# Test 1 Prep
## Semester 2

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# 1 Introduction

Here are Multi Notes to prepare for the first test

# 2 Double Integrals

Double integrals are integrals that represent the volumes under a surface, rather than **Definite Integrals**, which compute the area under a curve.
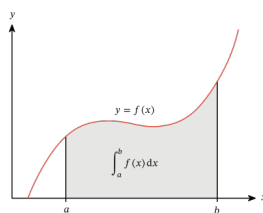
## 2.1 Definite Integrals Recap



Figure 1: Example of a definite Integral

Integrals are limits of a Riemann sum, which can be represented by the following equation:

$$\int_a^b f(x)\delta x = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + ... = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x_i \text{ ,where } \max \Delta x_i \to 0$$

## 2.2 Applying Definite Integrals to double integrals
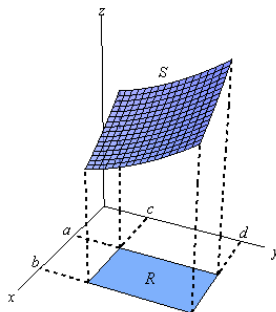


Figure 2: Example of a double Integral

Given that we have to find the area underneath the curve of a function $z = f(x, y)$, we will need to find **volume**, not area. We're going to define the rectangle area by:

$$R = \{(x,y)|a \le x \le b, c \le y \le d\} = [a,b] \times [c,d] \text{ (Cartesian Product)}$$

And thus we can define a solid $S$ using the equation:

$$S = \{(x, y, z)|(x, y) \in \mathbb{R}, 0 \le z \le f(x, y)\}$$

If one is to use equal subintervals to calculate, then you can use a double Riemann sum, with the area of a rectangle equal to $\Delta x \times \Delta y \times f(x, y)$. You can represent that sum, with

$$\lim_{m,n \to \infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta A$$

(**NOTE** the *'s mean that the values are at a specific point) Where m and n represent the number if subintervals. Double Riemann sums give volume, which when "limited" simplifies to

$$\iint_R f(x, y)\delta A = \iint_{[a,b] \times [c,d]} f(x, y)\delta A$$

## 2.3 Notes

1. If $f(x, y) \ge 0$ and $f(x, y) \in \mathbb{R}$ then $\iint_R f(x, yA) =$ is the volume of the solid bounded by x, y, and the surface

2. $\delta A = \delta x \delta y$ or $\delta y \delta x$. Order will impact evaluation but not result.

3. If $f(x, y)$ is continious in $R$, then the double integral exists. (**NOTE**: You can get away with some discontinuity, such as with steps and the like, but not with stuff like infinite behavior)

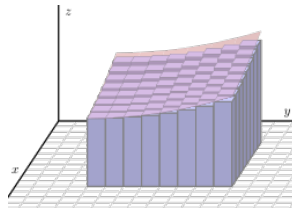# 3 Evaluating Double Integrals

## 3.1 Riemann Sums



Figure 3: Riemann Sums over Double Integrals

Just as with definite integrals, you can use Riemann-esque sums to estimate the area underneath a curve.

**Example Question**: Estimate $\iint_R 3x - y^2 \delta A$, where $R = [0, 2] \times [0, 2]$ using a double Riemann sum using the midpoint of each rectangle, with $m, n = 2$.
To solve, create a grid of a rectangle, and calculate $f(x^*, y^*)$, and multiply that for each rectangle by the $\Delta x$ and $\Delta y$.
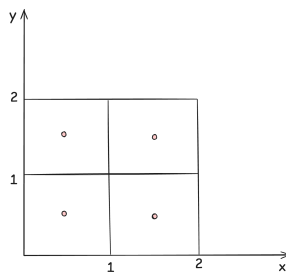


Figure 4: Example rectangle, with the points showing the $f(x^*, y^*)$ that is needed to be calculated

Once you have the graph, actually solving is relatively straightfoward:

$$A \approx f(0.5, 0.5)\Delta x \Delta y + f(1.5, 0.5)\Delta x \Delta y + f(0.5, 1.5)\Delta x \Delta y + f(1.5, 1.5)\Delta x \Delta y$$

$$= 1.25 \times 1 \times 1 - 0.75 \times 1 \times 1 + 4.25 \times 1 \times 1 + 2.25 \times 1 \times 1 = \boxed{7}$$

There are other ways such a question can be phrased, but the formula/idea should be similar.

## 3.2 Fubini's Theorem and Evaluating Double Integrals

Evaluating double integrals is like counting up loafs of bread, and Fubini's theorem says that no matter how one slices, you'll get the same amount of bread. Y will have to take the slices (integrals over the contours) of either x or y, getting a area function given the other variable, and then integrate that or its range.

Thus, Fubini's theorem states that if $f(x,y)$ is continious on $[a,b] \times [c,d]$ then $\iint_R f(x,y)\delta A$ equals:

$$V = \int_a^b B(x)\delta x = \int_a^b \left[ \int_c^d f(x,y)\delta y \right] \delta x \qquad\qquad B(x) = \int_c^d f(x,y)\delta x$$

$$V = \int_c^d A(y)\delta y = \int_c^d \left[ \int_a^b f(x,y)\delta x \right] \delta y \qquad\qquad A(y) = \int_a^b f(x,y)\delta y$$

# 4 The Very Nice Theorem

Interesting way to evauate integrals that can be separated into individual components.

$$\text{If } f(x,y) = g(x)h(y) \text{ and R} = [a,b] \times [c,d], \text{ then } \iint_R f(x,y)\delta A = \int_a^b g(x)\delta x \int_c^d h(y)\delta y$$

# 5 Double Integral Properties (over Rectangles)

- $\iint_R (f+g)\delta A = \iint_R f\delta A + \iint_R g\delta A$
- $\iint_R cf(x,y)\delta A = c\iint_R f(x,y)\delta A$
- If $f(x,y) \geq g(x,y)$, then $\iint_R f(x,y)\delta A \geq \iint_R g(x,y)$. (i.e. integrals preserve inequalities)