trading days starting from different time points. The results for the derivative IRS 3Y 7M are illustrated in figure 6. The development of the average hedge error is shown in the upper graphic. Error statistics are included in the second and last chart.

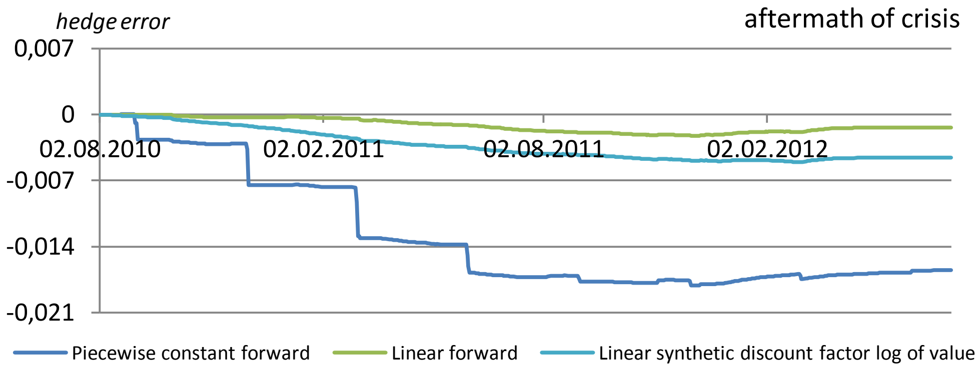
For the rebalancing strategy the result is clear. Over the entire considered period 2010-14, the constant interpolation lead to the worst results followed by the linear interpolation on the log of the discount factor. The lowest hedge error is derived under the linear interpolation on the forward.

In case of the ageing hedge, the results are slightly different. During the aftermath of crisis the linear interpolation on the log of synthetic discount factors shows the best performance which is replaced by the linear interpolation on the forward at the end of 2011.

For the ageing strategy we obtain once more that during the aftermath of the crisis the linear interpolation on the logarithm of discount factors is a reasonable alternative to the smoother linear interpolation on the forward. The piecewise constant interpolation leads to poor results in all cases except for the ageing strategy of the modified hedge product IRS 2Y quarterly where all interpolations imply nearly the same performances.

Moreover, we note that the L^1-errors as well as the the standard deviations during the aftermath of crisis are higher compared to the low interest rates market. That seems to be a reasonable finding as hedging under stable market conditions should outperform stronger fluctuating interest rates.

