Test 1 Prep
Semester 2

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Contents

1 Introduction

Here are Multi Notes to prepare for the first test

2 Double Integrals

Double integrals are integrals that represent the volumes under a surface, rather than **Definite Integrals**, which compute the area under a curve.

2.1 Definite Integrals Recap

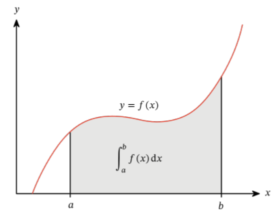


Figure 1: Example of a definite Integral

Integrals are limits of a Riemann sum, which can be represented by the following equation:

$$\int_a^b f(x)\delta x = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + ... = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i \text{ ,where } \max \Delta x_i \to 0$$

2.2 Applying Definite Integrals to double integrals

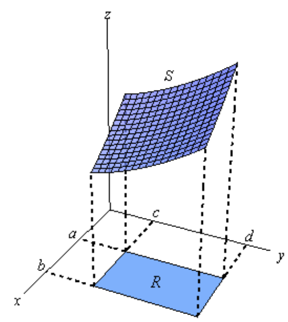


Figure 2: Example of a double Integral

Given that we have to find the area underneath the curve of a function $z = f(x, y)$, we will need to find **volume**, not area. We're going to define the rectangle area by:

$$R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\} = [a, b] \times [c, d] \text{ (Cartesian Product)}$$

And thus we can define a solid S using the equation:

$$S = \{(x, y, z)|(x, y) \in \mathbb{R}, 0 \leq z \leq f(x, y)\}$$

If one is to use equal subintervals to calculate, then you can use a double Riemann sum, with the area of a rectangle equal to $\Delta x \times \Delta y \times f(x, y)$. You can represent that sum, with

$$\lim_{m,n\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A$$

(**NOTE** the *'s mean that the values are at a specific point) Where m and n represent the number if subintervals. Double Riemann sums give volume, which when "limited" simplifies to

$$\iint_R f(x, y)\delta A = \iint_{[a,b]\times[c,d]} f(x, y)\delta A$$

2.3 Notes

1. If $f(x, y) \geq 0$ and $f(x, y) \in \mathbb{R}$ then $\iint_R f(x, yA) =$ is the volume of the solid bounded by x, y, and the surface
2. $\delta A = \delta x\delta y$ or $\delta y\delta x$. Order will impact evaluation but not result.
3. If $f(x, y)$ is continious in R, then the double integral exists. (**NOTE**: You can get away with some discontinuity, such as with steps and the like, but not with stuff like infinite behavior)

3 Evaluating Double Integrals

3.1 Riemann Sums

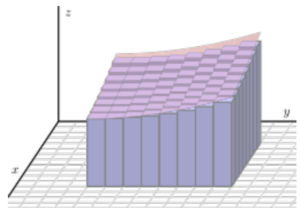


Figure 3: Riemann Sums over Double Integrals

Just as with definite integrals, you can use Riemann-esque sums to estimate the area underneath a curve.

Example Question: Estimate $\iint_R 3x - y^2 \delta A$, where $R = [0,2] \times [0,2]$ using a double Riemann sum using the midpoint of each rectangle, with $m, n = 2$.
To solve, create a grid of a rectangle, and calculate $f(x^*, y^*)$, and multiply that for each rectangle by the Δx and Δy.

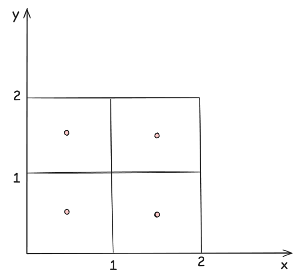


Figure 4: Example rectangle, with the points showing the $f(x^*, y^*)$ that is needed to be calculated

Once you have the graph, actually solving is relatively straightfoward:

$$A \approx f(0.5, 0.5)\Delta x \Delta y + f(1.5, 0.5)\Delta x \Delta y + f(0.5, 1.5)\Delta x \Delta y + f(1.5, 1.5)\Delta x \Delta y$$

$$= 1.25 \times 1 \times 1 - 0.75 \times 1 \times 1 + 4.25 \times 1 \times 1 + 2.25 \times 1 \times 1 = \boxed{7}$$

There are other ways such a question can be phrased, but the formula/idea should be similar.

