On The Branches of 3d $\mathcal{N} = 4$ Quiver Gauge Theories

Bachelor of Science Project

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Abstract

We use quivers to study the moduli spaces of supersymmetric field theories. Implementing tools from algebraic geometry, we use the Hilbert Series to count gauge invariant operators charged under the various symmetries of supersymmetric Lagrangians. We compare the Coulomb branch and the Higgs branch of these theories, and employ the monopole formula and Molien-Weyl projection respectively to compute the Hilbert Series for a range of quivers. The framework introduced provides a way to uncover the properties of the moduli space, such as the dimension and the global symmetries.

Declaration

All of the work was completed together with my project partner. All computations and calculations were worked on together, and we both contributed equally to researching the topic.

Acknowledgements

I would like to thank my supervisor, Amihay Hanany, for giving us the opportunity to undertake such an interesting project in a relatively advanced and technically challenging field. His guidance throughout the project has been inspiring, and I have the confidence to continue with theoretical physics for the foreseeable future. I am grateful to Torben Skrzypek for his early help in putting us on the right track with how to approach the project. Particular thanks to Guhesh Kumaran, for his continued support throughout the project, always being willing to meet and explain complex ideas and unfamiliar topics. Finally, many thanks to my project partner, for both his work towards this project, and for making it a very interesting and enjoyable experience.

"If you don't know what to do, just do something"

Dmitry Turaev, Imperial College London

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Chapter 1

Introduction

Symmetry underpins all of modern physics. The Standard Model (SM) of particle physics is built from the symmetry group $SU(3) \times SU(2) \times U(1)$. Einstein's theory of General Relativity is based on the equivalence principle [1], a symmetry of how we view forces, namely gravity, and non-inertial frames of reference. Special Relativity is described by the Poincaré group, which determines the symmetries under Lorentz transformations and space-time translations [2]. More recently, the Higgs boson, responsible for giving particles mass, does so via a mechanism known as "spontaneous symmetry-breaking"[3]. Based on these past triumphs, it seems natural to develop a theory by exploiting the properties of symmetries.

Over the past century, modern physics has been built from two theories: Quantum mechanics, describing the very small; and General Relativity, describing the very large. They are very successful in their own rights, with experiments matching theoretical models and predictions. However, the two theories are incompatible with each other. Physicists hope that a "Grand Unified Theory" can be found, reconciling the two at a fundamental level. Ultimately, this aims to unify gravity with the other three fundamental forces, in a similar manner to Abdus Salam, who, along with Sheldon and Glashow, developed a theory of Electroweak Unification [4], which combined electromagnetism with the weak nuclear force. This was a significant leap in creating the SM, and many more developments have been made since. However, there are still multiple discrepancies between the SM and experiments. It is an incomplete description, so we must probe further and develop new theories.

Supersymmetry (SUSY) is a prominent topic within recent popular science literature. SUSY is based on the idea that every elementary particle has a "superpartner" which is of the opposite particle type: fermions have bosonic superpartners and vice versa.

At this moment in time, no superpartners have been observed experimentally, though we may have expected to have seen them in the LHC given their masses. However, this does not disprove the theory as the supersymmetry is broken at these energies, and even higher energies would be required to produce these in particle interactions. Symmetry breaking is not uncommon; electroweak symmetry was broken shortly after the Big Bang, and so we see electromagnetism and the weak force as separate entities. Whether or not SUSY is found to be correct, these theories are nonetheless useful to us in our understanding of field theories, as we introduce new techniques and mathematics which will undoubtedly be useful in developing alternative models.

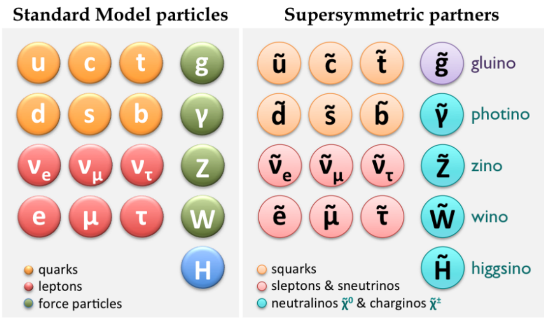

Figure 1.1: The current standard model particles, along with their superpartners. [5]

It also aids in trying to find ways to explain phenomena that are known to be inherent in our world. For example, QCD fails to give a definitive reason for quark confinement; it is simply something that falls out of the theory, which we confirm through simulations and experiments. However, Supersymmetric Quantum Chromodynamics (SQCD) provides a mechanism and predicts many related features accurately. SUSY is also related to ideas in string theory, with newer models like superstring theory and theories of supergravity (SUGRA). These aim to fix our understanding of gravity in the context of a more unified theory. Crucially, most string theories actually predict supersymmetry!

We see that supersymmetry is rich in novel ideas and new directions to advance modern physics.

In this project, we have investigated the properties of the moduli spaces of 3d $\mathcal{N} = 4$ Quiver gauge theories. The goal of this was to gain a greater understanding of supersymmetric field theories. Additionally, this was taken as an ideal opportunity to develop our knowledge and appreciation of key topics in applied mathematics and theoretical physics, such as Lie algebras, algebraic geometry, differential geometry, and more general ideas in field theories. To uncover properties of SUSY theories, we use quivers to represent the symmetries of Lagrangians. These are diagrams which allow us to easily see the gauge and global symmetries of a theory. To discover the structure of the moduli spaces of these theories, we use the Hilbert Series (HS), a tool from the field of algebraic geometry, to count gauge invariant operators (GIOs). Properties such as the dimension of the moduli space, the relations of its generators, and its global symmetry are determined. We consider the natural branching of the moduli space into the Coulomb and Higgs branches. The differences between these leads us to following different approaches to compute the HS on each branch. We implement the Molien-Weyl projection on the Higgs branch to count GIOs in a classical manner. On the Coulomb branch, quantum corrections lead us to using the Monopole formula. This project has revolved around computing the HS for a range of quivers on both branches, and learning newer techniques to extract information from these.

The structure of the report is as follows:

- Chapter 2 provides background information and mathematical preliminaries. We introduce the basics of the framework of SUSY, and the ideas of moduli spaces. We then move on to the tools of algebraic geometry.
- Chapter 3 introduces quivers, a pictorial way of representing SUSY Lagrangians which aid us in calculations and understanding symmetries of a theory.
- Chapter 4 considers the Coulomb branch of the moduli space, where we implement the Monopole formula to compute the HS for multiple quivers. We then probe the structure of the moduli space to determine its properties. This discussion allows us to understand the physical consequences of our calculations.
- Chapter 5 moves on to the Higgs branch. A more classical approach to computing the HS with the Molien-Weyl Projection is discussed.
- Appendix A introduces Lie algebras, the mathematical framework for describing symmetries and representations.

This report assumes a working knowledge of Lie algebras. Key results are summarised in Appendix A, though the reader is advised to consult: [6] & [7]. Familiarity with QFT is useful for understanding many of the ideas, though is not required. The mathematical background to supersymmetry presented is a small scratch on the surface, and skips over material that is not immediately relevant to the report. A full discussion can be found in [8] & [9].

Chapter 2

Background Material

2.1 Supersymmetry

In this section, we introduce the key ideas of supersymmetry with a more mathematical description of how superpartners arise and the consequences of this.

We begin with the Poincaré algebra, which describes the symmetries of spacetime. This algebra generates the Poincaré group.

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, P_\rho] &= i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho}\right) \end{aligned} \tag{2.1}$$

Where P^μ are the generators of spacetime translations, $M_{\mu\nu}$ generate Lorentz transformations (rotations and boosts), and $\eta_{\mu\nu}$ is the metric tensor.

2.1.1 Super-Poincaré Algebra

For a theory to be compatible with special relativity, it must adhere to the rules of the Poincaré algebra, so we use this as our foundation to build a new theory. We introduce operators, Q, which act on particles as so:

$$Q\left|boson\right\rangle = \left|fermion\right\rangle; Q\left|fermion\right\rangle = \left|boson\right\rangle. \tag{2.2}$$

We call these operators supercharges, which come in pairs. When we refer to an $\mathcal{N} = 1$ theory, this means we have 1 pair of supercharges. Our theories differ depending on the number of supercharges, where we say that a greater $\mathcal{N}$ has "more" supersymmetry. This allows us to provide a mathematical representation of how the superpartners can come to be. We note that these operators change the spin, from integer to half-integer or the reverse, and so must themselves carry spin. The supercharges are in fact Weyl Spinors. The majority of the framework is built on 4d $\mathcal{N} = 2$ theories. It turns out that 3d $\mathcal{N} = 4$ theories are reached from this by "dimensional reduction", so we will use $\mathcal{N} = 2$ and $\mathcal{N} = 4$ interchangeably for now.