Rebalancing hedge



Ageing hedge

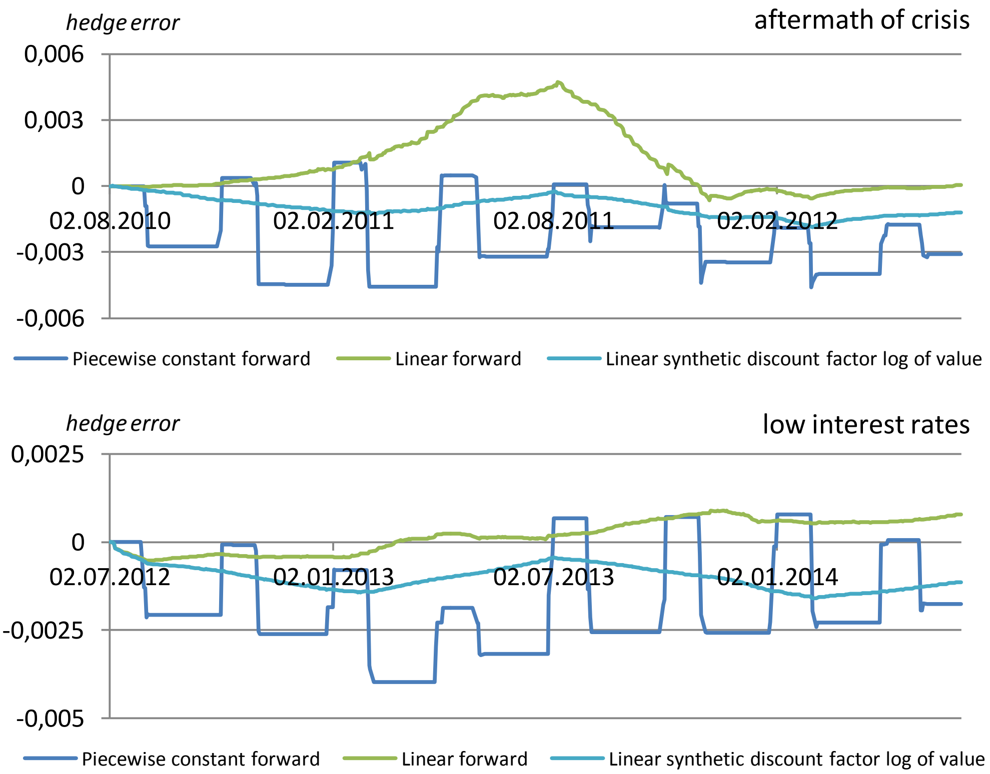
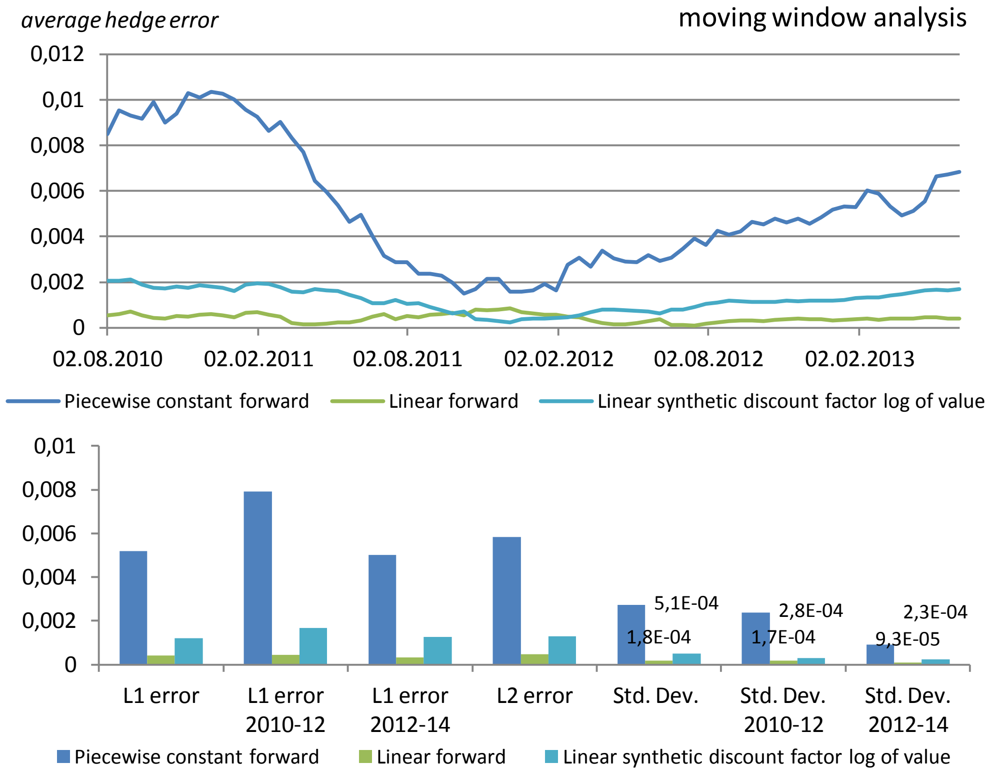


