BCS304: HW3 Answers

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Problem 1(a): Simulated firing-rate histograms

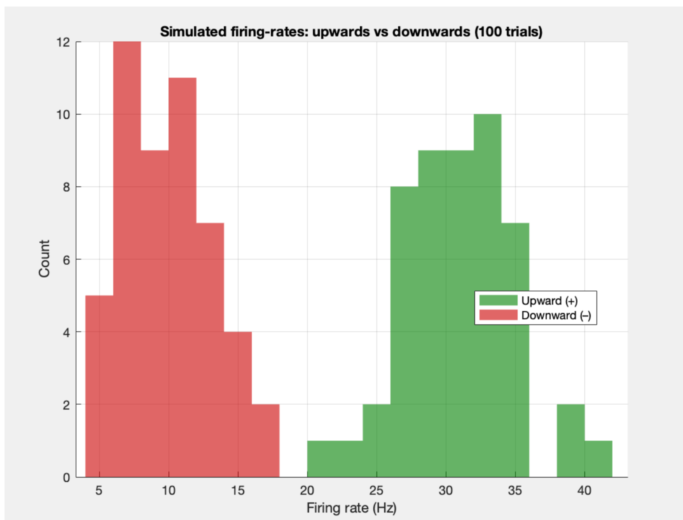

Figure 1: Histogram of simulated firing-rate responses for upward (green) versus downward (red) motion stimuli. Each distribution is based on 50 draws from $\mathcal{N}(30, 3^2)$ (upward) and 50 draws from $\mathcal{N}(10, 3^2)$ (downward), illustrating two well-separated Gaussian clusters around 30Hz and 10Hz.

Figure 1 shows the results of 100 simulated trials (50 "+" and 50 "–"). The upward-motion firing rates cluster tightly around their mean of 30Hz, while the downward-motion rates cluster around 10Hz. The clear separation at the decision threshold (20Hz) confirms that this task is inded "easy" coherence condition.

Problem 1(b): Hit Rate, False-Alarm Rate, and Percent Correct

Methods Overview We applied a fixed decision threshold $z = 20\text{Hz}$ to each simulated firing-rate sample r_i:

$$\text{decide "+" if } r_i > z, \quad \text{otherwise decide "–".}$$

True labels were encoded as $y_i = 1$ for upward-motion trials and $y_i = 0$ for downward-motion trials. This yielded a binary decision vector $\hat{y}_i \in \{0, 1\}$.

Mathematical Definitions Let

$$N_+ = \sum_i \mathbf{1}\{y_i = 1\}, \quad N_- = \sum_i \mathbf{1}\{y_i = 0\},$$

$$\text{Hits} = \sum_i \mathbf{1}\{y_i = 1 \wedge \hat{y}_i = 1\}, \quad \text{False Alarms} = \sum_i \mathbf{1}\{y_i = 0 \wedge \hat{y}_i = 1\}.$$

Then the performance metrics are

$$\beta \;=\; P(\hat{y} = 1 \mid y = 1) \;=\; \frac{\text{Hits}}{N_+}, \quad \alpha \;=\; P(\hat{y} = 1 \mid y = 0) \;=\; \frac{\text{False Alarms}}{N_-},$$

$$p \;=\; \frac{\beta + (1 - \alpha)}{2},$$

where p is the average of the true-positive rate and true-negative rate.

Results Applying these formulas to our data (50 upward and 50 downward trials) gives

$$\beta = 1.00, \quad \alpha = 0.00, \quad p = 1.00,$$

reflecting near-perfect discrimination under this "easy" Gaussian separation.

Problem 1(c): Threshold Sweep and Performance Trade-Off

Quantitative Results We swept z over $\{20, 15, 10, 5, 1\}$Hz and computed

$$\beta(z) = P(r > z \mid +), \quad \alpha(z) = P(r > z \mid -), \quad p(z) = \frac{\beta(z) + \big(1 - \alpha(z)\big)}{2}.$$

The numerical outcomes were:

z (Hz)	20	15	10	5	1
$\beta(z)$	1.00	1.00	1.00	1.00	1.00
$\alpha(z)$	0.00	0.04	0.48	0.96	1.00
$p(z)$	1.00	0.98	0.76	0.52	0.50