To see how these superpartners might behave, we need to determine how our supercharges interact with the Poincaré generators. We extend to the Super-Poincaré algebra by promoting the supercharges to generators and determining their commutator relations. We give the extension of the algebra for $\mathcal{N}=1$ as an example in (2.3) [10].

$$\begin{aligned} [M^{\mu\nu}, Q_\alpha] &= \frac{1}{2}(\sigma^{\mu\nu})^\beta_\alpha Q_\beta \\ [Q_\alpha, P^\mu] &= 0 \\ \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \end{aligned} \tag{2.3}$$

Where:

$$(\sigma^{\mu\nu})^\beta_\alpha = \frac{i}{4}(\sigma^\mu \bar{\sigma^\nu} - \sigma^\nu \bar{\sigma^\mu})^\beta_\alpha, \tag{2.4}$$

is the antisymmetrised product of the Pauli matrices. (This set of relations combined with (2.1), forms the full algebra.)

In quantum mechanics, we are familiar with multiplets being a set of states within a representation (think of the 2j+1 states in the j^{th} spin representation). These are built from the $\mathfrak{su}(2)$ Lie algebra. We can now use our Super-Poincaré algebra to construct states in an analogous way. Due to the chirality of the supercharges, we can build two different types of representations, called hypermultiplets (hplets) and vector multiplets (vplets). We will be interested in the representations under which these transform under given symmetries.

2.1.2 Moduli Spaces

The Standard Model of particle physics is governed by the Higgs field. This field, manifested by the Higgs boson, gives mass to particles by a mechanism known as symmetry-breaking. Without the Higgs, all particles would have the same mass; but the Higgs breaks this symmetry and forces different particles to have different masses. The interesting thing about the Higgs field is that it has a non-vanishing potential in a vacuum [3]. This may seem odd, as this is not expected from our experience with electromagnetism, for example. We know that the electric field is zero in a vacuum, and non-zero if we introduce a charge source. We refer to the field values in a vacuum as “Vacuum Expectation Values” (VEV). The non-vanishing VEV of the Higgs field explains the SM as we know it, where experiments match theory incredibly well. It is therefore an interesting task when building a field theory, to investigate vacuum states and the resulting fields.

As in ordinary QFT, we can describe our system with a Lagrangian, involving kinetic and potential terms as functions of the underlying fields. SUSY theories are based on “superfields”, and the Lagrangian includes a “superpotential” which is given by the interaction of these superfields. A moduli space exists when the scalar potential vanishes, and this will correspond to making the superpotential stationary. The representations of these superfields will be crucial later, as the superfields arise from the matter content of the vplets and hplets. The moduli space is parameterised by the VEVs, and can be defined as the space of inequivalent vacua. Mathematically, we can naturally treat these as algebraic varieties, or geometrically as manifolds. Our theories will involve a stationary superpotential as a result of contributions from the hplets or the vplets. We define the Higgs branch and Coulomb branches as the two separate spaces of solutions, and hence we branch the moduli space: $\mathcal{M} = \mathcal{M}_C \cup \mathcal{M}_H$. For 3d $\mathcal{N} = 4$ theories, these branches are both hyperkähler manifolds[1].

The Higgs branch is parameterised by scalars from the vplets, (i.e the contributions from the vplets vanish) while the Coulomb branch is parameterised by scalars from hplets. The two branches meet at the origin, where contributions from both the hplets and vplets vanish.

The key distinction between the two branches is due to a term in the Lagrangian which couples to the vplet contributions. This leads to quantum effects for the Coulomb branch, where the vplets don't vanish, unlike the Higgs branch which remains classical in this sense. When we move on to performing calculations, we will have to treat the two branches very differently as a consequence of this.

2.1.3 R-Symmetry

In general, the internal symmetries of a theory must commute with the symmetries of the Poincaré group. However, given that we have extended this to the Super-Poincaré group, the supercharges must be accounted for. For $\mathcal{N} = 1$ theories, we permit an internal U(1) symmetry which doesn't commute with the supercharges; this acts as so:

$$Q_\alpha \to e^{-i\lambda} Q_\alpha \quad \text{and} \quad \bar{Q}_{\dot{\alpha}} \to e^{i\lambda} \bar{Q}_{\dot{\alpha}} \tag{2.5}$$

This U(1) symmetry is called the R-symmetry, denoted by $U(1)_R$ with generator R. We then admit the following commutation relations:

$$[R, Q_\alpha] = -Q_\alpha \quad \text{and} \quad [R, \bar{Q}_{\dot{\alpha}}] = +\bar{Q}_{\dot{\alpha}} \tag{2.6}$$

For $\mathcal{N} > 1$, we instead have an SU($\mathcal{N}$) R-Symmetry. In our cases, the SU(4) symmetry decomposes and we only focus on an SU(2) R-symmetry. This will contribute to the global symmetry, and we aim to investigate operators charged under this symmetry.

2.2 Hilbert Series

The Hilbert Series is a mathematical tool from the field of algebraic geometry. It counts holomorphic functions on an algebraic variety by grading them by degree. In our case, we will be using them to count how many chiral ring operators there are at each charge under the global symmetries.

Definition 1. For an algebraic variety χ in $\mathbb{C}[x_1, \ldots, x_n]$ the Hilbert Series is given by [12]:

$$HS_\chi(t) = \sum_{i=0}^{\infty} dim_{\mathbb{C}}(\chi_i) t^i, \tag{2.7}$$

where χ_i is the i^{th} graded piece of the polynomial ring, and t is called a fugacity. This is a dummy variable used for counting.

[1]More on hyperkähler geometry can be found in [11].

It is a difficult task to determine what the chiral operators actually are, so we use the HS for achieving a simpler task of counting GIOs. We will then be able to find more about these GIOs and their properties. This in turn will provide a mathematical description of the moduli space.

2.2.1 HS for $\mathbb{C}^2/\mathbb{Z}_2$

We start with a simple example to avoid the abstraction of (1) and to explain how the HS is able to count with the fugacity t. Considering the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, we look at pairs of complex numbers, $(z_1, z_2) \in \mathbb{C}^2$, under the group action of $\mathbb{Z}_2$, i.e $z \mapsto \pm z$. The idea is to find combinations that are invariant and grade them by their degree with the exponent of t.

Degree	Invariant Monomials	Number of Monomials
0	1	1
1	...	0
2	z_1z_2,z_1^2,z_2^2	3
3	...	0
4	$z_1^2z_2^2$, $z_1^3z_2$, $z_2^3z_1$, z_1^4, z_2^4	5

Table 2.1: Summary of invariant monomials.

It is clear that invariant monomials on the polynomial ring must be of even degree, so for a monomial $z_1^pz_2^q$, p+q must be even. At even degrees m, there are m+1 independent invariant monomials, so using (1) we compute:

$$HS_{\mathbb{C}^2/\mathbb{Z}_2} = \sum_{m=0}^{\infty}(2m+1)t^{(2m)} = \frac{1-t^4}{(1-t^2)^3}. \tag{2.8}$$

We can see the use of a fugacity as a dummy variable now; the coefficient of t^m gives the number of invariant monomials at degree m. In general, a HS will be in the form of the quotient of two polynomials in t:

$$HS = P(t)/Q(t). \tag{2.9}$$

An algebraic variety is "freely generated" if P(t) =1 and Q(t) = $\prod_{i=1}^{\infty}(1-t_i^n)$ with $n_i \in \mathbb{Z}^+$. Alternatively, an algebraic variety is a "complete intersection" if Q(t) = $\prod_{i=1}^{\infty}(1-t_i^n)$ and P(t) = $\prod_{i=1}^{\infty}(1-t_i^m)$ [13].

Theorem 1. The complex dimension of a moduli space is given by the order of the pole of the HS at t=1 [14].

