ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2020
Instructor: Prof. Stanley H. Chan

PURDUE UNIVERSITY

Homework 2

Fall 2020
(Due: September 18, 2020 Friday)

Name: Elias Talcott Email: etalcott@purdue.edu

Homework is due at 11:59pm (midnight) Eastern Time. Please print this homework, write your solution, and scan the solution. Submit your homework through Gradescope. No late homework will be accepted.

Exercise 1.
Consider an experiment consisting of rolling a die twice. The outcome of this experiment is an ordered pair whose first element is the first value rolled and whose second element is the second value rolled.

(a) Find the sample space.

(b) Find the set A representing the event that the value on the first roll is greater than or equal to the value on the second roll.

(c) Find the set B corresponding to the event that the first roll is a six.

(d) Let C correspond to the event that the first valued rolled and the second value rolled differ by two. Find $A \cap C$.

Note that A, B, and C should be subsets of the sample space specified in Part (a).

A. $\Omega = \{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots,(6,6)\}$

B. $A = \{(1,1),(2,1),\ldots,(6,1),(2,2),\ldots,(6,2),\ldots,(3,3),\ldots,(6,3),\ldots,(6,6)\}$

C. $B = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

D. $C = \{(1,3),(2,4),(3,5),(4,6),(3,1),(4,2),(5,3),(6,4)\}$

$A \cap C = \{(3,1),(4,2),(5,3),(6,4)\}$

Exercise 2.
A space S is defined as $S = \{1, 3, 5, 7, 9, 11\}$, and three subsets as $A = \{1, 3, 5\}$, $B = \{7, 9, 11\}$, $C = \{1, 3, 9, 11\}$. Assume that each element has probability 1/6. Find the following probabilities: (a) $\mathbb{P}[A]$, (b) $\mathbb{P}[B]$, (c) $\mathbb{P}[C]$, (d) $\mathbb{P}[A \cup B]$, (e) $\mathbb{P}[A \cup C]$, (f) $\mathbb{P}[(A \backslash C) \cup B]$.

A. $\mathbb{P}[A] = 3(\frac{1}{6}) = 1/2$

B. $\mathbb{P}[B] = 3(\frac{1}{6}) = 1/2$

C. $\mathbb{P}[C] = 4(\frac{1}{6}) = 2/3$

D. $\mathbb{P}[A \cup B] = 6(\frac{1}{6}) = 1$

E. $\mathbb{P}[A \cup C] = 5(\frac{1}{6}) = 5/6$ $\quad A \cup C = \{1,3,5,9,11\}$

F. $\mathbb{P}[(A \backslash C) \cup B] = 0(\frac{1}{6}) = 0$ $\quad (A \backslash C) \cup B = \{5\} \cup \{7,9,11\} = \emptyset$

Exercise 3.
Let the events A and B have $\mathbb{P}[A] = x$, $\mathbb{P}[B] = xy$ and $\mathbb{P}[A \cup B] = x^2 z$. Find the following probabilities.

(a) $\mathbb{P}[A \cap B]$

(b) $\mathbb{P}[A^c \cap B^c]$

(c) $\mathbb{P}[A^c \cup B^c]$

(d) $\mathbb{P}[A \cap B^c]$

(e) $\mathbb{P}[A^c \cup B]$

A. $\mathbb{P}[A \cap B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cup B] = x + xy - x^2 z$

B. $\mathbb{P}[A^c \cap B^c] = 1 - \mathbb{P}[A \cup B] = 1 - x^2 z$

C. $\mathbb{P}[A^c \cup B^c] = 1 - \mathbb{P}[A \cap B] = 1 - x - xy + x^2 z$

D. $\mathbb{P}[A \cap B^c] = \mathbb{P}[A \backslash B] = \mathbb{P}[A] - \mathbb{P}[A \cap B] = x - x - xy + x^2 z = x^2 z - xy$

E. $\mathbb{P}[A^c \cup B] = 1 - \mathbb{P}[A \backslash B] = 1 - (x^2 z - xy) = 1 + xy - x^2 z$

$x-1<2$
$x<3$

$B = \{x \mid x<3\}$

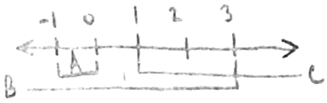
Exercise 4.
A number x is selected at random in the interval $[-1,3]$. Let the events $A = \{x \mid x<0\}$, $B = \{x \mid (x-1)^2 < 4\}$, $C = \{x \mid x>1\}$. Find (a) $\mathbb{P}[A \mid B]$, (b) $\mathbb{P}[B \mid C]$, (c) $\mathbb{P}[A \mid C^c]$, (d) $\mathbb{P}[B \mid C^c]$.

