Test 1
Semester 2

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1 Introduction

Here are Multi Notes to prepare for the first test

2 Double Integrals

Double integrals are integrals that represent the volumes under a surface, rather than **Definite Integrals**, which compute the area under a curve.

2.1 Definite Integrals Recap

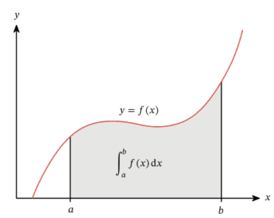

Figure 1: Example of a definite Integral

Integrals are limits of a Riemann sum, which can be represented by the following equation:

$$\int_a^b f(x)\delta x = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + ... = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i \text{ ,where } \max \Delta x_i \to 0$$

2.2 Applying Definite Integrals to double integrals

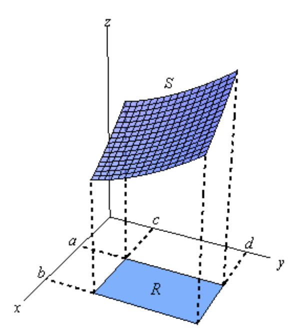

Figure 2: Example of a double Integral

Given that we have to find the area underneath the curve of a function $z = f(x, y)$, we will need to find **volume**, not area. We're going to define the rectangle area by:

$$R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\} = [a, b] \times [c, d] \text{ (Cartesian Product)}$$

And thus we can define a solid S using the equation:

$$S = \{(x, y, z)|(x, y) \in \mathbb{R}, 0 \leq z \leq f(x, y)\}$$

If one is to use equal subintervals to calculate, then you can use a double Riemann sum, with the area of a rectangle equal to $\Delta x \times \Delta y \times f(x, y)$. You can represent that sum, with

$$\lim_{m,n\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A$$

(**NOTE** the *'s mean that the values are at a specific point) Where m and n represent the number if subintervals. Double Riemann sums give volume, which when "limited" simplifies to

$$\iint_R f(x, y)\delta A = \iint_{[a,b]\times[c,d]} f(x, y)\delta A$$

2.3 Notes

1. If $f(x, y) \geq 0$ and $f(x, y) \in \mathbb{R}$ then $\iint_R f(x, yA) =$ is the volume of the solid bounded by x, y, and the surface
2. $\delta A = \delta x\delta y$ or $\delta y\delta x$. Order will impact evaluation but not result.
3. If $f(x, y)$ is continious in R, then the double integral exists. (**NOTE**: You can get away with some discontinuity, such as with steps and the like, but not with stuff like infinite behavior)

3 Evaluating Double Integrals

3.1 Riemann Sums

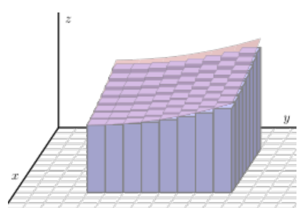

Figure 3: Riemann Sums over Double Integrals

Just as with definite integrals, you can use Riemann-esque sums to estimate the area underneath a curve.

Example Question: Estimate $\iint_R 3x - y^2\delta A$, where $R = [0, 2] \times [0, 2]$ using a double Riemann sum using the midpoint of each rectangle, with $m, n = 2$.
To solve, create a grid of a rectangle, and calculate $f(x^*, y^*)$, and multiply that for each rectangle by the Δx and Δy.

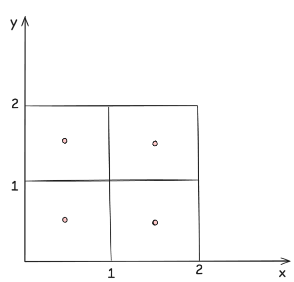

Figure 4: Example rectangle, with the points showing the $f(x^*, y^*)$ that is needed to be calculated

Once you have the graph, actually solving is relatively straightfoward:

$$A \approx f(0.5, 0.5)\Delta x\Delta y + f(1.5, 0.5)\Delta x\Delta y + f(0.5, 1.5)\Delta x\Delta y + f(1.5, 1.5)\Delta x\Delta y$$

$$= 1.25 \times 1 \times 1 - 0.75 \times 1 \times 1 + 4.25 \times 1 \times 1 + 2.25 \times 1 \times 1 = \boxed{7}$$

There are other ways such a question can be phrased, but the formula/idea should be similar.

3.2 Fubini's Theorem and Evalutating Double Integrals

Fubini's theorem like slicing a loaf of bread. Basically, you will have to take the slices (integrals over the contours) of either x or y, getting a area function given the other variable, and then integrate that or its range.

Thus, Fubini's theorem states that if $f(x, y)$ is continious on $[a, b] \times [c, d]$ then $\iint_R f(x, y)\delta A$ equals:

$$V = \int_a^b B(x)\delta x = \int_a^b \left[\int_c^d f(x, y)\delta y \right] \delta x \qquad B(x) = \int_c^d f(x, y)\delta x$$

$$V = \int_c^d A(y)\delta y = \int_c^d \left[\int_a^b f(x, y)\delta x \right] \delta y \qquad A(y) = \int_a^b f(x, y)\delta y$$