

Test 1 Prep
Semester 2

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March 1, 2025

Contents

1 Introduction 1

2 Double Integrals 1

2.1 Double Integrals in Rectangular Form 1

2.1.1 Fubini’s Theorem and Evaluating Double Integrals 2

2.1.2 The Very Nice Theorem 2

2.1.3 Double Integral Properties (over Rectangles) 2

2.1.4 Evaluating Double Integrals over Non-Rectangular Boundaries 2

2.1.5 Type 1 Shapes vs Type 2 Shapes 2

2.1.6 Integrating over Type 1 and Type 2 Bounds 3

2.1.7 Example 3

2.1.8 Helpful Extraneous Stuff 4

2.2 Double Integrals in Polar Form 4

3 Triple Integrals 4

3.1 Triple Integrals in Rectangular Form 4

3.1.1 Three Type of Regions of Space 5

3.2 Triple Integrals in Cylindrical Form 6

3.3 Triple Integrals in Spherical Form 6

4 Changing the Order of Integration 6

1 Introduction

The topics on the test are:

- Double Integrals in Rectangular and Polar form
- Triple Integrals in Rectangular form, Cylindrical and (maybe) Spherical form
- Changing the order of integration

2 Double Integrals

There are two main topics that we have to discuss regarding double integrals: first about rectangular and polar forms

2.1 Double Integrals in Rectangular Form

I’ll be copying stuff from the last test, cause this section is kinda simple.

2.1.1 Fubini's Theorem and Evaluating Double Integrals

Evaluating double integrals is like counting up loafs of bread, and Fubini's theorem says that no matter how one slices, you'll get the same amount of bread. You will have to take the slices (integrals over the contours) of either x or y, getting a area function given the other variable, and then integrate that over its range.

Thus, Fubini's theorem states that if $f(x, y)$ is continuous on $[a, b] \times [c, d]$ then $\iint_R f(x, y) \delta A$ equals:

$$\begin{aligned} V &= \int_a^b B(x) \delta x = \int_a^b \left[\int_c^d f(x, y) \delta y \right] \delta x & B(x) &= \int_c^d f(x, y) \delta y \\ V &= \int_c^d A(y) \delta y = \int_c^d \left[\int_a^b f(x, y) \delta x \right] \delta y & A(y) &= \int_a^b f(x, y) \delta x \end{aligned}$$

2.1.2 The Very Nice Theorem

Interesting way to evaluate integrals that can be separated into individual components.

$$\text{If } f(x, y) = g(x)h(y) \text{ and } R = [a, b] \times [c, d], \text{ then } \iint_R f(x, y) \delta A = \int_a^b g(x) \delta x \int_c^d h(y) \delta y$$

2.1.3 Double Integral Properties (over Rectangles)

- $\iint_R (f + g) \delta A = \iint_R f \delta A + \iint_R g \delta A$
- $\iint_R c f(x, y) \delta A = c \iint_R f(x, y) \delta A$
- If $f(x, y) \geq g(x, y)$, then $\iint_R f(x, y) \delta A \geq \iint_R g(x, y)$. (i.e. integrals preserve inequalities)

2.1.4 Evaluating Double Integrals over Non-Rectangular Boundaries

A big idea relating to evaluating double integrals over non-rectangular boundaries is how to find a bound, because it's more interesting/difficult to do so with these odd bounds. Thus we need to differentiate between type 1, type 2, and non-typical shapes.

2.1.5 Type 1 Shapes vs Type 2 Shapes

Type 1 Bound

Type one shapes have a $x \in [a, b]$ as boundaries, and are defined by two $y = g(x)$ functions.

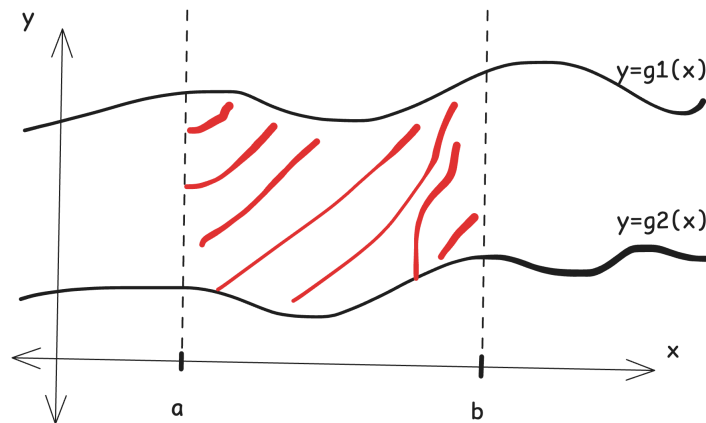


Figure 1: Example of a Type 1 bound

Type 2 Bound

Type 2 shapes have a $y \in [a, b]$ as boundaries, and are defined by two $x = h(y)$ functions

