Exercise 7.

The following result is known as the Bonferroni's Inequality.

(a) Prove that for any two events A and B, we have

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

(b) Generalize the above to the case of n events $A_1, A_2, \ldots, A_n$, by showing that

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots + \mathbb{P}(A_n) - (n-1).$$

Hint: You may use the generalized Union Bound $\mathbb{P}(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathbb{P}(A_i)$.

A. $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$

maximum of $\mathbb{P}(A \cup B) = 1$ (ranges $0 \to 1$)

$\therefore$ $\underline{\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1}$

B. $\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n)$

$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \sum_{i=1}^{n} \mathbb{P}(A_1) - \mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n)$

maximum of $\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = n-1$ $(0 \to n-1)$

$\therefore$ $\underline{\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots + \mathbb{P}(A_n) - (n-1)}$