Figure 5: Hedge error of the simple interpolations

Rebalancing hedge



Ageing hedge

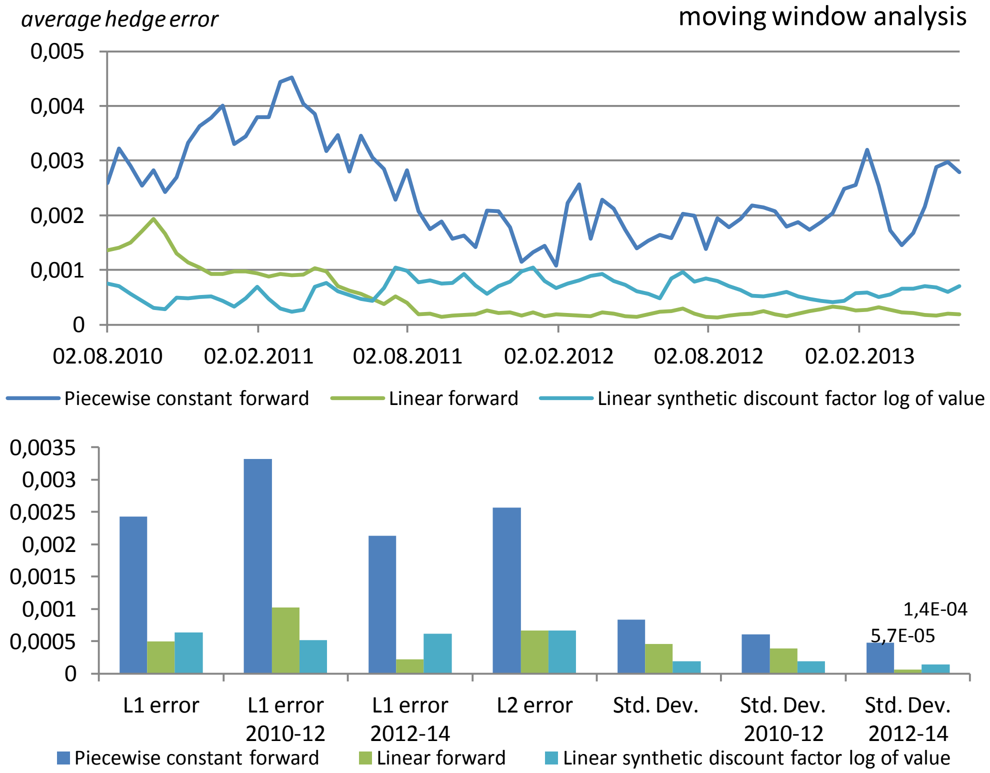


Figure 6: Moving window analysis of the simple interpolations

6.4 Rebalancing strategy

Aftermath of crisis and low interest rates market

The results of the sample derivative in the aftermath of crisis are displayed in figure 7. In the upper part, the hedge errors of the “best” interpolations are shown.[21] We note that any of the good performances are derived in terms of the interpolation entity synthetic discount factor. The cubic ones lead to the best results.

In the lower part of figure 7 we present three bar diagrams showing the statistics L^1-error, maximum absolute error and the standard deviation of the mean absolute hedge error. In particular, the statistics of the other interpolations are shown as well. They underline the results of the upper graphic showing that the entity synthetic discount factor on all splines is well performing.

The output for the low interest rates market is shown in figure 8. We observe again that the synthetic discount factor entities lead to better performances than the forward entities, but with exception for the linear interpolation. The latter on the log of synthetic discount factor leads to the worst result. We can also see that the harmonic interpolation on the forward is a bad choice.

Moving window analysis

If we have a look at the moving window analysis as displayed in graphic 9 it can be seen that the synthetic discount factor entities generally lead to better performances compared to the forward entities, except for the linear interpolation. In the latter case the interpolation on the log of synthetic discount factor implies the worst performance.

We note that the harmonic spline on the forward entities has the highest mean error and standard deviation, in particular during the aftermath of crisis. That means, this interpolation is quite sensitive to changes in the market[22].

Furthermore, we observe that the mean errors in the low interest rate market are significantly below the values in the aftermath of crisis which is a reasonable outcome regarding the development in the interest rate market. However, this effect is mitigated regarding the standard deviation.

6.5 Ageing strategy

As the results for the time frames after math of crises and low interest rates market are in line with the moving window analysis,[23] we only show the latter here. The statistics are presented in figure 10.