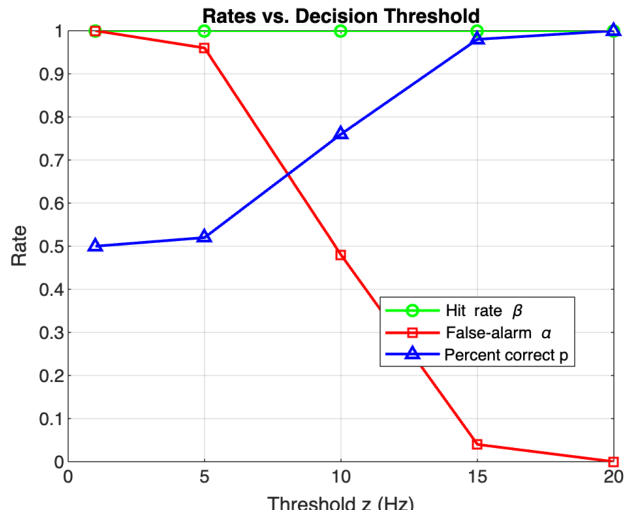

Figure 2: Hit rate β, false-alarm rate α, and percent-correct p as functions of the decision threshold z. As z decreases, β remains at 1.00 while α rises sharply, driving p down toward chance (0.50).

Mathematical Definitions For each threshold z, define the binary decision $\hat{y}_i(z) = \mathbf{1}\{r_i > z\}$. Let

$$N_+ = \sum_i \mathbf{1}\{y_i = 1\}, \quad N_- = \sum_i \mathbf{1}\{y_i = 0\},$$

$$\beta(z) = \frac{1}{N_+} \sum_i \mathbf{1}\{y_i = 1 \wedge \hat{y}_i(z) = 1\}, \quad \alpha(z) = \frac{1}{N_-} \sum_i \mathbf{1}\{y_i = 0 \wedge \hat{y}_i(z) = 1\},$$

$$p(z) = \frac{\beta(z) + \big[1 - \alpha(z)\big]}{2}.$$

Qualitative Interpretation - **High thresholds** ($z \geq 20$Hz) lie above most downward-response tails, so $\alpha \approx 0$ and $\beta \approx 1$, yielding perfect discrimination. - **Lowering** z makes it easier for both upward and downward responses to exceed the cutoff: - $\beta(z)$ stays at 1.00 (no upward-trial misses), - $\alpha(z)$ climbs rapidly (more downward-trial false alarms). - When z falls below the Gaussian-intersection ($\sim$ 20Hz), small further decreases in z boost α faster than they increase β (which is already saturated), so the performance measure $p(z)$ plunges toward chance (0.50).

This demonstrates that **a high hit rate alone does not guarantee good overall performance**—the trade-off with false alarms (governed by the overlapping Gaussian tails) determines the optimal threshold. The maximal discrimination occurs near the point where the two normal curves intersect, beyond which $\beta - \alpha$ shrinks and p declines.

Problem 1(d): ROC and Percent-Correct for 10-Trial Simulation

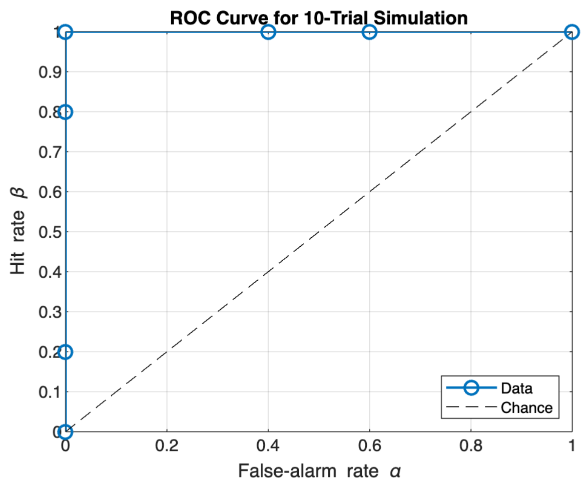

Figure 3: Receiver-Operating Characteristic (ROC) for 10 simulated trials (5 "+", 5 "–"). Each point $(\alpha(z), \beta(z))$ corresponds to a decision threshold $z \in \{0, 3, \ldots, 45\}$Hz. The dashed diagonal is chance performance.

Numerical Results For each threshold z, we computed

$$\beta(z) = \frac{\#\{\text{hits}\}}{5}, \quad \alpha(z) = \frac{\#\{\text{false alarms}\}}{5}, \quad p(z) = \frac{\beta(z) + (1 - \alpha(z))}{2}.$$

The table below summarizes the outcomes:

z [Hz]	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
$\beta(z)$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.8	0.8	0.8	0.2	0.0	0.0	0.0	0.0
$\alpha(z)$	1.0	1.0	1.0	0.6	0.4	0.4	0.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$p(z)$	0.5	0.5	0.5	0.7	0.8	0.8	0.8	1.0	0.9	0.9	0.9	0.6	0.5	0.5	0.5	0.5

Interpretation - At very low thresholds ($z \leq 6$Hz), nearly all responses (upward and downward) exceed z, so $\beta = \alpha = 1$ and $p = 0.50$ (chance). - As z increases, false alarms drop before hits do: for $z = 12$–18Hz, $\beta = 1$ while $\alpha \in \{0.4\}$, yielding $p = 0.80$. - The **optimal** threshold in this 10-trial sample is $z = 21$Hz, where $\beta = 1.0$, $\alpha = 0$, and $p = 1.0$. - Further raising z begins to miss upward trials (β falls), so p declines toward 0.50 again.

