Modelling the Solar System Using Ordinary Differential Equations

Computing Newton's second law according to ordinary Euler and Velocity Verlet

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Abstract

We simulate the solar system according to Newton's Newton's law of motion. Since the gravitational force is conservative, conservation of energy is an important principle that can act as a benchmark. It follows that the Euler method is not suitable, as the method, unlike the Velocity Verlet, does not conserve energy.
Firstly, we start simulating a hypothetical one-planet system, gradually adding more planets. The theoretical escape velocity, $v = \sqrt{2} \cdot 2\pi AU/yr$, for the Earth escaping the Sun-Earth system is computationally obtained. It is shown that the Velocity Verlet method is stable, using a step size of 10^{-4}, even for many planet systems. Based on Kepler's laws, a central force which is inverse-cube does not allow stable elliptical orbits. Planets affect each other by their gravitation. For the Sun-Earth-Jupiter case, the attraction is illustrated by increasing Jupiter's mass by first ten and then by thousand. Thereby, setting the centre of mass as the origin and not the Sun gives clearly different results. However, in reality, setting the Sun as the centre of mass is a justified approximation. Further, the perihelion angle of Mercury after a century, computed with a step size of 10^{-7}yr, is found to be $2.0932 \cdot 10^{-4}$, which has a relative error of 0.41% from the theoretical. The code for the computation can be found on Github repository.

Contents

1 Introduction

This report presents a simulation of the solar system by solving coupled ordinary differential equations. Ordinary differential equations (ODEs) appear in all types of physical problems. Here, we look at Newton's law of motion due to the gravitational force. By discretizing the continuous equations, we can solve them numerically. Creating a general object-oriented program which solves a one planet solar system, further planets or even astronomical objects can easily be added to extend the system. Thus, we take use of the central concept of object orientation: "write once and run many times".
The choice of the right algorithm for successful computation is crucial. The system is based on the general principle of conservation of energy, therefore the chosen algorithm should conserve the energy as well. As it turns out, the Euler method is not suitable as it does not conserve energy. Instead, the widely popular Velocity Verlet algorithm is used to solve the coupled ordinary differential equations.
To begin with, the report introduces three main types of solar systems. Starting with a hypothetical one planet solar system (system A), the system is extended to a three planet solar system (system B). Thereupon, all the planets of the solar system are added (system C). Our model also deals with the perihelion precision of Mercury. Thereafter, Newton's law of motion is presented as an ordinary differential equation and it is shown how the resulting coupled differential equations can be discretized. Then, the Euler method and Velocity Verlet method are introduced for solving the algorithm. Finally, the collected results are presented and discussed.

2 Methods

In the following, three main types of solar systems are introduced. These systems are solved with coupled ordinary differential equations, which are based on Newton's law of motion. The celestial bodies are assumed to be point objects

2.1 System A - One planet solar system

At first, we look at a hypothetical solar system with the Earth only, orbiting around the sun. The reason for doing that is to test our algorithm. The only force acting on the system is gravity. According to Newton's law of gravitation, the gravitational force, F_G is given by

$$F_G = \frac{GM_\odot M_{\mathrm{Earth}}}{r^2}, \tag{1}$$

where $M_\odot$ and M_{Earth} are the masses of the Sun and Earth respectively. The gravitational constant is G and r is the distance between the Earth and the Sun.

Circular motion Assuming that the Earth orbits the Sun in an almost circular motion, we can substitute $GM_\odot/r^2$ with the centripetal acceleration $a = v^2/r$. Thus eq. (1) can be rewritten to

$$F_G = \frac{M_{\mathrm{Earth}} v^2}{r}, \tag{2}$$

The speed of an object moving in uniform circular motion is related to the radius of the circle and the time to make one cycle around the circle. In our case, the radius is the average distance between the Sun and Earth, that is $1AU = 1.5 \times 10^{11}m$. Since the Earth circles the Sun in one year, the speed is given by

$$v = \frac{\text{circle's circumference}}{\text{time}} = \frac{2\pi\ 1AU}{1year}. \tag{3}$$

Thus, $GM_\odot$ can be expressed as

$$GM_\odot = v^2 r = 4\pi^2 AU^3/yr^2. \tag{4}$$

Scaling and vectorization Table 6 lists the masses for all relevant planets and their distance to the sun needed for our modulation. As the masses are so large, they can cause significant problems in computation. We get around the problem by scaling the equations. Therefore, the Sun's mass is set a reference, that is $M_\odot = 1$.

For a three dimensional system, the gravitational force becomes a vector. The scaled force due the Sun acting on the Earth is then

$$\mathbf{F_{Earth}} = GM_\odot \frac{M_{\text{Earth}}}{M_\odot} \frac{\mathbf{r_{Earth}} - \mathbf{r}_\odot}{|\mathbf{r_{Earth}} - \mathbf{r}_\odot|^3}, \tag{5}$$

where $\mathbf{r}$ stands for the position vector. However, we can compute the forces for each direction separately and add them in accordance with the superposition principle. For example, for the forces in the x-direction holds

$$\mathbf{F_{Earth,x}} = GM_\odot \frac{M_{\text{Earth}}}{M_\odot} \frac{\mathbf{r_{Earth,x}} - \mathbf{r_{\odot,x}}}{|\mathbf{r_{Earth}} - \mathbf{r}_\odot|^3}. \tag{6}$$

Since the Sun has a much larger mass than the Earth, the motion of the Sun could easily be neglected. Additionally, we assume that the orbit of the Earth around the Sun is co-planar, such as the model can be taken to the xy-plane. Discretizing and solving Newton's law is further explained in sections 2.4 and 2.5.

Energy conservation As the law of conservation of energy states, the total energy in a system which is not affected by outside forces remains constant over time. The same applies to conservative forces acting within the system, including the gravitational force.

The planets have both kinetic and potential energy given by their velocity, mass and their position related to the other planets and the Sun. The potential energy for a planet is defined as 0 when the distance between two planets reaches infinity. By calculating the total energy over time, we can determine how well our solver-program performs with both different methods and choice of delta-t. The applied methods are the ordinary Euler and Velocity Verlet method, presented in Section 2.5.

If the motion of the Earth is circular, see equation (3), the distance to the Sun is constant. Thus the potential energy, that is the gravitational force (equation (1)) is constant as well. Due to the conservation of energy, the kinetic energy E_{kin}

$$E_{kin} = \frac{1}{2} M_{Earth} v^2 \tag{7}$$

must also be constant. Since the the mass of the Earth, M_{Earth}, does not change, neither can the speed v of the Earth. In short, a circular orbit implies a constant kinetic energy.

Escape Velocity In order for the Earth to escape, its kinetic energy must be greater than or equal to its gravitation to the Sun. That is

$$\frac{1}{2} M_{Earth} v^2 \geq \frac{GM_\odot}{r^2}. \tag{8}$$

The idea behind is that, if the planet is escaping, its potential energy becomes zero as the distance to the Sun reaches infinity. Additionally, it is enough for the final velocity to be zero. Considering a circular orbit, the escape velocity is thus

$$v_{esc} \geq \sqrt{\frac{2GM_\odot}{r}}. \tag{9}$$

Since both the mass of the Sun, $M_\odot$, and the distance, r are set to one, and inserting the value for $G = 4\pi^2$, we get

$$v_{esc} \geq \sqrt{\frac{2 \cdot 4\pi^2 \cdot 1}{1}} = \sqrt{2} \cdot 2\pi. \tag{10}$$

Thus, the escape velocity is given by multiplying the Earth's initial circular velocity with $\sqrt{2}$. Furthermore, by plotting the case where the Earth's velocity is either a bit higher or smaller than the above value, we respectively expect the planet either ending in an extreme stretched elliptical orbit or the planet escaping it.