2.3 Plethystics Program

We have seen that there is some information encoded in the HS, for example the dimension of the moduli space. A range of tools and techniques have been developed to uncover other hidden properties. We introduce the plethystics program [15] to extract further information about the generators and their relations on a moduli space.

2.3.1 Plethystic Logarithm

Definition 2. The plethystic logarithm (PL) of a HS is given by:

$$PL[f(t, z_1, \ldots, z_n)] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} log(f(t^k, z_1^k, \ldots, z_n^k)), \tag{2.10}$$

where $\mu(k)$ is the Möbius function, defined for positive integers, k, as:

$$\mu(k) = \begin{cases} +1 & \text{if k is square free with an even number of prime factors,} \\ \text{-1} & \text{if k is square free with an odd number of prime factors,} \\ 0 & \text{if k has a squared prime factor,} \end{cases}$$

We note now that from this expression and our previous definitions that for a freely-generated HS, the PL is finite and contains positive coefficients only. Similarly, a complete intersection has a finite PL but with both positive and negative coefficients.

The PL provides information about generators and their relations, which is best shown by continuing with our HS for $\mathbb{C}^2/\mathbb{Z}_2$. We take the PL of our expression in (2.8):

$$PL[\frac{1-t^4}{(1-t^2)^3}] = 3t^2 - t^4. \tag{2.11}$$

The positive term tells us there are three generators at degree two (namely $z_1^2, z_2^2, z_1 z_2$). The negative term tells us that the generators satisfy one relation at degree four (namely $z_1^2 \cdot z_2^2 = (z_1 z_2)^2$). We call relations of this type "syzygies". In general we can always extract this information about a moduli space from the PL of the HS in the same manner. We note that the HS (2.8) is a complete intersection. Additionally, We see the origin of the term "freely-generated": there are no syzygies so the generators aren't constrained, and hence "free".

2.3.2 Plethystic Exponential

The PL has an analytic inverse which is used to count symmetric products [16].

Definition 3. The plethystic exponential (PE) is given by:

$$PE[f(t, z_1, \ldots, z_n)] = exp(\sum_{k=1}^{\infty} \frac{f(t^k, z_1^k, \ldots, z_n^k) - f(0, \ldots, 0)}{k}), \tag{2.12}$$

for a function f, satisfying f(0,. . . ,0) = 0.

We enjoy the property:

$$PE[f(t) + g(t)] = PE[f(t)]PE[g(t)], \tag{2.13}$$

as with ordinary exponentials. For convenience, we note from the definition of PE that:

$$PE[at^p - bt^q] = \frac{(1-t^q)^b}{(1-t^p)^a}. \tag{2.14}$$

As we will see later when we examine the Higgs branch, the ability to count symmetric products will be extremely useful.

Symmetric Products

When we take the tensor product of a space with itself, we combine the basis elements in ordered pairs to form a new basis. We will naturally double count several elements in this case, which we don't require. We therefore separate these products into a symmetric part (where we don't include repeated products) and an antisymmetric part, Λ, (consisting fully of just the parts that would've been double counted). Our tensor product is therefore split into the second symmetric product and the second antisymmetric product [17]:

$$R \otimes R = Sym^2(R) + \Lambda^2(R). \tag{2.15}$$

We take a simple example of a space of generators: the set of monomials, R = {a,b,c}. We can generate products of this set at a given degree, say 2^{nd}, by taking the tensor product of R with itself. We generate the two sets $\{a^2, ab, ac, b^2, bc, c^2\}$ & $\{ba \equiv ab, ca \equiv ac, cb \equiv bc\}$. We have evidently separated $R \otimes R$ into the symmetric and antisymmetric parts.

The PE generalises this to more complicated generators than simple monomials. We notice that the symmetric product of monomials at each degree can be found in the following product:

$$\frac{1}{1-a}\frac{1}{1-b}\frac{1}{1-c} = (1+a+a^2+a^3+\dots)(1+b+b^2+b^3+\dots)(1+c+c^2+c^3+\dots). \tag{2.16}$$

A simple rearrangement to group terms by their degree justifies our decision.

$$1+(a+b+c)+(a^2+b^2+c^2+ab+ac+bc)+(a^3+b^3+c^3+a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+abc)+\dots \tag{2.17}$$

At each degree k, we have generated the k^{th} symmetric product of {a,b,c}.

We now aim to generalise this process for an arbitrary set of generators or function. Beginning with the fact that $\frac{1}{1-x}$ has indicated a way to produce symmetric products, we continue:

$$\prod_{i=1}^{n}\frac{1}{1-x_i} = \exp(\log\prod_{i=1}^{n}\frac{1}{1-x_i}) = \exp(-\sum_{i=1}^{n}\log(1-x_i)) = \exp(\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{x_i^k}{k}) \tag{2.18}$$

Given that we started with an arbitrary x_i as a generator, we can replace this with a function $f(t_1,\dots,t_n) : \mathbb{R}^n \mapsto \mathbb{R}$, provided it vanishes at the origin; this results in (2.12). Given that the PE is the inverse of the PL, it can also simply be used to summarise information more succinctly. By writing the HS as the PE of its PL. We see that for $\mathbb{C}^2/\mathbb{Z}_2$, it is simpler to write:

$$HS_{\mathbb{C}^2/\mathbb{Z}_2} = PE[3t^2 - t^4], \tag{2.19}$$

instead of the full expression (2.8). This is especially useful when dealing with more complicated HS.

Chapter 3

Quivers

A Lagrangian for a SUSY theory is typically messy and complicated, with many terms obscuring the underlying symmetries and the consequent physics. We instead compact the information of the global and gauge symmetries in a “quiver” diagram [18]. These allow us to easily depict the representations under which the vplets and hplets transform, and hence we can see the effect on the moduli spaces which they parameterise.

For $\mathcal{N}$=2 theories, we construct these diagrams with a series of nodes connected with lines (edges). Circular nodes are used to represent gauge groups (and hence the gauge symmetries manifest in the Lagrangian). Square nodes give the flavour symmetries, which contribute to the global symmetry. The nodes depict the vplet contributions to the theory.

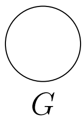

Figure 3.1: A single node for the gauge group G.

Edges in these diagrams denote the hplets. An edge between two nodes indicates that the hplet transforms in the bifundamental representation between those nodes. For example, we can depict a theory with a U(K) gauge symmetry and an SU(N) flavour symmetry, as seen in fig. 3.2.

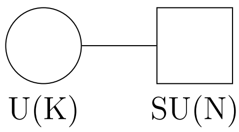

Figure 3.2: A gauge node connected to a flavour node by $N \times K$ hplets transforming in the bifundamental representation.

An edge that loops from a node back to itself indicates that the hplet transforms under the adjoint representation of that node.

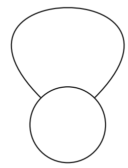

Figure 3.3: A node with a hplet transforming in the adjoint representation of the gauge group G.

We call quivers with only gauge nodes “unframed”, while “framed” quivers contain flavour nodes as well.

Chapter 4

Coulomb Branch

The study of the Coulomb branch is crucial in our understanding of gauge theories, as we improve our understanding of monopole operators and magnetic monopoles. This is another important task in theoretical physics, as the majority of modern gauge theories do in fact predict the existence of magnetic monopoles [19]. On the Coulomb branch, we suffer from quantum effects unlike the Higgs branch. We consequently must turn to the monopole formula to calculate the Hilbert Series.

4.1 Monopole Formula

We consider "monopole operators", and look at those that are gauge invariant. The monopole formula gives a way to compute the HS on the Coulomb branch [20].

$$HS_G(t,z) = \sum_{m \in \Gamma^{\star}_{\hat{G}}/\mathcal{W}_{\hat{G}}} z^{J(m)} t^{2\Delta(m)} P_G(t^2, m) \tag{4.1}$$

This expression is a sum over magnetic charges, m, which arise due to the monopole operators. Strictly, the sum is performed over charges in the Weyl Chamber, or over the weight lattice of the Langlands dual group modulo Weyl transformations, $\Gamma^{\star}_{\hat{G}}/\mathcal{W}_{\hat{G}}$. We use t as a fugacity to count R-charge, with a factor of 2 added for convenience; SU(2) irreps (irreducible representations) are labelled by half-integers, but it is preferable to count and work with integers. Another fugacity, z, is present too. This is due to a new symmetry which exists only on the Coulomb branch: topological symmetry. This is a hidden symmetry, as no terms in the Lagrangian are charged under it. Instead, there is a conserved current due to topological effects [21]. The topological charge is given by:

$$J(m) = \sum_{i=1}^{rank(G)} m_i. \tag{4.2}$$

$\Delta(m)$ is the conformal dimension which, for "good" and "ugly" theories, is the R-charge[1]. It is given by:

$$\Delta(m) = -\sum_{\alpha\in\Delta_+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{H}\sum_{\rho_i\in\mathcal{R}_i} |\rho(m)|, \tag{4.3}$$

where Δ_+ is the set of positive roots of the Lie algebra of the gauge group G, ρ_i is the weight of the representation $\mathcal{R}_i$ of the i^{th} hplet, and H is the number of hplets.