3.2 Fubini's Theorem and Evaluating Double Integrals

Evaluating double integrals is like counting up loafs of bread, and Fubini's theorem says that no matter how one slices, you'll get the same amount of bread. Y will have to take the slices (integrals over the contours) of either x or y, getting a area function given the other variable, and then integrate that or its range.

Thus, Fubini's theorem states that if $f(x, y)$ is continious on $[a, b] \times [c, d]$ then $\iint_R f(x, y)\delta A$ equals:

$$V = \int_a^b B(x)\delta x = \int_a^b \left[\int_c^d f(x, y)\delta y\right] \delta x \qquad B(x) = \int_c^d f(x, y)\delta x$$

$$V = \int_c^d A(y)\delta y = \int_c^d \left[\int_a^b f(x, y)\delta x\right] \delta y \qquad A(y) = \int_a^b f(x, y)\delta y$$

4 The Very Nice Theorem

Interesting way to evauate integrals that can be separated into individual components.

$$\text{If } f(x, y) = g(x)h(y) \text{ and R} = [a, b] \times [c, d]\text{, then } \iint_R f(x, y)\delta A = \int_a^b g(x)\delta x \int_c^d h(y)\delta y$$

5 Double Integral Properties (over Rectangles)

- $\iint_R (f+g)\delta A = \iint_R f\delta A + \iint_R g\delta A$
- $\iint_R cf(x,y)\delta A = c\iint_R f(x,y)\delta A$
- If $f(x,y) \geq g(x,y)$, then $\iint_R f(x,y)\delta A \geq \iint_R g(x,y)$. (i.e. integrals preserve inequalities)

6 Evaluating Double Integrals over Non-Rectangular Boundaries

A big idea relating to evaluating double integrals over non-rectangular boundaries is how to find a bound, because it's more interesting/difficult to do so with these odd bounds. Thus we need to differential between type 1, type 2, and non-typal shapes.

6.1 Type 1 Shapes vs Type 2 Shapes

Type 1 Bound
Type one shapes have a $x \in [a,b]$ as boundaries, and are defined by two $y = g(x)$ functions.

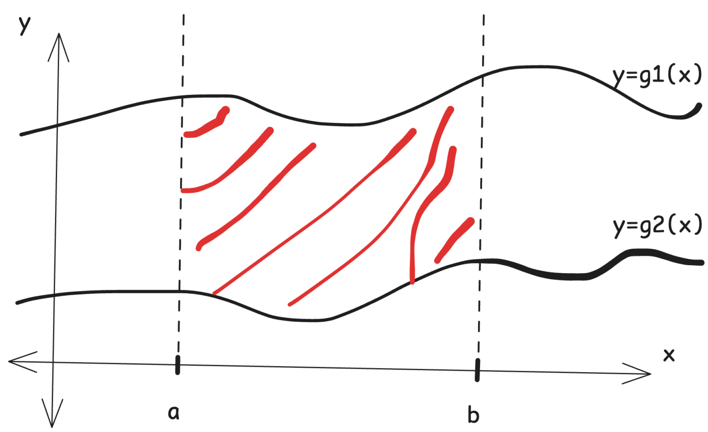


Figure 5: Example of a Type 1 bound

Type 2 Bound
Type 2 shapes have a $y \in [a,b]$ as boundaries, and are defined by two $x = h(y)$ functions