A. $\mathbb{P}[A|B] = \mathbb{P}[A \cap B] / \mathbb{P}[B] = 1/4$

B. $\mathbb{P}[B|C] = \mathbb{P}[B \cap C] / \mathbb{P}[C] = 1/2 / 1/2 = 1$

C. $\mathbb{P}[A|C^c] = \mathbb{P}[A \cap C^c] / \mathbb{P}[C^c] = 1/4 / 1/2 = 1/2$

D. $\mathbb{P}[B|C^c] = \mathbb{P}[B \cap C^c] / \mathbb{P}[C^c] = 0 / 1/2 = 0$



$\mathbb{P}[A] = 1/4$
$\mathbb{P}[B] = 1$
$\mathbb{P}[C]$ $1/2$, $\mathbb{P}[C^c] = 1/2$

Exercise 5.
Let A, B, C be events with probabilities $\mathbb{P}[A] = 0.3$, $\mathbb{P}[B] = 0.2$, $\mathbb{P}[C] = 0.5$. Find

(a) $\mathbb{P}[A \cup B]$ if A and B are independent

(b) $\mathbb{P}[A \cup B]$ if A and B are disjoint

(c) $\mathbb{P}[A \cup B \cup C]$ if A, B and C are independent

(d) $\mathbb{P}[A \cup B \cup C]$ if A, B and C are pairwise disjoint; Can this happen?

A. $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A]\mathbb{P}[B] = 0.5 - 0.06 = 0.44$

B. $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] = 0.5$

C. $\mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] - \mathbb{P}[A]\mathbb{P}[B] - \mathbb{P}[A]\mathbb{P}[C] - \mathbb{P}[B]\mathbb{P}[C] + \mathbb{P}[A]\mathbb{P}[B]\mathbb{P}[C]$
$= 0.3 + 0.2 + 0.5 - 0.06 - 0.15 - 0.1 + 0.03 = 0.72$

D. $\mathbb{P}[A \cup B \cup C] = \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] = 0.3 + 0.2 + 0.5 = 1$

This is possible since A, B, and C account for all events.
If $\mathbb{P}[A \cup B \cup C] > 1$, this would not be possible.

Exercise 6.

(a) By using the fact that $\mathbb{P}[A \cup B] \leq \mathbb{P}[A] + \mathbb{P}[B]$, show that $\mathbb{P}[A \cup B \cup C] \leq \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C]$.

(b) By using the fact that $\mathbb{P}\left[\bigcup_{k=1}^{n} A_k\right] \leq \sum_{k=1}^{n} \mathbb{P}[A_k]$, show that $\mathbb{P}\left[\bigcap_{k=1}^{n} A_k\right] \geq 1 - \sum_{k=1}^{n} \mathbb{P}[A_k^c]$.

A. Set $D = A \cup B$

$P[D \cup C] \leq \mathbb{P}[D] + \mathbb{P}[C]$ treat union of A and B as a single set D

$\therefore \underline{P[A \cup B \cup C] \leq \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C]}$

B. $\mathbb{P}[\bigcup_{k=1}^{n} A_k] \leq \sum_{k=1}^{n} \mathbb{P}[A_k]$

Exercise 7.
The following result is known as the Bonferroni's Inequality.

(a) Prove that for any two events A and B, we have

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

(b) Generalize the above to the case of n events $A_1, A_2, \ldots, A_n$, by showing that

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots + \mathbb{P}(A_n) - (n-1).$$

Hint: You may use the generalized Union Bound $\mathbb{P}(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathbb{P}(A_i)$.

A. $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$

maximum of $\mathbb{P}(A \cup B) = 1$ (ranges $0 \to 1$)

$\therefore \mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$

B. $\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n)$

$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \sum_{i=1}^{n} \mathbb{P}(A_1) - \mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n)$

maximum of $\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = n-1$ $(0 \to n-1)$

$\therefore \mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots + \mathbb{P}(A_n) - (n-1)$

Exercise 8.

