PURDUE UNIVERSITY

Homework 6

Fall 2020
(Due: Nov 6, 2020, Friday)

Name: Elias Talcott Email: etalcott@purdue.edu

Homework is due at 11:59pm (midnight) Eastern Time. Please print this homework, write your solution, and scan the solution. Submit your homework through Gradescope. No late homework will be accepted.

Exercise 1.
Flip a biased coin twice. Assume that $\mathbb{P}[\text{Head}] = p$. Denote "1" as head, and "0" as tail. Let X be the maximum of the two numbers, and let Y be the minimum of the two numbers.

(a) Find and sketch the joint PMF $p_{X,Y}(x, y)$.

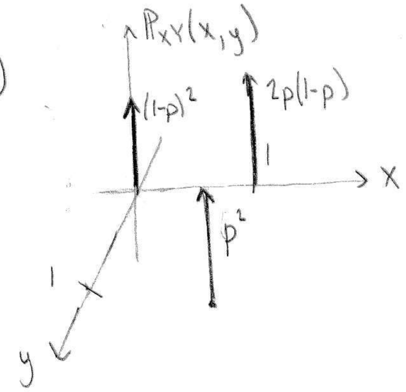
(b) Find the marginal PMF $p_X(x)$ and $p_Y(y)$.

(c) Find the conditional PMF $P_{X|Y}(x \mid y)$.

Caution: The setting of this problem is different from Video 7 Problem 1. There we have two persons throwing a coin 2 times per person (so totally 4 times). Here we throw the coin 2 times only.

$\mathbb{P}_{XY}(0,0) = (1-p)^2$, $\mathbb{P}_{XY}(1,0) = 2p(1-p)$, $\mathbb{P}_{XY}(1,1) = p^2$, $\mathbb{P}_{XY}(0,1) = 0$

A)



B) $p_X(x) = \begin{cases} (1-p)^2, & x = 0 \\ 2p(1-p) + p^2, & x = 1 \end{cases}$

$p_Y(y) = \begin{cases} (1-p)^2 + 2p(1-p), & y = 0 \\ p^2, & y = 1 \end{cases}$

c) $P_{X|Y}(x,y) = \dfrac{P_{XY}(x,y)}{P_Y(y)}$

$$P_{X|Y}(x,y) = \begin{cases} \dfrac{(1-p)^2}{(1-p)^2 + 2p(1-p)}, & x=0, y=0 \\ \dfrac{2p(1-p)}{(1-p)^2 + 2p(1-p)}, & x=1, y=0 \\ 0, & x=0, y=1 \\ 1, & x=1, y=1 \end{cases}$$

Exercise 2.

A stock market trader buys 50 shares of stock A and 100 shares of stock B. Let X and Y be the price changes to A and B, respectively, over certain period of time. Assume that the joint PMF $p_{X,Y}(x, y)$ is uniform over the set of integers satisfying

$$-1 \leq x \leq 3, \quad -1 \leq y - x \leq 1. \qquad x-1 \leq y \leq x+1$$

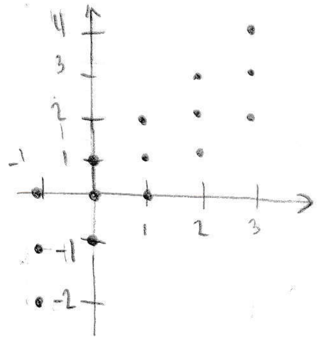
(We say that a discrete random variable X is uniformly distributed over a sequence of integers $\{N_a, \ldots, N_b\}$ if $p_X(x) = 1/(N_b - N_a + 1)$.)

(a) Find the joint PMF $p_{X,Y}(x, y)$, the marginal PMFs $p_X(x)$ and $p_Y(y)$.

(b) Find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(c) Determine the trader's profit.

A)



$$P_{X,Y}(x,y) = \frac{1}{15}, \quad -1 \leq x \leq 3, \quad x-1 \leq y \leq x+1$$

$$P_X(x) = \begin{cases} \frac{1}{5}, & -1 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} \frac{1}{15}, & y = -2, 4 \\ \frac{2}{15}, & y = -1, 3 \\ \frac{1}{5}, & 0 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

B)

$$\mathbb{E}[X] = \frac{-1 + 0 + 1 + 2 + 3}{5} \rightarrow \mathbb{E}[X] = 1$$

$$\mathbb{E}[Y] = \frac{1(-2) + 2(-1) + 3(0) + 3(1) + 3(2) + 2(3) + 1(4)}{15} \rightarrow \mathbb{E}[Y] = 1$$

C) $\text{Profit} = 50\,\mathbb{E}[X] + 100\,\mathbb{E}[Y] = 50(1) + 100(1)$

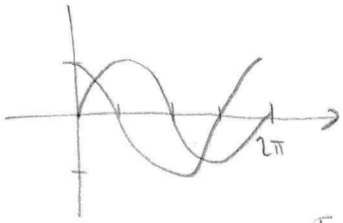
$\text{Profit} = 150$

Exercise 3.
Let $\Theta \sim \text{Uniform}[0, 2\pi]$.