[21] We refrain from plotting all 16 interpolations due to transparency reasons.
[22] Compare figure 2.
[23] The results are provided in [14].

Figure 7: Rebalancing hedge in the aftermath of crisis

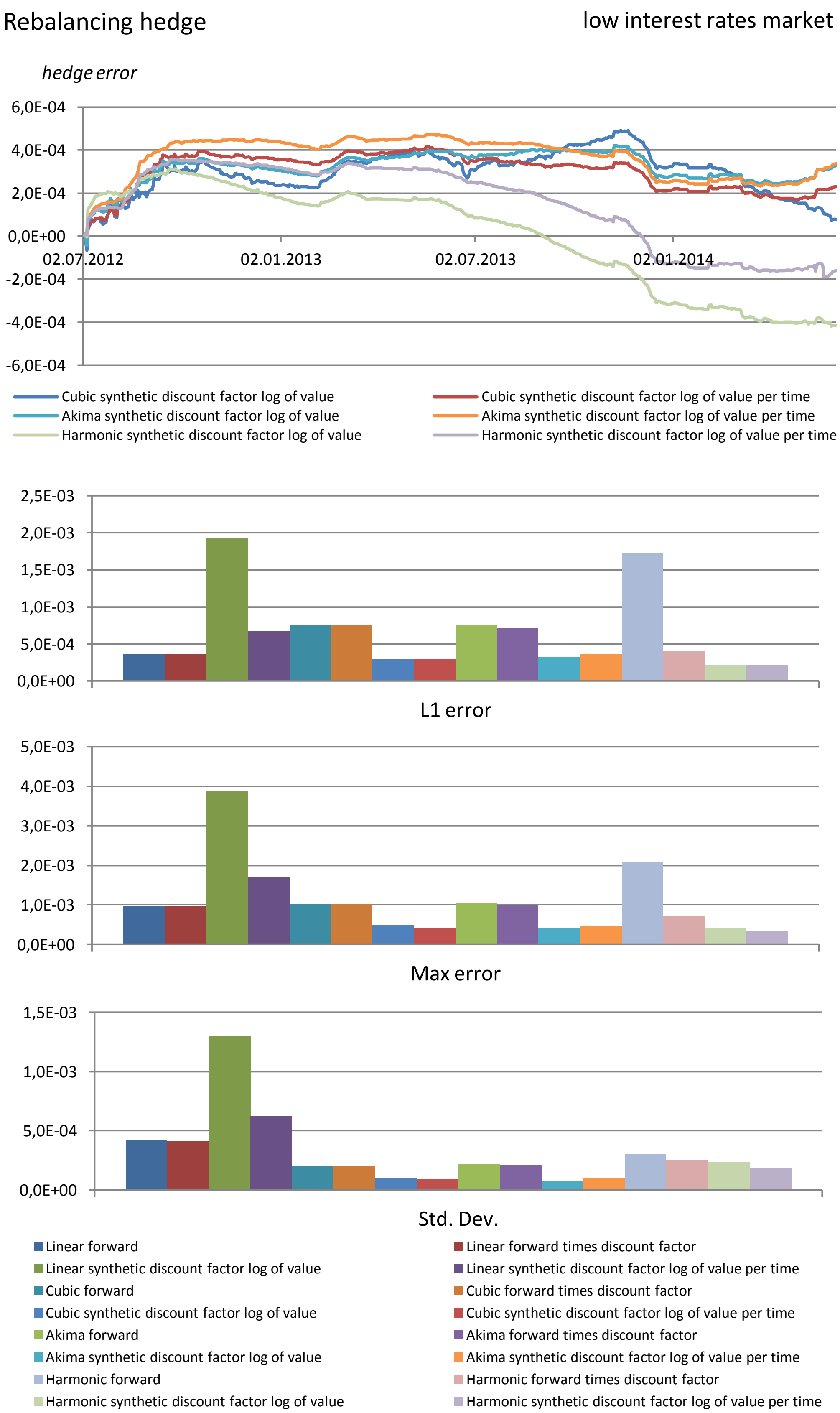


Figure 8: Rebalancing hedge in the low interest rates market