Testing different forms of the force Up to this point, we have assumed that the gravitational force can be modeled by an inverse-square force, eq. (1). In order to test the form of the force, we replace the inverse-square force with

$$F_G = \frac{GM_\odot M_{Earth}}{r^\beta}, \tag{11}$$

where we let β vary from to to three. For the Sun-Earth system with a circular orbit, nothing would change. That is because the Sun-Earth distance is $1AU$, which means $r^\beta = const$. For the elliptical orbit, an increased value of β means an increased change in the gravitational force when the distance between objects vary. Since the force is still conservative, the total energy and angular momentum are conserved.
Note that the potential energy has a different mathematical expression when changing β.

Perihelion precision of Mercury Like the other planets, Mercury travels around the Sun on a nearly elliptical orbit. The elliptical orbit is in reality not fixed, but gradually rotates. Thus, the perihelion of the orbit, which is Mercury's point of closest approach to the Sun, drifts. This drift is referred to as precision. Observations show that, when all classical effects (such as the perturbation of the orbit due to gravitational attraction from the other planets) are subtracted, the perihelion precision is 43 arc per century, denoted as 43´´. Generally, the perihelion angle θ_p, is given by

$$tan(\theta_p) = \frac{y_p}{x_p}, \tag{12}$$

where x_p (y_p) is the x (y) position of Mercury at perihelion. This observed phenomenon can be explained by taking into consideration the general theory of relativity, which states that mass interacts with space. More precise, a mass does not merely float in an unperturbed vacuum, but does shape the space around it. One could say that the space gets wrapped or curved. If a mass experiences the pull of gravity, it then simply follows a natural path in that curved space. Hence, the orbit is said to precess. The precision of the perihelion of Mercury is much greater than for other planets. This is due to Mercury being the closest planet to the sun, where the curvature of space-time is more pronounced.

In order to improve our closed elliptical orbits, which are a special feature of the Newtonian $1/r^2$ force, we add a general relativistic correction to the gravitational force

$$F_G = \frac{GM_\odot M_{Mercury}}{r^2}\left(1 + \frac{3l^2}{r^2c^2}\right), \tag{13}$$

where r denotes the distance between Mercury and the Sun. l stands for Mercury's orbital angular momentum per unit mass, and c is the speed of light in vacuum.

We will use that the speed of Mercury at perihelion is $12.44 AU/yr$ and that the distance to the Sun at perihelion is $0.3075 AU$. (Ref. [3], [6], [7])

2.2 System B - Three planet solar system

For this system, we introduce a third planet, Jupiter, to system A (Sun-Earth). We are interested in how Jupiter alters the Earth's motion as it is the most massive planet in the solar system. Its mass is about 1000 times smaller than that of the Sun. The Sun is still set as the center of mass of the system and the gravitational forces between the Earth and Jupiter can simply be added to system A. Similarly to eq. (5), the scaled gravitational force on Earth due to Jupiter reads

$$\mathbf{F}_{\mathbf{Earth}} = G\frac{M_{Jupiter}}{M_\odot}\frac{M_{\text{Earth}}}{M_\odot}\frac{\mathbf{r}_{\mathbf{Earth}} - \mathbf{r}_{\mathbf{Jupiter}}}{|\mathbf{r}_{\mathbf{Earth}} - \mathbf{r}_{\mathbf{Jupiter}}|^3}. \tag{14}$$

Hence the total force on the Earth is the sum of equations (5) and (14). The forces on Jupiter can be calculated in the same manner.

2.3 System C - Many planet solar system

Finally, we include all the planets in the simulation. When initializing our systems as we have until now, the center of mass of the system has a velocity. This is insignificant when the longest simulation is 12 years, which is approximately Jupiter's orbital period. When including all the other planets, the longest orbital period is about 165 years, which is that of Neptune. To avoid the systems center of mass drifting, we now change the frame of reference, such that the center of mass remains fixed. This is done by calculating where the center of mass is, and subtracting that from the initial position of all the planets. We also have to calculate the velocity of the center of mass, and subtracting that from the velocity of all the planets. The initial conditions are taken from NASA: `http://ssd.jpl.nasa.gov/horizons.cgi`.The generated data contain the x, y and z values as well as their corresponding velocities. The velocities are in units of AU per day. Note to change from OBSERVER to VECTOR when collecting the data.

2.4 Newton's law of motion as an ODE

For our simulation of the solar system, the only existing force is gravity given by Newtons second law. The second-order differential equation in one dimension states

$$m\frac{d^2x}{dt^2} = F(x(t)), \tag{15}$$

where m is the mass of the planet that is acted upon, x its position at a given time t and F is the gravitational force in the x-direction. Note that the differential equation is only dependent on one variable. Furthermore, the same equation applies to the y and z direction. Thus, for a three dimensional model, we can simply use the superposition principle.
Eq. (15) can be rewritten as two first-order differential equations. We define the position $x(t) = y^{(1)}(t)$ and its derivative, that is its velocity, as $v(t) = y^{(2)}(t)$

$$\frac{dx(t)}{dt} = \frac{dy^{(1)}(t)}{dt} = y^{(2)}(t). \tag{16}$$

Inserted in eq. (15), we get two first-order coupled differential equations

$$m\frac{dy^{(2)}(t)}{dt} = F(x(t)) = F(y^{(1)}(t)) \tag{17}$$

and

$$\frac{dy^{(1)}(t)}{dt} = y^{(2)}(t). \tag{18}$$

Discretizing Newton's second law Newton's second order differential equation, (15), can be solved by so called one-step methods. Their algorithm is rather simple. The starting point is a defined value, y_0, for the function $y(t)$ at t_0

$$y_0 = y(t = t_0). \tag{19}$$

Further, we like to solve the differential equation for a given time period: t_0 to t_{end}. Setting $t_0 = 0$, the step size, h, is then given by

$$h = \frac{t_{end} - t_0}{N} = \frac{t_{end}}{N}, \tag{20}$$

where N is the number of integration points. Thus, the time, t, is discretized by

$$t_i = i\Delta h, \qquad i = 0, 1, \ldots, N. \tag{21}$$

Making use of (17), (18) and the step size h, $y_{i+h} = y_{i+1}$ can be obtained by the next value of the function y. For y_1 we get

$$y_1 = y(t_1 = t_0 + h). \tag{22}$$

Generally,

$$y_{i+1} = y(t = t_i + h) = y(t_i). \tag{23}$$

Assuming that the function is rather well-behaved in the interval $[a, b]$, h is held constant. In other words, we use a fixed-step method.
In order to find an expression for (23), we Taylor expand our function y

$$\begin{aligned} y_{i+1} &= y(t = t_i + h) \\ &= y(t_i) + h(y'(t_i) + \cdots + y^{(p)}(t_i)\frac{h^{p-1}}{p!}) + O(h^{p+1}), \end{aligned} \tag{24}$$

with $O(h^{p+1})$ representing the truncation error. Abbreviating the derivatives in the parenthesis with

$$(y'(t_i) + \cdots + y^{(p)(t_i)}) = \Delta(t_i, y_i(t_i)), \tag{25}$$

we can rewrite eq. (24) to

$$\begin{aligned} y_{i+1} &= y(t = t_i + h) \\ &= y(t_i) + h\Delta(t_i, y_i(t_i)) + O(h^{p+1}). \end{aligned} \tag{26}$$

(Ref. [2])

2.5 Solving Newton's law of motion

2.5.1 Euler method

The Euler method is a simple first-order numerical method for solving ordinary differential equations with a given initial value. Continuing on from eq. (26), we define the first derivative of y_i as

$$y'(t_i) = f(t_i, y_i). \tag{27}$$

Truncating Δ at the first derivative, we get

$$y_{i+1} = y(t_i) + hf(t_i, y_i) + O(h^2) \tag{28}$$

$$= y_{i+h}. \tag{29}$$

where $O(h^2)$ represent the truncation error. In other words, for every step we make a round off error of the order of $O(h^2)$. Thus, the total error is the sum over all the steps N

$$NO(h^2) = \frac{t_{end}}{h}O(h^2) \approx O(h). \tag{30}$$

Euler's method is not recommended for precision calculation due to round of errors (explained in 6.2.1).