The first term in (4.3) is due to contributions from the vplts, while the second term is from the hplets. The final part of (4.1) is the "classical dressing factor", $P_G(t,m)$, given by:

$$P_G(t,m) = \prod_{i=1}^{r} \frac{1}{(1-t^{2i})^{\tilde{\lambda}_i(m)}}, \tag{4.4}$$

where $\tilde{\lambda}_i(m)$ are the Dual Young Tableaux. The dressing factor is added to ensure we count "dressed" rather than "bare" monopoles [20]. For multiple gauge groups, we find the dressing factor as the product of the individual dressing factors. That is, if $G = G_1 \times G_2 \times \cdots \times G_n$, then $P_G = P_{G_1}P_{G_2}\ldots P_{G_n}$.

4.2 Calculations

We now have the necessary tools and framework to compute the HS for specific quivers/theories.

4.2.1 U(1) with N Flavours

We begin with the quiver below, for a theory with a U(1) gauge symmetry and an SU(N) flavour symmetry. This in fact corresponds to the theory of SQED (Supersymmetric Quantum Electrodynamics).

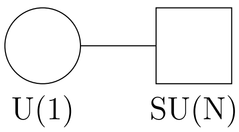

Figure 4.1: $\mathcal{N}=2$ quiver for a U(1) theory with N flavours.

We have a U(1) gauge group and an SU(N) flavour group, with N hplets transforming under the bifundamental representation. As U(1) has rank 1, there is one magnetic charge, m, to sum over.

We first consider $\Delta(m)$, given by (4.3). We note that U(1) is abelian, so has no root system. Therefore there is no vplet contribution. The fundamental representation of U(1) has a weight of 1. The N hplets therefore contribute:

$$\Delta(m) = \frac{N}{2}|m| \tag{4.5}$$

[1]We are only considering these, rather than "bad" theories. The distinction is based on the value of $\Delta(m)$ for certain operators. More on this can be found in [20].

With dressing factor:

$$P_{U(1)} = \frac{1}{1-t^2}, \tag{4.6}$$

we calculate:

$$HS_{U(1),N} = \frac{1}{1-t^2}\sum_{m\in\mathbb{Z}} z^m t^{N|m|}, \tag{4.7}$$

which evaluates to:

$$HS_{U(1),N} = \frac{1-t^{2N}}{(1-t^2)(1-zt^N)(1-z^{-1}t^N)}. \tag{4.8}$$

To determine the structure and properties of the moduli space, it is fruitful to choose N=2 for example, rather than continuing with a general N. The SU(2) flavour symmetry will be simpler and more insightful to investigate.

Hence for N=2, (4.8) becomes:

$$HS_{U(1),2}\frac{1-t^4}{(1-t^2)(1-zt^2)(1-z^{-1}t^2)} = 1+(z+1+\frac{1}{z})t^2+(z^2+z+1+\frac{1}{z}+\frac{1}{z^2})t^4+\dots \tag{4.9}$$

We perform the fugacity map $z \mapsto x^2$ using the Cartan Matrix of $\mathfrak{su}(2)$. As our fugacities are ultimately just dummy variables, we are free to do so.

$$HS_{U(1),2} = 1+(x^2+1+\frac{1}{x^2})t^2+(x^4+x^2+1+\frac{1}{x^2}+\frac{1}{x^4})t^4+\dots. \tag{4.10}$$

Now in a more suggestive form, we immediately recognise the coefficients of the t's as characters of SU(2).

$$HS_{U(1),2} = 1+[2]_{SU(2)}t^2+[4]_{SU(2)}t^4+\dots = \sum_{n=0}^{\infty}[2n]_{SU(2)}t^{2n} \tag{4.11}$$

The notation $[m]_{SU(2)}$ refers to the character of the representation of SU(2) with highest weight m. The form of (4.11) is known as a Character Expansion.

Theorem 2. The coefficient of t^2 in the unrefined HS is the character of the adjoint representation of the global symmetry [22].

The coefficient of t^2 for (4.11) is $[2]_{SU(2)}$, namely the character of the adjoint representation of SU(2), and hence this is the global symmetry of the moduli space. It is also interesting to see that this character originated from counting topological charge with z; the topological symmetry has been enhanced to a global symmetry. Theorem 2 is of particular use when the global symmetry is not immediately obvious from the quiver.

We can also "unrefine" the HS (so called as we lose information) by setting the z fugacities to 1, or equally by replacing the characters by their dimension.

$$HS^{unrefined}_{U(1),2} = \sum_{n=0}^{\infty} dim([2n]_{SU(2)})t^{2n} = \sum_{n=0}^{\infty}(2n+1)t^{2n} \tag{4.12}$$

This is sometimes performed to simplify expressions, when we are not interested in certain information for example. Crucially, we note that we have the same HS as (2.8). This means that our moduli space has 3 generators at degree 2 with a degree 4 syzygy. Geometrically, the moduli space is identical to the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. This is also shown by taking the PL of (4.8):

$$PL[HS_{U(1),N}] = -t^{2N} + t^2 + t^N(z + z^{-1}), \tag{4.13}$$

where the unrefined N=2 case matches (2.11). The information can also be encoded in the Highest-Weight Generating Function (HWG) [23]. Using Dynkin labels, we make the correspondence between characters and weights: $[n_1, n_2, \ldots, n_r] \longleftrightarrow \mu_1^{n_1}\mu_2^{n_2}\ldots\mu_r^{n_r}$ where the μ_i are new weight fugacities. We can therefore write $[2n]_{SU(2)}$ as μ^{2n}, yielding:

$$HWG_{U(1),2} = \sum_{n=0}^{\infty} \mu^{2n}t^{2n} = \frac{1}{1-\mu^2t^2}. \tag{4.14}$$

4.2.2 SU(2) with N flavours

We start with the quiver, and then calculate in the same manner as for U(1).

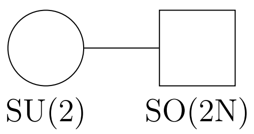

Figure 4.2: $\mathcal{N} = 2$ quiver for a SU(2) theory with N flavours.

For a SU(N) gauge group, we consider an SO(2N) flavour node. We calculate:

$$\Delta(m) = (N-2)|m|, \tag{4.15}$$

with the dressing factor:

$$P_{SU(2)}(t;m) = \begin{cases} \frac{1}{1-t^2} & \text{if m} = 0, \\ \frac{1}{1-t} & \text{if } m \neq 0, \end{cases} \tag{4.16}$$

There are two different cases for the dressing factor now, depending on the value of m. We can then calculate:

$$HS_{SU(2),N} = \frac{1}{1-t^2} + \frac{1}{1-t}\sum_{m=1}^{\infty} t^{(N-2)m} = \frac{1-t^{2N-2}}{(1-t^2)(1-t^{N-2})(1-t^{N-1})} \tag{4.17}$$

$$PL[HS_{SU(2),n}] = 1 + t^2 + t^{N-1} + t^{N-2} - t^{2N-2} \tag{4.18}$$

4.2.3 $\overline{s.regG_2}$

We now consider a more complicated quiver:

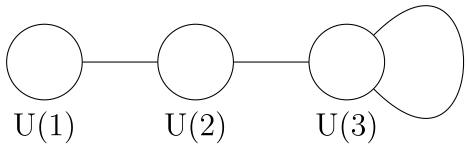

Figure 4.3: $\mathcal{N} = 2$ quiver for a $G = U(1) \times U(2) \times U(3)$ gauge theory.

