Particle Symmetry

2022

Lecturer: Professor Daniel J Waldram

Notes by Zhihao Zhang, Hao Yang, Zihan Wang, Steven Hsia,
Jieming Lin

Blackett Laboratory
Imperial College London
U.K.
2022

Contents

1 Introduction to Groups and Representation

1.1 Outline

Two Goals of the course:

- Introduction to Lie groups
- Application to particle physics

This course aims to do two things:

1. give an introduction to group theory and representations, specifically Lie groups and Lie algebras.
2. discuss the role off group theory in physics and give a short summary of the basic elements of particle physics and the symmetry of particles, including the standard model.

The outline is as follows:

1. basic introduction to abstract groups, matrix groups and the notion of representations.
2. summary of the particles in the standard model and their symmetries.
3. introduction to Lie groups and Lie algebra.
   - Lie algebras and the exponential map: topology.
   - Baker-Campbell-Hausdorff and homomorphisms
   - representations
   - Heisenberg groups
   - Poincare groups
   - spinor and Clifford algebras
4. classification and reoresentation of semi-simple Lie groups
   - Cartan decomposition, Killing form, Dynkin diagrams.
   - highest weight representations, characters, dimensions ...

Reading list books: main topic for Lie group

1. Jones: 'Groups, representation and physics'
2. Georgi: 'Lie algebras in particle physics'
3. Hall: 'Lie groups, Lie algebras and representations'
4. Stilwell: 'Naive Lie theory'
5. Carter, segal and MacDonaldL 'Lectures on Lie groups and lie algebras'

other physics books

1. Fuchs and Schwergert: 'symmetries, Lie algebras and representations'
2. Gilmore: 'Lie groups, Lie algebras and some of their applications'
3. Cornwell: 'Group theory in physics'

Pure maths books (differential geometry approach)

1. Helgason: 'Differential geometry, Lie groups and symmetry spaces'
2. Samelson: 'Notes on Lie algebras'
3. Fulton: 'Representation theory - a first course'
4. Nakahara: 'geometry, topology and physics'

For particle physics

1. Perkins: 'High energy physics'
2. Martin and Shaw: 'Particle physics'

Four Sections:

- Introduction to groups and representations
- Application to particle physics
- Lie Groups and Lie Algebras
- Classification of "semi-simple" Lie groups

1.2 Why Group Theory?

Set of symmetries $\Leftrightarrow$ Group

For a physicist the utility of group theory is that it is the mathematical way to encode symmetries and symmetries are ubiquitous in physics.

Example 1.1. The Poincare group encodes the symmetries of flat spacetime, namely transformations, rotations and Lorentz boosts. It leads to characterisation of particle by mass and spin.

Example 1.2. The wavefunction Ψ has a symmetry

$$\Psi \to e^{i\alpha}\Psi \tag{1.1}$$

such that physics are independent of phase $e^{i\alpha}$. Set of transformations forms group called U(1).

All known interactions are examples of gauge theories - local symmetries. For example if $\Psi \to e^{i\alpha(x)}\Psi$ depending on position x. Schrodinger equation is only invariant under this symmetry if we couple to electromagnetism. (local symmetries are natural if we are not to violate causality - spatially separated observers should be able to make their own transformations independently.) For standard model, the local symmetry is $SU(3) \times SU(2) \times U(1)$.

Important notion of symmetry breaking: theory itself may have a symmetry. For example pencil standing on its end.

(a) The system is symmetric around the pencil, but is bot a stable equilibrium.

(b) Pencil falls and point in some direction, ground state breaks symmetry

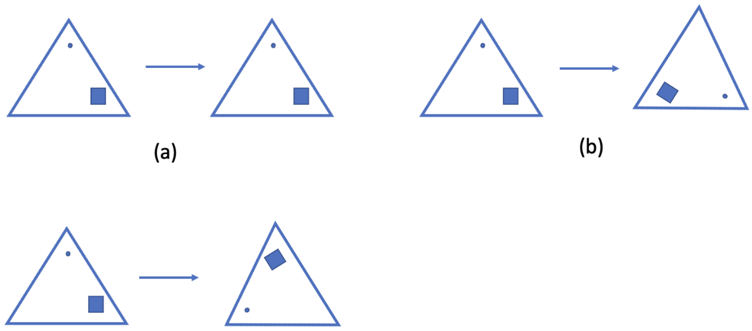
Same phenomena describes ferromagnetism and most importantly the Higgs mechanism.

1.3 Two Examples of Groups

Here we give two examples of simple groups:

- Rotation symmetry of a triangle $\mathbb{Z}_3$
- Rotation symmetry of a circle $\mathrm{U}(1) \backsimeq \mathrm{SO}(2)$

Consider the rotation symmetry of an equilateral triangle, we have three operations: These are rotations by 0, $\pi/3$ and $2\pi/3$. We can combine operators for



(a) (b) (c)

$x, y \in \{e, a, b\}$:

$$x \circ y = \text{do operation y followed by operation x.}$$

Then for multiplication table

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

where the rows are x and columns are y. However we could have got the same multiplication table from many other set of objects with a product.

For example:

1. $e = 0$, $a = 1$, $b = 2$ under addition modulo 3.
2. $e = 1$, $a = e^{2i\pi/3}$, $e^{4i\pi/3}$ under multiplication.
3. $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $a = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, $b = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ under matrix multiplication.

4. $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ under matrix multiplication.

So, we can define the **abstract group**

$$\mathbb{Z}_3 = \{e, a, b\}$$

with multiplication table as above. Note that this is a special case of a **cyclic group**

$$\mathbb{Z}_n = \{e, a, a^2, ..., a^{n-1}\}$$

which is the set with $ea = ae = e, e^2 = e, a^n = e$.

1.4 Abstract Group

From the examples above we are led to a definition:

Definition 1.1. An abstract group is a set $G = \{a, b, ...\}$ finite or infinite, together with a binary operation, the group product such that for all $a, b \in G$ satisfy:

1. closure: $ab = c$, $c \in G$
2. associativity: $a(bc) = (ab)c$
3. identity: $ea = ae = a$
4. inverses: $ab = ba = e$. The inverse element is usually written as a^{-1}.

You should think of the elements $a \in G$ as symmetry transformations. The product tells you how to combine symmetries. The identity e is the 'trivial' transformation that leaves the system unchanged.

There is a special class of groups:

Definition 1.2 (Abelian)**.** A group is **Abelian** if the product is commutative, that is $ab = ba$ for all a and b in group G.

Definition 1.3 (Group Order)**.** If the group is finite, the number of elements in the group is the **order** of the group.

Here we give some examples of finite groups:

Example 1.3. The real number $\mathbb{R}$ under addition is an Abelian infinite group. Under multiplication, $\mathbb{R} - \{0\}$ is also an Abelian group. Zero is excluded from the set because it has no inverse.

Example 1.4. Another example is U(1): unit norm complex numbers under multiplication:

$$\mathrm{U}(1) = \left\{e^{i\theta} : 0 \leq \theta < 2\pi\right\} \tag{1.2}$$

This is the symmetry group of wavefunction.

Example 1.5. Another example SO(2): Set of rotations by angle θ in the plane

$$SO(2) = \left\{R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \quad \theta \leq \theta < 2\pi\right\} \tag{1.3}$$

such that $R(\theta_1) R(\theta_2) = R(\theta_1 + \theta_2)$.

Example 1.6. General linear group $GL(2,\mathbb{R})$: a set of 2×2 invertible matrices

$$\mathrm{GL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \; ad - bc \neq 0, \; a,b,c,d \in \mathbb{R} \right\} \tag{1.4}$$

form group under multiplication.

1.5 Group Homomorphism

It is useful to have a notion of maps between groups that preserve the product. This allows us to define when two groups are the same.

Definition 1.4. A group **homomorphism** is a map $\phi : G \to H$ between two groups such that

$$\phi(a) \cdot \phi(b) = \phi(a \cdot b). \tag{1.5}$$

Example 1.7. Suppose we have $\mathbb{Z}_4 = \{e, a, a^2, a^3\}, \mathbb{Z}_2 = \{e', b\}$:

1) $\varphi_1 : \mathbb{Z}_2 \to \mathbb{Z}_4$ with $\varphi_1(e') = e, \quad \varphi_1(b) = a^2$. There exists a homomorphism.

2) $\sigma : \mathbb{Z}_2 \to \mathbb{Z}_4$ with $\sigma(e') = e, \quad \sigma(b) = a$. There is no homomorphism. $b^2 = e'$ but $\sigma(b)\sigma(b) = a - a = a^2 \neq \sigma(e')$..

3) $\varphi_2 : \mathbb{Z}_4 \to \mathbb{Z}_2$ with $\varphi_2(e) = e', \varphi_2(a) = b, \varphi_2(a^2) = e', \varphi_2(a^3) = b$. There exists a homomorphism.

Note that ϕ_1 is injective but ϕ_2 is surjective. We can further define the isomorphism to identify this relation.

Definition 1.5. An **isomorphism** is a homomorphism which is bijective (one-to-one). If two groups are isomorphic, we write $G \backsimeq H$.

Isomorphic groups are the same as abstract groups - see for example all the different isomorphic ways of defining $\mathbb{Z}_3$ in section 1.3.

Example 1.8. $\mathrm{SO}(2) \backsimeq \mathrm{U}(1)$: We have the isomorphism $\varphi : e^{i\theta} \mapsto R(\theta)$. Both groups describe rotations in the complex or real plane.

1.6 Group Products and Subgroups

One can also form a new group via product groups.

Definition 1.6. Given two Groups G, H, the product group G × H is the set of pairs

$$G \times H = \{(a, \alpha) : a \in G, \alpha \in H\} \tag{1.6}$$

with the product

$$(a, \alpha) \cdot (b, \beta) = (a \cdot b, \alpha \cdot \beta)$$

One can also find the subgroup:

Definition 1.7. Subgroup H of G is a subset H ⊂ G which forms a group under the product rule of G.

If H = G or H = $\{e\}$ then the subgroup is an improper subgroup.
Define the normal subgroup:

Definition 1.8. A normal subgroup is a subgroup N ⊂ G such that

$$ana^{-1} \in N, \quad \forall a \in G, n \in N \tag{1.7}$$

A group is **simple** if it has no proper normal subgroup.

Define the coset:

- Left coset: $aN = \{an : n \in N\}$
- Right coset: $Na = \{an : n \in N\}$

Definition 1.9. The **quotient group** G/N is the set of all left coset

$$G/N = \{aN : a \in G\} \tag{1.8}$$

under a set production.

Example 1.9.

$$SU(2)/\mathbb{Z}_2 \backsimeq SO(3) \tag{1.9}$$

Define the centre:

Definition 1.10. The centre of a group G is defined as $\mathbb{Z}(G)$ as

$$\mathbb{Z}(G) = \{h \in G : hg = gh for \forall g \in G\} \tag{1.10}$$

which is the set of elements that is commute with all elements.

1.7 Representation

Physically we need to "realise" the abstract symmetry group in some concrete way. The natural construct is via matrices.

Definition 1.11 (Representation)**.** A representation ρ of a group G is a homomorphism

$$\rho : G \mapsto GL(n, \mathbb{C}) \tag{1.11}$$

the integer n is the dimension of the representation.

Definition 1.12 (Faithful)**.** A representation is faithful if ρ is injective.

Definition 1.13 (Equivalent representation)**.** Two reps $\rho,\ \rho'$ are equivalent if

$$\rho'(a) = T^{-1}\rho(a)T$$

for some non-singular matrix T

Equivalent reps really just correspond to different choose of basis for the vector space on which the reps acts.

Definition 1.14. A **(left) G-module** is a vector space V over $\mathbb{C}$ with a map $G \times V \to V$ denoted as $a \cdot v,\ \forall a \in G, v \in V$ such that

- $e \cdot v = v$
- $a \cdot (b \cdot v) = (a \cdot b) \cdot v$
- $a \cdot (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 (a \cdot v_1) + \lambda_1 (a \cdot v_2)$

If we fixed a basis $\{e_i\}$, then a module define a representation. The vector in vector space can be written as:

$$v = v^i e_i$$

Consider the linearity,

$$\begin{aligned} a \cdot v = a \cdot (v^i e_i) &= v^i (a \cdot e_i) \\ &= v^i \rho(a)^j{}_i e_j \end{aligned}$$

Thus $v^i \to \rho(a)^i{}_j v^j$.

If we change the basis $e'_i = T^j{}_i e_j$ such that $v = v'^i e'_i$

$$v'^i = v^j (T^{-1})^i{}_j$$

Then, linearity gives

$$\begin{aligned} a \cdot v = v^i \rho(a)^j{}_i e_j &= v'^i \rho'(a)^j{}_i e'_j \\ &= v^k (T^{-1})^i{}_k \rho'(a)^j{}_i T^l{}_i \, e_l \end{aligned}$$

where we can identify ρ as,

$$\rho(a)^l{}_k = T^l{}_j \, \rho'(a)^j{}_i (T^{-1})^i{}_k$$

$$\rho(a)^j{}_i \to \rho'(a)^j{}_i = (T^{-1})^i{}_l \rho(a)^l{}_k T^k{}_j$$

thus $\rho(a)$ transforms as $\rho'(a) = T^{-1}\rho(a)T$, and gives a new rep ρ' that it is a equivalent rep to ρ.

1.8 Unitary Representation and QM

For physics key refinement:

Definition 1.15. A unitary representation is a homomorphism: $\rho : G \to \mathrm{U}(n)$, where

$$\mathrm{U}_n = \left\{ n \times n \text{ matrices} : M^\dagger M = \mathbb{1}_{n\times n} \right\}$$

More generally, if we have a Hilbert space $\mathcal{H}$ we have

$$\mathrm{U}(\mathcal{H}) = \{\text{unitary operator on } \mathcal{H}\}$$

In QM, we demand symmetry to be represented by unitary reps

$$\begin{aligned} |\psi\rangle &\in \mathcal{H} \\ \text{under symmetry } \ |\psi\rangle &\to |\psi'\rangle = S\,|\psi\rangle \end{aligned}$$

such that:

i) Linearity:

$$|\psi\rangle + |\chi\rangle \to S(|\psi\rangle + |\chi\rangle) = S\,|\psi\rangle + S\,|\chi\rangle$$

ii) Unitary operators preserve probabilities:

$$\begin{aligned} |\psi\rangle &\to S\,|\psi\rangle = |\psi'\rangle \\ |\chi\rangle &\to S\,|\chi\rangle = |\chi'\rangle \end{aligned}$$

then,

$$\langle\psi'|\chi'\rangle = \langle\psi|\chi\rangle$$

which implies that S satisfies

$$S^\dagger S = \mathbb{1}$$

is unitary

It appears that all symmetries of QM are represented by unitary operators. However, to preserve probability, we actually only needs:

$$|\,\langle\chi'|\psi'\rangle\,|^2 = |\,\langle\chi|\psi\rangle\,|^2$$

which can have arbitrary phase. So really we allow **projective unitary representation**.

- conventional: $\rho(a)\rho(b) = \rho(a \cdot b)$
- projective: $\rho(a)\rho(b) = e^{i\phi(a,b)}\rho(a \cdot b)$

One can show that any projective representation of a group G is defined by a convention representation of a (possibly) larger group $\hat{\mathrm{G}}$ that

$$\mathrm{G} = \hat{\mathrm{G}}/\mathrm{N}, \quad \mathrm{N} \subset \mathbb{Z}(\hat{\mathrm{G}})$$

where $\mathbb{Z}(\hat{\mathrm{G}})$ is the centre of the group $\hat{\mathrm{G}}$. We call $\hat{\mathrm{G}}$ the central extension of G. In practice, we consider the conventional representation of $\hat{\mathrm{G}}$ to exam projective representation of G.

Actually, there are two further possibilities:

- It is linear and projective unitary:

  linear:

  $$\rho(a)(\alpha\,|\psi\rangle + \beta\,|\chi\rangle) = \alpha\rho(a)\,|\psi\rangle + \beta\rho(a)\,|\chi\rangle$$

  projective unitary:

  $$\langle\rho(a)\psi|\rho(a)\chi\rangle = \langle\psi|\chi\rangle$$

- It is anti-linear and projective anti-unitary:

  anti-linear:

  $$\rho(a)(\alpha\,|\psi\rangle + \beta\,|\chi\rangle) = \alpha^*\rho(a)\,|\psi\rangle + \beta^*\rho(a)\,|\chi\rangle$$

  and projective anti-unitary:

  $$\langle\rho(a)\psi|\rho(a)\chi\rangle = \langle\psi|\chi\rangle^*$$

For a symmetry, the transformation needs to preserve the dynamics,

$$i\partial_t\,|\psi\rangle = \hat{H}\,|\psi\rangle$$

given $a \in G$, we have the transformation:

$$|\psi\rangle \to \rho(a)\,|\psi\rangle$$

- If there is no time reversal ($t \to -t$), *i.e.* $[S, \partial t] = 0$.

  $$\begin{aligned} i\partial_t\rho(a)\,|\psi\rangle &= \hat{H}\rho(a)\,|\psi\rangle \\ &= i\rho(a)\partial_t\,|\psi\rangle \end{aligned}$$

  and we can see that: to preserve dynamics we need $\rho^{-1}(a)\hat{H}\rho(a) = \hat{H}$.

- If we have time reversal. Then, consider the energy eigenstate: $\hat{H}|\psi\rangle = E|\psi\rangle$. The time evolution is given by $e^{-i\hat{H}t}|\psi\rangle$. Suppose S induce the time reversal, *i.e.* only brings $t \to -t$,

$$S|\psi\rangle = e^{i\hat{H}t}|\psi\rangle$$

  Thus, S acting as an anti-linear operator

$$S(\alpha|\psi\rangle + \beta|\chi\rangle) = \alpha^* S|\psi\rangle + \beta^* S|\chi\rangle$$

Wigner showed that for a symmetry operation $S(a)$ which induces time translation satisfies:

- S(a) is anti-linear;
- S(a) is (projective) anti-unitary:

$$\langle S(a)\psi|S(a)\chi\rangle = \langle\psi|\chi\rangle^* ;$$

- Preserve $\hat{H}$.

However, the time reversal is a Z_2 symmetry, so any group include Z_2 can be written as $\mathrm{Z}_2 \times G$ where

- Z_2 has anti-unitary representation,
- G has unitary representation.

Summary: for QM:

1) (projective) unitary (or anti-unitary for $t \to t$) representation of the symmetry group,

$$S(a)S(b) = e^{i\phi(a,b)}S(a \cdot b)$$

   is equivalent to the conventional unitary representation of $\hat{\mathrm{G}}$ where $\mathrm{G} = \hat{\mathrm{G}}/N$, $N \subset \mathbb{Z}(\hat{\mathrm{G}})$

2) It has to preserve dynamics:

$$S^{-1}(a)\hat{H}S(a) = \hat{H}$$

1.9 Lie Groups

Lie group is a group which is also a manifold with a smooth group product $G \times G \to G$. *i.e.* each element $a \in G$ is a point on the manifold. We can analyse Lie group via differential geometry, but we also have the Peter-Weyl theorem:

Theorem 1.1 (Peter-Weyl theorem). *Any compact Lie group is isomorphic to a subgroup of a unitary group.*

i.e. group defined by matrices (matrix Lie group)

Definition 1.16. A matrix Lie group is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$

For closed: For a group $G \subset \mathrm{GL}(n, \mathbb{C})$, take a sequence of matrices $A_p \in G$ such that,

$$\tilde{A} = \lim_{p \to \infty} A_p,$$

each component of A_p is converge in $\mathbb{C}$. Then the group G is closed if either $\tilde{A} \in G$ or $\tilde{A}$ is not invertable

Example 1.10. Examples of matrix Lie group:

- $\mathrm{U}(1) = \left\{ e^{i\theta} : 0 \leq \theta < 2\pi \right\} \backsimeq S^1$
- $\mathrm{SU}(2) = \{2 \times 2 \text{ unitary matrix } M,\ \det M = 1\} \backsimeq S^3$ (3-d sphere in 4-d space)
- $\mathrm{SU}(1,1) = \left\{ M \in GL(2, \mathbb{C}) : \det M = 1,\ M^T \eta M = \eta,\ \eta = \mathrm{diag}(1, -1) \right\}$ which is a non-compact Lie group.

Example 1.11. Consider Hermitian trace-less 2×2 matrices:

$$V = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix}$$

Act on V by an element of SU(2), M,:

$$V \to V' = M^\dagger V M$$

we can easily check that V' is still Hermitian and trace-less. And, $\det V = \det V'$,

$$a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2$$

Since the transformation is linear in V, it is also linear in a, b, c. Thus we can write the transformation as:

$$V' = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = O \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

such that $a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2$.