As a concrete example, lets take equations (17) and (18). Defining the intial values $y_0 = y^{(1)}(t=0)$ and $v_0 = y^{(2)}(t=0)$, the next steps are given by

$$y_1^{(1)} = y_0 + hv_0 + O(h^2) \tag{31}$$
$$y_1^{(2)} = v_0 - hy_0 F(y_0)/m + O(h^2). \tag{32}$$

Note that the next steps are always calculated with the information from the previous step. Thus, the Euler method is asymmetric in time. Moreover, the Euler method does not conserve energy.

The calculation is the same for the y- and z-direction. Below is a general code implementation of the Euler algorithm.

Algorithm 1: Euler algorithm

Result: Position and velocity of planet in 3D

1 initialization;

2 **for** *i=0,1, …, N-1* **do**

   `// x-direction`

3   $a_{x,i} = F_{x,i}/m$ `// acceleration in x-direction at time step i+1`

4   $r_{x,i+1} = r_{x,i} + hv_{x,i}$ `// position in x-direction at time step i+1`

5   $v_{x,i+1} = v_{x,i} + ha_{x,i}$ `// velocity in x-direction at time step i+1`

6

   `// y-direction`

7   $a_{y,i} = F_{y,i}/m$

8   $r_{y,i+1} = r_{y,i} + hv_{y,i}$

9   $v_{y,i+1} = v_{y,i} + ha_{y,i}$

10

   `// z-direction`

11   $a_{z,i} = F_{z,i}/m$

12   $r_{z,i+1} = r_{z,i} + hv_{z,i}$

13   $v_{z,i+1} = v_{z,i} + ha_{z,i}$

14 **end**

Looking at the equations for the x-direction (lines $3-5$) in the algorithm above, there are five FLOPs all together. We ignore the FLOPs for the force. In our program, we reduced them to 4.

2.5.2 Velocity Verlet method

For the Velocity Verlet method, we again approximate Newtons law of motion with a Taylor expansion. Now, we go up to the second derivative (third order). The Taylor expansion for the position reads

$$y_{i+1} = y(t_i) + hy^{(2)}(t_i) + \frac{h^2}{2} y^{(3)}(t_i) + O(h^3). \tag{33}$$

Likewise, the velocity, $y^{(2)}$, can be approximated with

$$y_{i+1}^{(2)} = y_i^{(2)}(t_i) + hy^{(3)}(t_i) + \frac{h^2}{2}y^{(4)}(t_i) + O(h^3). \tag{34}$$

We remind that the analytical expression for the acceleration, $y^{(3)}$, is known

$$y_i^{(3)} = \frac{d^2 x_i}{dt^2} = \frac{F(x_i, t_i)}{m}. \tag{35}$$

The first order Taylor expansion for the acceleration reads

$$y_{i+1}^{(3)} = y_i^{(3)} + hy_i^{(4)} + O(h^2). \tag{36}$$

The reason for not including higher orders is that the velocity, $y^{(2)}$, has a truncation error of $O(h^3)$. Approximating $hy_i^{(2)}$ with

$$hy_i^{(4)} \approx y_{i+1}^{(3)} - y_i^{(3)}, \tag{37}$$

the expression for the velocity (eq. (34)) can be rewritten to

$$y_{i+1}^{(2)} = y_i^{(2)} + \frac{h}{2}\left(y_{i+1}^{(3)} + y_i^{(3)}\right) y^{(3)} + O(h^3). \tag{38}$$

In summary, the equations for the position and the velocity respectively are

$$y_{i+1} = y(t_i) + hy^{(2)}(t_i) + \frac{h^2}{2}y^{(3)}(t_i) + O(h^3) \quad \text{and} \tag{39}$$

$$y_{i+1}^{(2)} = y_i^{(2)} + \frac{h}{2}\left(y_{i+1}^{(3)} + y_i^{(3)}\right) y^{(3)} + O(h^3). \tag{40}$$

It can be shown that the global error in the Velocity Verlet method is of the same order as in the basic Verlet method (described in Section 6.2.3), that is $O(h^2)$ (Ref. [5]). Consequently, the velocity Verlet method is an order better than the Euler method. In difference to the Euler method, the position in the velocity Verlet method is also dependent on the acceleration. Moreover, the velocity is dependent on both the current and the next acceleration. Generally, the energy is conserved when using Velocity Verlet.
A general implementation of the code is given below.

Algorithm 2: Velocity Verlet algorithm

Result: Position and velocity of planet in 3D

1 initialization;

2 **for** *i=0,1, ..., N-1* **do**

// x-direction

3 $a_{x,i} = F_{x,i}/m$

4 $r_{x,i+1} = r_{x,i} + hv_{x,i} + \frac{h^2}{2} a_{x,i+1}$

5 $a_{x,i+1} = F_{x,i+1}/m$

6 $v_{x,i+1} = v_{x,i} + \frac{h}{2}\left(a_{x,i} + a_{x,i+1}\right)$

7

// y-direction

8 $a_{y,i} = F_{y,i}/m$

9 $r_{y,i+1} = r_{y,i} + hv_{y,i} + \frac{h^2}{2} a_{y,i+1}$

10 $a_{y,i+1} = F_{y,i+1}/m$

11 $v_{y,i+1} = v_{y,i} + \frac{h}{2}\left(a_{y,i} + a_{y,i+1}\right)$

12

// z-direction

13 $a_{z,i} = F_{z,i}/m$

14 $r_{z,i+1} = r_{z,i} + hv_{x,i} + \frac{h^2}{2} a_{z,i+1}$

15 $a_{z,i+1} = F_{z,i+1}/m$

16 $v_{z,i+1} = v_{z,i} + \frac{h}{2}\left(a_{z,i} + a_{z,i+1}\right)$

17

18 **end**

Looking at the x-direction (lines $3-6$) the equations are calculated with 12 FLOPs. Again, we ignore the FLOPs of the force. In our program, the number of FLOPs are dependent on two loops. A generalisation of the number of FLOPs is therefore slightly more complicated.
The Velocity Verlet method is thus expected to be slower than the Euler method. (Ref. [2])

2.6 Kepler's second law - Conserved angular momentum

According to Kepler's second law, a connecting line, drawn from the Sun to the planet ("radius vector"), sweeps over areas of equal size in equal times. Figure 1 visualizes this. In other words, if the planet is closer to the sun on its orbit, it moves faster. If it is further away from the sun, it moves slower. This is due to the ellipse shape of planetary orbits, which makes the planet speed up when moving towards the sun, and slow down when moving away from it. If the orbit was circular, the speed of the planet would be constant.
Consider the area swept out in a small time interval. This can be expressed as

$$A = \frac{1}{2} r \cdot r d\theta, \tag{41}$$

where r is the distance between the sun and the planet, and $d\theta$ is the angle between position of the planet from the beginning and end of the time interval. The rate at which area is swept out is then

$$\frac{dA}{dt} = \frac{1}{2} r v_\theta, \tag{42}$$

where v_θ is the speed of the planet in the direction perpendicular to the radius vector. Lets now look at the definition of angular momentum,

$$L = m(\vec{r} \times \vec{v}) = m r v_\theta. \tag{43}$$

If we put the expression for angular momentum in equation (42), we end up with

$$\frac{dA}{dt} = \frac{L}{2m}. \tag{44}$$

Since we know from Kepler's second law that the speed of which area is swept out is constant, the angular momentum has to be conserved.