We have the gauge group $G = U(1) \times U(2) \times U(3)$, with a crucial difference to our previous quivers: we now have a hplet transforming in the adjoint representation of U(3). When considering the conformal dimension, we can deal with each node separately and then bring the pieces together. We have 6 magnetic charges, as rank(G)=6; there are K magnetic charges associated with a U(K) node.

We must first ungauge [24] as our quiver is unframed[2], to do so we set any magnetic charge to 0 and transform it to a square node. It is natural to choose the U(1) node for this, as there is only one charge on this node. In the process, it becomes an SU(1) flavour node. Our quiver becomes:

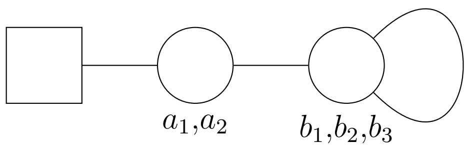

Figure 4.4: Ungauged quiver for the theory. The U(1) gauge node has become an SU(1) flavour node. We show the magnetic charges arising from the remaining unitary gauge nodes.

We have 5 magnetic charges now, with a's corresponding to the U(2) node and b's for the U(3) node. (We have used a's and b's, rather than just m's to make it clearer from where each magnetic charge arises.)

We calculate the conformal dimension with (4.3), noting that the contributions from the hplet in the adjoint representation cancels out the contributions from the U(3) vplet term.

$$\Delta(a_1, a_2, b_1, b_2, b_3) =$$
$$\frac{1}{2}(|a_1-b_1|+|a_1-b_2|+|a_1-b_3|+|a_2-b_1|+|a_2-b_1|+|a_2-b_3|+|a_1|+|a_2|)-|a_1-a_2|. \tag{4.19}$$

We must consider different cases for the dressing factors of U(2) and U(3), depending on whether the symmetries between the magnetic charges are broken or not.

[2]Without doing this, we would end up with infinities in the Monopole formula [25].

$$P_{U(2)} = \begin{cases} \frac{1}{(1-t^2)(1-t^4)} & a_1 = a_2, \\ \frac{1}{(1-t^2)^2} & a_1 < a_2, \end{cases} P_{U(3)} = \begin{cases} \frac{1}{(1-t^2)(1-t^4)(1-t^6)} & b_1 = b_2 = b_3, \\ \frac{1}{(1-t^2)^2(1-t^4)} & b_1 < b_2 = b_3 \text{ or } b_1 = b_2 < b_3, \\ \frac{1}{(1-t^2)^3} & b_1 < b_2, b_3, \end{cases} \tag{4.20}$$

We find the unrefined HS:

$$HS = \sum_{a_1 \leq a_2} \sum_{b_1 \leq b_2 \leq b_3} P_{U(2)} P_{U(3)} t^{2\Delta(m)}. \tag{4.21}$$

Given a much more challenging computation than previously, we turn to Mathematica [26]. Even then, we have 5 infinite series with 6 different cases (due to the possible combinations of the dressing factors) for each sum. The "perturbative approach" instead of direct calculation is therefore favourable. This involves performing the sums over a smaller number of terms, in hope of extracting the pattern of higher order terms and determining a closed form expression like (2.9). Luckily, there are a few more tricks which will allow us to know when we have found the answer in these cases.

Theorem 3. The dimension of the Coulomb branch is twice the rank of the gauge group [14].

Theorem 4. On a Calabi Yau surface, the numerator of the unrefined HS is always palindromic. The hyperkähler manifolds used also display this property [27].

The dimension of the moduli space, which can be read straight from the ungauged quiver, is 10. We compute the HS to order 10 (Summing from -10 to 10 on each sum), in order to exploit the palindromic property.

$$HS = 1 + 14t^2 + 104t^4 + 539t^6 + 2184t^8 + 7378t^{10} + \mathcal{O}(t^{12}) \tag{4.22}$$

We know from the dimension that the denominator of the closed form expression must be $(1-t^2)^{10}$. We hence multiply this by (4.22) to find the numerator, giving:

$$P(t) = 1 + 4t^2 + 9t^4 + 9t^6 + 4t^8 + t^{10} + \dots \tag{4.23}$$

We notice that this first set of terms in the series are palindromic, so by Theorem 4 we know immediately that this is the entire numerator. All further terms in the approximation will cancel out and we have found the HS.

$$HS = \frac{1 + 4t^2 + 9t^4 + 9t^6 + 4t^8 + t^{10}}{(1-t^2)^{10}} = \frac{(1+t^2)(1 + 3t^2 + 6t^4 + 3t^6 + t^8)}{(1-t^2)^{10}}. \tag{4.24}$$

To determine the global symmetry, we must now consider the refined HS, including fugacities for topological charge.

$$HS^{ref} = \sum_{a_1 \leq a_2} \sum_{b_1 \leq b_2 \leq b_3} z_1^{a_1+a_2} z_2^{b_1+b_2+b_3} P_{U(2)} P_{U(3)} t^{2\Delta(m)}. \tag{4.25}$$

As we are foremost interested in the coefficient of t^2, we can evaluate perturbatively to an even lower order.

$$HS = 1+(z_1^2z_2^2+z_1z_2^3+z_1z_2^2+z_1+z_2+\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_1z_2}+\frac{1}{z_1z_2^2}+\frac{1}{z_1z_2^3}+\frac{1}{z_1^2z_2^3})t^2+\dots \tag{4.26}$$

We can again use a fugacity map, this time mapping $z_1 \mapsto \frac{x_1^2}{x_2^3}$, $z_2 \mapsto \frac{x_2^2}{x_1}$.

$$HS = 1+(\frac{x_1^2}{x_2^3}+x_1+\frac{x_1}{x_2}+\frac{x_1}{x_2^2}+\frac{x_1}{x_2^3}+x_2+2+\frac{1}{x_2}+\frac{x_2^3}{x_1}+\frac{x_2^2}{x_1}+\frac{x_2}{x_1}+\frac{1}{x_1}+\frac{x_2^3}{x_1^2})t^2+\dots \tag{4.27}$$

We now recognise the coefficient as the character of the adjoint representation of G_2 (written with Dynkin labels as [0,1]), and hence identify this to be the global symmetry. With characters now known, we find the HWG:

$$HWG = \frac{1-t^{10}\mu_1^6\mu_2^4}{(1-t^2\mu_1^2)(1-t^3\mu_1^3)(1-t\mu_2)(1-t^5\mu_1^3\mu_2)(1-t^4\mu_2^2)}. \tag{4.28}$$

It can be found[3] that this quiver corresponds to the sub-regular nilpotent orbit of G_2 (written $\overline{s.regG_2}$).

[3] More advanced techniques such as the use of "Hasse Diagrams" can be used to identify this [28]. More on nilpotent orbits can be found in [13].

Chapter 5

Higgs Branch

Without quantum corrections, we are now able to count GIOs in a more classical manner. The following discussion and computations elucidate how operators arise on the moduli space, and how we can enumerate specifically those that are gauge invariant.

5.1 Molien-Weyl Projection

We first introduce the Molien-Weyl projection, and then explain the motivation behind it. We can compute a HS, now written as $g_{(G,N)}$, following the framework set out in [29]:

$$g_{(G,N)} = \int_G d\mu_G g^{\mathcal{F}_\flat} \tag{5.1}$$

Where $\int_G d\mu_G$ is the Haar measure of the gauge group G.

$$\int_G d\mu_G = \frac{1}{(2\pi i)^r} \oint_{|z_1|=1} \cdots \oint_{|z_r|=1} \frac{dz_1}{z_1} \cdots \frac{dz_r}{z_r} \prod_{\alpha^+} (1 - \prod_{l=1}^{r} z_l^{\alpha_l^+}) \tag{5.2}$$

where $r = rank(G)$ and α^+ are the positive roots of the Lie algebra of G.

Returning to the ideas of moduli spaces, we have stated previously that we aim to find a stationary superpotential. Our task now will be to find combinations of superfields which result in this. The set of these solutions is called the F-flat space [1], $\mathcal{F}_\flat$. We construct a generating function for this F-flat space, which we denote as $g^{\mathcal{F}_\flat}$ [30]; this is constructed by taking symmetric products of the superfield contributions from the relevant hplets and vplets. This is a HS in its own right, but this doesn't specifically count only GIOs, so we need to remove any other operators first before the HS can be of use. Given $g^{\mathcal{F}_\flat}$, we use (5.1) to project this onto the gauge group. The idea now is that when integrating over the gauge group, operators that aren't invariant under the group action will vanish, while those that are invariant will remain. This provides us with the HS, which can again be used to explore the structure of the moduli space.