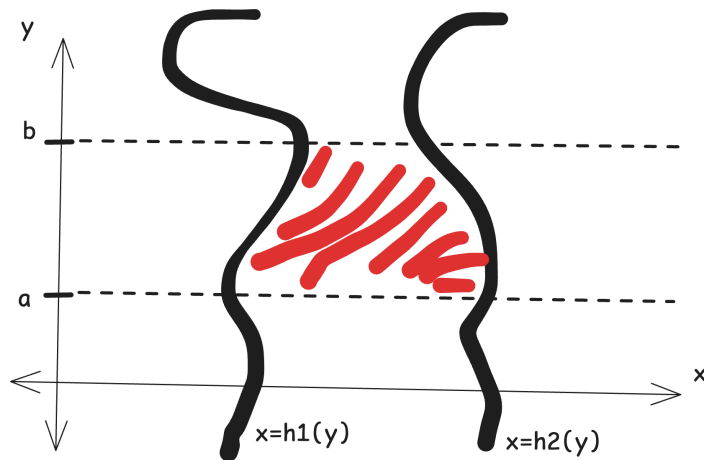


Figure 2: Example of a Type 2 Bound

NOTE: There are some surfaces that are BOTH type 1 and type 2 boundaries, such as a circle.

2.1.6 Integrating over Type 1 and Type 2 Bounds

There are *TWO OPTIONS*

- If the bounds are **functions of x**, or the base is a **type 1**,

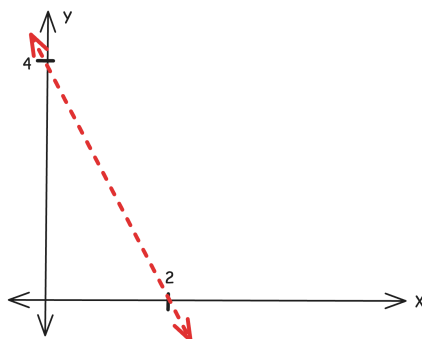
$$V = \iint_D f(x, y) \delta A = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y) \delta y \right] \delta x$$

- If the bounds are **functions of y**, or the base is a **type 2**,

$$V = \iint_D f(x, y) \delta A = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y) \delta x \right] \delta y$$

2.1.7 Example

Find the volume of $z = 8 - 4x - 2y$ over the boundary:



The bound is both a type 1 and a type 2 integral, but let's start with integrating as a type 1 integral.

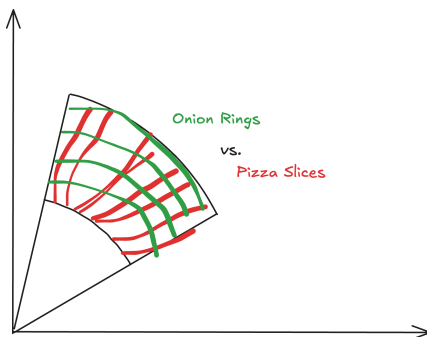
$$\begin{aligned} & \int_0^2 \left[\int_0^{-2x+4} f(x, y) \delta y \right] \delta x \\ &= \int_0^2 [8y - 4xy - y^2]_0^{-2x+4} \delta x \\ &= \int_0^2 [8(-2x+4) - 4(x)(-2x+4) - (-2x+4)^2] \delta x \\ &= \int_0^2 4x^2 - 16x + 16 \delta x = \left. \frac{4}{3}x^3 + 8x^2 + 16x \right|_0^2 \\ &= \boxed{32/3} \end{aligned}$$

2.1.8 Helpful Extraneous Stuff

- $\delta x \delta y$ refers to a horizontal distance, while $\delta y \delta x$ refers to a vertical distance
- To switch between δx and δy , one has to look at the resulting bound to try to switch if the region is both a type 1 and a type 2

2.2 Double Integrals in Polar Form

Double Integrals are used a lot, but for stuff like circles that can get unyieldly. To fix, you can use double integrals in terms of polar. Essentially, you convert the formula through changing x, y, and the bounds.