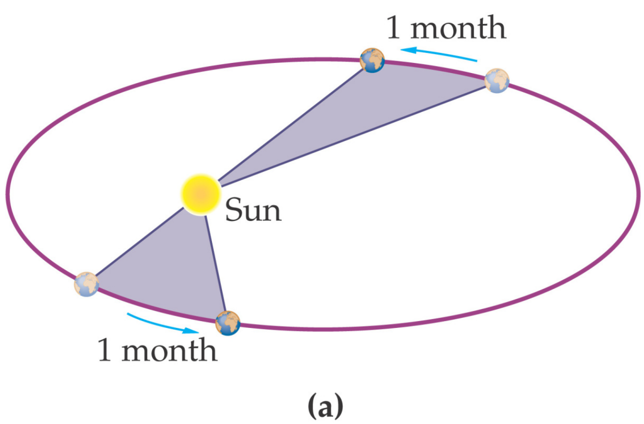


Figure 1: Kepler's second law visualized. The shaded areas are equal. Ref. [4]

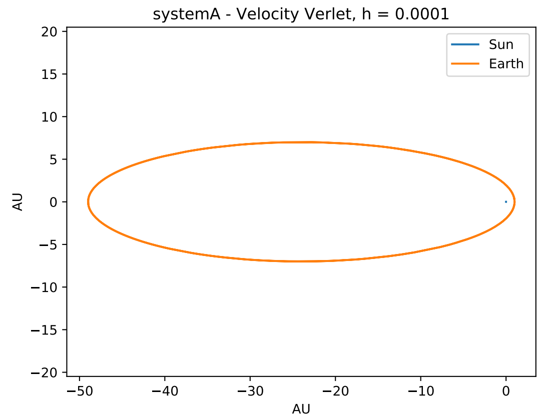
3 Results

For all the following results, the Velocity Verlet method is used. The reason for this is presented in Section 6.3.1, where the Euler method is compared to the Velocity Verlet method. The code for the computation and a README file can be found on Github repository.

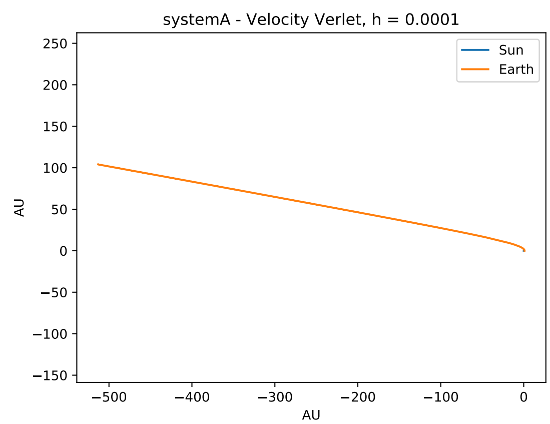
3.1 System A - One planet solar system

3.1.1 Escape Velocity

Figure 2 shows the orbit of a planet (Earth like) with different initial velocities, $v = 1.40 \cdot 2\pi AU/yr$ and $v = 1.42 \cdot 2\pi AU/yr$ respectively. Thus, the initial of velocity the planet must be larger than $v = 1.40 \cdot 2\pi \frac{AU}{yr}$, in order to escape from the Sun.



(a) planet not escaping the Sun, $v = 1.40 \cdot 2\pi \frac{AU}{yr}$



(b) planet escaping the Sun, $v = 1.42 \cdot 2\pi \frac{AU}{yr}$

Figure 2: Sun-Earth system: Orbit of a planet (Earth-like) with an initial distance of $1AU$ around the Sun with different initial velocities v.

3.1.2 Testing different forms of the force

Table 1 and 2 list the results for varying the exponent in the gravitational force, as described in paragraph 2.1, using system A for 5 years, varying Earth's initial velocity from circular to elliptical. The results include the relative difference in total energy, ϵ_E and the difference in angular momentum, ϵ_A.

Table 1: Sun-Earth system using Velocity Verlet for 5 years, $\Delta t = 0.0001$: Relative difference in total energy, ϵ_E, and angular moment, ϵ_A, between the start and end of the simulation when changing the gravitational force gradually from inverse-square to inverse-cube, exponent denoted by β. Earth's initial velocity is $2\pi AU/yr$.

β	ϵ_E	ϵ_A
2.0	$1.66 \cdot 10^{-14}$	$8.19 \cdot 10^{-15}$
2.2	$2.08 \cdot 10^{-12}$	$1.72 \cdot 10^{-14}$
2.4	$6.43 \cdot 10^{-13}$	$6.50 \cdot 10^{-15}$
2.6	$2.80 \cdot 10^{-12}$	$8.34 \cdot 10^{-15}$
2.8	$2.34 \cdot 10^{-11}$	$1.52 \cdot 10^{-14}$
3.0	$1.00 \cdot 10^{-14}$	$4.38 \cdot 10^{-15}$

Table 2: Sun-Earth system using Velocity Verlet for 5 years, $\Delta t = 0.0001$: Relative difference in total energy, ϵ_E, and angular moment, ϵ_A, between the start and end of the simulation when changing the gravitational force gradually from inverse-square to inverse-cube, exponent denoted by β. Earth's initial velocity is $5AU/yr$.

β	ϵ_E	ϵ_A
2.0	$4.99 \cdot 10^{-12}$	$1.42 \cdot 10^{-14}$
2.2	$3.51 \cdot 10^{-11}$	$1.74 \cdot 10^{-14}$
2.4	$7.73 \cdot 10^{-10}$	$1.35 \cdot 10^{-14}$
2.6	$1.60 \cdot 10^{-12}$	$1.68 \cdot 10^{-14}$
2.8	$6.48 \cdot 10^{-9}$	$6.75 \cdot 10^{-15}$
3.0	$1.45 \cdot 10^{4}$	$6.75 \cdot 10^{-15}$

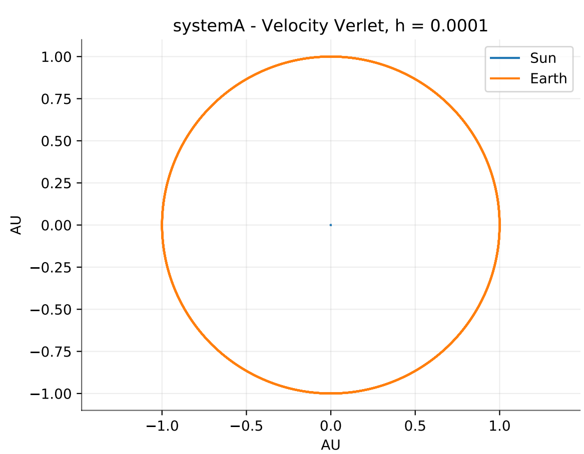


Figure 3: Simulation of the Sun-Earth system using Velocity Verlet, $\Delta t = 0.0001$. Varying the exponent, β, of the central force ($\beta \neq 2$ in eq. (1)) does not change Earth's orbit with initial circular velocity and 1AU distance to the Sun.

