PURDUE UNIVERSITY

Final

Fall 2020

Name: Elias Talcott   Session (circle one): Morning / Evening / DRC

Please copy and write the following statement:

I certify that I have neither given nor received unauthorized aid on this exam.

I certify that I have neither given nor received unauthorized aid on this exam.
(Please copy and write the above statement.)

Eli Talcott
(Signature)

Please state clearly each step you take to reach the final answer. A correct final answer without proper explanations will not receive the full credit.

Problem	Score
Q1	
Q2	
Q3	
Total	

© 2020 Stanley Chan. All Rights Reserved.

Exercise 1. (30 POINTS)

(a) (6 points)

$P_X(0) = \frac{1}{2}$, $P_X(1) = \frac{1}{2}$, $P_Y(0) = \frac{1}{2}$, $P_Y(1) = \frac{1}{2}$, $Z(t) = Xu(t-Y)$

i) $X = 0$, $y = 0$

ii) $x = 0$, $y = 1$

iii) $x = 1$, $y = 0$

iv) $x = 1$, $y = 1$

(b) (8 points)

Find PMF of $Z(\frac{1}{2})$

For case i) $Z(\frac{1}{2}) = 0$ $\mathbb{P}[\text{case}] = \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$

ii) $Z(\frac{1}{2}) = 0$ $\quad = \frac{1}{4}$

iii) $Z(\frac{1}{2}) = 1$ $\quad = \frac{1}{4}$

iv) $Z(\frac{1}{2}) = 0$ $\quad = \frac{1}{4}$

$\mathbb{P}[Z(\frac{1}{2}) = 1] = \frac{1}{4}$ $\quad \mathbb{P}[Z(\frac{1}{2}) = 0] = \frac{3}{4}$

© 2020 Stanley Chan. All Rights Reserved.

(c) (8 points)

Find $\mathbb{E}[Z(t)]$ $(\mu_Z(t))$ X = Bernoulli w/ $p=0.5$

Y = Bernoulli w/ $p=0.5$

$$\mathbb{E}[Z(t)] = \sum_{\text{cases}} X u(t-Y)\,\mathbb{P}[\text{case}]$$

$$= 0\left(\tfrac{1}{4}\right) + 0\left(\tfrac{1}{4}\right) + u(t)\left(\tfrac{1}{4}\right) + u(t-1)\left(\tfrac{1}{4}\right)$$

$$\mathbb{E}[Z(t)] = \tfrac{1}{4}u(t) + \tfrac{1}{4}u(t-1)$$

(d) (8 points)

Find joint PMF of $Z(\frac{1}{2})$ and $Z(\frac{3}{2})$

For case i) $Z(\frac{1}{2}) = 0$ $Z(\frac{3}{2}) = 0$ $\mathbb{P}[\text{case}] = \frac{1}{4}$

ii) $Z(\frac{1}{2}) = 0$ " $= 0$ " $= \frac{1}{4}$

iii) $Z(\frac{1}{2}) = 1$ " $= 1$ " $= \frac{1}{4}$

iv) $Z(\frac{1}{2}) = 0$ " $= 1$ " $= \frac{1}{4}$

$\mathbb{P}[Z(\frac{1}{2})=0 ; Z(\frac{3}{2})=0] = \frac{1}{2}$, $\mathbb{P}[Z(\frac{1}{2})=0, Z(\frac{3}{2})=1] = \frac{1}{4}$

$\mathbb{P}[Z(\frac{1}{2})=1, Z(\frac{3}{2})=0] = 0$, $\mathbb{P}[Z(\frac{1}{2})=1, Z(\frac{3}{2})=1] = \frac{1}{4}$

© 2020 Stanley Chan. All Rights Reserved.

Exercise 2. (30 POINTS)

(a) (10 points)

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & , t \geq 0 \\ 0 & , t < 0 \end{cases} \qquad F_T(t) = \begin{cases} 1 - e^{-\lambda t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

$U = \max\{T_1, T_2\}$, $V = \min\{T_1, T_2\}$, Find $f_U(t)$

$$F_U(t) = (1 - e^{-\lambda T_1})(u(T_1 - T_2)) + (1 - e^{-\lambda T_2})\cancel{(1 - u(T_1 - T_2))}$$

if $T_1 \geq T_2 : u(t_1 - t_2) = 1$
$T_1 < T_2 : u(t_1 - t_2) = 0$

$$\cancel{F_U(t) = u(T_1 - T_2) - e^{-\lambda T_1} u(T_1 - T_2) + 1 - u(T_1 - T_2) - e^{-}}$$

$$F_U(t) = \max\{(1 - e^{-\lambda T_1}), (1 - e^{-\lambda T_2})\}(t)$$

(b) (12 points)

$V = \min\{T_1, T_2\}$

$F_V(t) = \min\{(1-e^{-\lambda T_1}), (1-e^{-\lambda T_2})\}(t)$

© 2020 Stanley Chan. All Rights Reserved.

(c) (4 points)

$p_1 = \mathbb{P}[T_1 \geq T_2]$, $p_2 = \mathbb{P}[T_1 < T_2]$

$\mathbb{E}[U] = p_1(\frac{1}{\lambda}$ ...

(d) (4 points)

© 2020 Stanley Chan. All Rights Reserved.