A block of information is transmitted repeated over a noisy channel until an error-free block is received. Let $M \geq 1$ be the number of blocks required for a transmission. Define the following sets

(i) $A = \{M \text{ is even}\}$

(ii) $B = \{M \text{ is a multiple of } 5\}$

(iii) $C = \{M \text{ is less than or equal to } 7\}$

Assume that the probability of requiring one additional block is half of the probability without the additional block. That is:

$$\mathbb{P}[M = k] = \left(\frac{1}{2}\right)^k, \qquad k = 1, 2, \ldots.$$

Determine the following probabilities

(a) $\mathbb{P}[A]$, $\mathbb{P}[B]$, $\mathbb{P}[C]$, $\mathbb{P}[C^c]$

(b) $\mathbb{P}[A \cap B]$, $\mathbb{P}[A \setminus B]$, $\mathbb{P}[A \cap B \cap C]$

(c) $\mathbb{P}[A \,|\, B]$, $\mathbb{P}[B \,|\, A]$

(d) $\mathbb{P}[A \,|\, B \cap C]$, $\mathbb{P}[A \cap B \,|\, C]$

Hint: $\mathbb{P}[A] \neq \frac{1}{2}$. $\mathbb{P}[A] = \sum_{k=\text{even}} \mathbb{P}[M = k]$.

$10, 20$ $2, 4, 6, 8, n$

$\mathbb{P}[A] = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \ldots \left(\frac{1}{2}\right)^{2n}$

$\mathbb{P}[B] = \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^{10} + \ldots \left(\frac{1}{2}\right)^{5n}$

$\mathbb{P}[C] = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \ldots \left(\frac{1}{2}\right)^7$

A. $\mathbb{P}[A] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n}$, $\mathbb{P}[B] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{5n}$, $\mathbb{P}[C] = \sum_{n=1}^{7} \left(\frac{1}{2}\right)^{n}$, $\mathbb{P}[C^c] = \sum_{n=8}^{\infty} \left(\frac{1}{2}\right)^{n}$

B. $\mathbb{P}[A \cap B] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{10n}$, $\mathbb{P}[A \setminus B] =$, $\mathbb{P}[A \cap B \cap C] = 0$

C. $\mathbb{P}[A|B] = \dfrac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \dfrac{\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{10n}}{\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{8n}$, $\mathbb{P}[B|A] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{5n}$

D. $\mathbb{P}[A | B \cap C] = \dfrac{\mathbb{P}[A \cap B \cap C]}{\mathbb{P}[B \cap C]} = 0$, $\mathbb{P}[A \cap B | C] = \dfrac{\mathbb{P}[A \cap B \cap C]}{\mathbb{P}[C]} = 0$

Exercise 9. (PROGRAMMING)
Write a MATLAB / Python program to simulate the following experiments.

(a) Draw a dice 100 times. That is, generate a sequence of 100 random numbers from the set $\{1, \dots, 6\}$. Call this sequence $X_1, \dots, X_{100}$. Plot the histogram of $X_1, \dots, X_{100}$, with bin centers $\{1, \dots, 6\}$. Do not use a for-loop in your code. Submit your plot.

(b) Repeat (a) by drawing the dice 10000 times.

(c) Draw another dice 100 times. Call this sequence $Y_1, \dots, Y_{100}$. Let $Z_i = X_i + Y_i$ for $i = 1, \dots, 100$. Plot the histogram of $Z_1, \dots, Z_{100}$. Submit your plot.

(d) Repeat (c) by drawing the dices 10000 times.