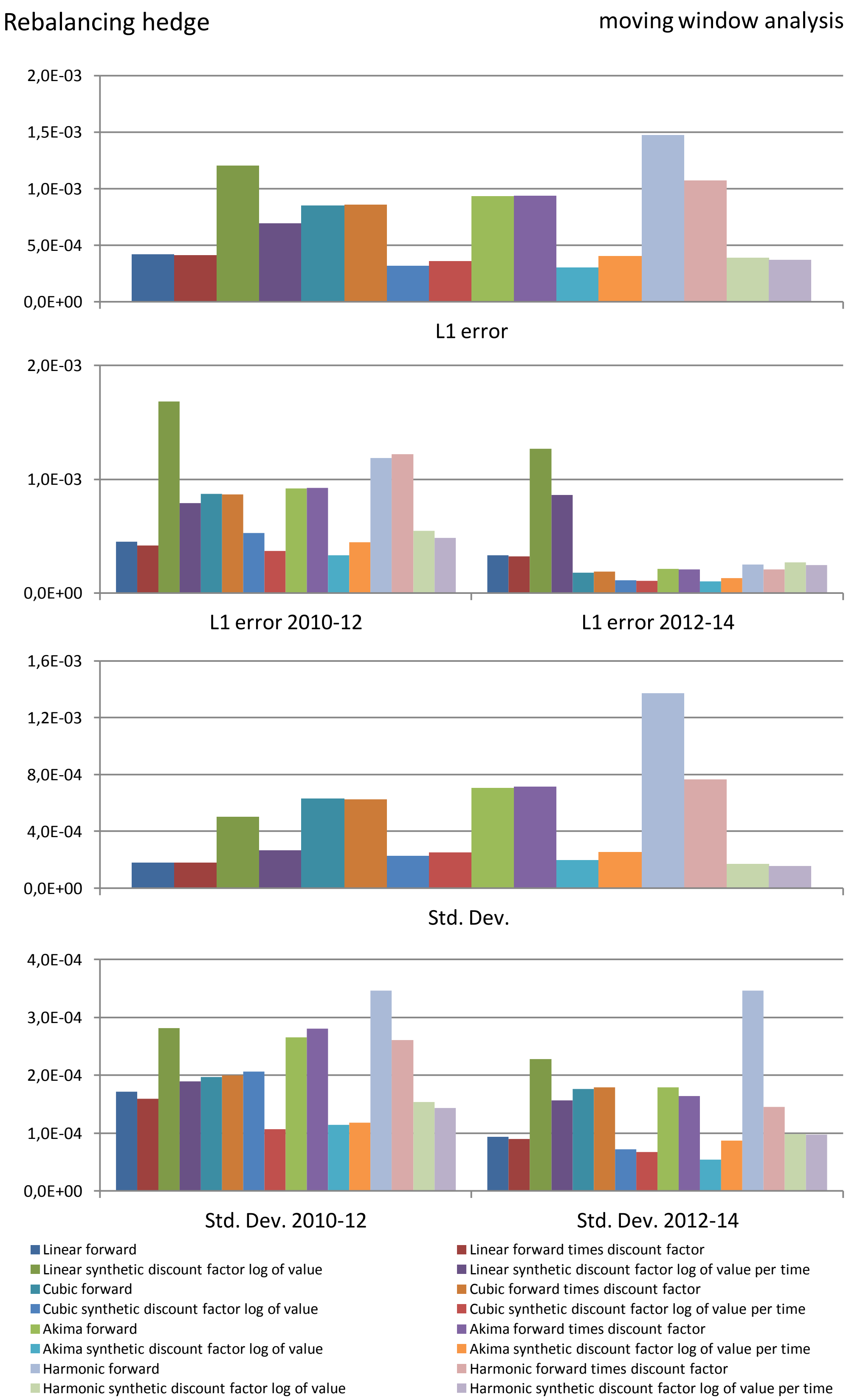


Figure 9: Error statistics moving window analysis of the rebalancing hedge

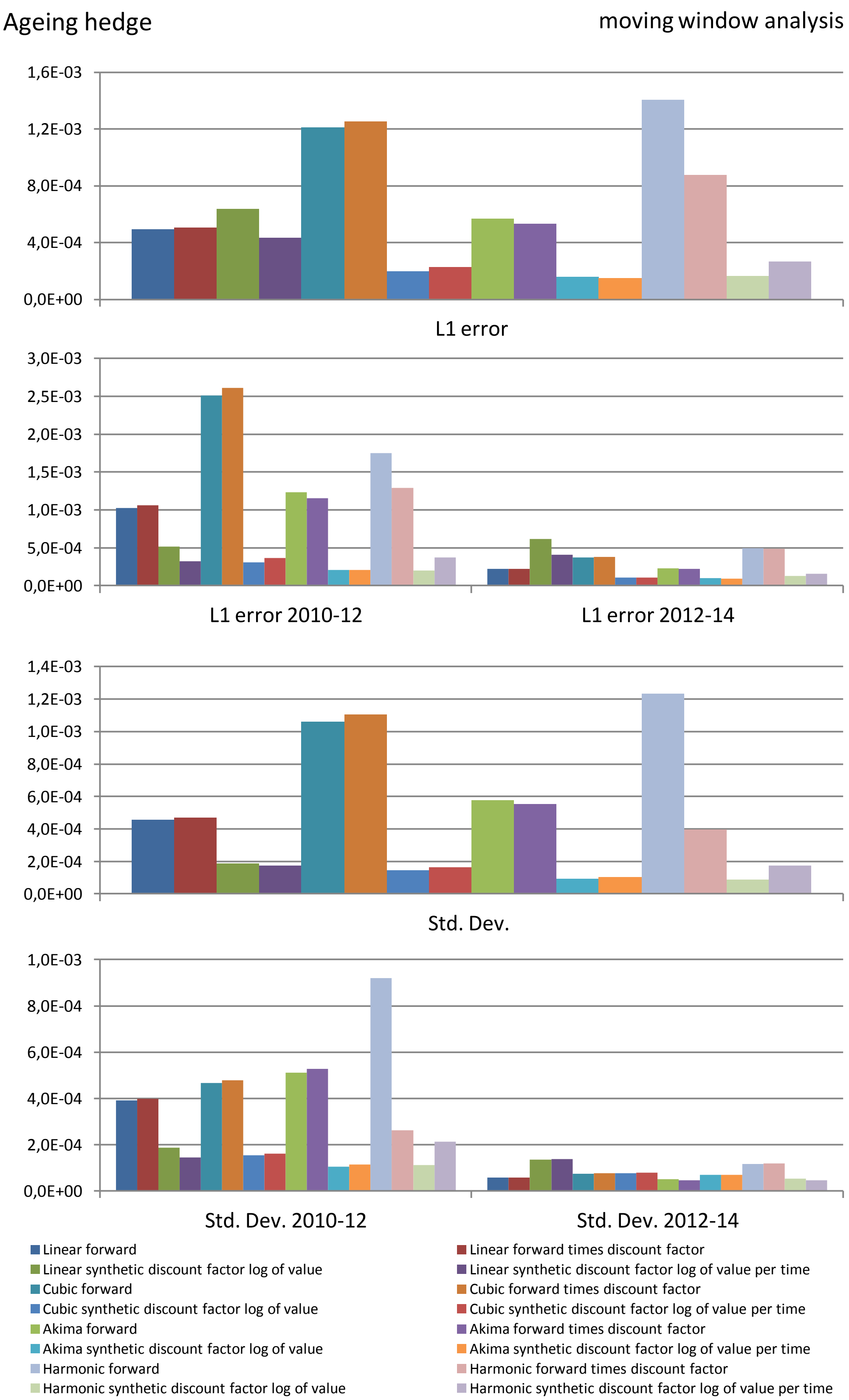


Figure 10: Error statistics moving window analysis of the ageing hedge

We observe the general result that the synthetic interpolation entities lead to better performances than the entities on the forward. This is especially marked in the aftermath of crisis for both the L^1-error and the standard deviation. In the low interest rate market the effect is strongly mitigated. However, the linear interpolation implies the reverse result. Here, the forward entities lead to better hedge performances.