[1]So called as this corresponds to a term in the Lagrangian known as the F-term being set to 0, and hence flat.

5.2 $\mathcal{N} = 1$ Quivers

Our previous quiver diagrams are based on $\mathcal{N} = 2$ theories. It preferable to use $\mathcal{N} = 1$ quivers on the Higgs branch, as we get a better picture of the symmetries at play and the superfields present.

We have a simple prescription for obtaining these new quivers from those that we are now familiar with:

- Each hplet between two nodes decomposes into two "half-hplets". We will treat these hplets as pairs transforming in the fundamental and antifundamental representations respectively. Pictorially, we split the edge into two oriented arrows. It is useful to note that each arrow has an R-charge of $\frac{1}{2}$.

- Each gauge group, originally transforming as an $\mathcal{N} = 2$ adjoint vplet, now decomposes into an adjoint vector multiplet and an adjoint chiral multiplet. We add an arrow starting and ending at the node corresponding to the gauge group in order to account for this.

5.3 Calculations

5.3.1 U(1) with N Flavours

Using our new quiver notation, our previous quiver used for the Coulomb branch (4.2.1) becomes:

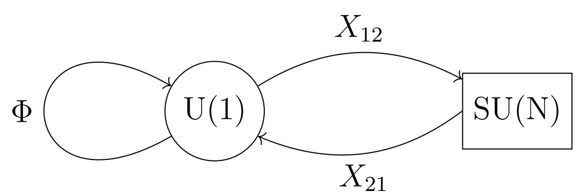

Figure 5.1: $\mathcal{N} = 1$ quiver for a U(1) theory with N flavours.

We have superfields arising from the gauge node and the hplets. Φ is due to the adjoint chiral multiplet from the vplet, while the original bifundamental hplet has decomposed into the two chiral multiplets X_{12} and X_{21}. Alternatively, we can see these superfields arising from closed loops on the $\mathcal{N} = 1$ quiver. By following different closed paths, we can generate different combinations of these superfields that are $\mathcal{N} = 1$ supersymmetric. Crucially, we want to be able to read the superpotential, $\mathcal{W}$, straight off the quiver. This is done by taking the full loop (i.e following all the arrows once). This construction is vital as it is also $\mathcal{N} = 2$ supersymmetric. If it weren't, the superpotential would change depending on the number of supercharges and hence wouldn't be invariant.

For this quiver, we read off the superpotential:

$$\mathcal{W} = Tr[X_{21} \cdot \Phi \cdot X_{12}], \tag{5.3}$$

where the trace is taken to ensure $\mathcal{W}$ is gauge invariant. For the F-flat condition, we must make this stationary, imposing:

$$\frac{\partial \mathcal{W}}{\partial \phi} = 0 \tag{5.4}$$

$$\frac{\partial \mathcal{W}}{\partial X_{12}} = 0 \tag{5.5}$$

$$\frac{\partial \mathcal{W}}{\partial X_{21}} = 0 \tag{5.6}$$

which is satisfied by superfields following the condition:

$$\mathcal{F}^{\flat} = \{\Phi = 0, X_{12} \cdot X_{21} = 0\}. \tag{5.7}$$

Vanishing Φ actually defines the Higgs branch, as this is the contribution from the vplets which vanishes. We don't impose that the hplet contribution vanishes, only that the 2^{nd} order combination does. Conversely, F-flatness can be met by imposing $X_{12} = 0 = X_{21}$ with no constraint on Φ. This choice corresponds to the Coulomb branch. However, the previously mentioned quantum corrections mean that we would no longer be able to continue with this method, as we will do with the Higgs branch.

By considering both the global and gauge symmetries, as presented in table 5.1, we can see how the representations of the superfields correspond to these symmetries.

		U(N)	
Symmetry	U(1)	SU(N)	U(1)
Fugacity	z	$x_1, x_2, \ldots x_{N-1}$	q
Φ	0	[0,...,0]	0
X_{12}	1	[1,0,...,0]	-1
X_{21}	-1	[0,...,0,1]	1

Table 5.1: Characters of Representations for gauge and Global Symmetries.

This allows us to build our generating function.

$$g^{\mathcal{F}^{\flat}}_{U(1),N} = (1-t^2)PE([1,0,...,0]wt + [0,0,...,1]\frac{t}{w}) \tag{5.8}$$

This construction arises from the following:

- Absorbing the U(1) global symmetry into the U(1) gauge symmetry, with a new fugacity $w = \frac{q}{z}$.
- The term (1-t^2) accounts for the second order relation $X_{12} \cdot X_{21} = 0$.
- $PE([1,0,...,0]wt)$ counts symmetric products of X_{12}, noting that we have the characters of the fundamental representations of SU(N) and U(1).
- $PE([0,0,...,1]\frac{t}{w})$ counts symmetric products of X_{21} in the antifundamental representation.

- The symmetric products are both graded at degree 1 by t.

We use the characters of SU(N), for the fundamental and antifundamental representations respectively:

$$[1, ..., 0] = x_1 + \frac{1}{x_{N-1}} + \sum_{k=2}^{N-1} \frac{x_k}{x_{k-1}} \tag{5.9}$$

$$[0, ..., 1] = \frac{1}{x_1} + x_{N-1} + \sum_{k=2}^{N-1} \frac{x_{k-1}}{x_k} \tag{5.10}$$

Therefore, using the properties of the plethystic exponential, we can write:

$$\begin{aligned} g^{\mathcal{F}}_{U(1),N}(t, x_1, \cdots, x_{N-1}, w) = \frac{(1-t^2)}{(1-x_1wt)(1-\frac{1}{x_{N-1}}wt)} \prod_{k=2}^{N-1} \frac{1}{1-\frac{x_k}{x_{k-1}}wt} \times \\ \frac{1}{(1-\frac{1}{x_1}\frac{t}{w})(1-x_{N-1}\frac{t}{w})} \prod_{k=2}^{N-1} \frac{1}{1-\frac{x_{k-1}}{x_k}\frac{t}{w}} \end{aligned} \tag{5.11}$$

We compute the Haar Measure of U(1):

$$\int_{U(1)} d\mu_{U(1)} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z}, \tag{5.12}$$

and then project onto the gauge group:

$$g_{U(1),N} = \frac{1}{2\pi i} \oint \frac{dw}{w} g^{\mathcal{F}}_{U(1),N}(t, x_1, \cdots, x_{n-1}, w). \tag{5.13}$$

We perform this integral using the residue theorem, noting that we have poles at:

$$w_1 = \frac{t}{x_1}, w_2 = \frac{x_1}{x_2}t, \ldots, w_{N-1} = \frac{x_{n-2}}{x_{n-1}}t, w_n = tx_{N-1}. \tag{5.14}$$

We take the unrefined HS for simplicity by setting $x_i = 1$, finding:

$$g_{U(1),N} = \frac{\sum_{m=0}^{N-1} \binom{N-1}{m}^2 t^{2p}}{(1-t^2)^{2(N-1)}}. \tag{5.15}$$

We immediately see that the moduli space has dimension 2(N-1). Considering the case N=2, as we did for the Coulomb branch, we obtain:

$$HS_{U(1),2} = \frac{1+t^2}{(1-t^2)^2} = \frac{1-t^4}{(1-t^2)^3}. \tag{5.16}$$

This is exactly the same as the HS for the Coulomb branch! The Coulomb branch and the Higgs branch are the same for this case.

Theorem 5. There exists a mirror symmetry between the Coulomb and Higgs branches. This duality arises due to allowed exchanges of branes [31].

The N=2 flavours example is a rare case of a self-mirror theory.

Chapter 6

Conclusion

Throughout this project, we have gained an understanding of supersymmetry, as well as an appreciation of many other important topics in applied maths and theoretical physics like Lie algebras, algebraic geometry, QFT and differential geometry.

Harnessing the techniques developed, such as the Hilbert Series, Plethystics Program, Character Expansion and HWG, we have been able to successfully perform a range of computations for multiple quivers on both the Higgs and Coulomb branches. This allowed us to explore the structure of the moduli spaces of these theories, discovering key properties such as the dimension and global symmetries. The use and application of quiver diagrams has also been crucial in our understanding; the results obtained have provided numerous insights into the properties and consequences of supersymmetric field theories.