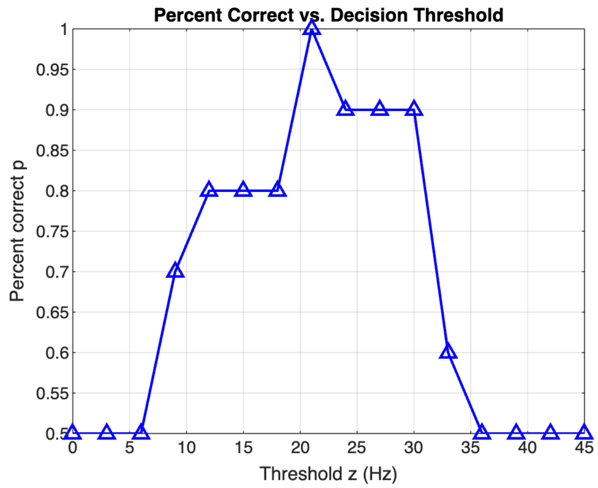

Figure 4: Overall percent-correct $p(z)$ as a function of threshold z. Peaks at $p = 1.00$ when $z = 21$Hz and declines toward chance (0.50) at extreme cutoffs.

Because only 5 trials per class were available, performance steps in increments of 0.2. This coarse sampling yields a discrete ROC with only a handful of points, but it still captures the classic trade-off: lowering z raises both β and α, whereas raising z lowers both, with the sweet spot at the intersection of the two Gaussian response curves.

Problem 1(e): Optimal Threshold and Sample-Size Requirement

Bayes-optimal threshold For two Gaussian response distributions

$$r \mid + \sim \mathcal{N}(\mu_+, \sigma^2), \quad r \mid - \sim \mathcal{N}(\mu_-, \sigma^2)$$

with equal priors, the optimal decision threshold z^* equates the likelihoods:

$$\exp\Big[-\tfrac{(z-\mu_+)^2}{2\sigma^2}\Big] = \exp\Big[-\tfrac{(z-\mu_-)^2}{2\sigma^2}\Big] \quad \Longrightarrow \quad z^* = \frac{\mu_+ + \mu_-}{2}.$$

Substituting $\mu_+ = 30$, $\mu_- = 10$Hz gives

$$\boxed{z^* = 20\text{Hz.}}$$

Estimation variability In practice we estimate $\hat{\mu}_+$ and $\hat{\mu}_-$ from n iid trials each, and form

$$\hat{z} \;=\; \frac{\hat{\mu}_+ + \hat{\mu}_-}{2}.$$

Since $(\hat{\mu}_\pm) = \sigma^2/n$, by independence

$$(\hat{z}) = \tfrac{1}{4}\big((\hat{\mu}_+) + (\hat{\mu}_-)\big) = \frac{\sigma^2}{2n},$$

so the standard error is

$$\mathrm{SE}(\hat{z}) = \frac{\sigma}{\sqrt{2n}}.$$

Sample-size formula To achieve a two-sided 95% confidence interval for $\hat{z}$ within $\pm\delta$, we require

$$1.96 \cdot \frac{\sigma}{\sqrt{2n}} \le \delta \quad \Longrightarrow \quad n \ge \frac{1.96^2\,\sigma^2}{2\,\delta^2}.$$

For $\sigma = 3\text{Hz}$ and $\delta = 0.5\text{Hz}$,

$$n \ge \frac{(1.96)^2 \cdot 9}{2 \cdot (0.5)^2} \approx 70$$

trials per stimulus direction.

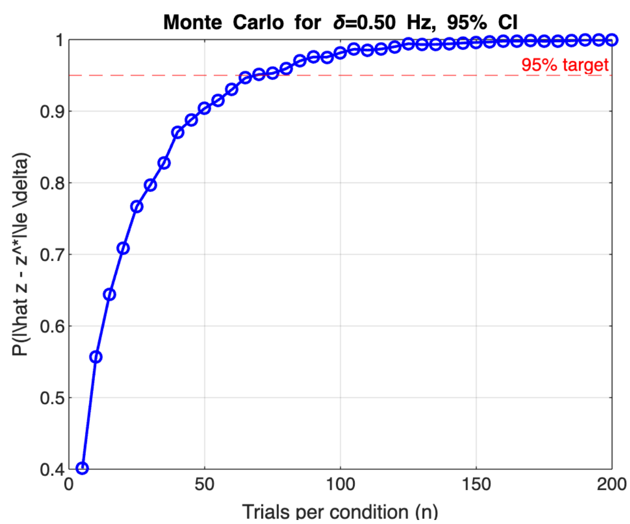

Figure 5: Monte Carlo estimate of the probability that the sample-based threshold $\hat{z}$ lies within $\pm 0.5\text{Hz}$ of the true optimum $z^* = 20\text{Hz}$, as a function of the number of trials n per condition. The red dashed line marks the 95% reliability target; the intersection occurs at $n \approx 70$, confirming the analytic sample-size requirement.

