Test 3 Prep
Semester 2

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1 Introduction

The topics for the test are:

- Refresher on Spherical and Cylindrical coordinates
- Refresher on Double and Triple Integrals
- The Jacobian
- Line Integrals
- FTC for line integrals
- Vector Fields
- Line integrals of Vector functions
- Conservative Vector Fields (esp. Tests for this stuff)
- Green's Theorem
- DIV and CURL

2 The Jacobian and Converting Between Coordinate Planes

For double integrals:

$$\iint_R f(x,y)\delta x\delta y = \iint_S f(x(u,v),y(u,v))\frac{\delta(x,y)}{\delta(u,v)}\delta u\delta v$$

$$\frac{\delta(x,y)}{\delta(u,v)} = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix}$$

3 Variable Jacobian:

$$\frac{\delta(x,y,z)}{\delta(u,v,w)} = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} & \frac{\delta x}{\delta w} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} & \frac{\delta y}{\delta w} \\ \frac{\delta z}{\delta u} & \frac{\delta z}{\delta v} & \frac{\delta z}{\delta w} \end{vmatrix}$$

3 Line Integrals

The area that can be taken under the cross section of a surface by a path.

3.1 W/R to dt

$$\int_C f(x,y)\delta s$$

δs represents a small arc length, and thus can be represented through a parameterization (in this example given a 2D function)

$$\delta s = \sqrt{(x'(t))^2 + (y'(t))^2}\delta t$$

thus

$$\int_{t_0}^{t_1} f(x,y)\delta s = \int_{t_0}^{t_1} f(x,y)\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\delta t$$

3.2 W/R to dx and dy

Projecting a path on the x or y axis rather than over the entire curve
Theorem 1: If $\vec{r}(t) =< x(t), y(t) >$ represents a smooth parameterization of C, then:

$$\int_C f(x,y)\delta x = \int_{t=a}^{t=b} f(x(t),y(t)) \cdot x'(t)\delta t$$
$$\int_C f(x,y)\delta y = \int_{t=a}^{t=b} f(x(t),y(t)) \cdot y'(t)\delta t$$

4 Vector Fields

A vector field is a function that can assign a vector to any point. Very similar but not completely equal to a gradient.
Calculating the "work" done by a vector field:

$$\int_c \vec{F}\vec{T}\delta s = \int_c \vec{F}\frac{\vec{r}(t)}{\|\vec{r'}(t)\|}\|\vec{r'}(t)\|\delta t = \int_c \vec{F} \cdot \vec{r'}(t)\delta t$$

4.1 Conservative Vector Fields

Conservative vector fields are always path independent, additionally they are the gradient of a function. One can use Clairaut's to find it.
Theorem: If $\vec{F}(x,y) =< P(x,y), Q(x,y) >$ is a vector field defined on an open connected set, D and if F is path independent in D then F is conservative in D.
That means that you can find a line integral by finding the original function and then plugging the beginning and end points.

4.2 Green's Theorem

(Equivalent to FTC)
Let C be a piecewise, sooth, positively oriented simple closed curve and let D be the region enclosed by C and supposed that P(x, y) and Q(x, y have continuous) partials on an open region containing D. Then

$$\int_C P(x,y)\delta x + Q(x,y)\delta y = \iint_D \frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\delta A$$

5 DIV and CURL

CURL os a measure the tendency of particles to rotate at a point. Think of it as measuring the spin of a top. ($F(x, y, z) =<$ $P(x, y, z), Q(x, y, z), R(x, y, z))$

$$\text{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta x & \delta y & \delta z \\ P & Q & R \end{vmatrix}$$

DIV is a difference of the flux leaving / entering a point.

$$\text{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F} = \delta x P + \delta y Q + \delta z R$$

For both, if conservative equal zero.