The techniques established have been very useful in a wider understanding of gauge and field theories. Having established a range of key ideas about field theories, we will be able to use these ideas in the future for developing new models and theories. Crucially, the idea of a moduli space, and its relation to GIOs, has been a salient take-away from this project.

Looking forward, there are many directions this project could continue. One key step to make would be to learn about branes and more topics in string theory. This would give a stronger understanding of how these relate to the underlying theories and quivers, as well as providing insight into the mirror symmetry duality between theories.

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Appendix A

Lie Algebras

Lie algebras and representation theory are crucial in describing symmetries in physics. We provide a succinct overview of the Cartan-Weyl Basis, roots, and weights, to aid the reader in understanding results used in the main text. We follow the results from [6] & [7]. This section is by no means complete, and the reader should refer to the above texts for a complete treatise on the subject.

A.1 Basics

Definition 4. A Lie algebra $\mathfrak{g}$ is a vector space endowed with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$, called the "Lie Bracket", satisfying:

- $[x, x] = 0, \forall x \in \mathfrak{g}$,
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity),

$\forall x, y, z \in \mathfrak{g}$,

where the Jacobi Identity with bilinearity immediately imply the asymmetry of the bracket. The Jacobi Identity is not automatic, as we might have expected from comparing Lie brackets to commutators.

For a finite or countably infinite dimensional lie algebra, where we take $d = dim_{\mathbb{C}}\mathfrak{g}$ as the dimension of $\mathfrak{g}$ as a vector space, we can construct a basis $\mathcal{B}$:

$$\mathcal{B} = \{T^a | a = 1, 2, \ldots, d\}, \tag{A.1}$$

with generators T^a which span $\mathfrak{g}$. The basis elements are so named as they can generate the entire lie algebra through the Lie bracket:

$$[T^a, T^b] = f^{ab}_c T^c, \tag{A.2}$$

where the coefficients f^{ab}_c are called "structure constants" which depend on the chosen basis. We note, due to the Jacobi Identity, that these must antisymmetric under exchange of upper indices. The structure constants completely define the Lie algebra.

$$f^{ab}_c = -f^{ba}_c \tag{A.3}$$

A.1.1 Morphisms

It is useful to be able to map between different algebras, and determine if there are any hidden equivalences. We also aim to introduce some common terminology to aid the reader.

Definition 5. We define a homomorphism $\phi : \mathfrak{g} \mapsto \mathfrak{h}$ which linearly preserves the Lie structure, i.e:

$$[x, y] \mapsto \phi([x, y]) = [\phi(x), \phi(y)], \forall x, y \in \mathfrak{g}. \tag{A.4}$$

Definition 6. If a homomorphism is an injective map (i.e has a trivial kernel) is it known as an monomorphism, or an embedding.

Definition 7. If a homomorphism is a surjective map it is known as an epimorphism.

Definition 8. A map that is both epimorphic and monomorphic is known as an isomorphism, denoted $\mathfrak{h} \cong \mathfrak{g}$. These are most important in our study of Lie Algebras, as they determine if two algebras are equivalent, and effectively identical under a relabelling (which is, of course, just a change of basis).

Definition 9. An endomorphism is a map from a vector space V to itself, i.e $\phi : V \mapsto V$ If the endomorphism is also an isomorphism, then it is called an automorphism.

A.1.2 Subalgebras and Ideals

A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ which is a Lie Algebra in its own right is a Lie subalgebra of $\mathfrak{g}$. A Lie Algebra naturally has at least two trivial subalgebras, itself and $\{0\}$. Any other subalgebras are known as proper subalgebras. We require that:

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \tag{A.5}$$

If the stronger condition:

$$[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}, \tag{A.6}$$

is satisfied, we call $\mathfrak{h}$ an ideal, or an invariant subalgebra.

Definition 10. As a vector space, $\mathfrak{g}$ can be written as the direct sum of its ideal subspaces $\mathfrak{g}_\mathfrak{i}$:

$$\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_\mathfrak{i} \equiv \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \mathfrak{g}_\mathfrak{n} \tag{A.7}$$

Definition 11. A simple Lie algebra is a non-abelian Lie algebra with no proper ideals.

Definition 12. A semisimple Lie algebra is a Lie algebra constructed by a direct sum of simple Lie algebras.

A.1.3 Representations

A representation is a homomorphism that preserves the structure of the algebra.

$$\begin{aligned} \rho : \mathfrak{g} \mapsto GL() \\ \rho([T^a, T^b]) = [\rho(T^a), \rho(T^b)] \end{aligned} \tag{A.8}$$

Definition 13. The adjoint representation is given by:

$$ad_x(y) = [x, y] \tag{A.9}$$

Although this definition may seem redundant, it will be easier to avoid excessive square brackets.

We can similarly use apply this map to the generators of a Lie algebras, where the structure constants determine the Lie brackets.

$$ad_{T^a}(T^b) = [T_a, T_b] = f_c^{ab} T^c. \tag{A.10}$$

A.1.4 Cartan-Weyl Basis

Definition 14. A Cartan subalgebra, $\mathfrak{g}_o$, is a maximal, abelian subalgebra of $\mathfrak{g}$, consisting of semisimple elements H^i. These H^i are eigenvectors of the adoint map, and therefore ad_H is diagonalisable.

$$\mathfrak{g}_o = span_{\mathbb{C}}\{H^i | i = 1, 2, \ldots, r\}, \tag{A.11}$$

where r $\equiv$ rank$\mathfrak{g} \equiv$ dim $\mathfrak{g}_o$.

The Cartan subalgebra isn't unique, but automorphisms relate the different choices.

We interest ourselves in the elements E^α such that:

$$[H^i, E^\alpha] = \alpha^i E^\alpha, i = 1, 2, \ldots, r \tag{A.12}$$

where $\{E^\alpha\}$ is distinct from $\{H^i\}$. Again, this gives us an eigenbasis.

Definition 15. We call the vector (α^i) a root of the subalgebra. There must be at least one non-zero eigenvalue α with respect to the adjoint map, $\alpha_i \neq 0$.

Definition 16. The set of all roots of $\mathfrak{g}$ is called the root system, denoted by $\Phi \equiv \Phi(\mathfrak{g})$.

Definition 17. The eigenspace of of the generators E^α is given by:

$$\mathfrak{g}_\alpha = span_{\mathbb{C}}\{E^\alpha\}. \tag{A.13}$$

Our Cartan-Weyl basis can therefore be constructed from the union of the generators of the Cartan subalgebra, and the eigenvectors of their adjoint maps:

$$\mathcal{B} = \{H^i | i = 1, 2, \ldots, r\} \cup \{E^\alpha | \alpha \in \Phi\} \tag{A.14}$$

This basis now has the following relations of generators:

- $[H^i, H^j] = 0$
- $[H^i, E^\alpha] = \alpha(H^i) E_\alpha = \alpha_i E^\alpha$
- $[E^\alpha, E^\beta] = N^{\alpha\beta} E^{\alpha+\beta}$ if $\alpha + \beta \neq 0$

A.1.5 Root Systems

Definition 18. The inner product on rootspace is given by:

$$(\alpha, \beta) \equiv \alpha^i \beta_i \tag{A.15}$$

Theorem 6. There exists a hyperplane which evenly divides root space, V, into two disjoint subspaces, $V_\pm$. This hyperplane can be chosen arbitrarily, as long as we remain in this convention. We choose V_+ to be the space in which the positive roots lie:

$$\Phi_+ = \{\alpha \in \Phi | \alpha\, 0\}, \tag{A.16}$$

with the converse for negative roots:

$$\Phi_- = \{\alpha \in \Phi | \alpha < 0\}. \tag{A.17}$$

Definition 19. A simple/fundamental root is a positive root that can't be formed from a linear combination of other positive roots. These can be found closest to the hyperplane.

This leads to several neat properties, which result in a natural diagrammatic representation of the system of roots.

If α and β are roots, then:

- $-\alpha$ and $-\beta$ are also roots
- $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer
- β - $2\alpha\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is also a root.