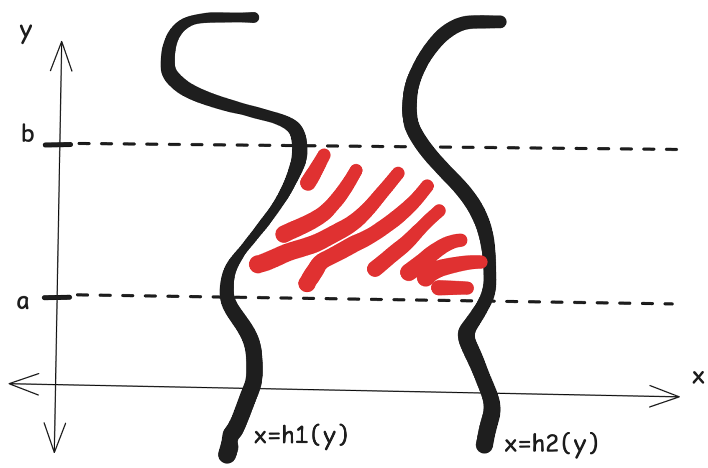


Figure 6: Example of a Type 2 Bound

NOTE: There are some surfaces that are BOTH type 1 and type 2 boundaries, such as a circle.

6.2 Integrating over Type 1 and Type 2 Bounds

There are *TWO OPTIONS*

- If the bounds are **functions of x**, or the base is a **type 1**,

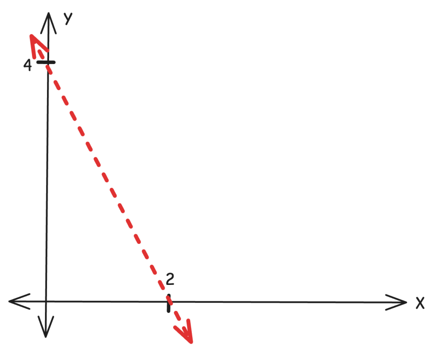
$$V = \iint_D f(x,y)\delta A = \int_a^b \Big[\int_{g_1(x)}^{g_2(x)} f(x,y)\delta y\Big]\delta x$$

- If the bounds are **functions of y**, of the base is a **type 2**,

$$V = \iint_D f(x,y)\delta A = \int_c^d \Big[\int_{h_1(y)}^{h_2(y)} f(x,y)\delta x\Big]\delta y$$

6.3 Example

Find the volume of $z = 8 - 4x - 2y$ over the boundary:



The bound is both a type 1 and a type 2 integral, but let's start with integrating as a type 1 integral.

$$\int_0^2 \Big[\int_0^{-2x+4} f(x,y)\delta y\Big]\delta x$$

$$= \int_0^2 \Big[8y - 4xy - y^2\Big|_0^{-2x+4}\Big]\delta x$$

$$= \int_0^2 \Big[8(-2x+4) - 4(x)(-2x+4) - (-2x+4)^2\Big]\delta x$$

$$= \int_0^2 4x^2 - 16x + 16\delta x = \frac{4}{3}x^3 + 8x^2 + 16x\Big|_0^2$$

$$= \boxed{32/3}$$

7 Helpful Extraneous Stuff

- $\delta x \delta y$ refers to a horizontal distance, while $\delta y \delta x$ refers to a vertical distance, e.g. trying to find the increase through the opposite bounds...
- To switch between δx and δy, one has to look at the resulting bound to try to switch if the region is both a type 1 and a type 2

8 Triple Integrals

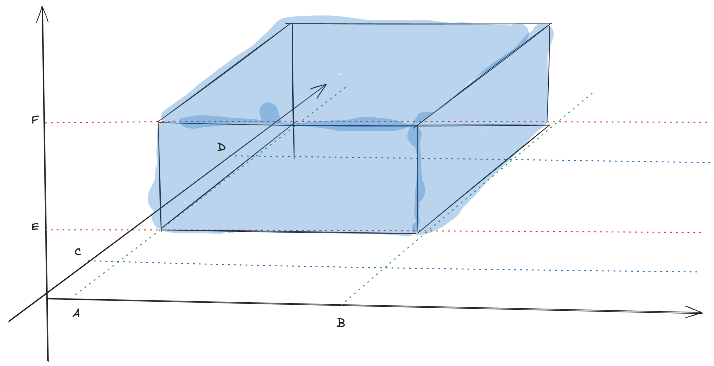


Figure 7: Three Dimensional Object

To integrate over three dimensional object, one can use triple integrals.