3.1.3 Perihelion precision of Mercury

Table 3 lists the numerical and theoretical perihelion angles for the procession of Mercury for the Sun-Mercury system after one, ten and hundred years using Velocity Verlet for different time steps. Mercury starts out $0.3075AU$ from the Sun with an initial velocity of $v = 12.44AU/yr$.

Table 3: Analytical, θ_A, and numerical, θ_N, perihelion angle for precision of Mercury for the Sun-Mercury system after 1, 10 and 100 years using Velocity Verlet for different step sizes Δt. Mercury's initial velocity is $v = 12.44AU/yr$ located $0.3075AU$ from the Sun.

time [year]	Δt	θ_A	θ_N
1	10^{-7}	$2.0847 \cdot 10^{-6}$	$2.0965 \cdot 10^{-6}$
1	10^{-8}	$2.0847 \cdot 10^{-6}$	$2.0916 \cdot 10^{-6}$
1	10^{-9}	$2.0847 \cdot 10^{-6}$	$2.0082 \cdot 10^{-6}$
10	10^{-7}	$2.0847 \cdot 10^{-5}$	$2.1491 \cdot 10^{-5}$
10	10^{-8}	$2.0847 \cdot 10^{-5}$	$2.0560 \cdot 10^{-5}$
100	10^{-6}	$2.0847 \cdot 10^{-4}$	$2.1592 \cdot 10^{-4}$
100	10^{-7}	$2.0847 \cdot 10^{-4}$	$2.0932 \cdot 10^{-4}$

3.2 System B - Three planet solar system

3.2.1 Comparing System B to System A

Next, we look at how Jupiter influences the Earth's orbit, as described in Section 4. Table 4 lists the relative difference in total energy between the start and end of simulation, ϵ_E, for System A and B. The Earth's initial velocity was changed from circular $v_c = 2\pi AU/yr$ to elliptical $v_e = 5AU/yr$. Jupiter's initial velocity was set to $v = 2\pi/\sqrt{5.1}$.
In order to get an impression of how the Earth's orbit gets changed, Figure 5 visualizes the absolute difference in the Earth's distance to the Sun for the two systems.

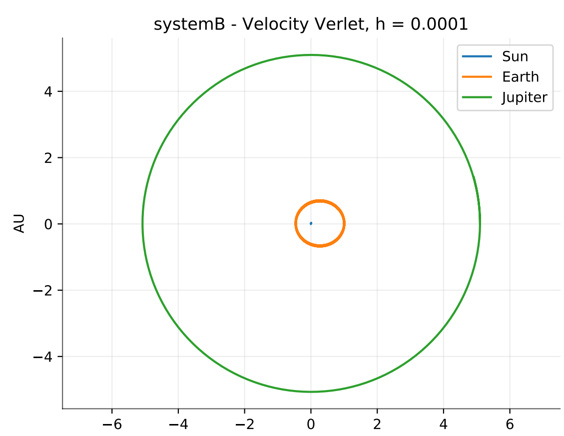
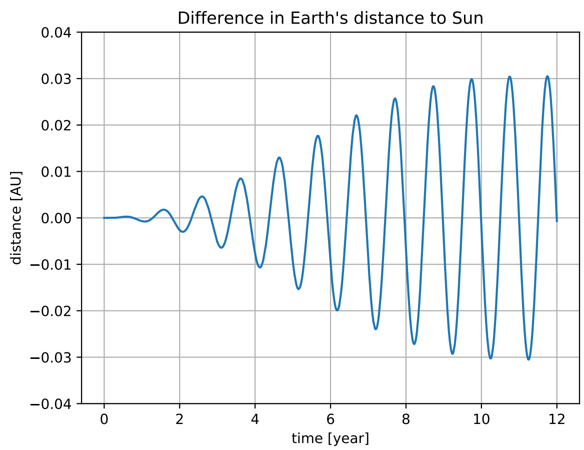


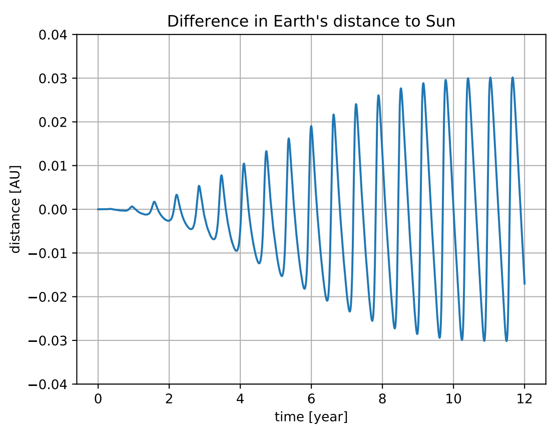
Figure 4: Simulation of Sun-Earth-Jupiter system for 12 years $\Delta t = 0.0001$. The initial velocity are chosen such that the orbits are circular.

Table 4: Sun-Earth system and Sun-Earth-Jupiter system using Veloctiy Verlet for 12 years, $\Delta t = 0.0001$: The relative difference in total energy between the start and end of the simulation ϵ_E where the Earth's initial velocity varies from $v_{circular} = 2\pi AU/yr$ to $v_{elliptical} = 5AU/yr$. Jupiter's velocity is $v = 2\pi/\sqrt{5.1}$.

system	Earth's orbitform	ϵ_E
system A	circular	$1.00 \cdot 10^{-14}$
system B	circular	$6.12 \cdot 10^{-14}$
system A	elliptical	$1.10 \cdot 10^{-9}$
system B	elliptical	$2.38 \cdot 10^{-11}$



(a) Earth's initial velocity is circular, $v = 2\pi AU/yr$



(b) Earth's initial velocity is elliptical, $v = 5AU/yr$

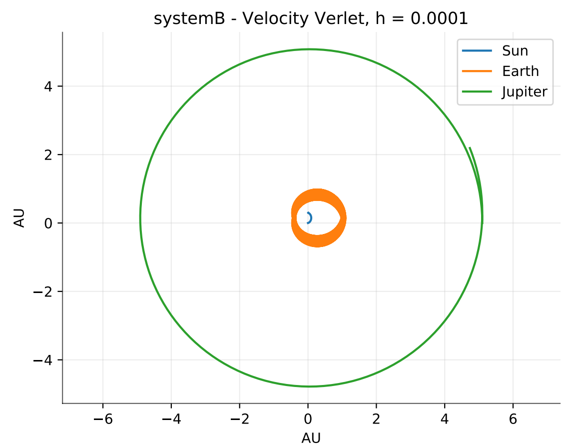
Figure 5: Sun-Earth system and Sun-Earth-Jupiter system using Velocity Verlet for 12 years, $\Delta t = 0.0001$: absolute difference in the Earth's distance to Sun for the two systems. Jupiter's initial velocity is set to $v = 2\pi/\sqrt{5.1}$ AU/yr.

3.2.2 Increasing the mass of Jupiter

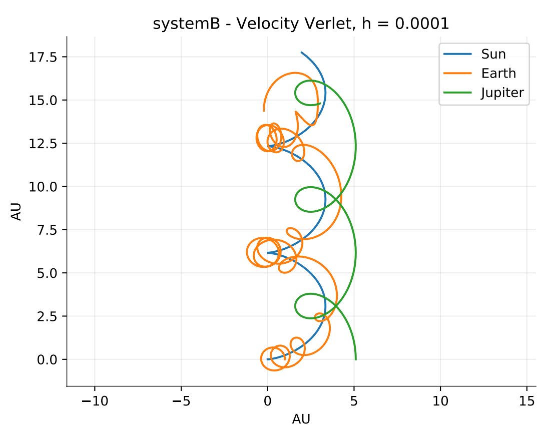
In Table 5, we look at the relative energy difference, ϵ_E, for system B, where the mass of Jupiter has been multiplied with first 10 and then with 100. Earth and Jupiter's initial velocity is is chosen such that the orbit is circular. Figure 6 shows the plots of these two simulations.