From the definition of a scalar product, we find the angle between the roots α & β to be:

$$cos(\phi) = \frac{(\alpha, \alpha)}{\sqrt{(\alpha, \alpha)(\beta, \beta)}} \tag{A.18}$$

From the above properties, we can only take on values:

$$cos^2(\phi) = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \tag{A.19}$$

corresponding to $\phi = 0°, 30°, 45°, 60°, 90°$.

Root Diagrams

We can encapsulate the above properties of root systems in root diagrams. We give that of G_2 below as an example.

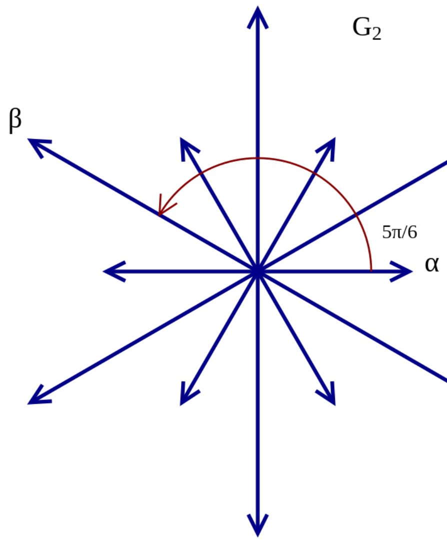

Figure A.1: The root system of G_2 [32].

Definition 20. The Weyl transformation, s_α is defined as:

$$s_\alpha(\beta) \equiv \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha, \tag{A.20}$$

with the corresponding Weyl group:

$$\mathcal{W} = \{s_\alpha | \alpha \in \Phi\} \tag{A.21}$$

The Weyl group gives the symmetries of the root diagram.

We encode the orthogonality relations of the simple roots in the Cartan Matrix A of $\mathfrak{g}$.

Definition 21. The Cartan Matrix is an $r \times r$ matrix with elements given by:

$$A_{ij} = 2\frac{(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(j)}, \alpha^{(j)})} \tag{A.22}$$

This completely defines the structure of a semisimple Lie algebra.

The Cartan Matrix has the following properties:

- The diagonal elements are always 2, seen trivially by setting i=j.
- Elements are positive integers due to the definition of the inner product.
- $detA > 0$.
- A is not block diagonal; it can't be decomposable into a block-diagonal as otherwise $\mathfrak{g}$ would possess vector subspaces and would therefore not be simple.
- As the inner product (,) is antisymmetric under exhange of indices, $A_{ij} = 0 \implies A_{ji} = 0$

From these properties, there is a limited number of possible Cartan Matrices that can be produced for a given rank r. One example is the Cartan matrix of G_2:

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

.

Killing Form

We introduce a metric on our root space, called the Killing form:

$$\kappa(x, y) \equiv tr(ad_x \circ ad_y), \tag{A.23}$$

Where "tr" is the trace, and $\circ$ is the composition of the adjoint maps. Note that it is bilinear and symmetric. Moreover, this can be written in terms of the structure constants of the Lie algebra:

$$\kappa^{ij} = f_k^{il} f_l^{jk} \tag{A.24}$$

Weights

Definition 22. Consider a representation $\rho : \mathfrak{g} \mapsto End(V)$ of $\mathfrak{g}$ on a vector space V and let $\lambda : \mathfrak{g}_0 \mapsto \mathbb{C}$ be a linear function. λ is a weight of the representation ρ if:

$$\rho(H)v = \lambda(H)v, \tag{A.25}$$

for non-zero eigenvectors v and $H \in \mathfrak{g}_0$.

We note that in the case that ρ is the adjoint representation, the weights are simply just the roots and 0. This equivalence between roots and weights is important, as we have the following result:

$$\begin{aligned} \rho(H)\rho(E^\alpha)v &= \rho(HE^\alpha)v \\ &= \rho(E^\alpha H + \alpha(H)E^\alpha)v = \rho(E^\alpha)\rho(H)v + \alpha(H)\rho(E^\alpha)v = (\lambda + \alpha(H))\rho(E^\alpha)v. \end{aligned} \tag{A.26}$$

So either $\rho(E^\alpha)v = 0$ or $\rho(E^\alpha)v$ is a new weight vector with weight $(\lambda + \alpha)$.

Definition 23. A co-root is defined:

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)}. \tag{A.27}$$

These coroots (or dual roots) live in the dual space to $\mathfrak{g}_o$, $\mathfrak{g}_o^\star$.

The dual space to the root space is called the weight space, containing weights of $\mathfrak{g}$

Definition 24. It is natural to use our coroots to create a basis for the rootspace:

$$\mathcal{B} = \{\alpha^{(i)\vee} | i = 1, 2, \dots, r\} \tag{A.28}$$

By standard procedure for a dual vector space, we construct a basis for the weight space.

$$\mathcal{B}^\star = \{\Lambda_{(i)} | i = 1, 2, \dots, r\} \tag{A.29}$$

where $\Lambda_{(i)}$ are called fundamental weights of $\mathfrak{g}$, satisfying:

$$\Lambda_{(i)}(\alpha^{(j)\vee}) = \delta_i^j, \qquad \text{for } i, j = 1, 2, \dots, r. \tag{A.30}$$

Definition 25. $\mathcal{B}^\star$ is known as the Dynkin Basis, and the components of the weights in this basis are called Dynkin labels.

A.1.6 Dynkin Diagrams

An alternative way of displaying the information encoded in the Cartan Matrix is through Dynkin Diagrams. These are built from nodes connected by lines. Each node represents a simple root, and the number of lines between two nodes is given by $max\{|A_{ij}|, |A_{ji}|\}$. Arrows between nodes dictates which corresponding root is larger.

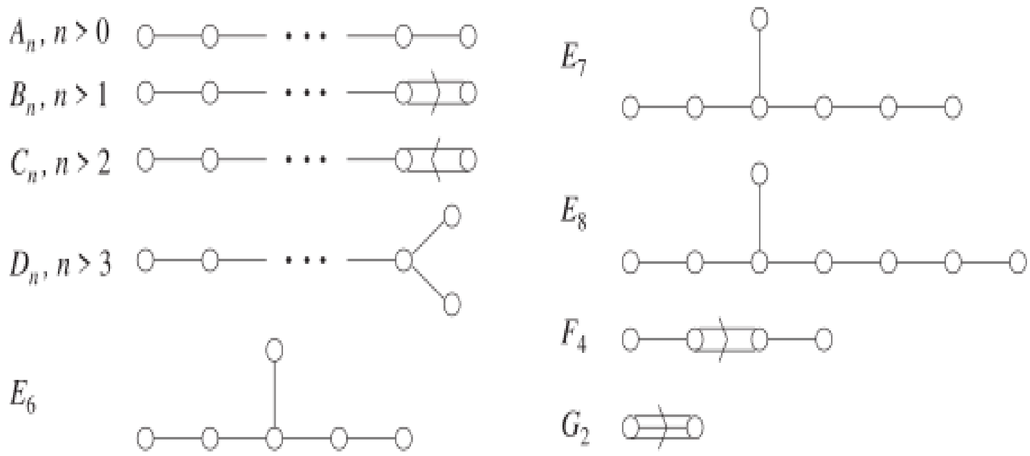

Figure A.2: Dynkin diagrams of classical and exceptional Lie algebras [33].

The i^{th} node of a Dynkin diagram can represent:

- The row index of the Cartan Matrix.
- The i^{th} simple root, $(\alpha^{(i})$.
- The i^{th} fundamental weight, $(\Lambda^{(i})$.
- The i^{th} generator of $\mathfrak{g}_o$, H^i.

A.1.7 Semisimple Lie Algebras

It is here that we explicitly define the simple lie algebras. There are four "classical" Lie algebras: $A_r, (r \geq 1)$; $B_r, (r \geq 3)$; $C_r, (r \geq 2)$; $D_r, (r \geq 4)$,
and five "Exceptional" Lie algebras (which don't follow a pattern like the classical cases):
E_6, E_7, E_8, G_2, F_2.

The classical Lie algebras obey the following equivalence relations:
$A_r \cong \mathfrak{sl}(r+1) \qquad B_r \cong \mathfrak{so}(2r+1) \qquad C_r \cong \mathfrak{sp}(r) \qquad D_r \cong \mathfrak{so}(r).$

With these root diagrams and dynkin diagrams, we have a sufficient framework to find roots and weights for the Lie algebras relevant to the report.