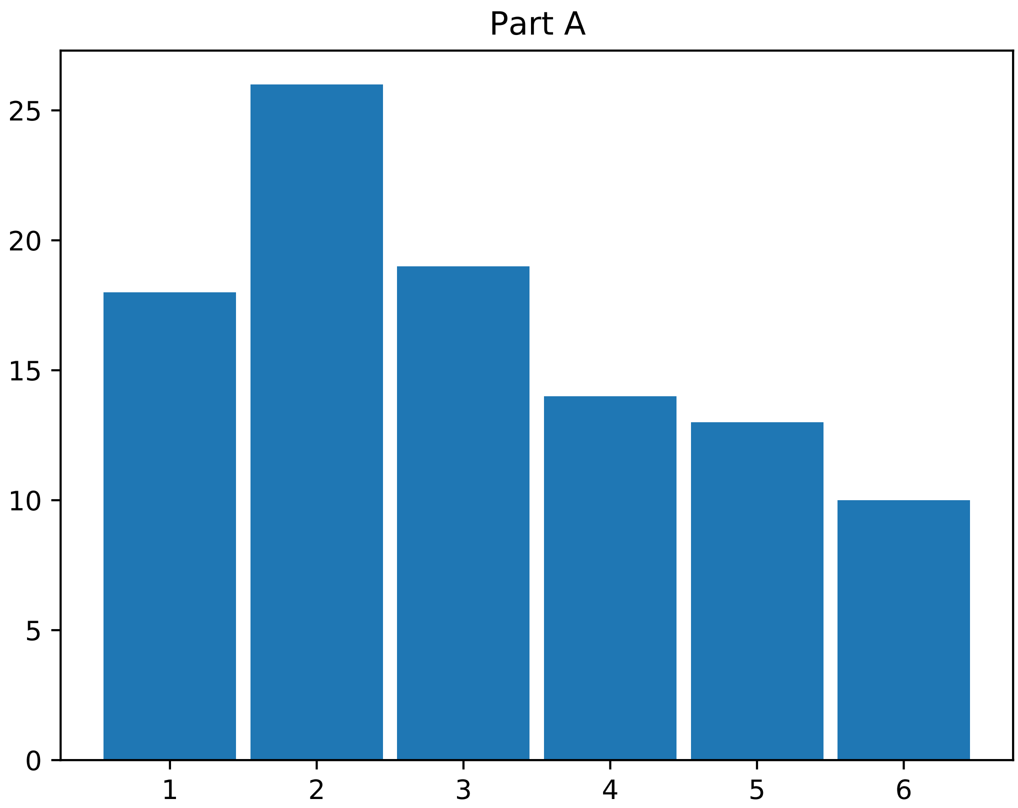
(e) Using the histogram found in (d), find the probability that $4 < Z_i \leq 7$.

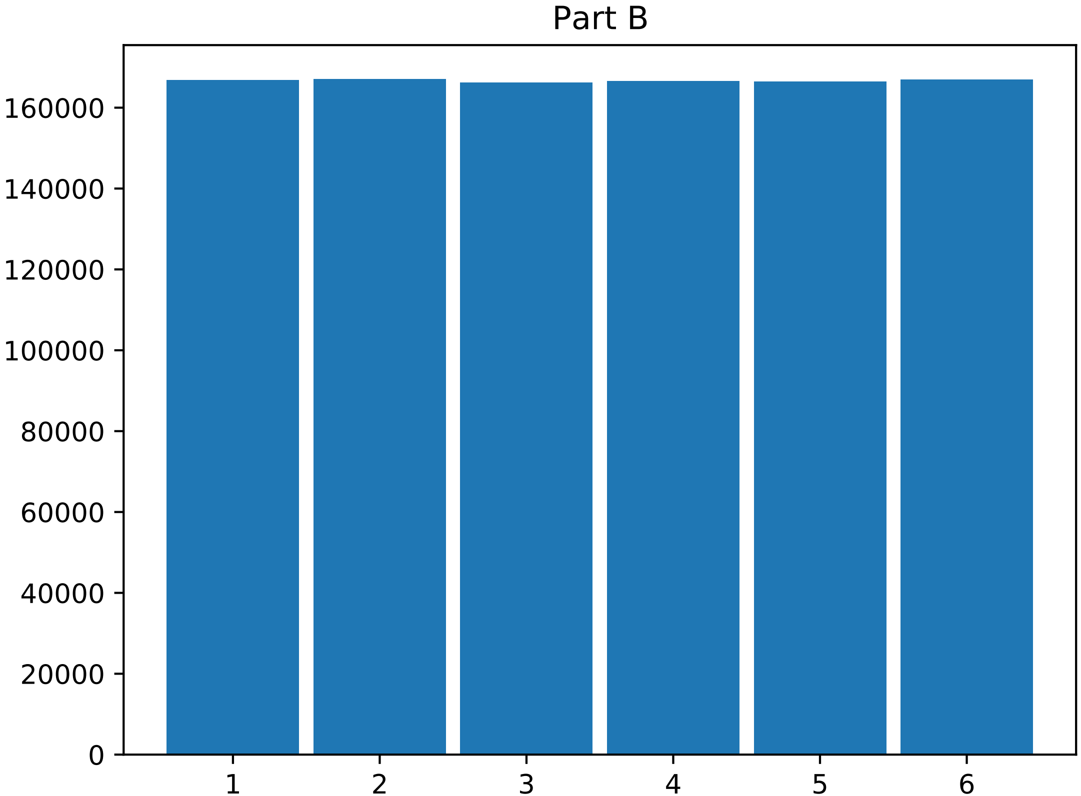
(f) In (c)-(d), Z_i is a sum of two random variables X_i and Y_i. What if we sum more random variables? That is, $Z_i = X_i^{(1)} + X_i^{(2)} + \dots + X_i^{(K)}$. Let $K = 10$. Plot the histogram of $\{Z_1, \dots, Z_{10000}\}$. Submit your histogram. Pay attention to the bin centers.