(a) If $X = \cos\Theta$, $Y = \sin\Theta$. Are X and Y uncorrelated?

(b) If $X = \cos(2\Theta)$, $Y = \sin(2\Theta)$. Are X and Y uncorrelated?

A)



$$\mathbb{E}[XY] = \mathbb{E}[\cos\theta \sin\theta]$$
$$= \mathbb{E}[\tfrac{1}{2}\sin(2\theta)]$$

$$\mathbb{E}[XY] = \frac{1}{4\pi}\int_0^{2\pi} \sin 2\theta \, d\theta = 0$$

$\therefore$ uncorrelated

B) $\mathbb{E}[XY] = \mathbb{E}[\cos(\theta/4)\sin(\theta/4)] = \mathbb{E}[\tfrac{1}{2}\sin(\theta/2)]$

$$\mathbb{E}[XY] = \frac{1}{4\pi}\int_0^{2\pi} \sin(\theta/2)\, d\theta = \frac{1}{4\pi}(4)$$

$\mathbb{E}[XY] = \frac{1}{\pi}$ $\quad\therefore$ correlated

Exercise 4.
Let

$$f_{X,Y}(x,y) = \begin{cases} ce^{-2x}e^{-y}, & \text{if } 0 \le y \le x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find c.

(b) Find $f_X(x)$ and $f_Y(y)$.

(c) Find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$, $\text{Var}[X]$ and $\text{Var}[Y]$.

(d) Find $\mathbb{E}[XY]$, $\text{Cov}(X,Y)$ and ρ.

A) $\iint_{R^2} f_{X,Y}(x,y)\,dx\,dy = 1$

$= \int_0^\infty \int_y^\infty ce^{-2x}e^{-y}\,dx\,dy = c\int_0^\infty e^{-y} \int_y^\infty e^{-2x}\,dx\,dy$

$= \frac{c}{2}\int_0^\infty e^{-y}e^{-2y}\,dy = \frac{c}{2}\int_0^\infty e^{-3y}\,dy = \frac{c}{6} = 1 \rightarrow c = 6$

B) $f_X(x) = \int_0^x 6e^{-2x}e^{-y}\,dy = 6e^{-2x}\int_0^x e^{-y}\,dy \rightarrow f_X(x) = 6e^{-2x}(-e^{-x}+1)\cdot \mathbb{1}_{(0,\infty)}(x)$

$f_Y(y) = 6e^{-y}\int_y^\infty e^{-2x}\,dx \rightarrow f_Y(y) = 3e^{-3y}\cdot \mathbb{1}_{(0,\infty)}(Y)$

C) $\mathbb{E}[X] = \int_0^\infty 6xe^{-2x}(-e^{-x}+1)\,dx = 6\int_0^\infty -xe^{-3x} + xe^{-2x}\,dx \rightarrow \mathbb{E}[X] = \frac{5}{6}$

$\mathbb{E}[Y] = \int_0^\infty 3ye^{-3y}\,dy \rightarrow \mathbb{E}[Y] = \frac{1}{3}$

$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\mathbb{E}[X^2] = \int_0^\infty 6x^2e^{-2x}(-e^{-x}+1)\,dx = \frac{19}{18} \rightarrow \text{Var}[X] = 13/36$

$\text{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$

$\mathbb{E}[Y^2] = \int_0^\infty 3y^2e^{-3y}\,dy = \frac{2}{9} \rightarrow \text{Var}[Y] = 1/9$

D) $\mathbb{E}[XY] = \iint_{\mathbb{R}^2} xy\, f_{X,Y}(x,y)\, dx\, dy$

$= \int_0^\infty \int_y^\infty xy\, 6e^{-2x} e^{-y}\, dx\, dy = 6 \int_0^\infty ye^{-y} \int_y^\infty xe^{-2x}\, dx\, dy$

$= \frac{3}{2} \int_0^\infty ye^{-y} \left(2ye^{-2y} + e^{-2y}\right) dy \rightarrow \mathbb{E}[XY] = 7/27$

$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$= 7/27 - (5/6)(1/3) \rightarrow \mathrm{Cov}(X,Y) = -1/54$

$\rho = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \dfrac{-1/54}{\sqrt{(13/36)(1/9)}} \rightarrow \rho = -0.0925$

Exercise 5.
Let X and Y be two random variables with joint PDF

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}.$$

X, Y independent

(a) Find the PDF of $Z = \max(X, Y)$.