Then we consider O,

i) $O \in \mathrm{GL}(3, \mathbb{R})$

ii) $O^T O = \mathbb{1}$, for $a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2$

iii) $\det O = 1$

and we can conclude that $O \in \mathrm{SO}(3)$.

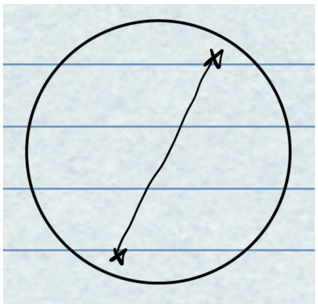
Thus, we have a map $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$. But, note that

$$\varphi(M) = \varphi(-M)$$

we call that the SU(2) is the **double cover** of SO(3),

$$\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$$

Since $\mathrm{SU}(2) \backsimeq S^3$, geometrically, SO(3) is S^3 with antipodal identified.



1.10 Classical Lie Group

There are three classical families: SO(n), SU(n), Sp(n), and we want to see how they emerges.

We ask the question of what are the possible generalisation of real/complex number?

- algebras with addition, subtraction, multiplication, division
- we can further drop commutativity and associativity.

These give us the **division algebra**. And furthermore the normed division algebra: *e.g.* on complex number $|z|^2 = x^2 + y^2$.

There are only four normed division algebras on $\mathbb{R}$:

$$\text{Real number } \mathbb{R}, \text{ Complex number } \mathbb{C}, \text{ Quaternion } \mathbb{H}, \text{ Octonion } \mathbb{O}.$$

For Real number $\mathbb{R}$: conjugate: $x^* = x$, norm: $|x|^2 = x^2$

- $\mathrm{GL}(n, \mathbb{R}) = \{n \times n \text{ real matrices} : \det \neq 0\}$
- conjugation of matrix: for $M_{ij} \in \mathrm{GL}(n, \mathbb{R})$, $M^\dagger_{ij} = M^*_{ji} = (M^T)_{ij}$
- orthogonal matrix: $M^T M = \mathbb{1}$
- $\mathrm{SO}(n) = \left\{M \in \mathrm{GL}(n, \mathbb{R}) : M^T M = \mathbb{1}, \det M = 1\right\}$ preserve their norm
- Acting M on a vector $v \in \mathbb{R}^n$, preserves the length $v^T v$

For Complex number $\mathbb{C}$: $z = x + iy$, $i^2 = -1$, conjugate: $z^* = x - iy$, norm: $|z|^2 = x^2 + y^2$.

- $\mathrm{GL}(n, \mathbb{C}) = \{n \times n \text{ camplex matrices} : \det \neq 0\}$
- conjugation of matrix: $M^\dagger_{ij} = M^*_{ji} = (M^*)^T_{ij}$
- unitary matrix: $M^\dagger M = \mathbb{1}$
- $\mathrm{SU}(n) = \left\{M \in \mathrm{GL}(n, \mathbb{C}) : M^\dagger M = \mathbb{1}, \det M = 1\right\}$ preserve their norm
- Acting M on a vector $v \in \mathbb{C}^n$, preserves the length $v^\dagger v$

For Quaternian $\mathbb{H}$: $z = u + iv + jx + ky$, $i^2 = j^2 = k^2 = -1$, conjugate: $z^* = u - iv - jx - ky$, norm: $|z|^2 = u^2 + v^2 + x^2 + y^2$. The relation between i, j, k are cyclic:

$$\begin{cases} i \cdot j = -j \cdot i = k \\ j \cdot k = -k \cdot j = i \\ k \cdot i = -i \cdot k = j \end{cases}$$

or can be written as $e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k$.

- $\mathrm{GL}(n, \mathbb{H}) = \{n \times n \text{ quaternion matrices} : \det \neq 0\}$
- conjugation of matrix: $M^\dagger_{ij} = M^*_{ji}$
- symplectic matrix: $(M^\dagger)_{ij} := M^*_{ji}$
- Symplectic group: $\mathrm{Sp}(n) = \left\{M \in \mathrm{GL}(n, \mathbb{H}) : M^\dagger M = \mathbb{1}, \det M = 1\right\}$ preserve their norm
- M preserve the length of $v \in \mathbb{H}^m$

We define the $\det M$ under $\mathbb{H}$ as the following:

$$\mathrm{GL}(n, \mathbb{H}) \backsimeq \left\{M \in \mathrm{GL}(2n, \mathbb{C}) : \Omega M \Omega^- 1 = M^*\right\}$$

Therefore we have the USp(2n) group:

$$\mathrm{Sp}(n) \backsimeq \mathrm{USp(2n)} = \left\{M \in \mathrm{GL}(2n, \mathbb{C}) : M^\dagger M = \mathbb{1}, M^T \Omega M = \Omega\right\}$$

and the $\det M$ is defined via the isomorphism:

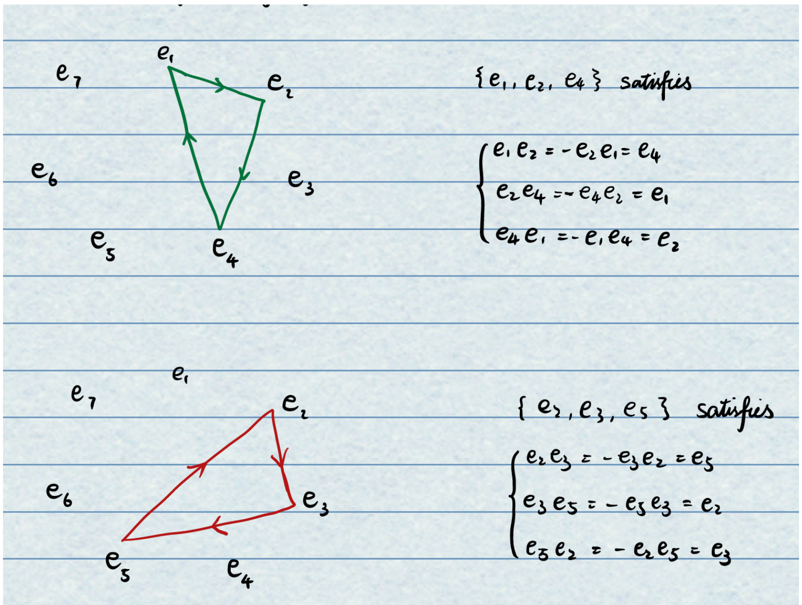
$$\det M = \det \varphi(M)$$

We will discuss these in the nest section.

For Octonion $\mathbb{O}$: $z = z_0 + \sum_{i=1}^7 z_i e_i$, conjugate: $z^* = z_0 - \sum_{i=1}^7 z_i e_i$, norm: $|z|^2 = \sum_{i=0}^7 z_i^2$. The basis follow the relation

$$e_i e_j = -\delta_{ij} + c_{ijk} e_k$$

The rule is: first jump 1, then 2, clockwise, and the connected elements forms a i, j, k quaternion. As a result, octonion is not associative. Hence $\mathrm{GL}(n, \mathbb{O})$ is not a group. Here we give some example:



However, there are exceptional groups from $\mathbb{O}$.

Example 1.12. Take quaternion $\mathbb{H}$ basis $e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k$. We perform the change of basis: $e'_i = M_i{}^j e_j$ while preserving the algebra such that:

$$e'_i e'_j = -\delta_{ij} + \epsilon_{ijk} e'_k$$

Thus implies:

$$\delta_{ij} M^i{}_k M^j{}_l = \delta_{kl} \Rightarrow M^T M = 1$$

$$\epsilon_{ijk} M^i{}_{i'} M^j{}_{j'} M^k{}_{k'} = \epsilon_{i'j'k'} \Rightarrow \det M = 1$$

Thus the symmetry preserving group is SO(3)

Likewise, for $\mathbb{O}$, the basis $e_i e_j = -\delta_{ij} + c_{ijk} e_k$ transform to $e' = Me$ such that:

$$e'_i e'_j = -\delta_{ij} + c_{ijk} e'_k$$

gives the constrains:

$$M^T M = 1 c_{ijk} M^i{}_{i'} M^j{}_{j'} M^k{}_{k'} = c_{i'j'k'}$$

Such M define the exceptional group G_2, which is a subgroup of SO(7).

We also have other exceptional groups F_4, E_6, E_7, E_8 defined via octonion.

Notice the norm implies that $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ are all compact Lie groups.

$$\begin{gathered}
\mathbb{R} : \mathbb{Z}_2 \\
\mathbb{C} : S^1 \\
\mathbb{H} : S^3 \\
\mathbb{O} : S^7
\end{gathered}$$

Therefore we have all the classical Lie groups:

$$G_{\mathbb{F}(n)} = \left\{ M \in \mathrm{GL}(n, \mathbb{F}) : M^\dagger M = 1, \det M = 1 \right\}$$

where $\mathrm{F} = \{\mathrm{R}, \mathrm{C}, \mathrm{H}\}$ gives all compact matrix Lie groups.

1.11 Symplectic Group

There is a more conventional way to define $\mathrm{Sp}(n)$. Using homomorphism of algebras: $\Psi : \mathbb{H} \to M(2, \mathbb{C})$:

$$\Psi(i) = \begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix}$$
$$\Psi(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\Psi(k) = \begin{pmatrix} 0 & i' \\ i' & 0 \end{pmatrix}$$

which gives:

$$u + ve_1 + xe_2 + ye_3 \to \begin{pmatrix} u + iv & x + iy \\ -x + iy & u - iv \end{pmatrix}$$

And we have the relation:

$$\Psi(z^*) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Psi(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then we define an isomorphism:

$$\mathrm{GL}(n, \mathbb{H}) \backsimeq \left\{ M \in \mathrm{GL}(2n, \mathbb{C}) : \Omega M \Omega^- 1 = M^* \right\}$$

where

$$\Omega = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

called the **symplectic form** such that $\Omega^T = -\Omega - \Omega^{-1}$ which is anti-symmetric.

For $\mathrm{Sp}(n)$ we need $M^\dagger M = 1$

$$\mathrm{Sp}(n) \backsimeq \mathrm{USp(2n)} = \left\{ M \in \mathrm{GL}(2n, \mathbb{C}) : M^\dagger M = \mathbb{1}, M^T \Omega M = \Omega \right\}$$

where we have used

$$M^T \Omega M = \Omega \Leftrightarrow \Omega M \Omega^{-1} = (M^T)^{-1} = M^*$$

There are other matrix group called symplectic group:

$$\mathrm{Sp}(2n, \mathbb{R}) = \left\{ M \in \mathrm{GL}(2n, \mathbb{R}) : M^T \Omega M = \Omega \right\}$$
$$\mathrm{Sp}(2n, \mathbb{C}) = \left\{ M \in \mathrm{GL}(2n, \mathbb{C}) : M^T \Omega M = \Omega \right\}$$

Note that, unlike USp(2n), these are non-compact.

Also, Ω can be defined as:

$$\Omega' = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}$$

such Ω' defines the isomorphic groups via:

$$\Omega = T\Omega' T^{-1}$$

for some non-singular T, and then M is related by a similarity transform $M = TM'T^{-1}$.

Physically, the group $\mathrm{Sp}(2n, \mathbb{R})$ appears in classical mechanics. A point in a phase space $\{q_1, \dots, q_n, p_1, \dots, p_n\}$ can be represented by a $2n$ vector v. We define the pairing between two vectors v, v':

$$\Omega(v, v') = v^T \Omega v' = (q_1 p_1' - q_1' p_1) + \cdots + (q_n p_n' - q_n' p_n)$$

Then $M \in \mathrm{Sp}(2n, \mathbb{R})$ preserves the pairing:

$$\Omega(Mv, Mv') = v^T M^T \Omega M v' = v^T \Omega v' = \Omega(v, v')$$

using $M^T \Omega M = \Omega$.

The pairing is appears to be the Poisson bracket:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial f}{\partial q_i} \right)$$

where

$$v = \begin{pmatrix} \frac{\partial f}{\partial q_1} \\ \frac{\partial f}{\partial p_1} \\ \vdots \end{pmatrix} \quad v' = \begin{pmatrix} \frac{\partial g}{\partial q_1} \\ \frac{\partial g}{\partial p_1} \\ \vdots \end{pmatrix}$$

Thus the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ is the group of transformation of $\{q_i, p_i\}$ that preserves Poisson bracket of classical mechanics. *i.e.* linear canonical transformation.

1.12 Representation, Reducibility and Decomposability

Recall that representation is a homomorphism $\rho : G \to \mathrm{GL}(n, \mathbb{C})$. For a matrix group we define the defining rep:

Definition 1.17. If $G \in \mathrm{GL}(n, \mathbb{C})$ is a matrix group, then we have the defining representation $\rho_{\mathrm{def}} : G \to \mathrm{GL}(n, \mathbb{C})$:

$$\rho(M) = M, \forall M \in G$$

For any group given a representation ρ, we can define the dual and conjugate representation:

Definition 1.18. If $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ is a representation of G, then,

$$\begin{aligned} \rho^*(a) &= \rho(a^{-1})^T, \forall a \in G \Rightarrow \text{dual} \\ \bar{\rho}(a) &= [\rho(a)]^*, \forall a \in G \Rightarrow \text{conjugate} \end{aligned}$$

define the dual and conjugate representation of G. One also have the dual, conjugate representation $\bar{\rho}^*$.

For unitary representation, one can show that $\rho^* \sim \bar{\rho}$.

In terms of modules (vector space that the representation acts on)

$$\begin{aligned} \rho : v^i \to \rho(a)^i{}_j v^j, &\qquad v^i \in V \simeq \mathbb{C}^n \\ \rho^* : w_i \to w_j \rho(a^{-1})^j{}_i, &\qquad w_i \in V^* \text{ dual vector space} \end{aligned}$$

therefore $w_i v^i$ is invariant

$$w_i v^i \to w_j \rho(a^{-1})^j{}_k \rho(a)^k{}_i v^i = w_i v^i$$

Given a complex vector space $V \ni v^i$, we have the conjugate vector space $\bar{V} \ni \bar{v}^{\bar{i}}$.

$$\begin{aligned} \bar{\rho} : \bar{v}^{\bar{i}} &\to \bar{\rho}(a)^{\bar{i}}{}_{\bar{j}} \bar{v}^{\bar{j}} \\ \bar{\rho}^* : \bar{w}_{\bar{i}} &\to \bar{w}_{\bar{j}} \bar{\rho}(a^{-1})^{\bar{j}}{}_{\bar{i}} \end{aligned}$$

similarly, $\bar{w}_{\bar{i}} \bar{v}^{\bar{i}}$ is invariant

$$\bar{w}_{\bar{i}} \bar{v}^{\bar{i}} \to w_j \bar{\rho}(a^{-1})^{\bar{j}}{}_{\bar{k}} \bar{\rho}(a)^{\bar{k}}{}_{\bar{i}} \bar{v}^{\bar{i}} = \bar{w}_{\bar{i}} \bar{v}^{\bar{i}}$$

We can construct new representation as module. First, we define the tensor product of modules:

Definition 1.19. Given G-module V and W, the **tensor product module** $V \otimes W$ is given by:

$$a \cdot v \otimes w = av \otimes aw$$

for all $a \in G$ and $v \otimes w \in V \otimes W$

In terms of representation,

G-module: $v^i \in V, \quad v^i \to \rho(a)^i{}_j v^j$, $\mathrm{GL}(n, \mathbb{C})$

G-module: $w^a \in W, \quad w^a \to \rho'(a)^a{}_b w^b$, $\mathrm{GL}(m, \mathbb{C})$

the new representation: $\rho'' : G \to \mathrm{GL}(nm, \mathbb{C})$ and the new G-module:

$$u^{ia} \in V \otimes W, \quad u^{ia} \to \rho(a)^i{}_j \rho'(a)^a{}_b u^{jb}$$

We can also take the direct sum of the modules:

Definition 1.20. GIven two G-modules V and W, one can define a new G-module $V \oplus W$ via:

$$a(v \oplus w) = av \oplus aw$$

where $v \oplus w \in V \oplus W$.

Notice that a given G-module may be reducible or decomposable.

Definition 1.21. A G-module V is said to be **reducible** if there exist a proper invariant subspace $W \subset V$, *i.e.*

1) $a \cdot w \in W$, $\forall w \in W$, $\forall a \in G$

2) $W \neq V$ and $W \neq 0$

Furthermore, W itself is a G-module.

In terms of representation, decomposible module correspond to the representation of the form:

$$T\rho(a)T^{-1} = \left(\begin{array}{c|c} \hat{\rho}(a) & A(a) \\ \hline 0 & B(a) \end{array}\right)$$

and

$$\left(\begin{array}{c} v \\ \hline 0 \end{array}\right) \in W$$

span W,

$$\left(\begin{array}{c|c} \hat{\rho}(a) & A(a) \\ \hline 0 & B(a) \end{array}\right) \left(\begin{array}{c} v \\ \hline 0 \end{array}\right) = \left(\begin{array}{c} \hat{\rho}(a)v \\ \hline 0 \end{array}\right)$$

is indeed an invariant subspace. Note that

$$\dim V \geq \dim W.$$

Definition 1.22. A decomposable G-module V is one such that $V = W_1 \oplus W_2$ where W_1, W_2 are proper invariant subspace.

In terms of representation, we have $\rho \sim \rho_1 \oplus \rho_2$:

$$T\rho(a)T^{-1} = \left(\begin{array}{c|c} \hat{\rho}_1(a) & 0 \\ \hline 0 & \hat{\rho}_2(a) \end{array}\right)$$

with

$$\left(\begin{array}{c} v \\ \hline 0 \end{array}\right) \in W_1, \left(\begin{array}{c} 0 \\ \hline v \end{array}\right) \in W_2$$

We can deduce that for every unitary representation, the reducible representation is decomposible, which is not generally the case.

$$\begin{aligned} \rho(a)^\dagger &= \rho(a^{-1}) \\ \Rightarrow A(a) &= 0 \end{aligned}$$

Definition 1.23. An irreducible G-module is one that has no proper invariant subspace, giving an irreducible representation (irrep).

The idea here is that knowing the irreps allows one to build all possible reps via direct sum.

1.13 Irreducible Representation of U(1)

There is a famous corollary:

Corollary 1.2. *Corollary to Schurs lemma: let V be a finite dimensional irreducible G-module, there exist a homomorphism $f : V \to V$, then f is proportional to the identity.*

This implies that all irreducible representation of Abelian group is 1-dimensional. Since U(1) is Abelian, so the irrep is 1-dimensional

$$\rho : \mathrm{U}(1) \to \mathrm{GL}(1, \mathbb{C}) \backsimeq \mathbb{C} - \{0\}$$

$$e^{i\theta} \to \rho(\theta)$$

this is a homomorphism $\rho(\theta_1)\rho(\theta_2) = \rho(\theta_1 + \theta_2)$.

By considering the infinitesimal around $\theta = 0$, and using the property of homomorphism, we can find the expression of ρ

$$\rho(\theta) = e^{in\theta}, n \in \mathbb{Z}$$

This integer n encodes the electric charge, for the wavefunction having U(1) symmetry.

In QM, we want unitary representation, hence the Hilbert space decompose into irreducible 1-dimensional representation labelled by n, which is the quantisation of electric charge.

1.14 Irreducible Representation of SU(2)

We want to find a way of constructing all SU(2). We will build some irreps out of the defining representation, this will actually gives all irreps.

Consider SU(2),

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} : xx^* + yy^* = 1, X, Y \in \mathbb{C} \right\}$$

- trivial representation: $\rho_1 : \mathrm{SU}(2) \to \mathrm{GL}(1, \mathbb{C})$ such that $\rho(a) = 1$ for all $a \in \mathrm{SU}(2)$.
- defining representation: $\rho_2 : \mathrm{SU}(2) \to \mathrm{GL}(2, \mathbb{C})$,

  $$\rho(a) = a = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix}$$

  act on module $V \backsimeq \mathbb{C}^2$. For $v^i \in V$,

  $$v^i \mapsto {\rho^i}_j(a) v^j, \; i = 1, 2$$

- dual and conjugate representation:

  $$\begin{aligned} \rho_2^*(a) &= (a^{-1})^T = a^8 = \bar{\rho}_2(a) \\ &= \begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix} \\ &= T \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} T^{-1}, \qquad \text{where } T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

  so we have $\rho_2(a) \sim \rho_2^*(a) \sim \bar{\rho}_2(a)$, which gets nothing new.

