Mid Term 2

Fall 2020

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Session (circle one): Morning / Evening / DRC

Please copy and write the following statement:

I certify that I have neither given nor received unauthorized aid on this exam.

I certify that I have neither given nor received unauthorized aid on this exam.

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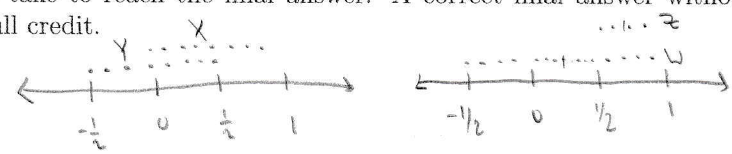
Eli Talcott

(Signature)

Please state clearly each step you take to reach the final answer. A correct final answer without proper explanations will not receive the full credit.

Exercise 1. (30 POINTS)



$X \sim \text{Uniform}(0,1)$, $Y \sim \text{Uniform}(-\frac{1}{2}, \frac{1}{2})$, $W = X+Y$, $Z = X-Y$

$\rightarrow W \sim \text{Uniform}(-\frac{1}{2}, 0) + 2\text{Uniform}(0, \frac{1}{2}) + \text{Uniform}(\frac{1}{2}, 1)$

$Z \sim \text{Uniform}(\frac{1}{2}, 1)$

$\mathbb{E}[W] = \frac{1}{4}$, $\mathbb{E}[Z] = \frac{3}{4}$ $\longrightarrow$ $\mathbb{E}[W]\mathbb{E}[Z] = \frac{3}{16}$

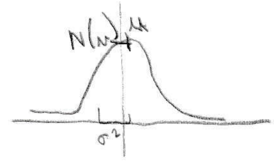
$\text{Cov}(W,Z) = \mathbb{E}[WZ] - \mathbb{E}[W]\mathbb{E}[Z] = \mathbb{E}[W,Z] - \frac{3}{16}$

$\mathbb{E}[WZ] = \int_{-1/2}^{1}\int_{1/2}^{1} wz\, p_{W,Z}(w,z)\, dz\, dw$

Exercise 2. (40 POINTS)

(a)

$X =$ fair coin flip $(-1,1)$, $N \sim \text{Gaussian}(\mu, \sigma^2)$



$$f_{Y|X}(y|+1) = \frac{p_{X,Y}(x,y)}{p_X(x)} = 2\,p_{X,Y}(x,y)$$

$$= 2p_{X,Y}(x, x+N) = 2p_{X,Y}(1, N+1)$$

$Y = \text{Gaussian}(\mu \pm 1, \sigma^2)$

$$f_{Y|X}(y|-1) = 2\,p_{X,Y}(-1, N-1)$$

(b)

$\mathbb{E}[Y|X=+1]$, $\mathbb{E}[Y|X=-1]$

$N(\mu) =$ value of N @ center point

$$\mathbb{E}[Y|X=+1] = \mathbb{E}[\text{Gaussian}(\mu, \sigma^2)] + 1$$

$$\rightarrow \mathbb{E}[Y|X=+1] = N(\mu) + 1$$

$$\mathbb{E}[Y|X=-1] = \mathbb{E}[\text{Gaussian}(\mu, \sigma^2)] - 1$$

$$\rightarrow \mathbb{E}[Y|X=-1] = N(\mu) - 1$$

$$\mathbb{E}[Y] = \tfrac{1}{2}(N(\mu)+1) + \tfrac{1}{2}(N(\mu)-1) \rightarrow \mathbb{E}[Y] = N(\mu)$$

(c)

probability of error = poe i/p = +1, o/p = -1 or i/p = -1, o/p = +1

$$poe = \mathbb{P}[Y < \tau \mid X = +1] + \mathbb{P}[Y \geq \tau \mid X = -1]$$

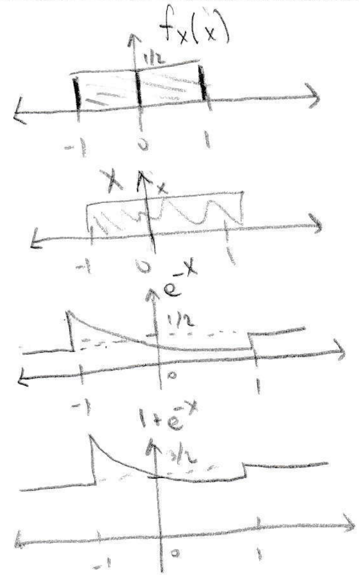
$$= F_{Y|X}[Y < \tau \mid +1] + F_{Y|X}[Y \geq \tau \mid -1]$$

$$= \frac{F_{X,Y}(+1, y)}{F_X(+1)} + \frac{F_{X,Y}(-1, y)}{F_X(-1)}$$

(d)