$$\iiint_E f(x,y,z)$$

Think about it as little cubes, with

$$\lim_{m,n,l\to\infty} \Sigma_{k=1}^{l}\Sigma_{j=1}^{m}\Sigma_{i=1}^{n} f(y_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta x \Delta y \Delta z$$

8.1 Three Type of Regions of Space

Type 1

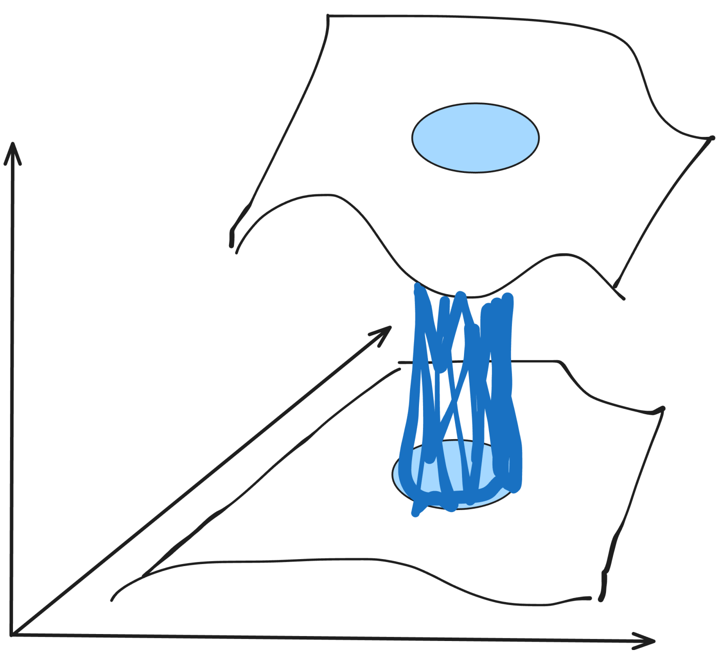


Figure 8: Example of a type 1 surface

$$\iiint_E f(x,y,z)\delta V = \iint_D \int_{g_2(x,y)}^{g_1(x,y)} g_2 - g_1 \delta z \delta A$$

Type 2

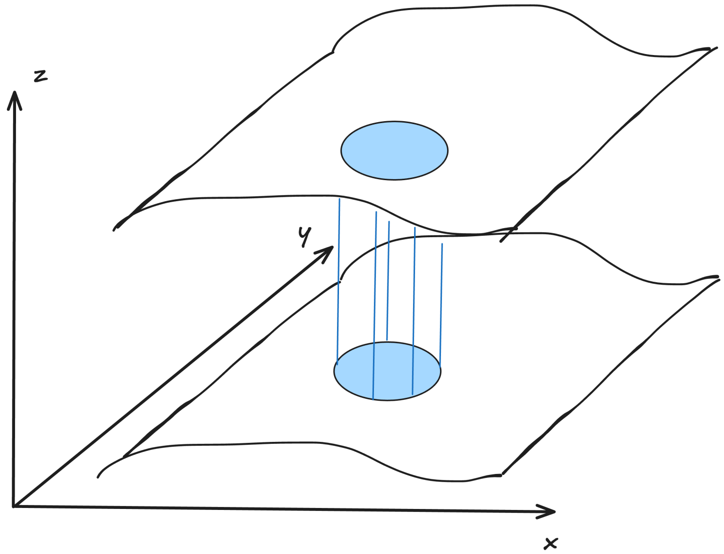


Figure 9: Example of a type 2 surface

$$\iiint_E f(x,y,z)\delta V = \iint_D \int_{h_2(x,y)}^{h_1(x,y)} h_2 - h_1 \delta y \delta A$$