We can also tensor product the defining representation: $\rho_2(a) \otimes \rho_2(a)$. It acts on the vector $w^{ij} \in W = V \otimes V$

$$w^{ij} = \begin{pmatrix} w^{11} \\ w^{12} \\ w^{21} \\ w^{22} \end{pmatrix}$$

has four components. By definition of tensor product, we have

$$\begin{aligned} w^{ij} &\mapsto {\rho^i}_k(a) {\rho^j}_l(a) w^{kl} \\ &= \begin{pmatrix} x^2 & -xy^* & -y^*x & y^*y^* \\ xy & xx^* & -y^*y & -y^*x^* \\ yx & -yy^* & x^*x & -x^*y^* \\ y^2 & yx^* & x^*y & x^*x^* \end{pmatrix} \begin{pmatrix} w^{11} \\ w^{12} \\ w^{21} \\ w^{22} \end{pmatrix} = {\tilde{\rho}^{ij}}_{kl}(a) w^{kl} \end{aligned}$$

which gives a new four dimensional representation.

However, this representation is decomposable. There are two invariant sub-spaces:

$$\begin{aligned} W_1 &= \{w \in W : w^{ij} = -w^{ji}\}, \quad \text{anti-symmetric} \\ W_3 &= \{w \in W : w^{ij} = w^{ji}\}, \quad \text{symmetric} \end{aligned}$$

For W_1, $W_1 \ni w^{ij} = \lambda \epsilon^{ij}$ where ϵ^{ij} is the total anti-symmetric tensor. Thus,

$$\begin{aligned} w^{ij} &\mapsto \rho^i{}_k(a) \rho^j{}_l(a) \lambda \epsilon^{kl} \\ &= \lambda(\rho(a) \epsilon \rho(a)^T)^{ij} = \lambda \epsilon^{ij} \in W_1 \end{aligned}$$

which is indeed invariant, and this is just the trivial representation.

For W_3 we get a new representation:

$$W_3 \ni w^{ij} = \begin{pmatrix} w^{11} & w^{12} \\ w^{12} & w^{22} \end{pmatrix}$$

from the tensor product we get

$$\begin{pmatrix} w^{11} \\ w^{12} \\ w^{22} \end{pmatrix} \mapsto \begin{pmatrix} x^2 & -2xy^* & y^{*2} \\ xy & xx^* - yy^* & -x^* y^* \\ y^2 & 2x^* y & x^{*2} \end{pmatrix} \begin{pmatrix} w^{11} \\ w^{12} \\ w^{22} \end{pmatrix}$$

this gives us a 3-dimensional representation $\rho_3(a)$.

Therefore, we have

$$\rho_2 \otimes \rho_2 \backsimeq \rho_1 \oplus \rho_3.$$

Furthermore, we can consider the tensor product of three defining modules: $\rho_2 \otimes \rho_2 \otimes \rho_2$. The module is therefore:

$$W = V \otimes V \otimes V$$

the vector $w^{ijk} \in W$ is 8-dimensional. Under the transformation

$$w^{ijk} \mapsto \rho^i{}_{i'}(a) \rho^j{}_{j'}(a) \rho^k{}_{k'}(a) w^{i'j'k'}$$

one can find three invariant sub-spaces $W = W_2 \oplus W_2' \oplus W_4$:

$$\begin{aligned} W_2 &= \{v \in W_2, v^{ijk} + v^{jik} = 0\} \\ W_2' &= \{v \in W_2, v^{ijk} + v^{ikj} = 0\} \\ W_2 &= \{v \in W_2, v^{ijk} \text{total symmetric}\} \end{aligned}$$

and

$$\rho_2 \otimes \rho_2 \otimes \rho_2 \backsimeq \rho_2 \oplus \rho_2 \oplus \rho_4$$

where the 4-dimensional ρ_4 is the new irreducible representation.

This procedure continuous: For tensor product of n defining representation $\rho_{2_1} \otimes \cdots \otimes \rho_{2_n}$, the module given by $W = V_1 \otimes \cdots \otimes V_n \ni w^{i_1 \cdots i_n}$ is decomposable. And the total symmetric invariant subspace define the new irreps of dimension $n+1$.

This gives all the irreps of SU(2).

$$\text{classified SU(2) irrpes} \Longleftrightarrow \text{ symmetric tensors}$$

$$\rho_n : \mathrm{SU}(2) \to \mathrm{GL}(n, \mathbb{C})$$

labelled by n, and the dimension is $\dim \rho_n = n$. From $\rho_2(a) \sim \rho_2^*(a) \sim \bar{\rho}_2(a)$ we have $\rho_n(a) \sim \rho_n^*(a) \sim \bar{\rho}_n(a)$.

We have shown that $\mathrm{SO}(3) \backsimeq \mathrm{SU}(2)/\mathbb{Z}_2$, thus each element of SO(3) is a left coset of SU(2), *i.e.* $\{a, -a\}$. For a we have,

$$w^{i_1 \cdots i_n} \to \rho_2(a)^{i_1}{}_{j_1} \cdots \rho_2(a)^{i_n}{}_{j_n} w^{j_1 \cdots j_n}$$

and for $-a$,

$$\begin{aligned} w^{i_1 \cdots i_n} &\to \rho_2(-a)^{i_1}{}_{j_1} \cdots \rho_2(-a)^{i_n}{}_{j_n} w^{j_1 \cdots j_n} \\ &= (-1)^n \rho_2(a)^{i_1}{}_{j_1} \cdots \rho_2(a)^{i_n}{}_{j_n} w^{j_1 \cdots j_n} \end{aligned}$$

Hence,

$$\rho_{n+1}(a) = (-1)^n \rho_{n+1}(-a)$$

since in SO(3) we identify a and $-a$, the representation of even n will give the same representation for a and $-a$.

1.15 Young Tableau and SU(n) Irreducible Representation

This symmetric and anti-symmetric structure of SU(2) can be generalised to SU(n). Consider the defining representation of SU(n):

$$\begin{aligned}\rho_{\text{def}} : \mathrm{SU}(n) &\to \mathrm{GL}(n, \mathbb{C})\\ M &\mapsto \rho_{\text{def}}(M) = M\end{aligned}$$

with module being $v^i \in V \backsimeq \mathbb{C}^n$, $i = 1, 2, \ldots, n$.

And the tensor product of them $V \otimes \cdots \otimes V \ni v^{i_1 i_2 \ldots i_p}$

$$v^{i_1 i_2 \ldots i_p} \mapsto {\rho_{\text{def}}}^{i_1}{}_{j_1} {\rho_{\text{def}}}^{i_2}{}_{j_2} \cdots {\rho_{\text{def}}}^{i_p}{}_{j_p} v^{j_1 \ldots j_p}$$

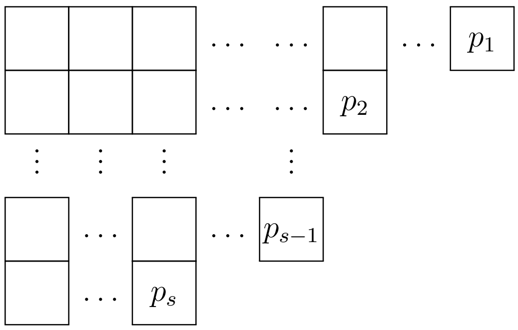
decompose into irreducible modules depending on the symmetry properties of $i_1 \ldots i_p$.

The goal then is to characterise all the symmetry properties of of p indices.

Definition 1.24. Given an ordered partition $\lambda = (p_1, \ldots, p_s)$ of a positive integer $p \in \mathbb{N}$ that is

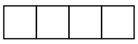
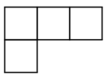
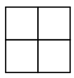
$$\begin{aligned}&p_1 + \ldots p_s = p, \qquad p_i \in \mathbb{N}\\ &p_1 \geq p_2 \geq \cdots \geq p_s\end{aligned}$$

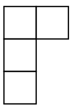
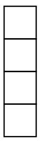
the associated Young Tableau $[\lambda]$ is a diagram



where the ith row has p_i boxes.

Example 1.13. $p = 4$

$\lambda = (4)$  $\lambda = (3, 1)$  $\lambda = (2, 2)$ 

$\lambda = (2, 1, 1)$  $\lambda = (1, 1, 1, 1)$ 

Then

Theorem 1.3. *The finite dimension irreducible representation of* SU(n) *are in 1-to-1 correspondence with Young tableau with* $s < n$. *The tableau encodes the symmetry of the tensor in* $W = V \otimes \cdots \otimes V$ *with the rules:*

1) anti-symmetries on columns

2) symmetries on rows

For example,

- $p = 0$, no copies of V hence it is trivial representation.
- $p = 1$, $W = V \ni v^i$ is the defining representation, there's no symmetric or anti-symmetric.
- $p = 2$, $W = V \otimes V \ni v^{ij}$

  $\boxed{i}\boxed{j}$ symmetrize on i, j, $(s \cdot v)^{ij} = \frac{1}{2}(v^{ij} + v^{ji}) := v^{(ij)}$

  $\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$ Anti-symmetrize on i, j, $(a \cdot v)^{ij} = \frac{1}{2}(v^{ij} - v^{ji}) := v^{[ij]}$

  Gives two irreducible representations.

- $p = 3$, $W = V \otimes V \otimes V \ni v^{ijk}$

  $\boxed{i}\boxed{j}\boxed{k}$ symmetrize on i, j, k,

  $$(s \cdot v)^{ijk} = \frac{1}{6}(v^{ijk} + v^{ikj} + v^{jki} + v^{jik} + v^{kij} + v^{kji}) := v^{(ijk)}$$

  $\begin{array}{|c|c|}\hline i & k \\ \hline j & \\ \cline{1-1}\end{array}$ Anti-symmetrize on i, j, following by symmetrizing on i, k

  $$\begin{aligned}(s \cdot a \cdot v)^{ijk} &= s \cdot \frac{1}{2}(v^{ijk} - v^{jik}) \\ &= \frac{1}{2}\left(\frac{1}{2}\left(v^{ijk} + v^{kji}\right) - \frac{1}{2}\left(v^{jik} + v^{jki}\right)\right) \\ &= \frac{1}{4}\left(v^{ijk} + v^{kji} - v^{jik} - v^{jki}\right)\end{aligned}$$

- Note that by convention we define to always do anti-symmetrize on colomn first and then symmetrize on rows.

Why is $s < n$? Consider a column of n indices:

$$\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \end{array} \\ \vdots \\ \begin{array}{|c|}\hline i_n \\ \hline \end{array}$$

The module is

$$v^{[i_1 i_2 \ldots i_n]} = \lambda \epsilon^{i_1 i_2 \ldots i_n},$$

where ϵ is the total anti-symmetric term. The representation,

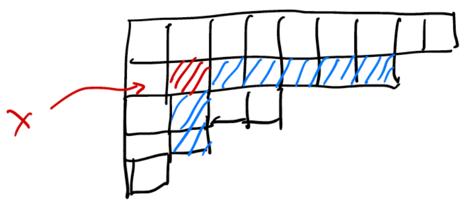
$$\begin{aligned} v^{[i_1 i_2 \ldots i_n]} &\mapsto {\rho_{\text{def}}}^{i_1}{}_{j_1} \cdots {\rho_{\text{def}}}^{i_n}{}_{j_n} \lambda \epsilon^{j_1 j_2 \ldots j_n} \\ &= (\det \rho_{\text{def}}) \lambda \epsilon^{i_1 i_2 \ldots i_n} \end{aligned}$$

For $\mathrm{SU}(n)$ the (det) $= 1$, so its a trivial map. So, any column with n-boxes can be replaced with trivial map.

What is the dimension of the irreducible representation?

Definition 1.25. Let $x = (i, j)$ be the jth box in the ith row of a Young Tableau $[\lambda]$ then,

$$\text{hoolc}(i, j) = \#\text{ boxes to the right of } x + \#\text{ boxes below } x + 1$$



hoolc(x) $= 8$

Theorem 1.4. *Let W be the irrep $\mathrm{SU}(n)$ module associate to $[\lambda]$. Then,*

$$\dim W = \prod_{(i,j) \in \lambda} \frac{n + j - i}{\mathit{hoolc(i,j)}}$$

Example 1.14. $\mathrm{SU}(7)$, $n = 7$,

- $\dim\left(\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ \\ \cline{1-1} \end{array} \right) = \dfrac{\begin{array}{|c|c|c|}\hline 7 & 8 & 9 \\ \hline 6 \\ \cline{1-1} \end{array}}{\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}} = 378$

Young Tableau can also keep track of the decomposition of tensor product.

Example 1.15. In SU(5), $v^i \in \ydiagram{1}$ is the defining representation.
Considering $v^{ijk} \in \ydiagram{2,1}$, $\dim(\ydiagram{2,1}) = \frac{5\times6\times4}{3\times1\times1} = 40$. What would be the tensor product of

$$\ydiagram{1} \otimes \ydiagram{2,1} = ?$$

The result should have four indices, hence we consider all the four indices diagram and match the dimension.

$$\ydiagram{1} \otimes \ydiagram{2,1} = \ydiagram{3,1} + \ydiagram{2,2} + \ydiagram{2,1,1}$$

The dimension:

$$\dim\left(\ydiagram{3,1}\right) = \frac{5\times6\times7\times4}{4\times2\times1\times1} = 105$$
$$\dim\left(\ydiagram{2,2}\right) = \frac{5\times6\times4\times5}{3\times2\times2\times1} = 50$$
$$\dim\left(\ydiagram{2,1,1}\right) = \frac{5\times6\times4\times3}{4\times1\times2\times1} = 45$$

And the LHS has dimension $= 5\times40 = 200$ which is equal to the sum of these three diagrams.

But Warning! Things get more complicated when dealing with tensor product like:

$$\ydiagram{1,1} \otimes \ydiagram{2,1}$$

Since $\rho^*_{\text{def}} \sim \bar{\rho}_{\text{def}}$, then $\rho^* \sim \bar{\rho}$ for all irreps defined by a Young Tableau. Given a ρ, the ρ^* can be determined from the Young Tableau.

Theorem 1.5. *Dual (or conjugate)* SU(n) *irreps correspond to Young Tableau that fit together as a rectangle of height n.*

Example 1.16. For SU(3),

$$\ydiagram{4,1}$$

is dual to

$$\ydiagram{4,3}$$

as they fit together to form a height 3 rectangle.

Example 1.17. Dual of defining representation of SU(n) $\Box$,

$$\Box^* = \begin{array}{c}\Box\\\Box\\\vdots\\\Box\end{array} \ni v^{[i_1 \dots i_{n-1}]}$$

with $n-1$ blocks. Why?

$$\begin{array}{ll}\Box \ni v^i & v^i \mapsto {\rho_{\mathrm{def}}}^i{}_j(a) v^j \\ \Box^* \ni w_i & w_i \mapsto w_j {\rho_{\mathrm{def}}}^j{}_i(a^{-1})\end{array}$$

The relation of $v^{[i_1 \dots i_{n-1}]}$ and w_j is

$$v^{[i_1 \dots i_{n-1}]} = \epsilon^{i_1 \dots i_{n-1} j} w_j$$

where ϵ is the total anti-symmetric tensor and it is an SU(n) invariance. We can see that

$$\begin{gathered} v^{[i_1 \dots i_{n-1}]} \mapsto {\rho_{\mathrm{def}}}^{i_1}{}_{j_1} \cdots {\rho_{\mathrm{def}}}^{i_{n-1}}{}_{j_{n-1}} v^{[j_1 \dots j_{n-1}]} \\ \Updownarrow \\ w_i \mapsto w_j ({\rho_{\mathrm{def}}}^j{}_i)^{-1} \end{gathered}$$

since

$${\rho_{\mathrm{def}}}^{i_1}{}_{j_1} \cdots {\rho_{\mathrm{def}}}^{i_n}{}_{j_n} \epsilon^{j_1 \dots j_n} = \det(\rho_{\mathrm{def}}) \epsilon^{i_1 \dots i_n} = \epsilon^{i_1 \dots i_n}$$

Consider $\rho_{\mathrm{def}} \otimes \rho^*_{\mathrm{def}}$

$$\Box \otimes \begin{array}{c}\Box\\\Box\\\vdots\\\Box\end{array}_{n-1} = \begin{array}{c}\Box\\\Box\\\vdots\\\Box\end{array}_{n} \oplus \begin{array}{ll}\Box & \Box\\\Box\\\vdots\\\Box\end{array}_{n-1} = 1 \oplus \begin{array}{ll}\Box & \Box\\\Box\\\vdots\\\Box\end{array}_{n-1}$$

Hence,

$$\dim \left(\begin{array}{ll}\Box & \Box\\\Box\\\vdots\\\Box\end{array}_{n-1} \right) = n^2 - 1 = \text{dim of the group}$$

is the **adjoint** representation.

For Lie groups there is always a special representation that has has the same dimension as the group, that is the adjoint representation.

$$\begin{array}{|c|c|}\hline i_1 & j \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} i_{n-1} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \ni v^{i_1 \dots i_{n-1} j} = \epsilon^{i_1 \dots i_{n-1} i} a^j{}_i$$

$a \in V \otimes V^*$ and

$$a^i{}_j \mapsto \rho_{\text{def}}(a)^i{}_k \; a^k{}_l \; \rho^{-1}_{\text{def}}(a)^l{}_j$$

where $\operatorname{Tr} a = 0$. So the adjoint representation is the traceless n x n matrices

Final comment: one can use refinements of Young Tableau to define the irreducible representation of the other classical groups $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$.

Now, we have to account for the extra invariant tensor.

$$\begin{aligned}&\text{metric } \delta_{ij} \quad \mathrm{SO}(n)\\ &\text{symplectic bilinear } \Omega_{ij} = -\Omega_{ji} \quad \mathrm{Sp}(n)\end{aligned}$$

Example 1.18. Given $\mathrm{SO}(n)$, the module $\square\square \ni v^{(ij)}$ further decompose into invariant subspaces

$$v^{(ij)} = \tilde{v}^{ij} + v_0 \delta^{ij}$$

symmetric traceless and trace parts.

Example 1.19. For $\mathrm{Sp}(n)$ the module $\begin{smallmatrix}\square\\\square\end{smallmatrix} \ni v^{[ij]}$ decompose into

$$v^{[ij]} = \tilde{v}^{ij} + v_0 \Omega_{ij}$$

where the $\tilde{v}^{ij}$ is orthogonal to the symplectic form $\tilde{v}_{ij}\Omega_{ij} = 0$

Finally, for the projection representation of $\mathrm{SO}(n)$, only the even dimension irreps of its universal cover is the projection representation.

2 Particle Physics

2.1 Physics and Unitary Irreducible Representation

Recall that physical symmetries in QM are realised by the unitary representations. So, in order to understand the symmetries of QM, we need to classify the unitary irreps.

Theorem 2.1 (Wigner's unitarian trick)**.** *If G is a compact Lie group, then every complex representation is equivalent to a unitary representation.*

Thus, we only need to classify the complex representations.

Theorem 2.2. *There are no faithful, finite-dimensional, unitary representations of non-compact Lie groups.*

i.e. a faithful unitary representation of non-compact Lie groups is ∞-dimensional.

For a ∞-dimensional irreps, correspond to the solutions $\psi(x)$ to wave equations on some manifold M. Effectively, we say that the non-compact Lie group construct a spacetime M.

i.e. Poincaré group $\sim$ Minkowski spacetime

We will be considering the following three cases:

- Spacetime symmetry: (non-compact)

  irreps of Poincaré symmetry

- Gauge symmetry: forces

  Higgs mechanism

- Global symmetry:

  quark model

2.2 Unitary Irreducible Representation and Poincare Group

There is a theorem due to Wigner:

Theorem 2.3 (Wigner's Theorem)**.** *Unitary irreducible representations of Poincaré group are in one-to-one correspondence with positive energy solutions to relativistic field equations.*

Equivalently, each (unitary irreducible) ***module*** *is a one-particle Hilbert space, which can be labelled by some fixed* ***mass*** *and* ***spin****.*

The Poincaré symmetry is the symmetry transform that preserves the metric of Minkowski: $ds^3 = \eta_{\mu\nu} dx^\mu dx^\nu$, which is given by:

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$$

and

$$\Lambda^\mu{}_\rho \Lambda^\mu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$$

The group can be written in:

$$\text{Poincaré} = \left\{ (\Lambda, a) : \ \Lambda^{-1} \eta \Lambda = \eta \right\}$$

The group production is indeed closed:

$$\begin{aligned}
(\Lambda, a) \cdot (\Lambda', a') \cdot x &= (\Lambda, a) \cdot (\Lambda' x + a') \\
&= \Lambda\Lambda' x + (\Lambda a' + a) \\
&= (\Lambda\Lambda', \Lambda a' + a) \cdot x
\end{aligned}$$

As a matrix group:

$$\text{Poincaré} = \left\{ \left(\begin{array}{c|c} \Lambda & a \\ \hline 0 & \mathbb{1} \end{array} \right) \in \mathrm{GL}(5, \mathbb{R}) : \ \Lambda \in \mathrm{O}(3,1) \right\}$$

where

$$\mathrm{O}(3,1) = \left\{ \Lambda \in \mathrm{GL}(4, \mathbb{R}) : \ \Lambda^{-1} \eta \Lambda = \eta \right\}$$

The subgroup of Poincaré,

1) Lorentz subgroup $= \{(\Lambda, 0) \in \text{Poincaré}\}$

2) Translation subgroup $= \{(\mathbb{1}, a) \in \text{Poincaré}\}$

Further subgroups of Lorentz:

- Orthochronous subgroup: $\mathrm{O}^+(3,1) = \{\Lambda \in \mathrm{O}(3,1) : \ \Lambda^0{}_0 \geq 0\}$

  i.e. there's no time reversal, but parity transform.