Moreover, the values of the statistics for the period 2012-14 are significantly lower than for the time frame 2010-12, the usual outcome in the multi-curve analyses.

6.6 Stability of the hedge simulation

A wide range of further analyses can be found in [14]. Particularly, the stability of the hedge error has been examined. Below we list some of the achieved results:

- The number of grid points used for the interpolation of curves does have an impact on the hedge performance. Whereas in the present calculation primarily swaps with yearly maturities are used,[24] we also consider a case where missing quarterly maturities are added to the applied lattice.[25] Except the broader range of hedge products implying an interpolation curve based on more grid points, the result are in line with the applied data set in the present case. However, a reduction of the number of calibration products leads to an interpolation curve constructed on a few grid points only. That particularly means shifts do not only have an impact nearby. Hence, these curve are not sufficiently robust and result into bad hedge performances, especially with respect to the short-term maturities up to 1 year.

- If we choose the bumping value as $\delta < 1.0\, e^{-8}$ the approximation of the differentiation is only a very poor estimate, whereas $\delta > 1.0\, e^{-4}$ results into errors in the numerical computation of the delta weight. Robust results are only derived for a delta shift in the range of $1.0\, e^{-8} \leq \delta \leq 1.0\, e^{-4}$.

- There is no risk arising from EONIA overnight index swaps in such a way that OIS hedge products would play a considerable role for hedging a three-month EURIBOR interest rate swap, even though the latter makes use of the EONIA overnight curve as discount curve.

[24] See table 1.

[25] If swaps with certain maturities are not available at the market we simply construct synthetic swaps in order to cover any quarterly maturity.

- Instead of a daily hedging frequency good result can also be achieved if the hedge portfolio is recomposed (for the rebalancing strategy) or extended (for the ageing strategy) on a weekly basis. It is particularly worth pointing out that hedging every second or third day can even imply a better performance than a daily portfolio reallocation.

- In order to reduce the computational effort for the ageing hedge strategy,[26] only those hedge products are purchased whose weights exceed a certain value. In the present case, we have used a boundary of $1.0\,e^{-6}$ allowing the computation time to be halved. The negligible impact on the hedge error has been in the range of $1.0\,e^{-8}$ or smaller.

7 Conclusion

Giving a brief overview of recent scientific papers in the field of interest rate curve interpolation, we have identified hedging as a significant assessment criterion due to its high relevance of application in the financial sector. Instead of considering locality of the hedge error only, we have enhanced the approach to a dynamic hedge simulation over a certain time frame by making use of two different hedging methodologies - the rebalancing and the ageing hedge.

We have adapted the multi-curve framework, i.e. a financial market model in which discount and forward curves have to be distinguished. In this context it seems to be reasonable to enhance the range of interpolation entities for the forward curve, mentioning *index times discount factor* for instance.

As based on delta-hedging the hedge simulation requires continuously differentiable interpolation functions for a stable performance without jumps. Therefore, we have presented continuous modifications of the Akima and the harmonic spline interpolation towards the existing definitions in the literature.

Numerical analysis were made possible by a Java-based implementation of the hedge simulation. In case of the discontinuous piecewise constant interpolation jumps and a bad performance has been the expected result. We have seen that the cubic interpolation methods generally lead to better results compared to simple linear interpolations. Moreover, the hedge simulation in an unstable market shows clearly worse results as compared to a calm market phase.