Exercise 3. (30 POINTS)

$$X \sim \text{Uniform}(-1,1) = \begin{cases} \frac{1}{2}, & -1 \le x \le 1 \\ 0, & \text{else} \end{cases}$$



$$Y = \frac{1}{1+e^{-X}}$$

Useful Identities

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots = \frac{1}{1-r}$
2. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
3. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$
4. $\sum_{k=1}^{\infty} kr^{k-1} = 1 + 2r + 3r^2 + \ldots = \frac{1}{(1-r)^2}$
5. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
6. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Common Distributions

Bernoulli	$\mathbb{P}[X=1] = p$	$\mathbb{E}[X] = p$	$\text{Var}[X] = p(1-p)$	$M_X(s) = 1 - p + pe^s$
Binomial	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$\mathbb{E}[X] = np$	$\text{Var}[X] = np(1-p)$	$M_X(s) = (1 - p + pe^s)^n$
Geometric	$p_X(k) = p(1-p)^{k-1}$	$\mathbb{E}[X] = \frac{1}{p}$	$\text{Var}[X] = \frac{1-p}{p^2}$	$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$\mathbb{E}[X] = \lambda$	$\text{Var}[X] = \lambda$	$M_X(s) = e^{\lambda(e^s - 1)}$
Gaussian	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mathbb{E}[X] = \mu$	$\text{Var}[X] = \sigma^2$	$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
Exponential	$f_X(x) = \lambda \exp\{-\lambda x\}$	$\mathbb{E}[X] = \frac{1}{\lambda}$	$\text{Var}[X] = \frac{1}{\lambda^2}$	$M_X(s) = \frac{\lambda}{\lambda - s}$
Uniform	$f_X(x) = \frac{1}{b-a}$	$\mathbb{E}[X] = \frac{a+b}{2}$	$\text{Var}[X] = \frac{(b-a)^2}{12}$	$M_X(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}$

Fourier Transform Table

	$f(t) \longleftrightarrow F(w)$		$f(t) \longleftrightarrow F(w)$
1.	$e^{-at}u(t) \longleftrightarrow \frac{1}{a+jw},\ a>0$	10.	$\text{sinc}^2(\frac{Wt}{2}) \longleftrightarrow \frac{2\pi}{W}\Delta(\frac{w}{2W})$
2.	$e^{at}u(-t) \longleftrightarrow \frac{1}{a-jw},\ a>0$	11.	$e^{-at}\sin(w_0 t)u(t) \longleftrightarrow \frac{w_0}{(a+jw)^2+w_0^2},\ a>0$
3.	$e^{-a\|t\|} \longleftrightarrow \frac{2a}{a^2+w^2},\ a>0$	12.	$e^{-at}\cos(w_0 t)u(t) \longleftrightarrow \frac{a+jw}{(a+jw)^2+w_0^2},\ a>0$
4.	$\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a\|w\|},\ a>0$	13.	$e^{-\frac{t^2}{2\sigma^2}} \longleftrightarrow \sqrt{2\pi}\sigma e^{-\frac{\sigma^2 w^2}{2}}$
5.	$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+jw)^2},\ a>0$	14.	$\delta(t) \longleftrightarrow 1$
6.	$t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+jw)^{n+1}},\ a>0$	15.	$1 \longleftrightarrow 2\pi\delta(w)$
7.	$\text{rect}(\frac{t}{\tau}) \longleftrightarrow \tau\text{sinc}(\frac{w\tau}{2})$	16.	$\delta(t-t_0) \longleftrightarrow e^{-jwt_0}$
8.	$\text{sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\text{rect}(\frac{w}{2W})$	17.	$e^{jw_0 t} \longleftrightarrow 2\pi\delta(w-w_0)$
9.	$\Delta(\frac{t}{\tau}) \longleftrightarrow \frac{\tau}{2}\text{sinc}^2(\frac{w\tau}{4})$		

Some definitions:

$$\text{sinc}(t) = \frac{\sin(t)}{t} \qquad \text{rect}(t) = \begin{cases} 1, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \qquad \Delta(t) = \begin{cases} 1 - 2|t|, & -0.5 \le t \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

Basic Trigonometry

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1.$$

$$\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \quad \sin A\sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$

$$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \quad \cos A\sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$