(b) Find the PDF of $Z = \min(X, Y)$.

You may leave your answers in terms of the $\Phi(\cdot)$ function.

A) $F_Z(z) = \mathbb{P}(\max(X,Y) \le z) = \mathbb{P}(x \le z \;\&\&\; y \le z)$

$= \mathbb{P}(x \le z)\mathbb{P}(y \le z) = F_X(z)F_Y(z) = (F_X(z))^2$ $\quad F_X(z) = F_Y(z)$

$f_Z(z) = \frac{\partial}{\partial z}(F_X(z))^2 = 2F_X(z)f_X(z)$

$f_Z(z) = 2\Phi(z)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$

B) $F_Z(z) = \mathbb{P}(x \le z \;||\; y \le z) = F_X(z) + F_Y(z) - F_X(z)F_Y(z)$

$= 2F_X(z) - (F_X(z))^2$

$f_Z(z) = \frac{\partial}{\partial z}(2F_X(x) - (F_X(z))^2) = 2f_X(z) - 2F_X(z)f_X(z)$

$f_Z(z) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}(1 - \Phi(z))$

Exercise 6.
Let $Y = X + N$, where X is the input, N is the noise, and Y is the output of a system. Assume that X and N are independent random variables. It is given that $\mathbb{E}[X] = 0$, $\text{Var}[X] = \sigma_X^2$, $\mathbb{E}[N] = 0$, $\text{Var}[N] = \sigma_N^2$.

(a) Find the correlation coefficient ρ between the input X and the output Y.

(b) Suppose we estimate the input X by a linear function $g(Y) = aY$. Find the value of a that minimizes the mean squared error $\mathbb{E}[(X - aY)^2]$.

(c) Express the resulting mean squared error in terms of $\eta = \sigma_X^2/\sigma_N^2$.

A) $$\rho = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\text{Cov}(X, X+N)}{\sqrt{\text{Var}(X)\text{Var}(X+N)}} = \frac{\mathbb{E}[X(X+N)] - \mathbb{E}[X]\mathbb{E}[X+N]}{\sqrt{\text{Var}(X)(\text{Var}(X)+\text{Var}(N)+2\text{Cov}(X,N))}}$$

$$= \frac{\mathbb{E}[X^2] + \cancel{\mathbb{E}[XN]}^{0} - \cancel{\mathbb{E}[X]}^{0}\mathbb{E}[X+N]}{\sqrt{\text{Var}(X)(\text{Var}(X)+\text{Var}(N)+2(\cancel{\mathbb{E}(XN)}^{0} - \mathbb{E}(\cancel{X})^{0}\mathbb{E}(N)))}} = \frac{\mathbb{E}[X^2]}{\sqrt{\text{Var}(X)(\text{Var}(X)+\text{Var}(N))}}$$

$$= \frac{\mathbb{E}[X^2]}{\sqrt{\sigma_X^4 + \sigma_X^2\sigma_N^2}}$$

B) To minimize MSE, $aY \to X$

$$X = Y - N \rightarrow aY = Y - N \rightarrow a = \frac{Y-N}{Y} = 1 - \frac{N}{Y}$$

$$a = 1 - \frac{N}{Y}$$

C)

Exercise 7.
Let X and Y have a joint PDF

$$f_{X,Y}(x,y) = c(x+y),$$

for $0 \le x \le 2$ and $0 \le y \le 2$.

(a) Find c, $f_X(x)$, $f_Y(y)$, and $\mathbb{E}[Y]$.

(b) Find $f_{Y|X}(y|x)$.

(c) Find $\mathbb{P}[Y > X \mid X > 1]$.

(d) Find $\mathbb{E}[Y|X = x]$.

(e) Find $\mathbb{E}[\mathbb{E}[Y|X]]$, and compare with the $\mathbb{E}[Y]$ computed in (a).