- Proper subgroup: $\mathrm{SO}(3,1) = \{\Lambda \in \mathrm{O}(3,1) : \det\Lambda = 1\}$
  i.e. has TP transform but no T or P transform.
- Proper Orthochronous subgroups: $\mathrm{SO}^+(3,1) = \{\Lambda \in \mathrm{O}(3,1) : \det\Lambda = 1, \Lambda^0{}_0 \geq 0\}$
  i.e. neither TP, T, P.

Note that we have $\mathrm{ISO}^+(3,1)$ as the proper orthochronous Poincare group: (Λ, a) such that $\det\Lambda = 1$, $\Lambda^0{}_0 \geq 0$.

$$\mathrm{ISO}^+(3,1) = \left\{ \left(\begin{array}{c|c} \Lambda & a \\ \hline 0 & \mathbb{1} \end{array} \right) \in \mathrm{GL}(5,\mathbb{R}) : \ \Lambda \in \mathrm{SO}^+(3,1) \right\}$$

Furthermore, what we really want is the projective unitary irreps:

$$\text{proj. irreps} \Longleftrightarrow \text{irreps of } \mathrm{Spin}^+(3,1)$$

where $\mathrm{SO}^+(3,1) = \mathrm{Spin}^+(3,1)/\mathbb{Z}_2$. Then really we need is the unitary irreps of $\mathrm{ISpin}^+(3,1)$.

$$\mathrm{ISpin}^+(3,1) = \left\{ \left(\begin{array}{c|c} \Lambda & a \\ \hline 0 & \mathbb{1} \end{array} \right) \in \mathrm{GL}(5,\mathbb{R}) : \ \Lambda \in \mathrm{Spin}^+(3,1) \right\}$$

2.3 Spin-zero Irreducible Representation of $\mathrm{ISO}^+(3,1)$

Since $\mathrm{ISpin}^+(3,1)$ is non-compact (boosts + translation are non-compact), it has a infinite dimensional unitary rep according to Wigner's theorem. So we expect:

- module = ∞-dimensional Hilbert space $\mathcal{H}$
- representations:

$$\begin{aligned} S : \mathrm{ISpin}^+(3,1) &\to \mathrm{U}(\mathcal{H}) \\ (\Lambda, a) &\mapsto S(\Lambda, a) \end{aligned}$$

Thus, we need to construct $\mathcal{H}$ and $S(\Lambda, a)$.

Consider a complex Klein–Gordon equation:

$$(\partial^2 + m^2)\psi(x) = 0 \tag{2.1}$$

Let $\mathcal{H} = \{\text{functions } \psi(x) : \ (\partial^2 + m^2)\psi(x) = 0\}$ This is a function space.

Choose a set of basis: $\{\psi_p(x) = e^{ip\cdot x}\}$ such that $p^2 = m^2$. Then, any solution can be written in:

$$\psi(x) = \int \frac{\mathrm{d}^3\vec{\mathbf{p}}}{2E(\vec{\mathbf{p}})}\psi(p)e^{ip\cdot x}$$

or wirte in bra-ket notation;

$$\begin{aligned} |p\rangle &= e^{ip\cdot x} \\ |\psi\rangle &= \int \frac{\mathrm{d}^3\vec{\mathbf{p}}}{2E(\vec{\mathbf{p}})}\psi(p)e^{ip\cdot x} \end{aligned}$$

The norm on $\mathcal{H}$:

$$\langle p|p'\rangle = (2\pi)^3 2E(\vec{\mathbf{p}})\delta^{(3)}(\vec{\mathbf{p}} - \vec{\mathbf{p}}')$$

is Lorentz covariant.

Define a transformation:

$$\psi(x) \to \psi'(x) = \psi(\Lambda^{-1}(x-a)) \tag{2.2}$$

In the plane wave solutions (basis):

$$\begin{aligned} e^{ip\cdot x} \to e^{ip\cdot\Lambda^{-1}(x-a)} &= e^{i\Lambda p\cdot(x-a)} \\ &= e^{-i(\Lambda p)\cdot a}e^{i(\Lambda p)\cdot x} \end{aligned}$$

where

$$
\begin{aligned}
p \cdot \Lambda^{\text{-}1} x &= \eta_{\mu\nu} p^{\mu} (\Lambda^{\text{-}1})^{\nu}{}_{\rho} x^{\rho} \\
&= p^{\mu} x^{\rho} (\eta_{\mu\nu} (\Lambda^{\text{-}1})^{\nu}{}_{\rho}) \\
&= p^{\mu} x^{\rho} (\Lambda^{\sigma}{}_{\mu} \eta_{\sigma\rho}) \\
&= \eta_{\sigma\rho} (\Lambda^{\sigma}{}_{\mu} p^{\mu}) x^{\rho} \\
&= (\Lambda p) \cdot x
\end{aligned}
$$

hence, we have:

$$|p\rangle \mapsto e^{-i(\Lambda p)\cdot a} |\Lambda \cdot p\rangle$$

And we define the unitary representation:

$$S(\Lambda, a) |p\rangle = e^{-i(\Lambda p)\cdot a} |\Lambda p\rangle$$

such that $\langle p| S^{\dagger} S |q\rangle = \langle p|q\rangle$.

It turns out that this Hilbert space is irreducible, but we can decompose it in

$$\mathcal{H} = \mathcal{H}_{m^2,+} \oplus \mathcal{H}_{m^2,-}$$

which correspond to positive and negative energy.

What we already know about the irreps:

- Transnational subgroup: $(\mathbb{1}, a)$

  This irrep is 1-dimensional: $\mathcal{H}_p = \{\lambda |p\rangle : \lambda \in \mathbb{C}\}$. We have $|p\rangle \mapsto e^{-ip\cdot a} |p\rangle$, therefore the one-dimensional representation is

  $$\mathrm{U}(a) = e^{-ip\cdot a}$$

  Note that this is not a faithful finite dimensional rep.

- The big Hilbert space is built out of one-dimensional irrep of translation

  $$\mathcal{H}_{m^2,+} = \bigoplus_{\substack{p \\ p^2=m^2 \\ p^0>0}} \mathcal{H}_p$$

2.4 General Irreducible Representations of Poincare Group

Now, we look at another wave equation: (Proca equation)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.3}$$

$$\partial^\mu F_{\mu\nu} + m^2 A_\nu = 0 \tag{2.4}$$

We have:

$$\partial^\nu (\partial^\mu F_{\mu\nu} + m^2 A_\nu) = 0$$

since $F_{\mu\nu}$ is anti-symmetric, we then have

$$\partial^\nu A_\nu = 0$$

which is the Lorentz gauge freedom. Substituting gives:

$$\partial^2 A_\nu + m^2 A_\nu = 0$$

which are four Klein–Gordon equations.

We then have the plane wave solution:

$$A_\mu(x) = \varepsilon_\mu e^{-ip\cdot x},\ p^2 = m^2,\ p^\mu \varepsilon_\mu = 0$$

generally,

$$A_\mu(x) = \int \frac{\mathrm{d}^3\vec{\mathbf{p}}}{2E(\vec{\mathbf{p}})} \tilde{A}_\mu(p) e^{ip\cdot x},\ p^\mu \tilde{A}_\mu = 0 \tag{2.5}$$

Space of solutions form a Hilbert space.

For each p we have a set of vectors expanded in polarisation:

$$\tilde{A}_\mu(p) = \psi_i(p) \varepsilon^i_\mu$$

$$p = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \varepsilon^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ \varepsilon^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ \varepsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

which satisfy $\varepsilon^i_\mu \varepsilon^{j\mu} = \delta^{ij}$ and $p^\mu \varepsilon_\mu = 0$. Therefore, we have

$$|A_\mu\rangle = \int \frac{\mathrm{d}^3\vec{\mathbf{p}}}{2E(\vec{\mathbf{p}})} \psi_i(p) \left|p, \varepsilon^i\right\rangle$$

where $|p, \varepsilon^i\rangle = \varepsilon^i_\mu e^{ip\cdot x}$. We can also check the normalisation

$$\left\langle p', \varepsilon^i \middle| p, \varepsilon^j \right\rangle = \delta^{ij} (2\pi)^3 2E(\vec{\mathbf{p}}) \delta^{(3)}(\vec{\mathbf{p}} - \vec{\mathbf{p}}')$$

Then, we write the unitary representation: $S(\Lambda, a)$

$$\begin{aligned}\left|p, \varepsilon^i\right\rangle \mapsto S(\Lambda, a)\left|p, \varepsilon^i\right\rangle \\ = e^{-i(\Lambda p)\cdot a}\left|\Lambda \cdot p, \Lambda \cdot \varepsilon^i\right\rangle\end{aligned}$$

fix $p^0 > 0$ again gives a **irreducible unitary representation**.

Consider the polarisation ε^i.

Definition 2.1. We define the "little group", or "stabilised group" H_p that leave p invariant.

i.e. the rotation in this case

$$\mathrm{SO}^+(3,1) \supset \mathrm{H}_p = \left\{\Lambda^\mu{}_\nu \in \mathrm{SO}^+(3,1) : \Lambda^\mu{}_\nu p^\nu = p^\mu\right\}$$

notice that $\mathrm{H}_p \backsimeq \mathrm{SO}(3)$ is the rotation group and the little group for any time-like vector p are isomorphic.

The ε^i form a basis of the vector of the form b

$$v^\mu = \begin{pmatrix} o \\ v^x \\ v^y \\ v^z \end{pmatrix} = \left(\begin{array}{c} 0 \\ \hline \vec{\mathbf{v}} \end{array}\right)$$

Under the little group: $i = 1, 2, 3$

$$v^i = R^i{}_j v^j$$

where v^i transforming under the defining rep of H_p,

$$\mathrm{H}_p \ni \Lambda = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & R^i{}_j \end{array}\right)$$

The polarisation vectors form a basis of a module for the little group, which is the defining rep of SO(3). Thus, the basis for Proca equation solution:

$$\left|p, \varepsilon^i\right\rangle \in \mathcal{H}_p \otimes V$$

where V is the defining module.

We see that Proca Hilbert space generalises from the Klein-Gordon Hilbert space by choosing an irrep of the little group for each

$$|p\rangle \in \mathcal{H}_p \longrightarrow \left|p, \varepsilon^i\right\rangle \in \mathcal{H}_p \otimes V$$

Actually, we want $\mathrm{ISpin}^+(3,1)$

- project representation of $\mathrm{ISO}^{+}(3,1) \sim$ irrep of $\mathrm{ISpin}^{+}(3,1)$
- project representation of $\mathrm{SO}(3,1) \sim$ irrep of $\mathrm{SU}(2)$

Thus, we expect wave equation with each irreps of SU(2)

There is a Wigner classification of SU(2) irreps;

Wave equation	spin	irrep of SU(2)	dimension
Klein-Gordon	spin-0	trivial	1
Dirac	spin-1/2	$\square$	2
Proca	spin-1	$\square\square$	3
Massive 3/2-spin	spin-3/2	$\square\square\square$	4
Pauli-Fierz	spin-2	$\square\square\square\square$	5

Thus, we have the Hilbert space labelled by the mass and spin as,

$$\mathcal{H}_{p,s} = \mathcal{H}_p \otimes V_s$$

where V_s is the spin s module. The spin s module has representation as a symmetric tensors with $n-1$ indices, $n = 2s+1$.

Recall the translation subgroup: 1-d irreps

$$\mathbb{C} \backsimeq \mathrm{H}_p \ni |p\rangle\,, \quad \mathrm{U}(\mathbb{1}, a) = e^{-ip\cdot a}\,|p\rangle$$

For spin s, we can write

$$\mathcal{H}_{p,s} = \mathcal{H}_p \otimes V_s \backsimeq \mathbb{C}^{2s+1} \ni |p, i\rangle\,, i = 1, 2,\ \ldots, 2s+1$$

Thus we construct the big Hilbert space of solution of wave equations;

$$\mathcal{H} = \bigoplus_{\substack{p \\ p^2=m^2 \\ p^0>0}} \mathcal{H}_{p,s}$$

e.g. For Klein-Gordon takes continuous value of p

$$\mathcal{H} \ni |p\rangle = \int \frac{\mathrm{d}^3\vec{\mathbf{p}}}{2E(\vec{\mathbf{p}})}\psi(p)e^{ip\cdot x}.$$

How about the massless case?

We follow the same procedure, but get a different little group. We can boost any p into the form

$$p^\mu = \begin{pmatrix} E \\ E \\ 0 \\ 0 \end{pmatrix}$$

And we find that the little group correspond to this is

$$\mathrm{H}_p \backsimeq \mathrm{ISO}(2)$$

which can be regarded as the translation + rotation in the 2-d subspace *i.e.* boost in y, z direction and rotation $\mathrm{SO}(2) \subset \mathrm{SO}^+(3)$ in y, z plane. Now each Poincaré irrep correspond to

1) mass zero: $m^2 = 0$
2) irrep of the little group ISO(2)

To understand the irreps of ISO(2), we can use the same strategy as for $\mathrm{ISO}^+(3)$

1) fix a "momentum" for the "translation" group (in 2-d subspace)
2) find the little group of ISO(2) (little little group)

For the translation subgroup,

$$\mathcal{H}_p \ni |k\rangle \to e^{-ik\cdot c}, \quad k = (k^1, k^2), \; c = (c^1, c^2)$$

For the "little^2 group",

if $k \neq 0$, there is no rotation that leave it invariant,

if $k = 0$, then the little little group is SO(2).

i.e. rotation of a vector in plane is invariant unless the vector is 0.

So, there are two possibilities:

1) $k = 0$:

Since $\mathrm{SU}(2) \backsimeq \mathrm{U}(1)$, we have the rep of U(1), which is labelled by an integer n. We define the helicity $\alpha = \frac{1}{2}n$ with the helicity irrep.

$$\mathcal{H} = \bigoplus_{\substack{p \\ p^2=m^2 \\ p^0>0}} \mathcal{H}_p \otimes V_\alpha$$

where V_α is the 1-d irreps of the U(1) with helicity α.

$$\begin{cases} \alpha = 0, & \text{massless Klein-Gordon} \\ \alpha = \pm\frac{1}{2}, & \text{Chiral mode of massless Dirac} \\ \alpha = \pm 1, & \text{two polarisation of EM} \end{cases}$$

2) $k = (k^1, k^2) \neq 0$:

$$\mathcal{H} = \bigoplus_{\substack{p^2=m^2 \\ p^0>0 \\ k^2=t^2}} \mathcal{H}_p \otimes \mathcal{H}_{k^i}$$

where t^2 is a fixed value. This module depends on two continuous variables p^μ, k^i. Thus, it correspond to solutions of wave equations in 6-d. This is called the continuous spin irrep.

2.5 Quark Model

In the Standard Model, there are three families of leptons and quarks

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}$$

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}$$

The interactions are

- strong force: SU(3)
- Electroweak: SU(2) $\times$ U(1) and spontaneously broken to U(1) for Electromagnetism.

Because of confinement, we can only observe quarks in bound states. The dominant interaction of bound states is strong force, which treats all quarks equally. Hence, if we take u, d, s quarks, there is a approximate symmetry of flavour as $\mathrm{SU}(3)_{\text{flavour}}$ when strong force dominant.

Therefore we have the quark model, which approximate hadrons by combination of valence quarks, corresponding to conserved quark number with respect to the approximate symmetry. The aim is to understand hadrons using approximate symmetry.

We have the SU(3) flavour symmetry

$$\text{triplet: } \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

which is in the defining rep of SU(3). The big Hilbert space

$$\mathcal{H} = V \times \mathcal{H}_{m,s=1/2}$$

where V is the defining module of SU(3), $V \backsimeq \mathbb{C}$. In addition, each quark is in the defining rep of SU(3) of colour,

$$\mathcal{H} = V_{\text{colour}} \otimes V_{\text{flavour}} \otimes \mathcal{H}_{\mathrm{s}=1/2}$$

In bound state, we can think of the quarks at rest.Instead of the full Poincare group, we only need to focus on the little group SU(2) of spin because of the fixed momentum.

So, we simplity the quark state as in the space:

$$\mathcal{H} = V_{\text{colour}} \otimes V_{\text{flavour}} \otimes V_{\text{spin}}$$

which are defining rep os $\mathrm{SU}(3)_{\text{colour}}$, $\mathrm{SU}(3)_{\text{flavour}}$, and $\mathrm{SU}(2)_{\text{spin}}$, respectively.

Given a quark state $q^{ia\alpha} \in \mathcal{H}$, we have

- $i = 1, 2, 3$ flavour
- $a = 1, 2, 3$ colour
- $\alpha = 1, 2$ spin$-1/2$

gives the representation:

$$q^{ia\alpha} \mapsto {\rho_{3f}}^{i}{}_{j}{\rho_{3c}}^{a}{}_{b}{\rho_{2s}}^{\alpha}{}_{\beta}q^{jb\beta}$$

we denote it as (3,3,2) or $(\square_3, \square_3, \square_2)$

First we construct the baryon consisting 3 quarks.

$$B^{ijk\ abc\ \alpha\beta\gamma} \in \mathcal{H} \times \mathcal{H} \times \mathcal{H}$$

denoting:

$$(\square_3 \otimes \square_3 \otimes \square_3,\ \square_3 \otimes \square_3 \otimes \square_3,\ \square_2 \otimes \square_2 \otimes \square_2)$$

Performing the tensor product,

$$\begin{aligned}\square \otimes (\square \otimes \square) &= \square \otimes \left(\ydiagram{2} \oplus \ydiagram{1,1}\right) \\ &= \ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}\end{aligned}$$

- For SU(3),
$$\ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \mathbb{1}$$
with dimension 10, 8, 8, 1.
- For SU(2),
$$\ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1}$$
with dimension 4, 2, 2.

We also have two physical implementations

- Confinement: baryon is neutral with respect to the strong charge, which is in the trivial representation of $\mathrm{SU}(3)_{\text{colour}}$. *i.e.* $\mathrm{SU}(3)_{\text{colour}}$ must in $\ydiagram{1,1,1}$, which is the total anti-symmetric one.

- Pauli-statistics: As quarks are Fermions, the wavefunction must be anti-symmetric under exchange of $ia\alpha$, $jb\beta$, $kc\gamma$.

Hence, given the $\mathrm{SU}(3)_\mathrm{c}$ singlet, we can write the baryon as

$$B^{ijk\ \alpha\beta\gamma}\epsilon^{abc}$$

and $B^{ijk\ \alpha\beta\gamma}$ must be symmetric under exchange of $i\alpha$, $b\beta$, $c\gamma$. which means that $\mathrm{SU}(3)_\mathrm{f}$ and $\mathrm{SU}(2)_\mathrm{s}$ module must have the same symmetry.