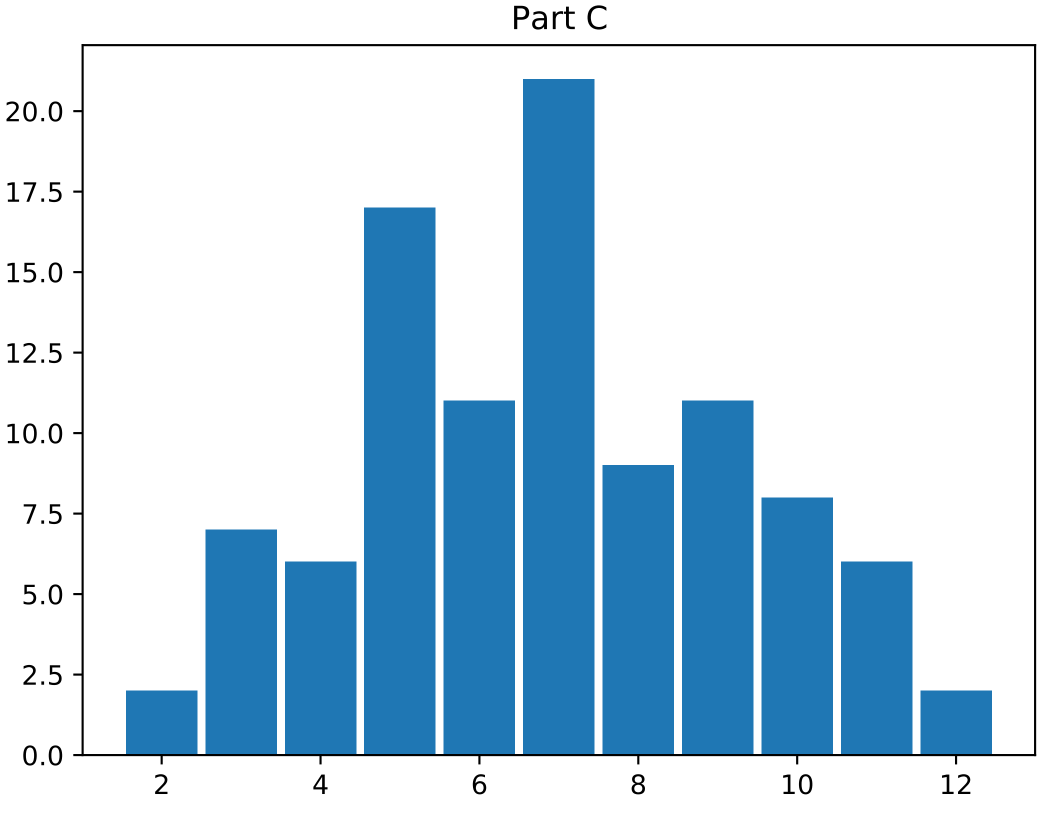
(g) Repeat (f) by setting $K = 100$. Plot the histogram of $\{Z_1, \dots, Z_{10000}\}$. Submit your histogram. Pay attention to the bin centers.