A) $\iint_{\mathbb{R}^2} cx + cy \, dx\,dy = c\int_0^2 x + y \, dx\,dy = 2c\int_0^2 y + 1 \, dy = 8c \rightarrow c = \frac{1}{8}$

$f_X(x) = \frac{1}{8}\int_0^2 x+y\,dy = \frac{1}{8}(2x+2) \rightarrow f_X(x) = \frac{x+1}{4}$

$f_Y(y) = \frac{1}{8}\int_0^2 x+y\,dx = \frac{1}{8}(2y+2) \rightarrow f_Y(y) = \frac{y+1}{4}$

$\mathbb{E}[Y] = \int_0^2 y\left(\frac{y+1}{4}\right)dy \rightarrow \mathbb{E}[Y] = 7/6$

B) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{\frac{1}{8}(x+y)}{\left(\frac{x+1}{4}\right)} \rightarrow f_{Y|X}(y|x) = \dfrac{x+y}{2(x+1)}$

C) $\mathbb{P}[Y > X \mid X > 1] = \dfrac{\mathbb{P}[Y>X] \cap \mathbb{P}[X>1]}{\mathbb{P}[X>1]} = \dfrac{\mathbb{P}[1 \le X \le Y]}{1 - F_X(1)}$

$= \dfrac{\frac{1}{8}\int_0^2\int_1^y x+y\,dx\,dy}{1 - \int_0^1 \left(\frac{x+1}{4}\right)dx} \xrightarrow[5/8]{3/16} \mathbb{P}[Y > X \mid X > 1] = \frac{3}{10}$

D) $\mathbb{E}[Y|X=x] = \int_0^2 y\, f_{Y|X}(y|x)\,dy = \int_0^2 y\left(\frac{x+y}{2(x+1)}\right)dy$

$= \int_0^2 \dfrac{xy+y^2}{2x+2}\,dy \rightarrow \mathbb{E}[Y|X=x] = \dfrac{3x+4}{3(x+1)}$

E) $\mathbb{E}[\mathbb{E}[Y|X]] = \int_0^2 \mathbb{E}[Y|X=x]\, f_X(x)\,dx$

$= \int_0^2 \left(\frac{3x+4}{3(x+1)}\right)\left(\frac{x+1}{4}\right) dx \rightarrow \mathbb{E}[\mathbb{E}[Y|X]] = 7/6$

Same as result for $\mathbb{E}[Y]$

Exercise 8.
This is part 1 of a (pretty long) programming exercise.

Image classification is an important problem in computer vision and (probably) the most widely used test bed problem in artificial intelligence. In this project, we are going to study a simplified version of a common image classification problem by classifying foreground and background regions in an image. This may sound challenging as we have not taught these in class. However, the tools you need are actually ready, e.g., conditional probability, Bayes' rule, normal distribution.

The image that we are going to play with can be downloaded from the course website. You tasks for this exercise are:

(a) Read the image and display the image. Go to the course website, you will see a data file. Download this `.zip` file and decompress it into your current working directory. You will be able to see an image file `cat_grass.jpg`. To read the image and display the image, you can use `imread` and `im2double` in MATLAB, or `plt.imread` in Python. Submit the displayed image and your code.

(b) Extract an 8×8 patch from the image. To access to the (i, j)th pixel of the image, you can type `Y(i,j)`. To access the an 8×8 patch at pixel (i, j), you can do `Y(i:i+7, j:j+7)` for some index `i` and `j`. Extract all the available 8×8 patches from the image, and store them as a $64 \times K$ matrix where K is the number of patches. The following code will be useful.

```
for i=1:M-8
    for j=1:N-8
        z = Y(i+[0:7], j+[0:7]);
        ... % other steps; your job.
    end
end
```

Here, `M` and `N` are the number of rows and columns of the image, respectively. No need to worry about the boundary pixels; just drop them. Hint: Use the command `reshape` in MATLAB (and Python) to turn an 8×8 patch into a 64×1 column vector.

Submit the first 2 columns and the last 2 columns of your data matrix.

(c) Compute the mean vector

$$\boldsymbol{\mu} = \frac{1}{K}\sum_{k=1}^{K} \boldsymbol{x}_k, \tag{1}$$

where $\boldsymbol{x}_k$ is the k-th column of your data matrix. Submit your calculated mean vector. Repeat the calculation using the command `mean` (MATLAB) or `numpy.mean` (Python). Submit the mean returned by this command.

(d) Compute the covariance matrix

$$\boldsymbol{\Sigma} = \frac{1}{K}\sum_{k=1}^{K} (\boldsymbol{x}_k - \boldsymbol{\mu})(\boldsymbol{x}_k - \boldsymbol{\mu})^T. \tag{2}$$

Submit your calculated mean vector. Repeat the calculation using the command `cov` (MATLAB) or `numpy.cov` (Python). Submit the covariance returned by this command.

Please attach your results after this page.