Consider the $B^{ijk\ \alpha\beta\gamma}$,

$$B^{ijk\ \alpha\beta\gamma} \sim \left(\ydiagram{3}_3 \oplus \ydiagram{2,1}_3 \oplus \ydiagram{2,1}_3 \oplus \mathbb{1}_3,\ \ydiagram{3}_2 \oplus \ydiagram{2,1}_2 \oplus \ydiagram{2,1}_2\right)$$

Pauli statistic tells us that the only possibilities are

- $\dim = 10 \times 4 = 40$

$$(\ydiagram{3}_3, \ydiagram{3}_2)$$

- $\dim = 8 \times 2 = 16$

$$\left(\ydiagram{2,1}_3, \ydiagram{2,1}_2\right)$$

For each pair of $(i\alpha)$ we have six values to take. Hence, symmetric in $i\alpha$, $b\beta$, $c\gamma$ correspond to $\ydiagram{3}_6$ which has dim $= 56$. Thus, we have

$$(\ydiagram{3}_3, \ydiagram{3}_2) \oplus \left(\ydiagram{2,1}_3, \ydiagram{2,1}_2\right)$$

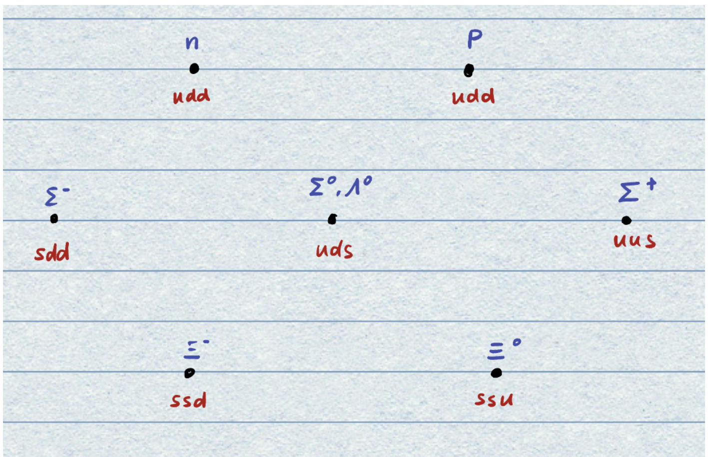
has the exact dim of 56.

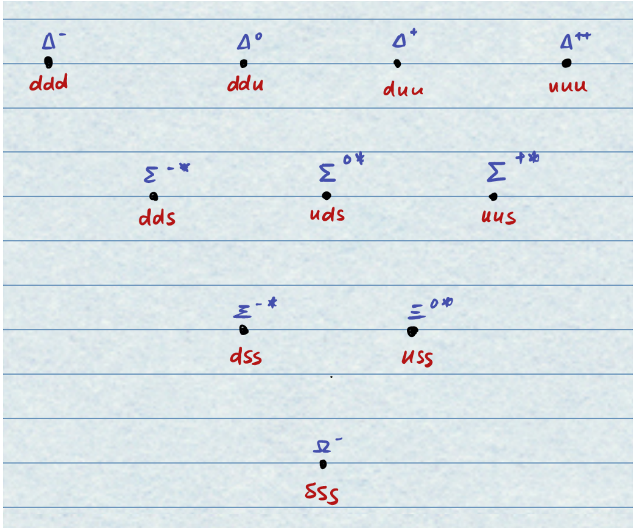
Then we have two types of baryons:

i) (10,1,4) - spin 3/2

ii) (8,1,2) - spin 1/2

and we expect 10 spin 3/2 particles and 8 spin 1/2 particles.





We can see that there is a good symmetry for $u \leftrightarrow d$, but s is heavier. The observation of Δ^{++} gives a strong evidence of the quantum number: colour.

Then we look at mesons. To construct the meson we need the anti-quarks, which is in the dual/conjugates representation: $(\bar{3}, \bar{3}, \bar{2}) \sim \left(\ydiagram{1,1}_3, \ydiagram{1,1}_3, \square_2\right)$.

The meson is also a colour singlet: $q\bar{q}$.

$$q\bar{q} \sim \square_3 \otimes \ydiagram{1,1}_3 = \ydiagram{1,1,1}_3 \oplus \ydiagram{2,1}_3 = \mathbb{1} \oplus \ydiagram{2,1}_3$$

Since $q\bar{q}$ are different, there's no Pauli exclusion.

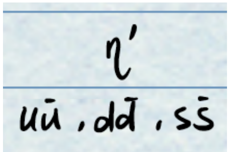
The representation:

$$(\mathbb{1}, 3 \times \bar{3}, 2 \times \bar{2}) = \left(\mathbb{1}, \mathbb{1}_3 \oplus \ydiagram{2,1}_3, \ydiagram{2}_2 \oplus \ydiagram{1,1}_2\right)$$

i) $(1, 1, 1) \oplus (1, 8, 1)$ spin-0

ii) $(1, 8, 3) \oplus (1, 1, 3)$ spin-1

correspond to four types of mesons. Here we give two examples of pseudo-scalar meson (spin-0): octet (1,8,1) and singlet (1,1,1). The other two have the similar structures.



We can write the state with the done label the dual, we have

$$M^{i\ a\ \alpha}_{\ j\ b\ \beta} = M^{i\ \alpha}_{\ j\ \beta}\delta^{a}_{\ b}$$

for a $\mathrm{SU}(3)_c$ singlet.

For spin-0:

$$M^{i\ \alpha}_{\ j\ \beta} = M^{i}_{\ j}\delta^{\alpha}_{\ \beta}$$

for $\mathrm{SU}(2)_s$ singlet. We can further decompose $M^{i}_{\ j}$ in to trace-less part and trace

$$M^{i}_{\ j} = M_{o\ j}^{\ i} + M_0\delta^{i}_{\ j} \sim \text{ octet} + \text{singlet}$$

Note that

- η' should looks like $\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$
- πs and η should look like the diagonal trace-less

$$i.e.\ \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix}, \ldots$$

However, actually, what we observed are not consistent with what we predicted.

- π^0 is almost $u\bar{u} + d\bar{d}$
- η'/η are almost $s\bar{s}$

The group theory apparently cannot tell us everything.

There are other possible bound state: $i.e.$ pentaquark: $qqq\bar{q}\bar{q}$

3 Lie Algebra and Semi-simple Lie Group

3.1 Exponential Map and Lie Algebra

Lie algebra captures the structure of a Lie group in the infinitesimal neighbourhood of $\mathbb{1}$, which is almost enough to reproduce the whole Lie group (up to global structure). One can define Lie algebra geometrically but recalling

Theorem 3.1 (Peter-Weyl Theorem)**.** *Any compact Lie group is isomorphic to a subgroup of a unitary group.*

So we will only focus on matrix Lie groups. Let us start with the exponential map for matrices.

Definition 3.1. The exponential of a matrix M is

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

where $M^0 = 1$. It converges for any M and defines a smooth map.

The exponential e^M satisfies the properties:

- $e^0 = \mathbb{1}$
- $(e^X)^{-1} = e^{-X}$
- $(e^X)^* = e^{X^*}$
- $(e^X)^T = e^{X^T}$
- $e^{\alpha X + \beta X} = e^{\alpha X} e^{\beta X}, \forall \alpha, \beta \in \mathbb{C}$
- $e^{PXP^{-1}} = Pe^X P^{-1}$

Notice that in general $e^X e^Y \neq e^{X+Y}$ does not hold for all matrices X and Y. we will see why in the next section 3.2.

Definition 3.2. Let G be a matrix group. The Lie algebra of G is given by

$$\mathfrak{g} \equiv \text{Lie}(G) := \{X : e^{tX} \in G, \forall t \in \mathbb{R}\}$$

We may expand the exponential e^{tX} around $t = 0$,

$$e^{tX} = \mathbb{1} + tX + O(t^2)$$

Since e^{tX} is a smooth function of t by definition, we can find its derivative at $t = 0$.

$$\left.\frac{d}{dt} e^{tX}\right|_{t=0} = X$$

Proposition 3.1. $\det\left(e^X\right) = e^{\mathrm{Tr}(X)}$

Proof. Suppose the matrix X can be diagnosed as $X = PDP^{-1}$ where D is a diagonal matrix. Then we have

$$\det\left(e^X\right) = \det\left(Pe^D P^{-1}\right) = \det\left(e^D\right) = \mathrm{Tr}\{D\} = \mathrm{Tr}\{X\}$$

where we use the fact that the trace is invariant under similarity transformations. □

Let us see some examples of Lie algebra.

Example 3.1. $\mathrm{SO(n) = \{M \in GL(n, \mathbb{R}) : M^T M = \mathbb{1}, \det\{M\} = 1\}}$
If $M = e^{tX}$, then $M^T = e^{tX^T}, M^{-1} = e^{-tX}$, hence we require

$$M^T = M^{-1} \iff tX^T = -tX \iff X^T = -X$$

Hence, we obtain

$$\mathfrak{so}(n) = \mathrm{Lie}(SO(n)) = \{X \in \mathbf{GL(n}, \mathbb{R}) : X^T = -X\}$$

One may notice that there is also the other requirement

$$1 = \det(M) = \det\left(e^{tX}\right) = e^{\mathrm{Tr}(X)} \Rightarrow \mathrm{Tr}(X) = 0$$

but which is already implied by $X^T = -X$. Hence, we see that $\mathfrak{so}(n) = \mathrm{Lie}(SO(n))$ and $\mathfrak{o}(n) = \mathrm{Lie}(O(n))$ are the same.

Example 3.2. $\mathrm{SU(n) = \{M \in GL(n, \mathbb{C}) : M^\dagger M = \mathbb{1}, \det(M) = 1\}}$
Again considering $M = e^{tX}$, then $M^\dagger = e^{tX^\dagger}, M^{-1} = e^{-tX}$, we have

$$M^T = M^{-1} \iff tX^\dagger = -tX \iff X^\dagger = -X$$

Also $\det(M) = 1$ requires $\mathrm{Tr}\, X = 0$, hence we obtain the Lie algebra

$$\mathfrak{su}(n) = \mathrm{Lie(SU(n))} = \left\{\mathrm{X} \in \mathbf{GL}(\mathrm{n}, \mathbb{C}) : \mathrm{X}^\dagger = -\mathrm{X}, \mathrm{Tr}\,\mathrm{X} = 0\right\}$$

(In physics, it is preferred to write $M = e^{itX}$ such that X is Hermitian, i.e. $X^\dagger = X$.)

Proposition 3.2. *The algebra $\mathfrak{g}$ of a Lie group G is a vector space over $\mathbb{R}$ with a bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

i) $aX + bY \in \mathfrak{g}, \forall X, Y \in \mathfrak{g}, \forall a, b \in \mathbb{R}$

ii) $[X, Y] = -[X, Y] \in \mathfrak{g}, \forall X, Y \in \mathfrak{g}$

Note that for a matrix Lie group,

$$[X, Y] = XY - YX$$

Definition 3.3. Given a Lie algebra $\mathfrak{g}$ one can choose a basis $\{T_i\}$ such that $X = X^i T_i$ for all $X \in \mathfrak{g}$ then

$$[T_i, T_j] = f_{ij}{}^k T_k$$

where $f_{ij}{}^k$ are called the "**structure constants**" of $\mathfrak{g}$. They uniquely encode the bracket and hence the Lie algebra.

Let us see some examples.

Example 3.3. One may write the Lie algebra of SU(2) in the following form.

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ia_3 & a_1 + ia_2 \\ -a_1 + ia_2 & -ia_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}$$

in which the elements are anti-Hermitian and traceless. It is easy to check that the three matrices

$$T_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

form a basis of $\mathfrak{su}(2)$ and satisfy the relation

$$[T_i, T_j] = 2\varepsilon_{ijk} T_k$$

where ε_{ijk} is the Levi-Civita symbol.

Example 3.4. The Lie algebra $\mathfrak{so}(3)$ can be given by

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & a_1 & -a_2 \\ a_1 & 0 & a_3 \\ a_2 & -a_3 & 0 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}$$

in which the elements are anti-symmetric.

$$\widetilde{T}_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{T}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \widetilde{T}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

give a basis of $\mathfrak{so}(3)$ and satisfy

$$[\widetilde{T}_i, \widetilde{T}_j] = \varepsilon_{ijk} \widetilde{T}_k$$

From above two examples, one can find a bijective map, $T_i \mapsto 2\widetilde{T_i}$, hence $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ have the same structure and are the same Lie algebra. Recall that $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$, geometrically its the 3-sphere with antipodal points identification. However, the antipodal identification does not affect the local geometry near $\mathbb{1}$, they still have the same Lie algebra.

These examples motivate us to define the homomorphism for Lie algebras. First, we abstract the definition of the Lie algebra.

Definition 3.4. A finite-dimensional real Lie algebra $\mathfrak{g}$ is a finite dimensional vector space over $\mathbb{R}$ with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

1) $[\cdot,\cdot]$ is bilinear,

2) $[X, Y] = -[Y, X]$ (anti-symmetry),

3) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

Here is an important theorem about Lie algebra isomorphism.

Theorem 3.2. *Every abstract real Lie algebra is isomorphic to the Lie algebra of some Lie group.*

And we also define the homomorphism and isomorphism.

Definition 3.5. A Lie algebra homomorphism between two Lie algebras, $\mathfrak{g}$ and $\mathfrak{h}$ is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$$

for all $X, Y \in \mathfrak{g}$. If φ is a bijection then it defines an isomorphism and we write $\mathfrak{g} \backsimeq \mathfrak{h}$.

Geometric meaning of Lie algebras

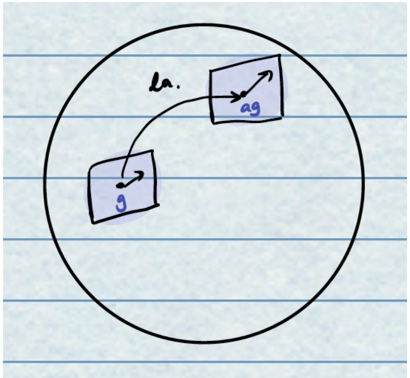
Really, Lie algebras are about geometry. Here we give a geometric meaning of the Lie algebra.

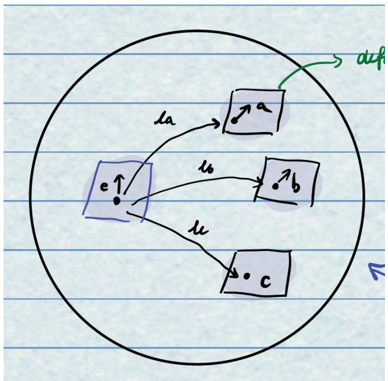
The Lie group is a manifold with a smooth product law.

We define the Left-action:

$$\begin{aligned} l_a : G &\to G \\ g &\to ag, \ \forall a, g \in G \end{aligned}$$

which also gives a map on the tangent space $T_gG \to T_{ag}G$ By running a through all element in G we can take a vector $v \in T_eG$ and map it to a vector $l_a v$ at any T_aG.





This gives a vector field for each vector on T_eG, which forms the **left-invariant** vector field.

Therefore, the Lie algebra is the is a set of left-invariant vector fields on G. And locally every neighbourhood looks the same.

Definition 3.6. A Lie sub-algebra is a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that

$$[X, Y] \in \mathfrak{h}$$

for all $X, Y \in \mathfrak{h}$. If a Lie group is a sub-group of another Lie group, *i.e.* $H \subset G$, then $\text{Lie}(H) \subset \text{Lie}(G)$.

Definition 3.7. Suppose we have two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. One can construct a new Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the bracket

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2])$$

for all $X_1, Y_1 \in \mathfrak{g}_1$ and $X_2, Y_2 \in \mathfrak{g}_2$. Moreover, if $G = G_1 \times G_2$, then $\text{Lie}(G) = \text{Lie}(G_1) \oplus \text{Lie}(G_2)$.

3.2 Lie Theorem

We have introduced the idea of Lie algebra, but how can we go from Lie algebra to Lie group? An important theorem provides the idea.

Theorem 3.3. *Let G and H be Lie groups with Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$. Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, then there exists a unique Lie group homomorphism $\Phi : G \to H$ such that*

$$\Phi(e^X) = e^{\varphi(X)}$$

for all $X \in \mathfrak{g}$.

Corollary 3.4. *Let G and H be simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}$. Then if $\mathfrak{g} \backsimeq \mathfrak{h}$, then $G \backsimeq H$.*

i.e. up to isomorphism, there is a unique simply-connected Lie group for each Lie algebra.

This means that to understand all simply connected Lie groups, we just need to understand all Lie algebras. There are two ingredients to show hos this works.

(1) The Lie bracket essentially defines the group product.

Proposition 3.3 (Baker-Campbell-Hausdorff formula)**.** *Any two elements, X and Y, of a Lie algebra satisfy*

$$e^X e^Y = e^Z$$

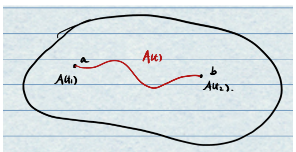
where

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{4}[X.[X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots$$

where $[\cdot, \cdot]$ is the Lie bracket.

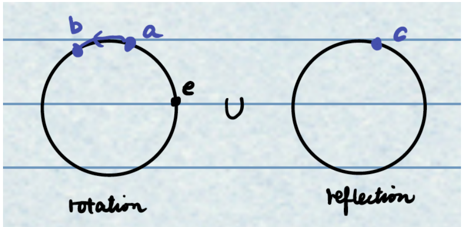
(2) Topological restrictions, *e.g.* "simply connected" on G, so that each $\mathfrak{g}$ gives a unique G.

Definition 3.8. A Lie group G as a manifold is (path-)connected if for any two points $a, b \in G$ there exists a continuous path $A(t)$, $t_1 \leq t \leq t_2$ in G such that $A(t_1) = a$ and $A(t_2) = b$.

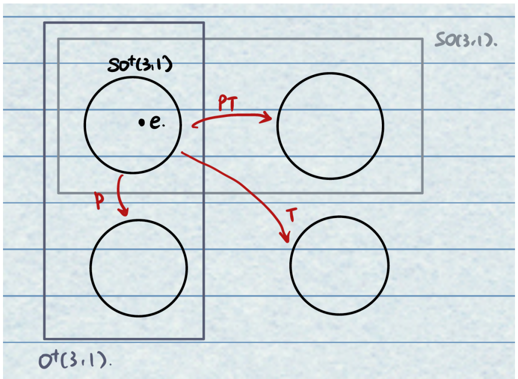


Example 3.5.

$$\mathrm{O}(2)=\left\{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}:0\le\theta\le 2\pi\right\}\bigcup\left\{\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}:0\le\theta\le 2\pi\right\}$$



Example 3.6. For $\mathrm{O}(3,1)$, there are four not connected components.



This is generally true:

- $G_{conn}\subset G$ is the connected component contains the identity. It is a normal subgroup.
- $\Gamma = G/G_{conn}$ is a discrete group.

Since the Lie algebra determines the structure near the identity, we would only expect to reconstruct G_{conn}.

Definition 3.9. A Lie group G, viewed as a manifold, is simply connected if it is connected and every loop in G can be continuously shrunk to a point. (Roughly speaking, there is no hole on the manifold G.)

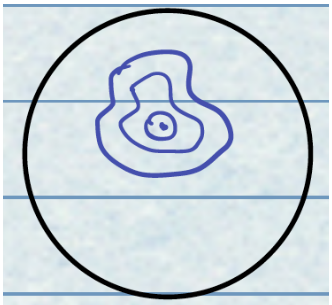
A more formal description is that: Consider a set of loops $\{A(s,t) : 0\le s\le 1, t_1\le t\le t_2\}$ such that

1) for fixed s, $A(s,t_1)=A(s,t_2)$,

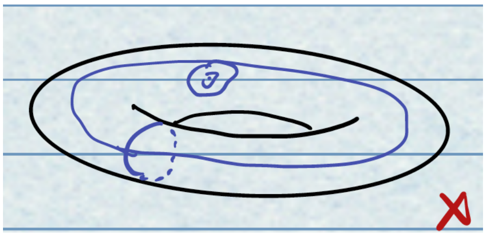
2) for $s=0$, $A(0,t)=A(t)$ where $A(t)$ is the original loop,

3) for $s = 1$, $A(1, t) = c$ where c is a fixed point.

Example 3.7. The sphere is simply connected.



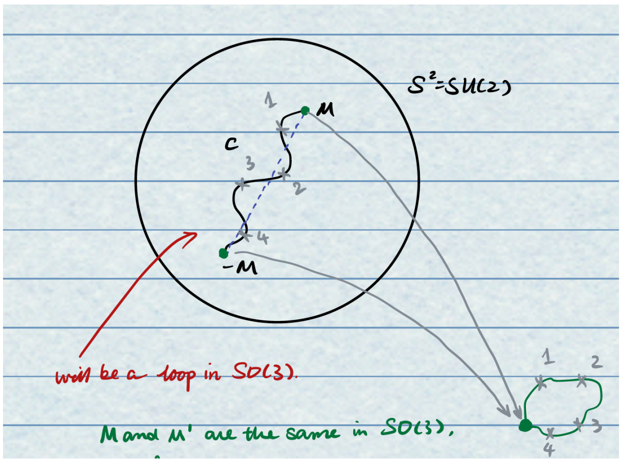
Example 3.8. The torus is not simply connected.



Example 3.9. The Lie group

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} : \mathrm{xx}^* + \mathrm{yy}^* = 1 \right\} \backsimeq \mathrm{S}^2$$

is simply connected. $\mathrm{SO}(3) \backsimeq \mathrm{SU}(2)/\mathbb{Z}_2$ is not simply connected.