Table 5: Sun-Earth-Jupiter system using Velocity Verlet for 12 years, $\Delta t = 0.0001$: The relative difference in total energy between the start and end of the simulation, ϵ_E, with circular initial velocity.

$Mass_{Jupiter}$ [kg]	Earht's orbitform	ϵ_E
$M_{Jupiter} \cdot 10$	circular	$9.59 \cdot 10^{-13}$
$M_{Jupiter} \cdot 1000$	circular	$1.89 \cdot 10^{-9}$



(a) Jupiter's mass increased by a factor of 10.



(b) Jupiter's mass increased by a factor of 1000. Centre of mass moves up.

Figure 6: Simulation of Sun-Earth-Jupiter system for 12 years ($\Delta t = 0.0001$) with Jupiter's mass increased with 10 and 1000 respectively in (a) and (b).

3.3 System C - Many planet solar system

3.3.1 Three body solar system

Before we include all the planets of the solar system, we want to compare system B to the Sun-Earth-Jupiter system setting the centre of mass as the origin. Thus, the Sun is no longer the centre of mass. As the difference is difficult so see, we multiplied Jupiter's mass with 1000 again.

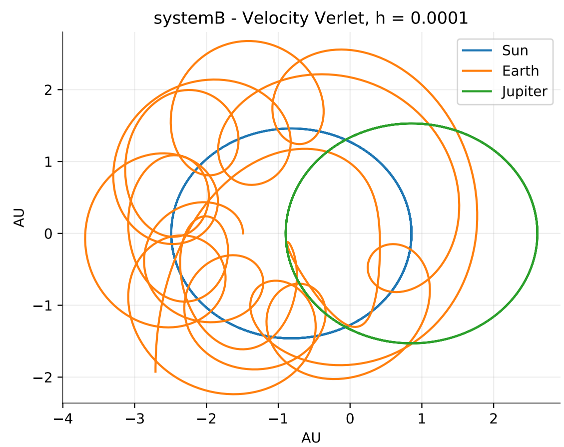


Figure 7: Simulation of Sun-Earth-Jupiter system for 12 years and $\Delta t = 0.0001$. Jupiter's mass increased with 1000. Centre of mass set as the origin.

3.3.2 Adding further planets

Building on on the three body solar system, Figure 8 shows the orbits of all the planets in the solar system using NASA's initial conditions.

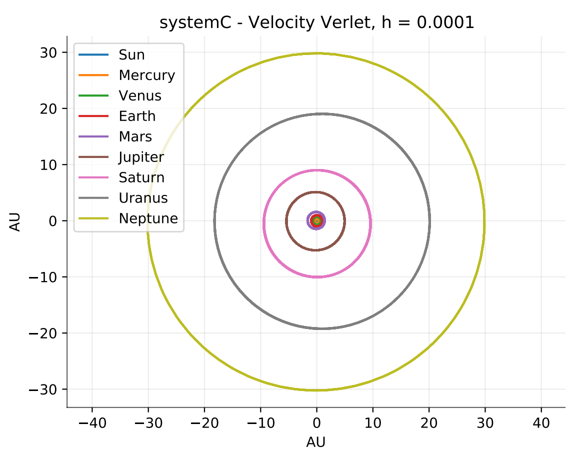


Figure 8: Orbits of planets of the Solar system using Velocity Verlet for 1000 years, $\Delta t = 0.0001$. The relative error in the difference of energy between the start and end simulation is of the order of 10^{-11}

4 Discussion

4.1 System A - One planet solar system

4.1.1 Escape Velocity

As mentioned in section 2.1, the escape velocity of any object in orbit is the velocity for a circular orbit, multiplied by $\sqrt{2} \approx 1.41$. Figure 2 shows the object remaining in orbit for the initial velocity of $v = 1.40 \cdot 2\pi AU/yr$, and escaping orbit for the initial velocity of $v = 1.42 \cdot 2\pi AU/yr$, which is exactly as theory suggest. An interesting note, is that the orbital time of the object in figure (2a) was over 200 years.

4.1.2 Testing different forms of the force

In Section 2.6, it is shown that the angular momentum should be conserved. This still applies when we change the gravitational force from being inverse-square related to the distance, to being inverse-cube related. Note that the inverse-cube force is still conservative as it is only dependent on the distance.
Table 1 and 2 show that the angular momentum is conserved as we gradually vary β from 2 to 3. For Table 1, where the Earth's initial velocity is circular, shows that the total energy is conserved for all β. The relative error varies between $\epsilon_E = 10^{-14}$ to 10^{-11}. We expected no considerable difference, as described in paragraph 2.1, this is related to the units. The Earth has an average distance of $1AU$ to the Sun. For a circular orbit, the distance is constant and $(1AU)^2$ is no different from $(1AU)^3$.
For the case where the Earth's velocity is elliptical (Table 2), the energy is conserved only for $\beta < 3$. The relative error until there is minimally greater than for the circular case. This might be due to the gravitational constant, G, being determined by a circular orbit. Even though fluctuations for different β in the relative energy difference er visible, there are minimal. Round-off errors must be considered.
However for $\beta = 3$, the relative error in the difference in energy abruptly increases tremendously for the elliptical case. Looking at the Earth's orbit, one sees that the Earth escapes the Sun. Thus, the system is not stable anymore. It can be shown that almost any other force law than inverse-square leads to orbits that do not "close". That is, they do not come back to the places that they started with the original velocities. Furthermore, despite all central forces allow circular orbits, they are not always stable. This coincides with Kepler's laws. Generally, Newton's inverse square law is deduced from Kepler's third law of planetary motion. A inverse-cube force does not allow elliptical orbits. (Ref [1]).

The most interesting part of this problem, is looking at what happens when the orbit is circular and the distance between the objects is $1AU$. As also mentioned in section 2.6, the gravitational force does not change with β for this distance. This is a results of units chosen for the simulations. If we did not use $1AU$, and instead used the initial distance of $1.5 \cdot 10^{11}m$, the gravitational force would not remain constant, and figure 3 would look different, even though $1AU = 1.5 \cdot 10^{11}m$.

4.1.3 Perihelion precision of Mercury

In order to solve this problem, we had to rewrite most of the solver function. In the previous simulation of the solar system, we stored all the values of position and velocity. For a step size of $10^{-4}yr$, this is not a problem, but for the perihelion precision of Mercury, we needed a much smaller step size. This purposed a memory problem, because having 12 arrays with a length of $10^{1}0$ needs $12 \cdot 10^{10} \cdot 8$ bytes $= 960$Gb of memory. We rewrote the function so that it didn't store more than three positions at the time. Another thing we noticed, was that writing the position to file was highly unnecessary, because we weren't interested in plotting the orbit. When we didn't remove the function that writes to file, the program ran very slowly, and we ended up with files with sizes of more than 100Gb. After we noticed the size of the file with position, we removed the function, and the program ran 20 times faster. Another solution to this, could have been only writing every $1000th$ or so position to file, which probably would have been more than enough to make a nice plot.