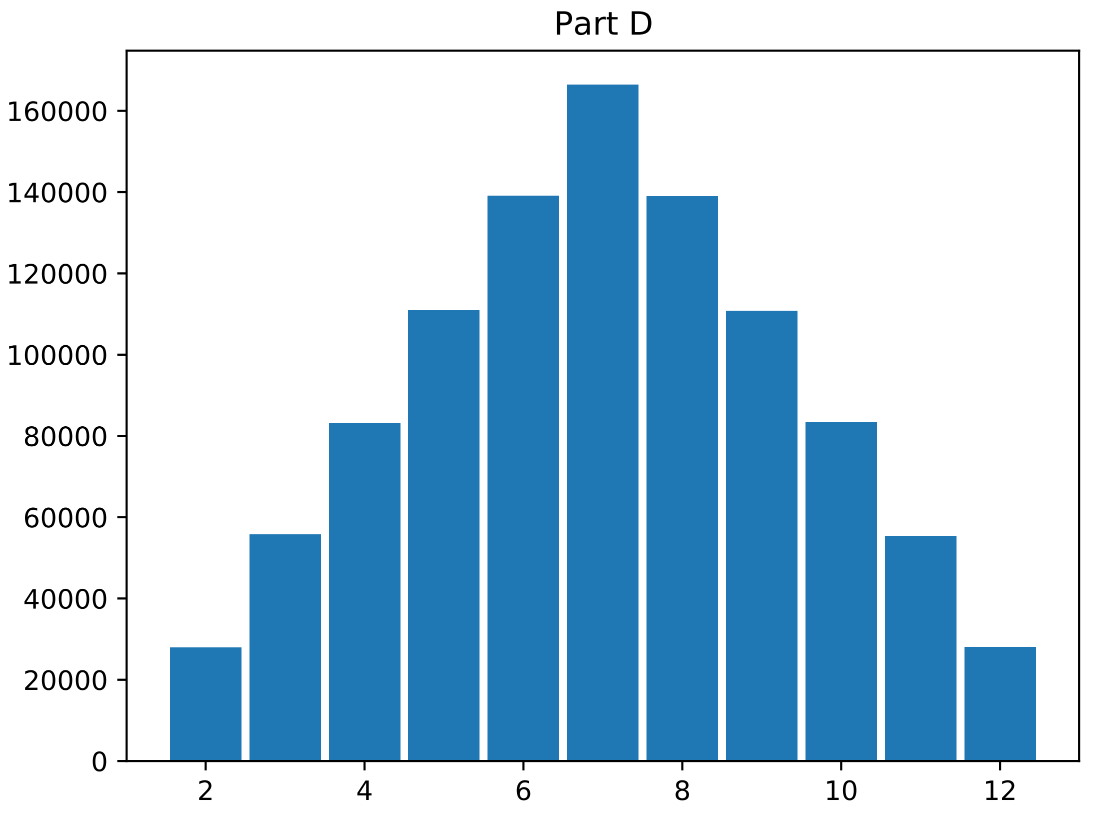
Please put your plots after this page.

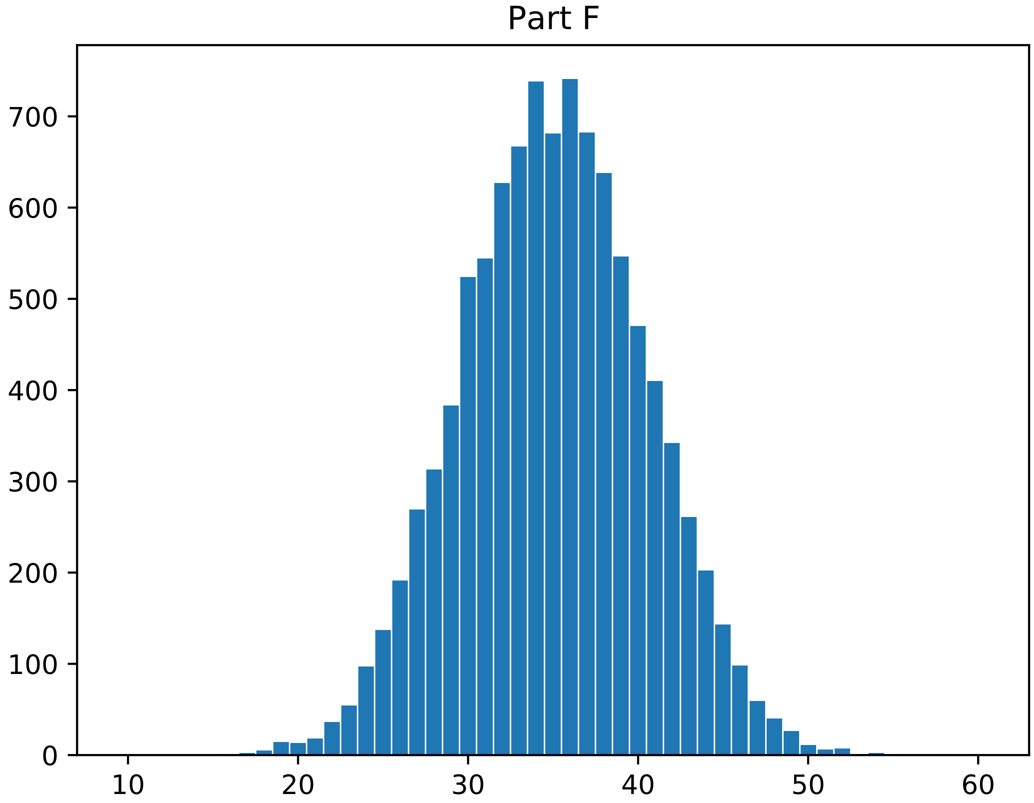
E. Probability of $4 < Z_i \leq 7$: 0.416