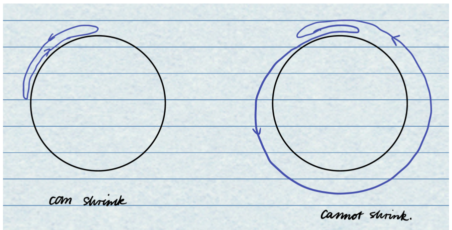
Example 3.10. The Lie group

$$\mathrm{U}(1) = \left\{\mathrm{e}^{\mathrm{i}\theta} : 0 \leq \theta < 2\pi\right\} \backsimeq \mathrm{S}^1$$

with Lie algebra

$$\mathfrak{u}(1) = \mathrm{Lie}\{\mathrm{U}(1)\} = \{\mathrm{ix} : \mathrm{x} \in \mathbb{R}\} \backsimeq \mathbb{R}$$

S^1 is not simply connected.



So what is the simply connected Lie group with Lie algebra $\mathfrak{u}(1)$? The answer is real numbers under addition, $(\mathbb{R}, +)$, or positive real numbers under multiplication, $(\mathbb{R}_{\geq 0}, \times)$. These two are isomorphic via the exponential map

$$\mathbb{R} \to \mathbb{R}_{\geq 0}, \lambda \mapsto e^{\lambda}$$

Moreover, there is a discrete subgroup $\mathfrak{u}(1) = \mathbb{R}/\mathbb{Z}$ where $\mathbb{Z} \subset \mathbb{R}$ is normal.

Example 3.11. In general, for the classical Lie groups,

- SO(n) is not simply connected,
- SU(n) is simply connected,
- Sp(n) is simply connected.

Definition 3.10. The spin group Spin(n) is the simply-connected Lie group with Lie algebra Lie{SO(n)}. In general, $\mathrm{SO(n)} \backsimeq \mathrm{Spin(n)}/\mathbb{Z}_2$.

For small n, there are some "accidental" isomorphism.

- $\mathrm{Spin}(2) \backsimeq \mathbb{R}$ under addition or $\mathbb{R}^+$ under multiplication
- $\mathrm{Spin}(3) \backsimeq \mathrm{SU}(2)$
- $\mathrm{Spin}(4) \backsimeq \mathrm{SU}(2) \times \mathrm{SU}(2)$
- $\mathrm{Spin}(5) \backsimeq \mathrm{Sp}(2)$
- $\mathrm{Spin}(6) \backsimeq \mathrm{SU}(4)$

We can define the same thing for SO(n − 1, 1): Spin(3, 1) $\backsimeq$ SL(2, $\mathbb{C}$).

The relation between simply-connected and non-simply-connected groups with the same Lie algebra is as bellows.

Definition 3.11. Let G be a connected Lie group. The **universal cover** of G is a simply connected Lie group $\widetilde{G}$ with a Lie group homomorphism $\Phi : \widetilde{G} \to G$ such that the associated Lie algebra homomorphism $\phi : \mathrm{Lie}(\widetilde{G}) \to \mathrm{Lie}(G)$ is an isomorphism.

Theorem 3.5. *The universal cover always exists (Though the universal cover of a matrix Lie group is not necessarily a matrix Lie group). Furthermore, one can define the Kernal*

$$\ker \Phi := \{a \in \widetilde{G} : \Phi(a) = e\}$$

then $\ker \Phi$ *is a normal, Abelian, discrete subgroup of* $\widetilde{G}$*, called the fundamental group* $\pi_1(G)$ *of* G*, and* $G = \widetilde{G}/\ker \Phi$*.*

Example 3.12. Take G = SO(3), $\tilde{G}$ = SU(2), $\Psi : \tilde{G} \to G$. Since $\Psi(M) = \Psi(-M)$, ker$\Psi = \{\mathbb{1}, -\mathbb{1}\}$. Therefore, SO(3) = SU(2)/$\mathbb{Z}_2$.

Summary:

- There is a one-to-one correspondence between abstract Lie algebras with simply-connected Lie groups.
- Furthermore, every abstract Lie algebra has a corresponded Lie group.
- Connected Lie groups arise from the simply-connected Lie group via the universal cover.
- Disconnected Lie group arise from the semi-direct product of connected Lie group with discrete group. *i.e.* $G = G_{dis}/\Gamma$

3.3 Representation of Lie Algebra

Definition 3.12. A complex finite-dimensional representation of a Lie algebra $\mathfrak{g}$ is a homomorphism $\hat{\rho} : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ such that

$$\hat{\rho}([X, Y]) = [\hat{\rho}(X), \hat{\rho}(Y)]$$

for all $X, Y \in \mathfrak{g}$.

Similar to Lie groups, the representation of a Lie algebra can be given by the module. Hence, we define the module for a Lie algebra.

Definition 3.13. A (left) $\mathfrak{g}$-module is a vector space V over $\mathbb{C}$ with a product $\mathfrak{g} \times V \to V$ denoted by $X \cdot v$ such that

1) $[X, Y] \cdot v = X \cdot (Y \cdot v) - Y \cdot (X \cdot v), \forall X, Y \in \mathfrak{g}$

2) $X \cdot (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 X \cdot v_1 + \lambda_2 X \cdot v_2, \forall \lambda_1, \lambda_2 \in \mathbb{C}$

We can import the same notions from the representation of Lie groups. For example,

- equivalence: If $\hat{\rho} = T\hat{\rho}' T^{-1}$, then $\hat{\rho} \sim \hat{\rho}'$.
- reducible: The $\mathfrak{g}$-module V is reducible if there exists a proper invariant subspace $W \subset V$.
- decomposible: A $\mathfrak{g}$-module V is decomposible if it can be written as a direct sum of invariant subspace.

Given a representation of a Lie group G, how can we find the representation of its Lie algebra $\mathfrak{g}$? Let $\rho : G \to \mathrm{GL}(\mathrm{n}, \mathbb{C})$ be a representation of G, then

$$\frac{d}{dt}\rho(e^{tX})\bigg|_{t=0} := \hat{\rho}(X)$$

defines a representation of $\mathfrak{g}$, or equivalently,

$$\rho(e^{tX}) = \rho(\mathbb{1} + tX + \cdots) = \mathbb{1} + t\hat{\rho}(X) + \cdots$$

In particular, this implies that

- if we have a tensor product representation ρ of G as a module $V \otimes W$, then

$$a \cdot (v \otimes w) = a \cdot v \otimes a \cdot w \Rightarrow u^{ia} = v^i \otimes w^a \mapsto \rho^i{}_j \rho^a{}_b u^{jb}$$

- for the Lie algebra this defines a module: $a = e^{tX}$,

$$X \cdot (v \otimes w) = X \cdot v \otimes w + v \otimes X \cdot w \Rightarrow u^{ia} = v^i \otimes w^a \mapsto \hat{\rho}^i{}_j u^{ja} + \hat{\rho}^a{}_b u^{ib}$$

When does a representation $\hat{\rho}$ of $\mathfrak{g}$ give a representation ρ of G?

Corollary 3.6. *For simply-connected Lie groups G, there is a one-to-one correspondence between the representations of G and representations of* $\mathrm{Lie}(G)$.

Example 3.13. Consider SU(2) and SO(3),

- As Lie algebra: $\mathfrak{su}(2) \backsimeq \mathfrak{so}(3)$, hence they share the same representation.
- As Lie group: $\mathrm{SU}(2) \backsimeq \mathrm{SO}(3)/\mathbb{Z}_2$, hense only the odd dimension irrep of SU(2) is the same as SO(3)

Definition 3.14. If we view the Lie algebra $\mathfrak{g}$ as a vector space, it defines a $\mathfrak{g}$-module $V \backsimeq \mathfrak{g}$,

$$X \cdot v = [X, v] \in V$$

for all $X \in \mathfrak{g}$ and $v \in V \backsimeq \mathfrak{g}$, $\dim V = \dim \mathfrak{g}$. This is the **adjoint module**. It is a homomorphism

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\dim \mathfrak{g}, \mathbb{R}).$$

Check that this is a module:

- Linearity:

$$\begin{aligned} X \cdot (\lambda_1 v_1 + \lambda_2 v_2) &= [X, \lambda_1 v_1 + \lambda_2 v_2] \\ &= \lambda_1 [X, v_1] + \lambda_2 [X, v_2] \\ &= \lambda_1 X \cdot v_1 + \lambda_2 X \cdot v_2 \end{aligned}$$

- Homomorphism:

$$\begin{aligned} [X, Y] \cdot v &= [[X, Y], v] \\ &= [X, [Y, v]] - [Y, [X, v]] \quad \leftarrow \textit{using Jacobi identity} \\ &= [X, Y \cdot v] - [Y, X \cdot v] \\ &= X \cdot (Y \cdot v) - Y \cdot (X \cdot v) \end{aligned}$$

Therefore, for all Lie groups, one can extend the Lie algebra adjoint representation to a group representation.

Definition 3.15. For matrix Lie group G, viewing $\mathfrak{g}$ as a vector space of matrices. We have the adjoint $\mathfrak{g}$-module: $G \times V \to V$:

$$M \cdot v = MvM^{-1}, \quad \forall M \in G,\ \forall v \in V \backsimeq \mathfrak{g}$$

We choose a basis $\{T_i\}$ to show this explicitly: $X = X^i T_i$, $v = v^i T_i$. One have:

$$\begin{aligned}[X, v] &= X^i v^j [T_i, T_j] \\ &= X^i v^j {f_{ij}}^k T_k\end{aligned}$$

we have the map of v^i as:

$$v^i \mapsto (X^k {f_{kj}}^i) v^i \equiv \hat{\rho}_{\text{adj}}(X)^i{}_j v^j$$

thus,

$$\hat{\rho}_{\text{adj}}(X)^i{}_j = X^k {f_{kj}}^i \tag{3.1}$$

Given the adjoint map, one can define the Killing form:

Definition 3.16. The Killing for for a Lie algebra $\mathfrak{g}$ is the bilinear map: $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$

$$\langle X, Y \rangle = \operatorname{Tr} \hat{\rho}_{\text{adj}}(X) \hat{\rho}_{\text{adj}}(Y), \quad \forall X, Y \in \mathfrak{g}.$$

By definition, $\langle X, Y \rangle = \langle Y, X \rangle$.

We shall see later that it us useful to complexify Lie algebra:

Definition 3.17. Let $\mathfrak{g}$ be finite dimensional real Lie algebra. We define the complexification $\mathfrak{g}_{\mathbb{C}}$:

$$\mathfrak{g}_{\mathbb{C}} := \{X_1 + iX_2 : X_1, X_2 \in \mathfrak{g}\}$$

such that:

1) $i(X_1 + iX_2) = iX_1 - X_2$
2) $[X_1 + iX_2, Y_1 + iY_2] = [X_1, Y_1] - [X_2, Y_2] + i[X_2, Y_1] - i[X_1, Y_2]$

Example 3.14. Consider $\mathfrak{su}(n)$

$$\mathfrak{su}(n) = \left\{X : X^\dagger = -X, \operatorname{Tr} X = 0\right\}$$

Construct the complexified Lie algebra $X + iY \in \mathfrak{su}(n)_{\mathbb{C}}$:

$$(X + iY)^\dagger = (X^\dagger - iY^\dagger) = -X + iY \neq -X - iY$$

$$\operatorname{Tr} X + iY = \operatorname{Tr} X + i \operatorname{Tr} Y = 0$$

Therefore $\mathfrak{su}(n)_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$ are $n \times n$ trace-less, complex matrices.

One can also find that:

- $\mathfrak{so}(n)_{\mathbb{C}} = \mathfrak{so}(n, \mathbb{C})$: $n \times n$ trace-less, anti-symmetric complex matrices with $X^T = -X$
- $\mathfrak{sp}(n)_{\mathbb{C}} = \mathfrak{sp}(2n, \mathbb{C})$: $2n \times 2n$ trace-less, complex matrices with $\Omega X + X^T \Omega = 0$, where Ω is the symplectic tensor.

In conclusion, we have the following properties:

- If G is simply-connected, then the finite dimensional irrep of G is in one-to-one correspondence to the finite dimensional irrep of $\mathfrak{g}_{\mathbb{C}}$. (Lie Theory)
- If G is simply-connected and compact, the finite dimensional, unitary irrep of G is in one-to-one correspondence to the finite dimensional irrep of $\mathfrak{g}_{\mathbb{C}}$. (Wigner's Trick)

3.4 Semi-simple Lie Algebra and Roots

We can define a special class of Lie groups:

Definition 3.18. A group is **simple** if it has no (connected) proper, normal subgroups. It is **semi-simple** if all proper normal subgroups are non-Abelian.

For Lie algebra, we have an equivalent definition:

Definition 3.19. An ideal of a Lie algebra $\mathfrak{g}$ is a sub-algebra $\mathfrak{h} \subset \mathfrak{g}$ such that:

$$[X, H] \subset \mathfrak{h}, \qquad \forall X \in \mathfrak{g}, \forall H \in \mathfrak{h} \tag{3.2}$$

If H $\subset$ G is normal, then Lie(H) is an ideal of Lie(G)

Then, we can define the semi-simple Lie algebra:

Definition 3.20. A **simple Lie algebra** is one where the only ideals are $\mathfrak{g}$ and $\mathfrak{o}$. And the semi-simple Lie algebra can be written as:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus ... \oplus \mathfrak{g}_n \tag{3.3}$$

where $\mathfrak{g}_i$ are simple Lie algebras.

Similarly a semi-simple group G is a product of simple groups,

$$\mathrm{G} = \mathrm{G}_1 \otimes \mathrm{G}_2 \otimes ... \otimes \mathrm{G}_n \tag{3.4}$$

Amazingly, by Killing and Cartan in their PhD works:

Theorem 3.7. *The complexified semi-simple Lie algebra can be* ***completely classified*** *along with all their finite dimensional irreducible representation.*

Furthermore, we have the theorem:

Theorem 3.8. *A complex Lie algebra is* ***semi-simple*** *iff it is isomorphic to the complexification of the Lie algebra of a* ***compact, simply-connected Lie group****.*

This effectively classified all compact Lie groups, because one can always obtains those connected ones via quotient.

Generally, a compact Lie group can be written in the form:

$$\mathrm{G} = \mathrm{G}_{sc} \times \mathrm{U}(1)^n \tag{3.5}$$

as the simply-connected part and the U(1) factors, which are the only Abelian factors one can have.

Note that there is a equivalent way of defining the semi-simple Lie algebra:

Definition 3.21. A semi-simple Lie algebra is one where the killing form $\langle \cdot, \cdot \rangle$ is non-degenerate. *i.e.* if $\langle X, Y \rangle = 0$, $\forall Y$, then $X = 0$.

So how does the classification works?

The first ingredient is the **Cartan sub-algebra**

Definition 3.22. A Cartan sub-algebra of a semi-simple Lie algebra $\mathfrak{g}$ is a sub-algebra $\mathfrak{h} \subset \mathfrak{g}$ such that:

1. Abelian $[H_1, H_2] = 0 \quad \forall H_1, H_2 \in \mathfrak{h}$
2. Maximal: if $[X, H] = 0 \quad \forall H \in \mathfrak{h}$, then $X \in \mathfrak{h}$

The **rank** of $\mathfrak{g}$ is defined to be the dimension of $\mathfrak{h}$

Recall that $\mathfrak{g}$ is a $\mathfrak{g}$-module it self, which is a adjoint module. Since $\mathfrak{h}$ is a sub-algebra of $\mathfrak{g}$, then $\mathfrak{g}$ is also an $\mathfrak{h}$-module. And since $\mathfrak{h}$ is Abelian, all the irreducible representations of $\mathfrak{h}$ are 1 dimension. Thus, we can decompose $\mathfrak{g}$-module into 1 dimensional $\mathfrak{h}$-modules. This is called the **root decomposition**.

Since all the irreps of Abelian groups are 1-dim, so are the reps of the Abelian Lie algebras. Hence all irreps of $\mathfrak{h}$ is 1-dim.

In detailed, let's look at the Abelian group U(1) representation,

$$\rho_n(\theta) = e^{in\theta}$$

for $\mathrm{U}(1) \times \mathrm{U}(1) \times ... \times \mathrm{U}(1)$ $\mathrm{U}(1)^r$ we have representation

$$\rho_{n_1...n_r}(\theta_1...\theta_r) = e^{in_1\theta_1}...e^{in_r\theta_r}$$

for a general Lie algebra:

$$\mathfrak{h} = diag(ix^1, ..., ix^r)$$

The representation of the Lie algebra $\hat{\rho}\,(x)$, using $\rho(e^{tx}) = e^{t\hat{\rho}(x)}$

$$\hat{\rho}\,(x) = ix^1 + ... + ix^r = n_i(ix^i)$$

We have $ix^i \in \mathfrak{h}$, then $n_i \in \mathfrak{h}^*$, where h^* is the dual space of $\mathfrak{h}$.

Note that the irreps are labelled by n_i and are complex. Hence, in order to decompose $\mathfrak{g}$ as $\mathfrak{h}$-module, we need to complexify $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$.

Definition 3.23. A **root** of $\mathfrak{g}$ is a linear function $\alpha : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ such that there exist a non-zero element $X \in \mathfrak{g}_{\mathbb{C}}$ such that:

$$\mathrm{ad_H} \cdot \mathrm{X} = [\mathrm{H}, \mathrm{X}] = \alpha(\mathrm{H})\mathrm{X}, \ \forall \mathrm{H} \in \mathfrak{h}_{\mathbb{C}}$$

Thus we can define the root space

Definition 3.24. Given a Cartan sub-algebra, $\mathfrak{h} \subset \mathfrak{g}$, a **root space** is a subspace $\mathfrak{g}_\alpha$ given by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} : \ [H, X] = \alpha(H)X, \quad \forall H \in \mathfrak{h}_{\mathbb{C}}\} \tag{3.6}$$

where $\alpha : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$, or equivalently $\alpha \in \mathfrak{h}^*_{\mathbb{C}}$, is the **root**. This root space $\mathfrak{g}_\alpha$ is a one-dimensional $\mathfrak{h}$-module.

If we choose a basis $H = H^i T_i$, $i = 1, ..., r$, $\alpha(H) = \alpha_i H^i$. Thus, $\mathfrak{g}_\alpha$ is a invariant subspace of $\mathfrak{h}$ and is one-dimensional. One can also find that α is in the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$, since it contract with $\mathfrak{h}$ to give a number.

We then have a decomposition of $\mathfrak{g}_{\mathbb{C}}$:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \tag{3.7}$$

one can treat $\mathfrak{h}$ as the $\alpha = 0$ root, where R is the Cartan decomposition.

Example 3.15. For $\mathfrak{su}(2)$,

$$\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a, b, c \in \mathbb{C} \right\}$$

The Cartan sub-algebra is :

$$\mathfrak{h}_{\mathbb{C}} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad \lambda \in \mathbb{C} \right\}$$

which has rank = 1.

Then we find the root spaces

$$\mathfrak{g}_{\alpha_1} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad b \in \mathbb{C} \right\}$$

find $\alpha(H)$,

$$\left[\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right] = 2\lambda \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

thus we find $\alpha_1 = 2$ and

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad b \in \mathbb{C} \right\}$$

Following the same procedure we find $\mathfrak{g}_{\alpha_2}$ and $\alpha_2 = -2$

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \quad c \in \mathbb{C} \right\}$$

Pick the basis of $\mathfrak{h}_\mathbb{C}, \mathfrak{g}_2, \mathfrak{g}_{-2}$:

$$J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}_\mathbb{C}, \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_2, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}_{-2},$$

they form a basis of $\mathfrak{su}(2)$, which satisfy

$$[J_3, J_\pm] = \pm 2 J_\pm$$
$$[J_+, J_-] = J_3$$

Thus, as we expected $\mathfrak{su}(2) = \mathfrak{h}_\mathbb{C} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}$

Example 3.16. For $\mathfrak{su}(3)$, The Cartan sub-algebra is :

$$\mathfrak{h}_\mathbb{C} = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

which has rank $= 2$.