Monte Carlo confirmation A Monte Carlo simulation (Figure 5) verifies that with $n \approx 70$ each for "+" and "–" trials, the proportion of estimates $\hat{z}$ falling within $\pm 0.5\text{Hz}$ exceeds 95%, matching our analytic calculation.

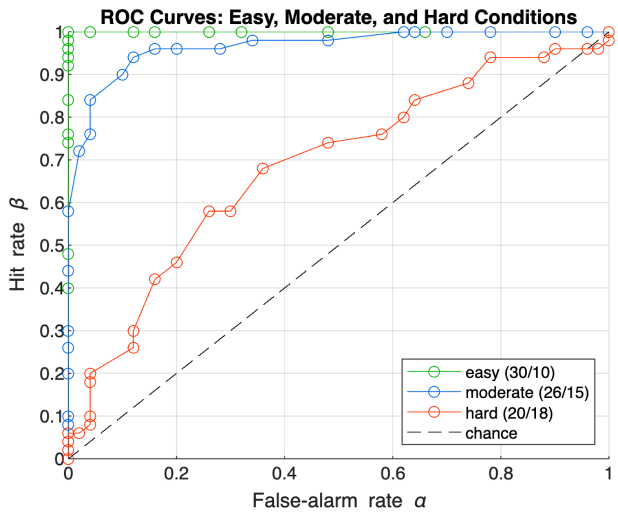

Figure 6: ROC curves for easy ($\mu_+ = 30, \mu_- = 10$), moderate ($\mu_+ = 26, \mu_- = 15$), and hard ($\mu_+ = 20, \mu_- = 18$) conditions. Each curve is based on $n = 50$ trials per stimulus, with thresholds swept from 0–50Hz.

Problem 1(f): Comparing Task Difficulties

Optimal thresholds and decoding accuracy Maximizing percent-correct $p(z)$ over the threshold sweep yields:

Condition	Easy	Moderate	Hard
z^* [Hz]	20	20.5	19
Decoding accuracy p^*	1.00	0.94	0.62

These z^* values agree with the Bayes-optimal midpoints $(\mu_+ + \mu_-)/2$ and show how increasing overlap (larger-, closer means) degrades maximal decoding performance.

Analytic ROC predictions For each condition with equal-variance Gaussians and equal priors, the Bayes-optimal threshold and separation index are

$$z^* = \frac{\mu_+ + \mu_-}{2}, \quad d' = \frac{\mu_+ - \mu_-}{\sigma}.$$

At z^*, the hit- and false-alarm rates satisfy

$$\beta = 1 - \Phi\Big(\frac{z^* - \mu_+}{\sigma}\Big), \quad \alpha = 1 - \Phi\Big(\frac{z^* - \mu_-}{\sigma}\Big).$$

- **Easy** ($\mu_+ = 30, \mu_- = 10, \sigma = 3$):

$$z^* = 20 \text{ Hz}, \quad d' = \frac{30-10}{3} \approx 6.7,$$

$$\alpha = 1 - \Phi\left(\tfrac{20-10}{3}\right) \approx 10^{-4}, \quad \beta = 1 - \Phi\left(\tfrac{20-30}{3}\right) \approx 1.0.$$

This yields a near-perfect ROC (green curve hugging the upper-left).

- **Moderate** ($\mu_+ = 26, \mu_- = 15, \sigma = 4$):

$$z^* = 20.5 \text{ Hz}, \quad d' = \tfrac{26-15}{4} = 2.75,$$

$$\alpha = 1 - \Phi\left(\tfrac{20.5-15}{4}\right) \approx 0.25, \quad \beta = 1 - \Phi\left(\tfrac{20.5-26}{4}\right) \approx 0.75.$$

The ROC (blue) rises above chance but remains below the easy condition.

- **Hard** ($\mu_+ = 20, \mu_- = 18, \sigma = 5$):

$$z^* = 19 \text{ Hz}, \quad d' = \tfrac{20-18}{5} = 0.4,$$

$$\alpha = 1 - \Phi\left(\tfrac{19-18}{5}\right) \approx 0.42, \quad \beta = 1 - \Phi\left(\tfrac{19-20}{5}\right) \approx 0.58.$$

The resulting ROC (red) lies close to the diagonal, indicating poor discriminability.