Exercise 3. (40 POINTS)

(a) (4 points)

$X(t,\xi) = A(\xi)t + B(\xi)$, $-\infty < t < \infty$

$\mu_X = \mathbb{E}[X(t)] = \mathbb{E}[A(\xi)t + B(\xi)] = t\mathbb{E}[A(\xi)] + \mathbb{E}[B(\xi)]$

$\mu_X(t) = \mu_A t + \mu_B$

(b) (12 points)

Find $R_X(t_1,t_2)$, Is it WSS?

$R_X(t_1,t_2) = \mathbb{E}[X(t_1)X(t_2)] = \mathbb{E}[(A(\xi)t_1 + B(\xi))(A(\xi)t_2 + B(\xi))]$

$= \mathbb{E}[A^2(\xi)t_1t_2 + A(\xi)t_1B(\xi) + A(\xi)t_2B(\xi) + B^2(\xi)]$

$= t_1t_2\mathbb{E}[A^2(\xi)] + t_1\mathbb{E}[A(\xi)B(\xi)] + t_2\mathbb{E}[A(\xi)B(\xi)] + \mathbb{E}[B^2(\xi)]$

$= \mu_A^2 t_1t_2 + \mu_A\mu_B t_1 + \mu_A\mu_B t_2 + \mu_B^2$

$= (\mu_A t_1 + \mu_B)(\mu_A t_2 + \mu_B)$

Since $\mu_X(t)$ is not constant, X is NOT WSS

© 2020 Stanley Chan. All Rights Reserved.

(c) (14 points)

Find $M_X(s)$ $\quad X(t) = A(\varepsilon)t + B(\varepsilon)$

$$M_A(s) = e^{\mu_A s + \frac{\sigma_A^2 s^2}{2}}$$

$$M_B(s) = e^{\mu_B s + \frac{\sigma_B^2 s^2}{2}}$$

A, B are independent $\quad$ For $Z = X+Y$, $M_Z(s) = M_X(s) M_Y(s)$

$$\therefore \; M_X(s) = t e^{\mu_A s + \frac{\sigma_A^2 s^2}{2}} \, e^{\mu_B s + \frac{\sigma_B^2 s^2}{2}}$$

© 2020 Stanley Chan. All Rights Reserved.

(d) (5 points)

Find $f_X(t)$

$$F_X(t) = \int_{-\infty}^{\infty} A(\varepsilon)t + B(\varepsilon)\, dt$$

$$= \left[\tfrac{1}{2}A(\varepsilon)t^2 + B(\varepsilon)t\right]_{-\infty}^{\infty}$$

(e) (5 points)

Find $\mathbb{P}[|X(t)| > \alpha]$ where $\alpha > 0$

© 2020 Stanley Chan. All Rights Reserved.

Useful Identities

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r}$
2. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$
4. $\sum_{k=1}^{\infty} kr^{k-1} = 1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}$
5. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
6. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Common Distributions

Bernoulli	$\mathbb{P}[X=1] = p$	$\mathbb{E}[X] = p$	$\text{Var}[X] = p(1-p)$	$M_X(s) = 1 - p + pe^s$
Binomial	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$\mathbb{E}[X] = np$	$\text{Var}[X] = np(1-p)$	$M_X(s) = (1-p+pe^s)^n$
Geometric	$p_X(k) = p(1-p)^{k-1}$	$\mathbb{E}[X] = \frac{1}{p}$	$\text{Var}[X] = \frac{1-p}{p^2}$	$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{E}[X] = \lambda$	$\text{Var}[X] = \lambda$	$M_X(s) = e^{\lambda(e^s-1)}$
Gaussian	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{E}[X] = \mu$	$\text{Var}[X] = \sigma^2$	$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x) = \lambda \exp\{-\lambda x\}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\text{Var}[X] = \frac{1}{\lambda^2}$	$M_X(s) = \frac{\lambda}{\lambda - s}$
Uniform	$f_X(x) = \frac{1}{b-a}$	$\mathbb{E}[X] = \frac{a+b}{2}$	$\text{Var}[X] = \frac{(b-a)^2}{12}$	$M_X(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}$

Fourier Transform Table

	$f(t) \longleftrightarrow F(w)$		$f(t) \longleftrightarrow F(w)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw},\ a>0$	10.	$\text{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{w}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-jw},\ a>0$	11.	$e^{-at}\sin(w_0 t)u(t) \longleftrightarrow \frac{w_0}{(a+jw)^2+w_0^2},\ a>0$
3.	$e^{-a\|t\|} \longleftrightarrow \frac{2a}{a^2+w^2},\ a>0$	12.	$e^{-at}\cos(w_0 t)u(t) \longleftrightarrow \frac{a+jw}{(a+jw)^2+w_0^2},\ a>0$
4.	$\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a\|w\|},\ a>0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2 w^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2},\ a>0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+jw)^{n+1}},\ a>0$	15.	$1 \longleftrightarrow 2\pi\delta(w)$
7.	$\text{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\text{sinc}(\frac{w\tau}{2})$	16.	$\delta(t-t_0) \longleftrightarrow e^{-jwt_0}$
8.	$\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\text{rect}(\frac{w}{2W})$	17.	$e^{jw_0 t} \longleftrightarrow 2\pi\delta(w-w_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\text{sinc}^2(\frac{w\tau}{4})$		

Some definitions:

$$\text{sinc}(t) = \frac{\sin(t)}{t} \qquad \text{rect}(t) = \begin{cases} 1, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \qquad \Delta(t) = \begin{cases} 1 - 2|t|, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Basic Trigonometry

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1.$$

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \quad \sin A \sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \quad \cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$

© 2020 Stanley Chan. All Rights Reserved.