The root spaces are:

$$\mathfrak{g}_{\alpha_1} = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{C} \right\}, \quad \alpha_i \lambda^i = \lambda_1 - \lambda_2, \quad \alpha = (1, -1)$$

$$\mathfrak{g}_{\alpha_2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{C} \right\}, \quad \alpha_i \lambda^i = \lambda_1 + 2\lambda_2, \quad \alpha = (1, 2)$$

$$\mathfrak{g}_{\alpha_3} = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{C} \right\}, \quad \alpha_i \lambda^i = 2\lambda_1 + \lambda_2, \quad \alpha = (2, 1)$$

and the corresponding $\mathfrak{g}_{-\alpha_1}, \mathfrak{g}_{-\alpha_2}, \mathfrak{g}_{-\alpha_3}$

Definition 3.25. The inner product on $\mathfrak{h}_{\mathbb{C}}$ via Killing form, which is equivalence to taking the trace:

$$\langle H, H\rangle = \mathrm{Tr}\left(H^2\right)$$

In the example of $\mathfrak{su}(3)$, one can find the metric,

$$\mathrm{Tr}\left(H^2\right) = \mathrm{Tr}\begin{pmatrix} (\lambda_1)^2 & & \\ & (\lambda_2)^2 & \\ & & (\lambda_1+\lambda_2)^2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = g_{ij}\lambda^i\lambda^j$$

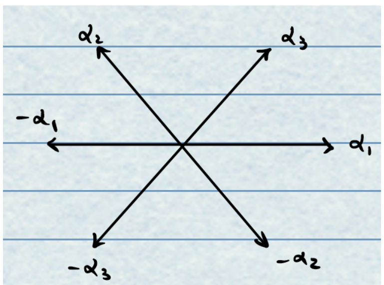
here we take the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ as the metric, taking the inverse metric,

$$g^{ij} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

thus we could define the inner products of α_i to find the length and angle between them,

$$\begin{aligned} &\alpha_1 + \alpha_2 = \alpha_3 \\ &\alpha_1 \cdot \alpha_3 = \alpha_2 \cdot \alpha_3 = 1 \\ &\alpha_2 \cdot \alpha_3 = -1 \end{aligned}$$

using this metric, we have the following diagram in the root space of $\mathfrak{su}(3)$:



Example 3.17. We then take a look at the $\mathfrak{sp}(2)$,

$$\mathfrak{sp}(2)_{\mathbb{C}} = \mathfrak{sp}(4,\mathbb{C}) = \left\{ X = \left(\begin{array}{cc|cc} \lambda_1 & a & c & d \\ b & -\lambda_1 & e & f \\ \hline -f & d & \lambda_2 & a' \\ e & -c & b' & -\lambda_2 \end{array}\right), 4\times 4 \text{ matrices}, \quad X^{\boldsymbol{T}}\Omega + \Omega X = 0, \right\}$$

where,

$$\Omega = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}, \; \mathfrak{h}_{\mathbb{C}} = \left\{ \begin{pmatrix} \lambda_1 & & & \\ & -\lambda_1 & & \\ & & \lambda_2 & \\ & & & -\lambda_2 \end{pmatrix}, \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

$$\mathfrak{g}_{\alpha_1} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \alpha_1 = (2,0)$$

$$\mathfrak{g}_{\alpha_2} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \alpha_2 = (1,-1)$$

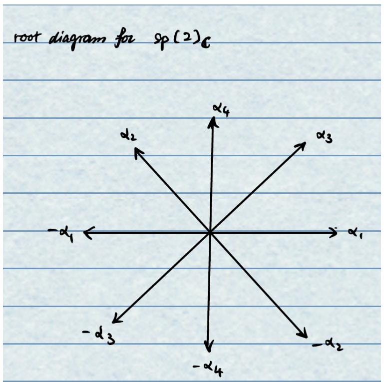
$$\mathfrak{g}_{\alpha_3} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \alpha_3 = (1,1)$$

$$\mathfrak{g}_{\alpha_4} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \alpha_4 = (0,2)$$

where $g_{ij}\lambda^i\lambda^j = \mathrm{Tr}(H^2) = 2\lambda_1^2 + 2\lambda_2^2$

$$g_{ij} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, g^{ij} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Root diagram for $\mathfrak{sp}(2)_{\mathbb{C}}$



Theorem 3.9. *One can reconstruct $\mathfrak{g}$ and its brackets from the root space $\mathfrak{h}^*$.*

To see that, we have

1. The vector space

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

2. Bracket

$$\begin{aligned}
&[\mathfrak{h}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}] = 0, \quad \text{(1-d Abelian)} \\
&[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha \\
&[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin R \\ \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha + \beta \in R \end{cases}
\end{aligned}$$

Why $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$? Using Jacobi identity:

$$\begin{aligned}
[\mathfrak{h}_{\mathbb{C}}, [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]] &= [[\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_\alpha], \mathfrak{g}_\beta] + [\mathfrak{g}_\alpha, [\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_\beta]] \\
&= \alpha(\mathfrak{h}_{\mathbb{C}}) [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] + \beta(\mathfrak{h}_{\mathbb{C}}) [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \\
&= (\alpha + \beta)(\mathfrak{h}_{\mathbb{C}}) [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \\
&= (\alpha + \beta)(\mathfrak{h}_{\mathbb{C}}) \mathfrak{g}_{\alpha+\beta}
\end{aligned}$$

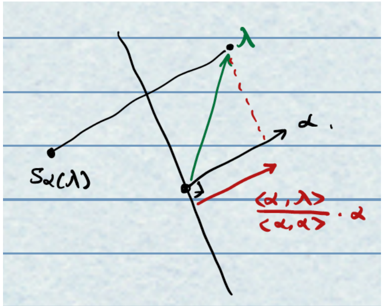
There's a very explicit chose of basis of $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{g}_\alpha$ called the Cheralley basis with all commutators give explicit in terms of roots.

Recall for the semi-simple Lie algebra the inner product on $\mathfrak{h}_{\mathbb{C}}$ is positive definite.

Theorem 3.10. *Let α be a root, and $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ is a reflection in the plane orthogonal to α.*

$$s_\alpha(\lambda) = \lambda - \frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \tag{3.8}$$

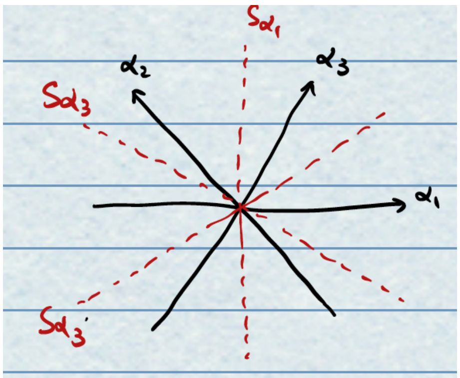
Then, if β is a root, $s_\alpha(\beta)$ is also a root. *i.e.* s_α is an automorphism of R.



Definition 3.26. The Weyl group W is the finite group generated by s_α for all $\alpha \in R$.

Example 3.18. For example, the Weyl group in $\mathfrak{su}(3)$ is isomorphic to S(3), the symmetric group corresponding to all the possible permutations of three elements, or equivalently, the symmetry group of an equilateral triangle.

$$\{e, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_1}s_{\alpha_2}, s_{\alpha_2}s_{\alpha_3}\}$$



3.5 Dynkin Diagram and Semi-simple Lie Algebra

Definition 3.27. A set of fundamental roots:

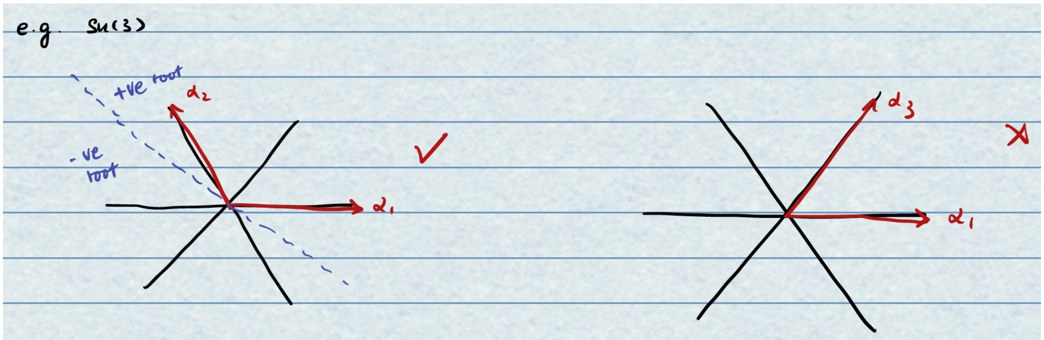
$$\Pi = \{\alpha_1, ..., \alpha_r\} \subset R$$

where r is the rank of $\mathfrak{g}$

Notice that:

1. $\alpha_1, ..., \alpha_r$ forms a basis of $\mathfrak{h}^*$.
2. any roots α can be written as $\alpha = n^i \alpha_i$ with either $n^i \geq 0$ (+ve roots) or $n^i \leq 0$ (-ve roots).

Taking $\mathfrak{su}(3)$ as example:

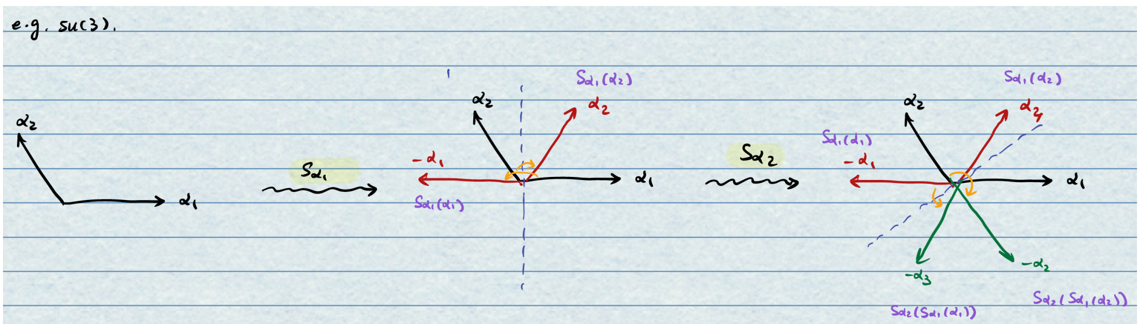


One can find:

- The set of Π is unique up to the action of the Weyl group.
- Can generate all the roots from Π by s_α where $\alpha \in \Pi$.

  i.e. knowing Π means knowing R.

Again, with $\mathfrak{su}(3)$, we can construct the whole R by acting s_α on α:



Thus, to characterise the semi-simple Lie algebra, we need to know all the possible Π.

Definition 3.28. Given a set of fundamental roots $\Pi = \{\alpha_1, ..., \alpha_r\}$, we define the **Cartan matrix**:

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}, \quad \forall \alpha_i, \alpha_j \in \Pi \tag{3.9}$$

In general, $A_{ij} \neq A_{ji}$ and $A_{ii} = 2$.

Note that:

1) $A_{ij} \in \mathbb{Z}$, since $\alpha(H) \in \mathbb{Z}$

2) $A_{ij} \leq 0$, for that the positive coefficients condition forced the angle between the fundamental roots to be greater than $90°$

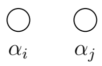
Considering

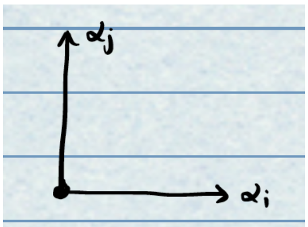
$$K = A_{ij}A_{ji} = \frac{4\langle \alpha_i, \alpha_j \rangle^2}{\langle \alpha_i, \alpha_i \rangle \langle \alpha_j, \alpha_j \rangle} = 4\cos^2\theta_{ij} \tag{3.10}$$

since $A_{ij} \in \mathbb{Z}$, then K can only take four possible values: $K = 0, 1, 2, 3$, which correspond to $\theta = 1/2\pi, 2/3\pi, 3/4\pi, 5/6\pi$, respectively. Then, there are only **four** possibilities of relations between fundamental roots, and we use joint line to represent each case:

(a) α_i, α_j are orthogonal: $K = 0$

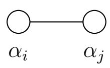
$$\theta = 1/2\pi$$
$$A_{ij} = A_{ji} = 0$$

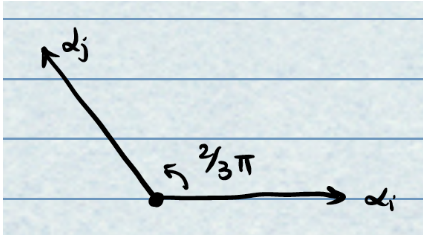
associate with diagram: 



(b) α_i, α_j, $K = 1$, satisfy

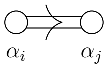
$$\langle \alpha_i, \alpha_i \rangle = \langle \alpha_j, \alpha_j \rangle, \quad \theta = 2/3\pi$$
$$A_{ij} = A_{ji} = -1$$

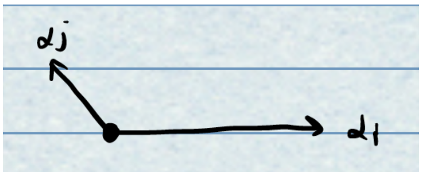
associate with diagram: 



(c) α_i, α_j, $K = 2$, satisfy

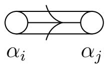
$$\langle \alpha_i, \alpha_i \rangle = 2 \langle \alpha_j, \alpha_j \rangle, \quad \theta = 3/4\pi$$
$$A_{ij} = -1, \quad A_{ji} = -2$$

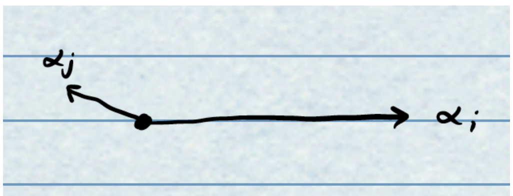
associate with diagram: 



(d) α_i, α_j, $K = 3$, satisfy

$$\langle \alpha_i, \alpha_i \rangle = 3 \langle \alpha_j, \alpha_j \rangle, \quad \theta = 5/6\pi$$
$$A_{ij} = -1, \quad A_{ji} = -3$$

associate with diagram: 



For a given set of fundamental roots, we can encode the geometry in a **Dynkin diagram**. It is also worth noticed that not all the Dynkin diagrams are allowed. We have the theorem that:

Theorem 3.11. *If $\mathfrak{g}$ is a semi-simple Lie algebra, there exist a non-degenerate invariant inner product $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$.*

1) $\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = 0, \ \forall X,Y,Z \in \mathfrak{g}$.

2) $\langle X, Y\rangle = 0, \ \forall Y$, then $X = 0$.

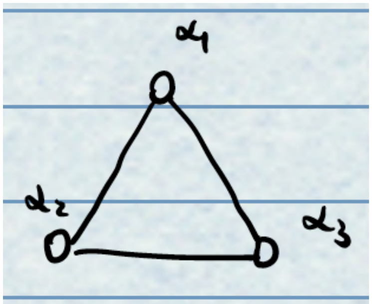
The Killing form $\langle X, Y\rangle = \mathrm{Tr}\,\hat{\rho}_{\mathrm{adj}}(X)\hat{\rho}_{\mathrm{adj}}(Y)$ is an example of this. If $\mathfrak{g}$ is a matrix Lie algebra, one can also use the defining representation: $\langle X, Y\rangle = \mathrm{Tr}\,\hat{\rho}_{\mathrm{def}}(X)\hat{\rho}_{\mathrm{def}}(Y) = \mathrm{Tr}\,XY$

If $\mathfrak{g}$ is simple, then $\langle\ ,\ \rangle$ is unique up to a scale. And we have the key theorem:

Theorem 3.12. *The restriction of $\langle\ ,\ \rangle$ to a (suitable) real Cartan sub-algebra $\mathfrak{h}$ can be fixed to be positive definite. i.e. $\langle H, H\rangle \geq 0, \ \forall H \in \mathfrak{h}$.*

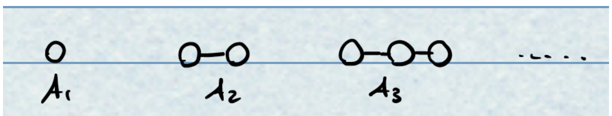
This condition is very restricted on the possible Dynkin diagrams. *i.e.* the off-diagonal elements are ≤ 0, having too much off-diagonal terms will violet this condition.

For example, the following diagram is not allowed, as we can see we cannot construct such bases in $\mathbb{E}^3$

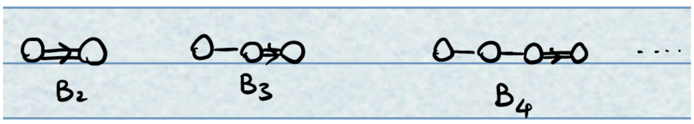


We have found a set of Dynkin diagrams for simple Lie algebras and the simply-connected Lie groups associated to those simple Lie algebras,

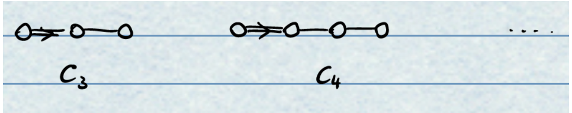
- A_n series: $\mathfrak{su}(n+1)_{\mathbb{C}}$



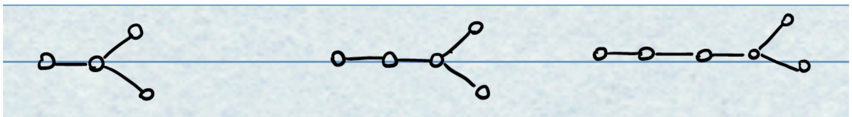
- B_n series: $\mathfrak{so}(2n+1)_{\mathbb{C}}$



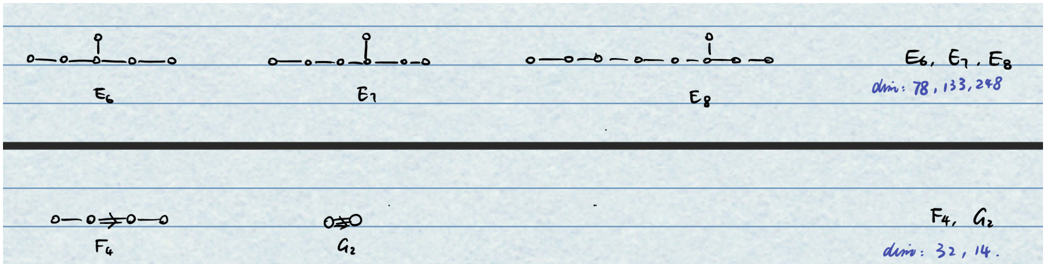
- C_n series: $\mathfrak{sp}(n)_{\mathbb{C}}$



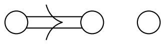
- D_n series: $\mathfrak{so}(2n)_{\mathbb{C}}$



- Exceptional



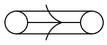
These Dynkin diagram gives all the simple Lie algebras. One can write them together to give the direct sum:

Example 3.19. $\mathfrak{so}(5) \oplus \mathfrak{su}(2)$ associated to the diagram:  which correspond to the Lie group $\text{Spin}(5) \otimes \text{SU}(2)$

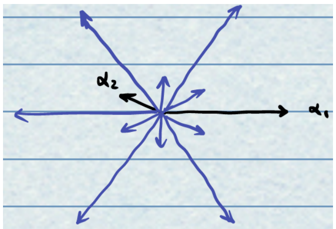
We can also see the "accidental isomorphism" from the diagram:

- $B_1 \sim A_1$ as $\underset{\alpha_i}{\bigcirc}$ which indicates that $\text{Spin}(3) \cong \text{SU}(2)$
- $C_1 \sim A_1 \to \text{Spin}(3) \cong \text{Sp}(1)$
- $C_2 \sim B_2 \to \text{Spin}(5) \cong \text{Sp}(2)$
- $D_3 \sim A_3 \to \text{Spin}(6) \cong \text{SU}(4)$
- $D_2 \sim A_1 \oplus A_1 \to \text{Spin}(4) \cong \text{SU}(2) \otimes \text{SU}(2)$

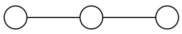
Here we give some examples of constructing the root space from the Dynkin diagram:

Example 3.20. G2:  α_i α_j have the Cartan matrix and the corresponding root diagram

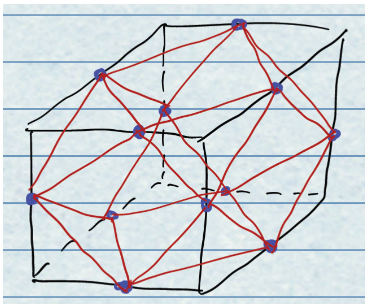
$$A_{ij} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

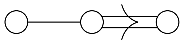


we can check that #root + #rank = 14 = dim(G2)

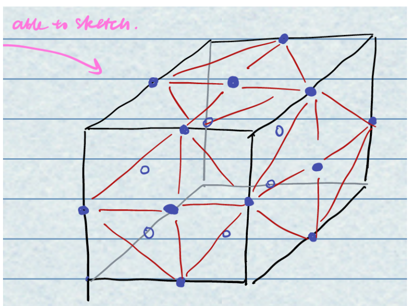
Example 3.21. $\mathfrak{su}(4)_{\mathbb{C}}$: 

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$



Example 3.22. $\mathfrak{so}(7)_{\mathbb{C}}$: 

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$



3.6 Irreducible Representation of Semi=simple Lie Algebras

We can use the root diagram to understand the finite dimensional irreducible representation as well. The rule of constructing the irreducible representation is analogue to what we do in the spin representation of SU(2)

Theorem 3.13. *Let $\mathfrak{g}$ be semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan sub algebra. Then, any $\mathfrak{g}$-module can be decomposed into:*

$$V = \bigoplus_w V_w \tag{3.11}$$

where

$$V_w = \{v \in V \mid H \cdot v = w(H) \cdot v \quad \forall H \in \mathfrak{h}\} \tag{3.12}$$

We called w the weight of V.