Part A
0
5
10
15
20
25
1
2
3
4
5
6

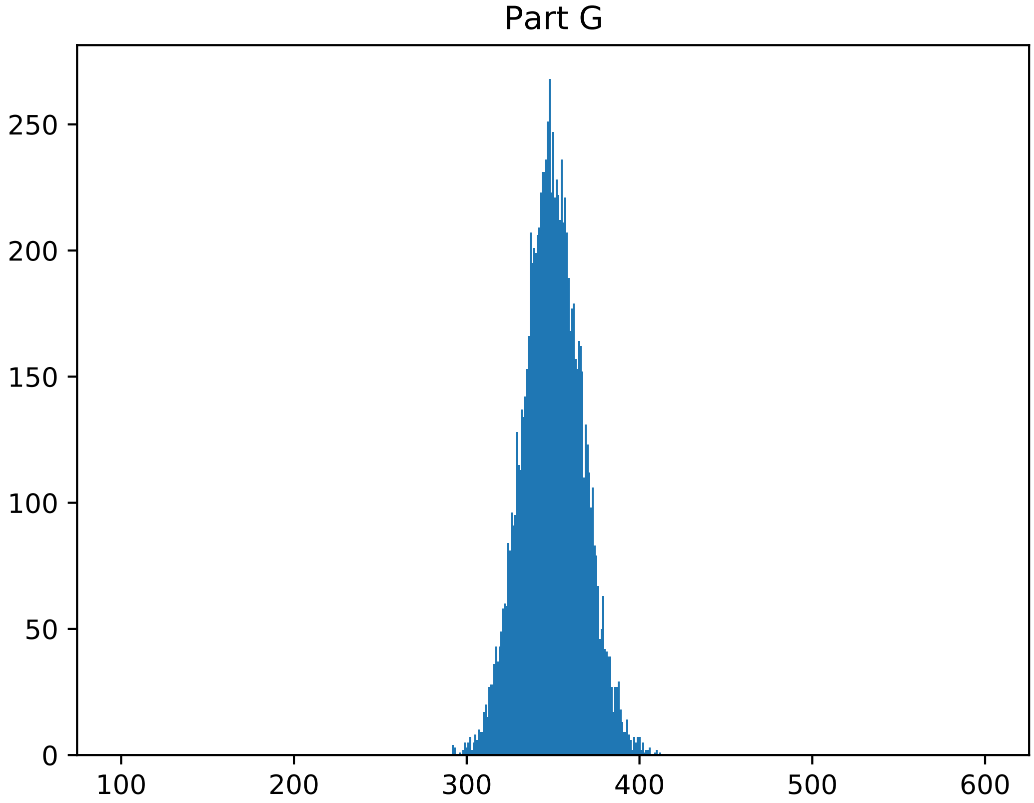

Part B
0
20000
40000
60000
80000
100000
120000
140000
160000
1
2
3
4
5
6


Part C
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
2
4
6
8
10
12


Part D
0
20000
40000
60000
80000
100000
120000
140000
160000
2
4
6
8
10
12



Part F
0
100
200
300
400
500
600
700
10
20
30
40
50
60



Part G
0
50
100
150
200
250
100
200
300
400
500
600