Table 3 shows the calculation of the perihelion angle for Mercury. It includes both the theoretical and numerical solutions, as well as the step size and length of the simulation. We see that for a 100 years, and a step size of 10^{-7}, the numerically calculated perihelion angle of Mercury is $\theta_N = 2.0932 \cdot 10^{-7}$, which is is a relative error of $\approx 3.57\%$. If we look at a step size ten times smaller, the relative error is $\approx 0.41\%$. When trying to run the program with a step size of $10^{-9}yr$, we got errors. We suspected a memory problem, but considering that we are not storing a lot of values, and we don't write a lot of values to file, this is most likely not the case. In the future, this problem could be looked more into, in order to calculate the perihelion angle at an even higher precision to see if we can get more accurate results.

4.2 System B - Three Planet solar system

4.2.1 Comparing System B to System A

When adding a third object to the simulation, we did not see much difference in the orbits. Figure 4, that shows the three body system, is not very different from Figure 3, which shows a circular orbit for system A. The most obvious difference is that Earth's orbit not longer has the Sun in the center. Figure 5, which visualizes the difference in distance between system A and B, shows that adding Jupiter does indeed affect the Earth. The effect, which is difficult to see in Figure 4, is that the distance between the Sun and the Earth actually fluctuates around what would have been a completely circular orbit (constant distance). The fluctuation starts out very small, and then increases gradually in magnitude until about the 10^{th} Earth orbit, from which the fluctuation looks constant. This could be due to a property of the three body system, which might be complicated to analyze. At most, the fluctuation is about 3% of the initial Sun-Earth difference of $1AU$, which is pretty significant.

Apparent from Table 4, an elliptical orbit is less stable than a circular one. That is no surprise since our method finding the gravitational constant, G, is based on a circular orbit. Furthermore, adding more planets imply more calculations and round-off errors must be considered. We remind that the Velocity Verlet method has a global error of the order of $O(h^2)$.

The difference in stability between the three-body and two-body system is still negligible.

4.2.2 Increasing the mass of Jupiter

Table 5 shows the relative change in total energy from the beginning and end of the simulation. Theoretically, the total energy shouldn't change at all, as the only force involved is conservative. Our results agree with theory. If Jupiter's mass is multiplied by 1000, the relative error in energy-difference is still of the order of 10^{-9}. The fact that the relative error remained almost unchanged, shows that the Velocity Verlet method still is stable at the same number of integration points, when Jupiter's mass is increased.
When increasing the mass of Jupiter, the simplification that the Sun's motion can be neglected is not applicable anymore. Figure 6 a) clearly shows that the Sun's orbit gets affected by Jupiter. The Sun starts moving. If Jupiter's mass is multiplied by 1000, we must consider that Jupiter's mass becomes almost equal to the Sun's mass. Apparent from 6 b), the centre of mass moves up, following the motion of the planets. Until now, we have assumed the Sun to be the centre of mass, which becomes a poor approximation if the mass of the Jupiter increases so much.

4.3 System C - Many planet solar system

4.3.1 Three body solar system

Comparing Figure 6 b) to 7, where Jupiter's mass is multiplied by a thousand, it is obvious that setting the centre of mass as the origin does change the orbits. In reality, this effect is barely visible as the Sun is so much heavier. Our earlier approximation, setting the Sun as the centre of mass is thus justified.

4.3.2 Adding further planets

As further planets are added to the solar system, each orbit automatically gets more affected. Figure 8 shows the solar system after a 1000 years-simulation, where the planets orbit the Sun. The orbits seem reasonable, with Neptune being with around 30 AU farthest away from the Sun. Table 6 lists the distances to the Sun for the different planets. As the planets are affected by each other, the orbits do minimally change, thus the orbit appears thicker in the plot. As expected, the orbits are not circular, but more elliptical in shape.
The Velocity Verlet algorithm is still stable, as the relative error in energy difference between the start and end of the simulation is of the order of 10^{-11}.

5 Conclusion

Unlike the Euler method, the Velocity Verlet method is suitable for modeling the solar system, since it conserves energy.
For our model, the gravitational constant, $G = 4\pi^2 AU^3/yr^2$ was determined by assuming circular orbits of the planets and measuring the masses in solar masses. Thus, changing the system to elliptical orbits implies a minimally larger relative error in the difference in the total energy. For the Sun-Earth system, simulated for 5 years, the relative error in the energy difference is of the order of 10^{-14} if the Earth's initial velocity is circular and $\Delta t = 10^{-4}$. By contrast, using an initial velocity which is elliptical, $v = 5AU/yr$, the relative error increases to the order of 10^{-12}.

Generally, the relative error is quite stable for different systems and different time-simulation using a step size of 10^{-4}.
Furthermore, changing the gravitational force from being inverse-square related to the distance, to being inverse-cube related shows that the inverse case is more stable for elliptical orbits. In accordance with the theory, elliptical orbits become unstable for an inverse-cube force. A circular orbit was not affected, which is due to the chosen units.
Adding Jupiter's mass to the Sun-Earth case, does alter the Earth's orbit of the order of $10^{-2}AU$. Multiplying Jupiter's mass with a thousand, such as its mass has the same magnitude as of the Sun's, changes the form of the orbits greatly. For example, the Sun starts moving considerably and the relative error in the difference of energy increases considerably. Adjusting the centre of mass as the origin and not the Sun improves the results. However, in reality, setting the Sun as the centre of mass is justified.
Considering again the Sun-Earth system, the theoretical escape velocity of the Earth, $v = \sqrt{2} \cdot 2\pi$, was also computationally obtained.
Generally, future studies should include how planet-type, initial velocity and the duration of the time-simulation influence the orbits of the planets. When calculating the perihelion angle of Mercury, we needed a much higher precision than earlier, which meant having to rewrite the program to not store positions in memory. After simulating a 100 years with a step size of 10^{-7}years, we found the perihelion angle of Mercury to be $2.0932 \cdot 10^{-4}$, which is an error of 0.41% from the theoretical value. Writing all values to file also took up both unnecessary amounts of memory and computation time. The analysis should also include the stability of the Velocity Verlet method when changing the step size.
Finally, we have assumed the celestial bodies to be point objects. This is a great simplification and further studies could consider expansion of the objects. Additionally, further celestial bodies such as moons could be included.

6 Appendix

6.1 Planet's properties

Table 6: Mass (in kg) of Sun and of relevant planets plus their distance to the Sun (in AU)

Celestial body	Mass [kg]	Distance to Sun [AU]
Sun	2.0×10^{30}	0.00
Earth	6.0×10^{24}	1.00
Jupiter	1.9×10^{27}	5.10
Mars	6.6×10^{23}	1.52
Venus	4.9×10^{24}	0.72
Saturn	5.5×10^{26}	9.54
Mercury	3.3×10^{23}	0.39
Uranus	8.8×10^{25}	19.19
Neptun	1.0×10^{26}	30.06
Pluto	1.3×10^{22}	39.53

6.2 Solving Newton's law of motion

6.2.1 Euler method - round of errors

It is seems plausible that by decreasing the step size h, Euler's method should increase in accuracy. However, the risk of round-off errors also increases. Using the two-step formula to numerically compute the derivative f

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h), \tag{45}$$

round-off errors can occur if $f(x+h) - f(x) \approx 0$. This is the reason why the Euler method is not recommended for precision calculation.

6.2.2 Euler-Cromer method

The Euler-Cromer method is an improved modification of the Euler method. The only difference is, when computing $y^{(1)}_{n+1}$, we use the already computed next velocity $y^{(2)}_{n+1}$.
Generally, the equations read

$$y^{(1)}_{n+1} = y^{(1)}_n + hy^{(2)}_{n+1} + O(h^2) \tag{46}$$

$$y^{(2)}_{n+1} = y^{(2)}_n + ha_n + O(h^2). \tag{47}$$

Here, a_n stands for the acceleration, $F(y_n)/m$, at the n^{th} integration step. Likewise to the Euler method, the Euler-Cromer method has an local error of $O(h^2)$.
A general code implementation is shown below.