A very interesting result of the study is the well-performing entity *synthetic discount factor*. This approach has been motivated by the pre-crisis single-curve relation where forwards could be expressed in terms of discount factors. However, in a multiple curve market framework discount and forward curves

[26] The portfolio of hedge products for the ageing strategy is consistently growing over time as new hedge products can be added to the existing portfolio on each trading. Therefore, a large amount of products has to be evaluated at each time requiring a high calculation effort and long computation times.

are strictly separated. The good results could therefore create some confusion. So it is worth pointing out that the synthetic discount factors are simply another (fictive) interpolation entity and not linked to the pre-crises relation between discount factors and forwards at all.

Among the linear interpolations, the entities *forward* and *forward times discount factor* (*forward value*) are performing better than *synthetic discount factors*.[27]

In summary, the enhanced hedge criterion based on a dynamic hedge simulation seems to be more meaningful than just a local hedge consideration. On the one hand, this is underpinned by the practical appliance of hedging in the financial industry. On the other hand, due to the dynamic of the hedge simulation, curves have to be constructed for each trading date. The hedge error and its statistics therefore reflect not only one single curve but instead comprise all constructed curves. This approach can be further improved by studying the moving windows.

[27] Considering the rebalancing hedge, this result holds for all situations. For the ageing hedge, it holds for the 2012-2014 sample data. See Figure 6, 8 and 9.

A Appendix

A.1 Linear interpolation on log of discount factor

The linear interpolation on the log of discount factor (respective synthetic discount factor) coincides with a piecewise linear "smoothing" of a piecewise constant interpolation for the forward rates.

Assume that the forward is given in terms of (synthetic) discount factors, i.e.

$$F(T) = \frac{1}{d}\left(\frac{D(T)}{D(T+d)} - 1\right).$$

We denote the interpolation entity by $A(T) := \log(D(T))$. Let $F(T_i)$ and $F(T_{i+1})$ be two adjacent forward values which have been calibrated. As the calibration times[28] are $T_i + d$ and $T_{i+1} + d$, we have the grid points $A(T_i + d)$ and $A(T_{i+1} + d)$. We take $T \in [T_i, T_{i+1}]$ and distinguish two cases.

If $T > T_i + d$, the linear interpolation implies

$$\begin{aligned} A(T) &= \frac{(T_{i+1}+d) - T}{(T_{i+1}+d) - (T_i+d)}\, A(T_i+d) + \frac{T - (T_i+d)}{(T_{i+1}+d) - (T_i+d)}\, A(T_{i+1}+d) \\ &= \frac{(T_{i+1} - T) + d}{T_{i+1} - T_i}\, A(T_i+d) + \frac{(T - T_i) - d}{T_{i+1} - T_i}\, A(T_{i+1}+d), \end{aligned}$$

and analogously

$$A(T+d) = \frac{T_{i+1} - T}{T_{i+1} - T_i}\, A(T_i+d) + \frac{T - T_i}{T_{i+1} - T_i}\, A(T_{i+1}+d).$$

We obtain

$$A(T+d) - A(T) = \frac{d}{T_{i+1} - T_i}\big(A(T_i+d) - A(T_{i+1}+d)\big) = \text{constant},$$

which proves

$$F(T) = \frac{1}{d}\left(\frac{e^{-A(T)}}{e^{-A(T+d)}} - 1\right) = \frac{1}{d}\big(e^{A(T+d)-A(T)} - 1\big) = \text{constant}.$$

If $T < T_i + d$, then $A(T)$ lies in between the calibration values $A(T_{i-1} + d)$ and $A(T_i + d)$. So we have

$$A(T) = \frac{(T_i+d) - T}{T_i - T_{i-1}}\, A(T_{i-1}+d) + \frac{T - (T_{i-1} - d)}{T_i - T_{i-1}}\, A(T_i+d),$$

[28] If the forward is calibrated via the discount curve, the discount factor at maturity of the forward determines the calibration time. For more details we refer to [14] once more.

and in particular the forward is not constant.

Clearly,

$$\lim_{T \nearrow T_i + d} A(T) = A(T_i + d) = \lim_{T \searrow T_i + d} A(T),$$

which shows the continuity.

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