Type 3

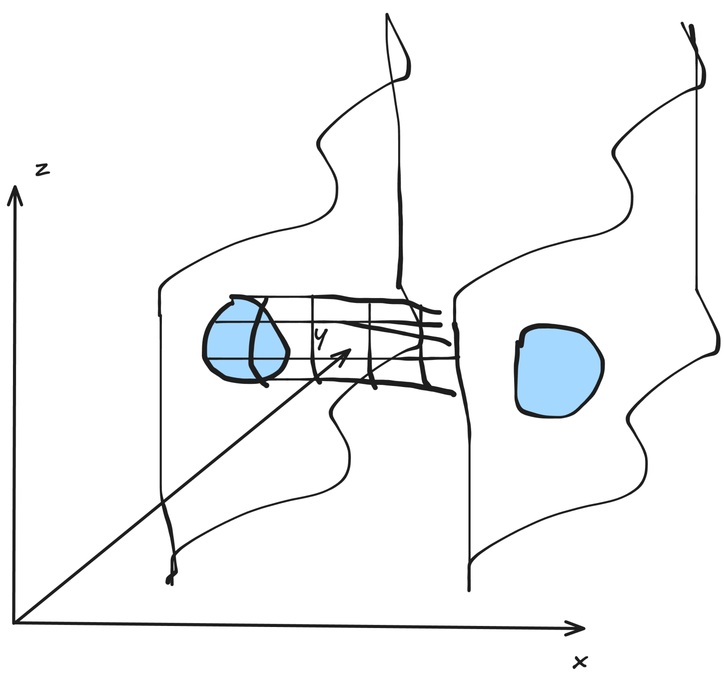


Figure 10: Example of a type 3 surface

$$\iiint_E f(x,y,z)\delta V = \iint_D \int_{k_2(x,y)}^{k_1(x,y)} k_2 - k_1 \delta x \delta A$$

9 Changing the Bounds of Integrals

To change the bounds of an integral, all you do is to try to visualize the graph of the function, and then change the integrands to that.

Example

$$\int_0^1 \int_0^7 \delta x \delta y = \int_0^1 \int_x^1 f(x,y) \delta y \delta x$$

10 Odd stuff with two and Three Dimensional Integrals

10.1 Odd stuff with 2x Integrals

- Graphing is usually good, remember when it would be good to split something up

10.2 Off stuff with 3x Integrals

- Bounding Issues: Remember to list out bounds for the things well; it often helps to set some variables to zero

11 Vector Value Functions

Vector value functions are parametrics that create three D vectors.

$$\vec{r} : R \to R^3$$

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

11.1 Odd stuff about VV functions

- $\vec{r}'(t)$ will be the tangent to curve provided by $\vec{r}'(t)$, but not to $\vec{r}(t)$ itself
- $\vec{r}'(t)$ implies a cusp
- Usual properties apply, order for cross product matters

11.2 Unit Tangent and the Normal

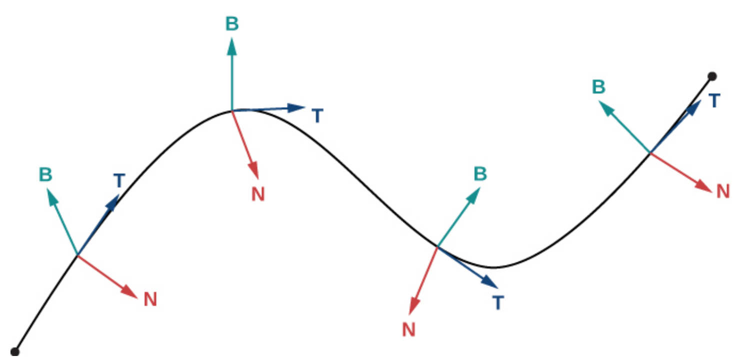


Figure 11: Graph showing binormals, normals, and tangents

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$\vec{N}(t) = \frac{\delta}{\delta t} \left[\frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|} \right]$$

$$\vec{B} = \vec{T} \times \vec{N}$$

- Normal Plane, the plane orthogonal to the curve's direction: $\vec{B} \times \vec{N}$
- Osculating Plane, best approximates curve to a point: $\vec{N} \times \vec{T}$
- Rectifying Plane, twisting behavior: $\vec{T} \times \vec{B}$