Algorithm 3: Euler-Cromer algorithm

Result: Position and velocity of planet in 3D

initialization;

for *i=0,1, . . ., N-1* **do**

    // x-direction

    $a_{x,i} = F_{x,i}/m$

    $v_{x,i+1} = v_{x,i} + ha_{x,i}$

    $r_{x,i+1} = r_{x,i} + hv_{x,i+1}$

    // y-direction

    $a_{y,i} = F_{y,i}/m$

    $v_{y,i+1} = v_{y,i} + ha_{y,i}$

    $r_{y,i+1} = r_{y,i} + hv_{y,i+1}$

    // z-direction

    $a_{z,i} = F_{z,i}/m$

    $v_{z,i+1} = v_{z,i} + ha_{z,i}$

    $r_{z,i+1} = r_{z,i} + hv_{z,i+1}$

end

6.2.3 Verlet method

For the Verlet method, we again approximate Newton's law of motion with a Taylor expansion. Now, we go up to the second derivative (third order), which is known via Newton's second law

$$y^{(3)}(t) = a(x,t). \tag{48}$$

Additionally, we approximate both in the forward, $(t+h)$, and in the backward, $(t-h)$, direction. The corresponding equations are

$$t+h: \qquad y_{i+1} = y(t_i) + hy^{(2)}(t_i) + \frac{h^2}{2}y^{(3)}(t_i) + O(h^3), \tag{49}$$

$$t-h: \qquad y_{i-1} = y(t_i) - hy^{(2)}(t_i) + \frac{h^2}{2}y^{(3)}(t_i) + O(h^3). \tag{50}$$

Combining equations (49) and (50) reads

$$y_{i+1} = 2y(t_i) - y_{i-1} + h^2y^{(3)}(t_i) + O(h^4). \tag{51}$$

Note that the truncation error has improved to $O(h^4)$, since the $y^{(2)}$ terms cancel. Thus, the velocity is not a part of the equation, because the force is conservative; that is only dependent on position. Additionally, the method is not self-starting. For $i = 0$, both the values y_i and the factious y_{-1} must be known. This problem can be solved by using the Velocity Verlet method.

6.3 System A - One planet solar system

6.3.1 Tests of the algorithms

In order to test the stability of our algorithm, Table 7 lists the absolute difference in total energy between the beginning and end of the simulation for system A. The simulations were done for one year, varying Δt from 10^{-1} to 10^{-6} and using both the Euler and Velocity Verlet method. Figure 9 shows the Earth's orbit around the Sun, where the Earth has a circular initial velocity, $v = 2\pi AU/yr$, using the Euler method with $\Delta t = 0.001$.

Table 7: Sun-Earth system using Euler and Velocity Verlet for 8 years: Absolute difference in total energy between the beginning and end of the simulation, ϵ_E, for different time steps, Δt. Theoretically, the difference should be zero.

$\Delta t[yr]$	ϵ_E $[AU^2/yr^2]$	ϵ_{VV} $[AU^2/yr^2]$
10^{-1}	$1.54 \cdot 10^{-4}$	$1.99 \cdot 10^{-5}$
10^{-2}	$1.09 \cdot 10^{-4}$	$4.04 \cdot 10^{-10}$
10^{-3}	$5.89 \cdot 10^{-5}$	$1.41 \cdot 10^{-14}$
10^{-4}	$1.21 \cdot 10^{-5}$	$9.32 \cdot 10^{-18}$
10^{-5}	$1.39 \cdot 10^{-6}$	$5.27 \cdot 10^{-17}$
10^{-6}	$1.40 \cdot 10^{-7}$	$9.65 \cdot 10^{-17}$

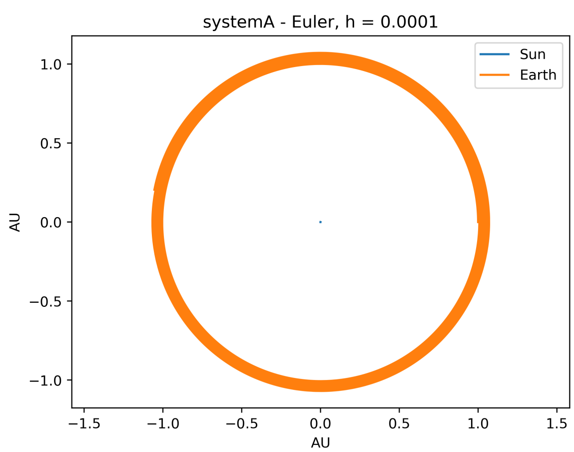


Figure 9: Sun-Earth system using Euler for 8 years, $\Delta t = 0.0001$: The Earth's orbit around the Sun is not a circle if the Earth's initial velocity is circular, $v = 2\pi AU/yr$. This is due Euler method not conserving energy.

Discussion As explained in 2.1, the total energy for a circular orbit should be conserved. Thus the energy difference, shown in Table 7, should theoretically be zero. Generally, the absolute difference in total energy reduces with decreasing Δt. Furthermore, the Velocity Verlet method beats the Euler method significantly. This is no surprise as the Euler method, unlike the Velocity Verlet method, does not conserve energy. For $\Delta t = 0.0001$, the relative error is 6.3% for the

Euler method and $6.0 \cdot 10^{-4}\%$ for the Velocity Verlet. Figure 9 using the Euler method shows this clearly. The Earth's orbit was supposed to be solely a circle not a spiral. Additionally, the global mathematical error for the Euler method is one order less than for the Velocity Verlet ($O(h)$ vs $O(h^2)$). A more detailed comparison is provided in Section 2.5.2. Hence, the Velocity Verlet method is clearly more appropriate for our task and therefore used in further computation. The machine precision is limited to about 10^{-15} to 10^{-18} using double precision. We therefore choose $\Delta t = 10^{-4}$, which gives an absolute error with the order of 10^{-14}, as our future step size.

References

[1] *Astronomy Workshop:Central force*. Oct. 2009. URL: https://janus.astro.umd.edu/front/pages/links/Centralforce1.html. (accessed: 25.10.2020).

[2] *FYS3150 Lectures Ordinary differential equations*. Oct. 2017. URL: http://compphysics.github.io/ComputationalPhysics/doc/pub/ode/pdf/ode-print.pdf. (accessed: 12.10.2020).

[3] *FYS3150 Project3*. Oct. 2020. URL: http://compphysics.github.io/ComputationalPhysics/doc/Projects/2020/Project3/pdf/Project3.pdf. (accessed: 20.10.2020).

[4] *Kepler's second law*. Oct. 2020. URL: https://eloisechen.wordpress.com/2013/02/20/conservation-of-angular-momentum-and-keplers-second-law/. (accessed: 13.10.2020).

[5] Binbin Li. *Error Prooagation of Verlet Algorithm*. URL: https://www.researchgate.net/profile/Binbin_Li2/publication/265908915_Error_Propagation_of_Verlet_Algorithm/links/542118150cf241a65a1e59db/Error-Propagation-of-Verlet-Algorithm.pdf. (accessed: 13.10.2020).

[6] *Mercury Orbit Precession*. URL: https://ccnmtl.columbia.edu/projects/mmt/frontiers/web/chapter_2/8898.html. (accessed: 20.10.2020).

[7] *Precession of the perihelion of Mercury*. Mar. 2000. URL: https://aether.lbl.gov/www/classes/p10/gr/PrecessionperihelionMercury.htm. (accessed: 20.10.2020).