Mira Mors and Elias Tidemand Ruud

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Department of Physics
University of Oslo
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Since $\mathbf{U}$ is orthogonal holds $\mathbf{U^T U} = I$

$$w_i^T w_j = (Uv_i)^T (Uv_j) = v_i^T U^T U v_j = v_i^T I v_j = v_i^T v_j \tag{1}$$

3.1 Jacobi Method

The Jacobi method guarantees a solution for all real symmetric matrices. The method rotates the initial matrix using similarity transformations. By discarding the off-diagonal elements, the eigenvalues stay preserved. How does this work? Firstly, we look at the transformation matrix.
The transformation matrix is orthogonal since its inverse is equal to its transpose

$$\mathbf{S}^T = \mathbf{S}^{-1}.$$

In $\mathbb{R}^2$, the transformation matrix is given by

$$\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

and θ denotes the degree of rotation in the plane.
In $\mathbb{R}^3$, the rotation along the x, y and z axis respectively are

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix} \quad R_y = \begin{bmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{bmatrix} \quad R_z = \begin{bmatrix} cos(\theta) & -sin(\theta) & \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Generally, a rotation i $\mathbb{R}^n$ is given by

$$S(k,l,\theta) = \begin{array}{c} \begin{array}{ccccccc} & & l & & k & & \end{array} \\ \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & cos(\theta) & \dots & -sin(\theta) & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & sin(\theta) & \dots & cos(\theta) & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ l \\ \\ k \\ \\ \\ \end{array}$$

$cos(\theta)$ and $sin(\theta)$ are now placed in the l$^{\text{th}}$ row and l$^{\text{th}}$ column and k$^{\text{th}}$ row and k$^{\text{th}}$ column.
The similarity transformation

$$\mathbf{A}_{new} = \mathbf{S}^T(k,l,\theta) \cdot \mathbf{A} \cdot \mathbf{S}(k,l,\theta) \tag{2}$$

rotates row and column k and l of $\mathbf{A}$ an angle θ such as the entries $\mathbf{A}_{new}(k,l)$ and $\mathbf{A}_{new}(l,k)$ become zero.

The new entries are given with

$$\begin{aligned}
b_{ii} &= a_{ii} \qquad i \neq k, i \neq l \\
b_{ik} &= a_{ik}cos(\theta) - a_{il}sin(\theta) \qquad i \neq k, i \neq l \\
b_{il} &= a_{il}cos(\theta) + a_{ik}sin(\theta) \qquad i \neq k, i \neq l \\
b_{kk} &= a_{kk}cos^2(\theta) - 2a_{kl}cos(\theta)sin(\theta) + a_l l sin^2(\theta) \\
b_{ll} &= a_{ll}cos^2(\theta) + 2a_{kl}cos(\theta)sin(\theta) + a_{kk}sin^2(\theta) \\
b_{kl} &= (a_{kk} - a_{ll})cos(\theta)sin(\theta) + a_{kl}(cos^2(\theta) - sin^2(\theta))
\end{aligned} \tag{3}$$

The aim is to get zero for all non-diagonal elements b_{kl}. For each iteration, θ must be chosen accordingly. The Jacobi Method is an iterative method, thus this procedure is continued until the sum over the squared non-diagonal matrix elements, $off(\mathbf{A})$, are less than a prefixed test, ϵ, (ideally equal zero). That is

$$off(\mathbf{A}) = \sqrt{\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} = |a_{ij}|^2} < \epsilon \tag{4}$$

Since this is quite a time-consuming test, it can be replaced by finding the largest (absolute) value of the off-diagonal elements

$$max|a_{ij}| < \epsilon \qquad i \neq j. \tag{5}$$

Given a real symmetric matrix, the algorithm is set to succeed. However, the large amount of iterations (about n^3) make the method very slow.