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\iint f(x, y) \delta A = \iint f(r \cos \theta, r \sin \theta) r \delta r \delta \theta$$

Some restrictions

- $0 \leq \theta < 2\pi$
- r has to be positive.

3 Triple Integrals

3.1 Triple Integrals in Rectangular Form

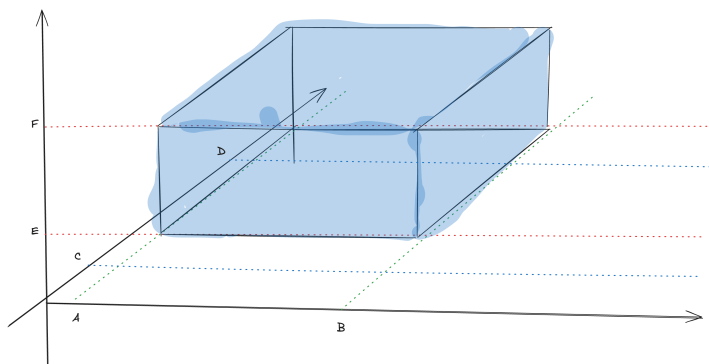


Figure 3: Three Dimensional Object

To integrate over three dimensional object, one can use triple integrals.

$$\iiint_E f(x, y, z)$$

Think about it as little cubes, with

$$\lim_{m, n, l \rightarrow \infty} \sum_{k=1}^l \sum_{j=1}^m \sum_{i=1}^n f(y_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta x \Delta y \Delta z$$

3.1.1 Three Type of Regions of Space

Type 1

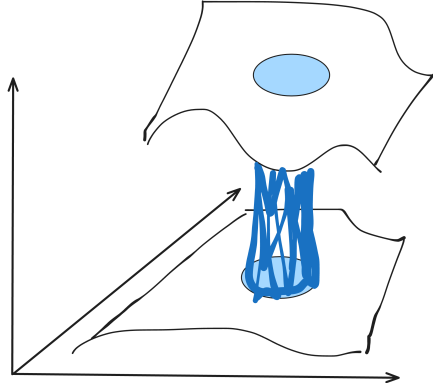


Figure 4: Example of a type 1 surface

$$\iiint_E f(x, y, z) \delta V = \iint_D \int_{g_2(x, y)}^{g_1(x, y)} g_2 - g_1 \delta z \delta A$$

Type 2

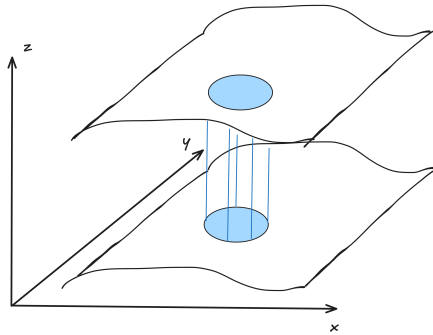


Figure 5: Example of a type 2 surface

$$\iiint_E f(x, y, z) \delta V = \iint_D \int_{h_2(x, y)}^{h_1(x, y)} h_2 - h_1 \delta y \delta A$$

Type 3

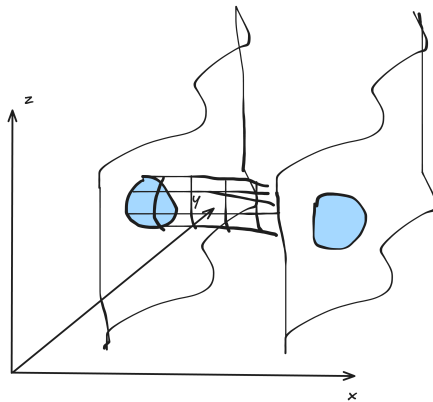


Figure 6: Example of a type 3 surface

$$\iiint_E f(x, y, z) \delta V = \iint_D \int_{k_2(x, y)}^{k_1(x, y)} k_2 - k_1 \delta x \delta A$$

3.2 Triple Integrals in Cylindrical Form

3.3 Triple Integrals in Spherical Form

4 Changing the Order of Integration

To change the bounds of an integral, all you do is to try to visualize the graph of the function, and then change the integrands to that.

Example

$$\int_0^1 \int_0^7 \delta x \delta y = \int_0^1 \int_x^1 f(x, y) \delta y \delta x$$