Example 3.23. Take $\mathfrak{g}_{\mathbb{C}} = \mathfrak{su}(2)_{\mathbb{C}}$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}$$

we choose a basis:

$$J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}, \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_2, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}_{-2}$$

such that:

$$[J_3, J_\pm] = \pm 2 J_\pm, \quad [J_+, J_-] = J_3 \tag{3.13}$$

In the defining module: $V \cong \mathbb{C}^2$, the defining representation $\hat{\rho}_2(X) = X$, then

$$\hat{\rho}_2(J_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\rho}_2(J_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\rho}_2(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

decompose in $\mathfrak{h}$-module:

$$J_3 \cdot v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$$

thus,

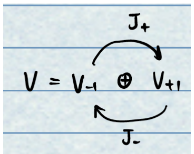
$$V = V_{+1} \oplus V_{-1},$$
$$V_{+1} = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \right\}, \quad V_{-1} = \left\{ \begin{pmatrix} 0 \\ b \end{pmatrix} \right\}$$

we also have:
if $v \in V_{-1}$

$$J_+ \cdot v \in V_{+1}, \quad J_- \cdot v = 0$$

if $v \in V_{+1}$

$$J_- \cdot v \in V_{+1}, \quad J_+ \cdot v = 0$$



For higher n dimension irreducible representation:

$$v^{i_1 \ldots i_n} \to \hat{\rho}_2{}^{i_1}{}_j v^{j i_2 \ldots i_n} + \cdots + \hat{\rho}_2{}^{i_n}{}_j v^{i_1 \ldots i_{n-1} j}$$

For $\hat{\rho}_2(J_3) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ this counts the (number of $i = 1$ indices - number of $i = 2$ indices) as the module's eigenvalue.

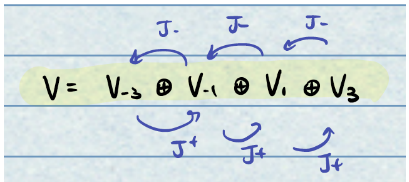
Take $n = 2$ adjoint representation: $\square\square$

$$v^{11} \neq 0 \to \text{eigenvalue} = 2 - 0 = 2$$
$$v^{12} \neq 0 \to \text{eigenvalue} = 1 - 1 = 0$$
$$v^{22} \neq 0 \to \text{eigenvalue} = 0 - 2 = -2$$

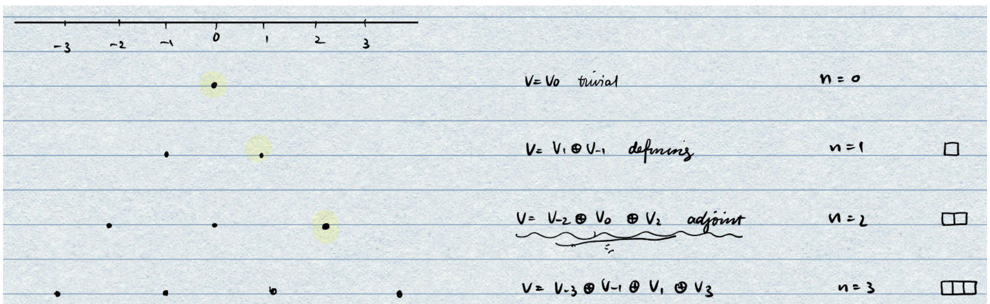
$$V = V_{-2} \oplus V_0 \oplus V_2$$

Take $n = 3$ irreducible representation: $\square\square\square$

$$v^{111} \neq 0 \to \text{eigenvalue} = 3 - 0 = 3$$
$$v^{112} \neq 0 \to \text{eigenvalue} = 2 - 1 = 1$$
$$v^{122} \neq 0 \to \text{eigenvalue} = 1 - 2 = -1$$
$$v^{222} \neq 0 \to \text{eigenvalue} = 0 - 3 = -3$$



In general we get the weight space:



Definition 3.29. If $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ is a set of fundamental roots, we define a set of fundamental weight $\{w_1, \ldots, w_r\}$ via:

$$\alpha_i = \sum_j A_{ji} w_j \tag{3.14}$$

and the weight lattice:

$$\Gamma = \left\{ n^i w_i \mid n^i \in \mathbb{Z} \right\}$$

we also define the dominant integral weight λ

Definition 3.30. A dominant integral weight λ is a element of weight lattice Γ

$$\lambda = \sum_i n_i w_i, \quad \text{with } n_i w_i \text{ non-degenerate}$$

Then we have the theorem:

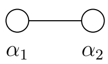
Theorem 3.14. *There is a one-to-one correspondence between the dominant integral weight λ and the finite dimensional module v of $\mathfrak{g}$.*

In the decomposition

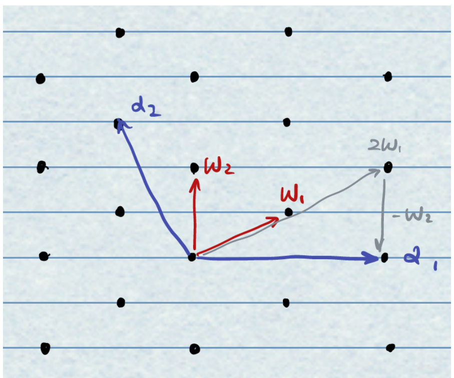
$$V = \bigoplus_w V_w$$

each w has the form $w = \sum_i m_i \alpha_i$, where m_i is non-negative integer, λ is called the highest weight.

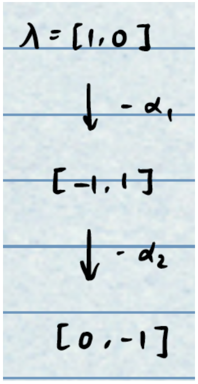
Furthermore, the allowed weights are in $\mathfrak{su}(2)$ modules for each root α_i

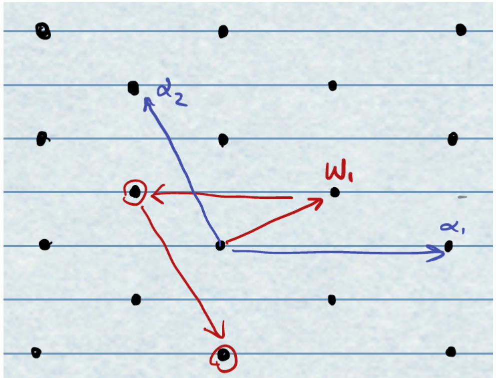
Example 3.24. Consider $\mathfrak{su}(3)$ 

$$A_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \Rightarrow \begin{cases} \alpha_1 = 2w_1 - w_2 \\ \alpha_2 = -w_1 + 2w_2 \end{cases} \Rightarrow \begin{cases} w_1 = \frac{1}{3}(2\alpha_1 + \alpha_2) \\ w_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2) \end{cases}$$



We take a look at the modules for $\lambda = w_1$, denoted as $[1, 0]$. Under this notation, $\alpha_1 = [2, -1]$, $\alpha_2 = [-1, 2]$. We build the allowed wights by subtracting the α_i from λ.





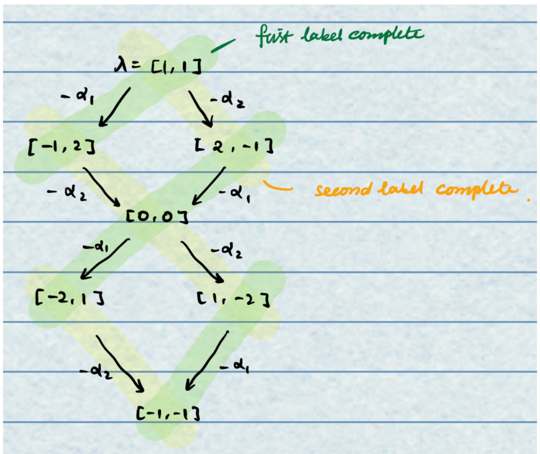
We can see that the ith label connected by the arrow of subtracting α_i forms a $\mathfrak{su}(3)$ module.

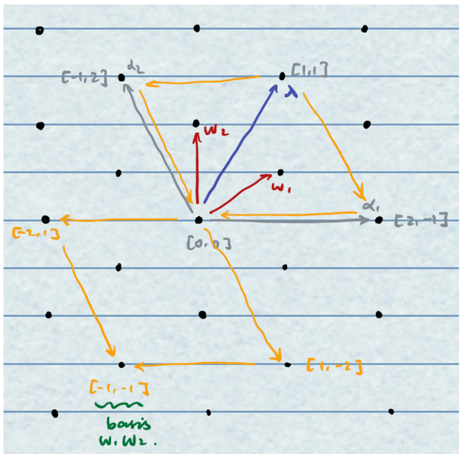
Therefore $\mathfrak{su}(3)$ decomposed as expected:

$$V = V_{[1,0]} \oplus V_{[-1,1]} \oplus V_{[0,-1]}$$

which is the 3-dimensional defining representation of $\mathfrak{su}(3)$.

Example 3.25. Now consider $\lambda = [1, 1]$,





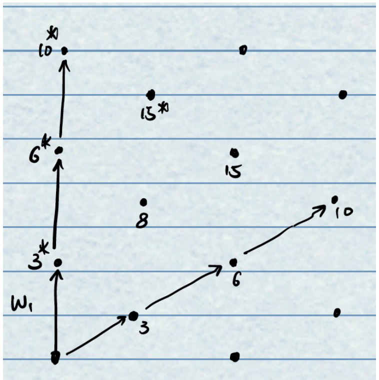
This gives us the decomposition:

$$\begin{aligned} V &= V_{[1,1]} \oplus V_{[-1,2]} \oplus V_{[2,-1]} \oplus V_{[0,0]} \oplus V_{[-2,1]} \oplus V_{[1,-2]} \oplus V_{[-1,1]} \\ &= \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha \end{aligned}$$

which is the adjoint representation of $\mathfrak{su}(3)$.

Note that the Cartan $V_{[0,0]}$ here has degeneracy, *i.e.* $\dim(V_{[0,0]}) = \dim(\mathfrak{h}) = 2$, so we still have 8 degree of freedoms for $\mathfrak{su}(3)$.

For $\mathfrak{su}(3)$, we have $\lambda = n_1 w_1 + n_2 w_2$ with n_1, n_2 non-negative. And we can associate a Young tableau with each of these λ



$$\Rightarrow \quad \begin{array}{cccccccc} \boxed{} & \boxed{} & \cdots & \boxed{} & \boxed{1} & \boxed{2} & \cdots & \boxed{n_1} \\ \boxed{1} & \boxed{2} & \cdots & \boxed{n_2} & & & & \end{array}$$

Summary:

- Compact Lie group is in one-to-one correspondence to semi-simple complex Lie algebra.
- For groups:

  Dynkin diagram $\to$ Fundamental roots $\to$ All roots $\to$ Complexified Lie algebra $\to$ Campact Lie group
- For representations:

  Dominant integral λ $\to$ Weight of irreps $\to$ Lie algebra complex irreps $\to$ Lie group complex irreps $\to$ Unitary irreps

Example 3.26. Consider $\mathfrak{su}(4)_{\mathbb{C}} = \mathfrak{sl}(4, \mathbb{C})$ $\underset{\alpha_1}{\bigcirc}\!\!-\!\!-\!\!-\!\!\underset{\alpha_2}{\bigcirc}\!\!-\!\!-\!\!-\!\!\underset{\alpha_3}{\bigcirc}$

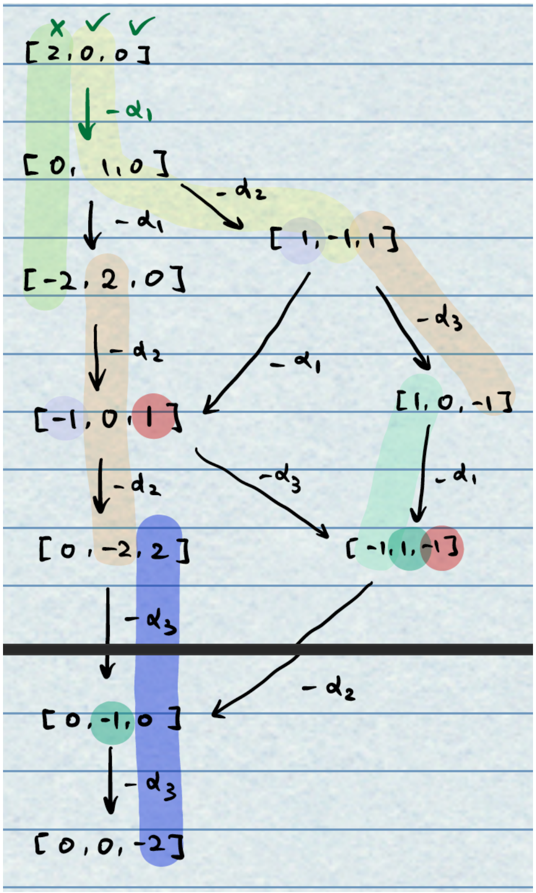
We determine the Cartan matrix from the Dynkin diagram

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Then, write the fundamental roots α_i in fundamental weights w_i:

$$\begin{cases} \alpha_1 = [2, -1, 0] \\ \alpha_2 = [-1, 2, -1] \\ \alpha_3 = [0, -1, 2] \end{cases}$$

Take the dominant integral weight $\lambda = [2, 0, 0]$, and subtract α_i from it accordingly, we obtained all the possible weights for decomposition as shown below.



Thus, the dimension of the module is 10 if there's no degeneracy, which correspond to the Young tableau: $\boxed{}\boxed{}$

There's a way to determine the degeneracy purely from the dominant integral weight λ.

First, we introduce the character:

Definition 3.31. Given a $\mathfrak{g}$-module with decomposition

$$V = \bigoplus_w V_w,$$

one can find a polynomial:

$$\mathrm{char}(V)(x_1 \dots x_r) = \sum_{w \in \Gamma} \dim(V_w) e(w) \tag{3.15}$$

where for $w = n^i w_i$,

$$e(w) = x_1^{n_1} x_2^{n_2} \dots x_r^{n_r} \tag{3.16}$$

Example 3.27. For $\mathfrak{su}(3)$,

- $\lambda = [1,0]$,

  $$V = V_{[1,0]} \oplus V_{[-1,1]} \oplus V_{[0,-1]}$$

  and all module has dim $= 1$. We have:

  $$\mathrm{char}(V) = x_1 + \frac{x_2}{x_1} + \frac{1}{x_2} \tag{3.17}$$

- $\lambda = [0,1]$, is switching x_1 and x_2, which is the dual representation

  $$\mathrm{char}(V) = x_2 + \frac{x_1}{x_2} + \frac{1}{x_1} \tag{3.18}$$

- $\lambda = [1,1]$,

  $$V = V_{[1,1]} \oplus V_{[-1,2]} \oplus V_{[2,-1]} \oplus V_{[0,0]} \oplus V_{[-2,1]} \oplus V_{[1,-2]} \oplus V_{[-1,1]}$$

  and all module has dim $= 1$, except $\dim(V_{[0,0]}) = 2$ We have:

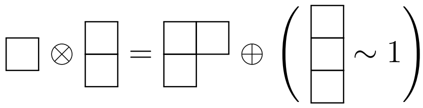
  $$\mathrm{char}(V) = x_1 x_2 + \frac{x_2^2}{x_1} + \frac{x_1^2}{x_2} + 2 + \frac{x_2}{x_1^2} + \frac{x_1}{x_2^2} + \frac{x_2}{x_1} \tag{3.19}$$

For characters, we have the composition rules:

- $\mathrm{char}(V_1 \oplus V_2) = \mathrm{char}(V_1) + \mathrm{char}(V_2)$
- $\mathrm{char}(V_1 \otimes V_2) = \mathrm{char}(V_1) \cdot \mathrm{char}(V_2)$

Consider $V_{\lambda=[0,1]} \otimes V_{\lambda=[1,0]}$:

$$\begin{aligned}
\mathrm{char}(V_{\lambda=[0,1]} \otimes V_{\lambda=[1,0]}) &= (x_2 + \frac{x_1}{x_2} + \frac{1}{x_1})(x_1 + \frac{x_2}{x_1} + \frac{1}{x_2}) \\
&= x_1 x_2 + \frac{x_2^2}{x_1} + \frac{x_1^2}{x_2} + 2 + \frac{x_2}{x_1^2} + \frac{x_1}{x_2^2} + \frac{x_2}{x_1} + 1 \\
&= \mathrm{char}(V_{\lambda=[1,1]}) + \mathrm{char}(V_{\lambda=[0,0]})
\end{aligned}$$

which written in Young tableau is: 

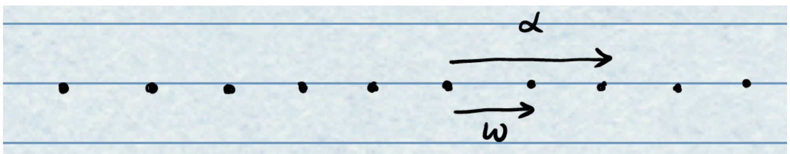
At this point we only need a formula to obtain character directly from λ:

Theorem 3.15 (Weyl character formula)**.** *Let λ be a dominant integral weight, and V_λ be the corresponding $\mathfrak{g}$-module, then*

$$\mathrm{char}(V_\lambda) = \frac{\sum_{s\in W}(\det(s))e(s(\lambda+\rho))}{\sum_{s\in W}(\det(s))e(s(\rho))} \tag{3.20}$$

where $\rho = \sum_i w_i$, W is the Weyl group and $\det(s) = \pm 1$ *(for rotation/reflection)*

Example 3.28. Consider $\mathfrak{su}(2)$, the weight space is shown below:



if we take $\lambda = nw$, the posible weights will be $[n], [n-2], \ldots, [-n+2], [-n]$. Therefore,

$$\begin{aligned}
\mathrm{char}(V_\lambda) &= x^n + x^{n-2} + \cdots + x^{-n} \\
&= \frac{x^{n+1} - x^{-(n+1)}}{x - x^{-1}}
\end{aligned}$$

For the character formula:

- $\lambda = nw$,

- $\rho = w$,
- Weyl group $= \{s_\alpha, e\} \cong \mathbb{Z}_2$, where s_α is a reflection.

Therefore we have $\lambda + \rho = [n+1]$, $s_\alpha(\lambda+\rho) = [-(n+1)]$

$$e(\lambda+\rho) + (-1)e(s_\alpha(\lambda+\rho)) = x^{n+1} - x^{-(n+1)}$$
$$e(\rho) + (-1)e(\rho) = x - x^{-1}$$

which gives the same result.

Notice that if we set $x_1, \ldots, x_r$ to 1, the character will become the dimension of the module. Thus, we have a simple formula:

$$\dim(V) = \frac{\prod_{\alpha\in R^+} \langle \alpha, \lambda+\rho\rangle}{\prod_{\alpha\in R^+} \langle \alpha, \rho\rangle} \tag{3.21}$$

where R^+ is the set of positive roots.
e.g. for $\mathfrak{su}(3)$, $\lambda = n_1 w_1 + n_2 w_2$

$$\dim(V_\lambda) = \frac{1}{2}(n_1+1)(n_2+2)(n_1+n_2+2)$$

Summary

The idea has been to understand symmetries in the content of QM.

- Symmetry $\to$ unitary representation on Hilbert space
- Lie groups $\longleftrightarrow$ Manifold $\sim$ smooth and can take derivatives

  There are two types of Lie groups:

  - Compact Lie group: finite dimensional, unitary representations $\to$ internal symmetry (*i.e.* Standard model symmetry)
  - Non-compact Lie group: infinite dimensional, unitary representations $\longleftrightarrow$ wave equations on spacetime, be think of as a spacetime symmetry.
- Non-compact Lie groups: irreps of Poincare group $\Longleftrightarrow$ Particles
- Compact Lie group $\Longleftrightarrow$ semi-simple Lie algebra which we can completely classified along with all irreps.