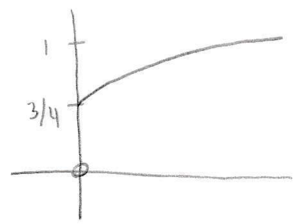
Homework 5

Fall 2020
(Due: October 23, 2020, Friday)

Name: Eliza Talcott Email: etalcott@purdue.edu

Homework is due at 11:59pm (midnight) Eastern Time. Please print this homework, write your solution, and scan the solution. Submit your homework through Gradescope. No late homework will be accepted.

Exercise 1.
A random variable X has CDF:



$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - \frac{1}{4}e^{-2x}, & \text{if } x \geq 0. \end{cases}$$

(a) Find $\mathbb{P}[X \leq 3]$, $\mathbb{P}[X = 0]$, $\mathbb{P}[X < 0]$, $\mathbb{P}[4 < X < 7]$ and $\mathbb{P}[X > 8]$.

(b) Find $f_X(x)$.

A) $\mathbb{P}[X \leq 3] = F_X(3) = 1 - \frac{1}{4}e^{-2(3)} \rightarrow \mathbb{P}[X \leq 3] = 1 - \frac{1}{4}e^{-6}$

$\mathbb{P}[X = 0] = F_X(0) = 1 - \frac{1}{4}e^{0} \rightarrow \mathbb{P}[X = 0] = 3/4$

$\mathbb{P}[X < 0] = 0$

$\mathbb{P}[4 < X < 7] = F_X(7) - F_X(4) = (1 - \frac{1}{4}e^{-2(7)}) - (1 - \frac{1}{4}e^{-2(4)}) \rightarrow \mathbb{P}[4 < X < 7] = \frac{1}{4}(e^{-8} - e^{-14})$

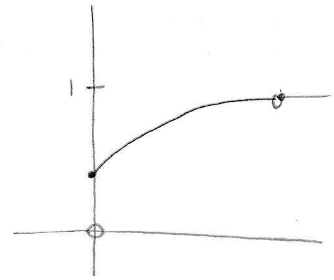
$\mathbb{P}[X > 8] = 1 - F_X(8) = 1 - (1 - \frac{1}{4}e^{-2(8)}) \rightarrow \mathbb{P}[X > 8] = \frac{1}{4}e^{-16}$

B) $$f_X(x) = \frac{d}{dx}(F_X(x)) = \begin{cases} 0, & x < 0 \\ 3/4, & x = 0 \\ -\frac{1}{4}(-2)e^{-2x}, & x > 0 \end{cases}$$

$$f_X(x) = \begin{cases} 0, & x < 0 \\ 3/4, & x = 0 \\ \frac{1}{2}e^{-2x}, & x > 0 \end{cases}$$

Exercise 2.
A continuous random variable X has a cumulative distribution

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 0.5 + c\cos^2(\pi x/2 + \theta), & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$



(a) Find the minimum θ, assuming $0 \le \theta \le 2\pi$.

(b) Based on your result in (a), find c.

(c) Find $f_X(x)$.

A) $\cos(\theta) = 0$ @ $\theta = \pi/2 \rightarrow \min(\theta) = \pi/2$

B) For $\theta = \pi/2$: $0.5 + c\cos^2(\pi x/2 + \pi/2) = 1$ when $x = 1$

$c\cos^2(\pi(1)/2 + \pi/2) = 0.5$

$c\cos^2(\pi) = 0.5 \rightarrow c(1) = 0.5$

$c = 0.5$

C) $$f_X(x) = \begin{cases} 0 & , x < 0 \\ 0.5 & , x = 0 \\ \frac{d}{dx}\left(0.5 + c\cos^2(\pi x/2 + \theta)\right) & , 0 < x < 1 \\ 1 - \frac{d}{dx}\left(0.5 + c\cos^2(\pi/2 + \theta)\right) & , x = 1 \\ 0 & , x > 1 \end{cases}$$

$$\frac{d}{dx}\left(0.5 + c\cos^2(\pi x/2 + \theta)\right) = 2c\cos(\pi x/2 + \theta)\left(-\sin(\pi x/2 + \theta)\right)(\pi/2)$$
$$= -\pi c\cos(\pi x/2 + \theta)\sin(\pi x/2 + \theta)$$

$$f_X(x) = \begin{cases} 0 & , x < 0 \\ 0.5 & , x = 0 \\ -\pi c\cos(\pi x/2 + \theta)\sin(\pi x/2 + \theta) & , 0 < x < 1 \\ 1 + \pi c\cos(\pi/2 + \theta)\sin(\pi/2 + \theta) & , x = 1 \\ 0 & , x > 1 \end{cases}$$

Exercise 3.
Let X be a Gaussian random variable with $\mu = 5$, $\sigma^2 = 16$.

(a) $\mathbb{P}[X > 3]$, $\mathbb{P}[2 \le X \le 6]$.

(b) If $\mathbb{P}[X < a] = 0.7910$, find a.

(c) If $\mathbb{P}[X > b] = 0.1635$, find b.

(d) If $\mathbb{P}[13 < X \le c] = 0.0011$, find c.

A) $\mathbb{P}[X>3] = 1-\mathbb{P}[X \le 3] = 1-\Phi\left(\frac{3-\mu}{\sigma}\right) = 1-\Phi(-0.5)$

$\mathbb{P}[X>3] = 0.6915$

$\mathbb{P}[2 \le X \le 6] = \mathbb{P}[X \le 6] - \mathbb{P}[X < 2] = \Phi\left(\frac{1}{4}\right) - \Phi\left(\frac{-3}{4}\right) = 0.5987 - 0.2266$

$\mathbb{P}[2 \le X \le 6] = 0.3721$

B) $\Phi\left(\frac{a-5}{4}\right) = 0.7910 \rightarrow a = (0.81(4)) + 5 \rightarrow a = 8.24$

C) $1-\Phi\left(\frac{b-5}{4}\right) = 0.1635 \rightarrow b = (0.98(4)) + 5 \rightarrow b = 8.92$

D) $\Phi\left(\frac{c-5}{4}\right) - \Phi\left(\frac{13-5}{4}\right) = 0.0011 \rightarrow \Phi\left(\frac{c-5}{4}\right) = 0.9783$

$c = (2.02(4)) + 5 \rightarrow c = 13.08$

Exercise 4.
Compute $\mathbb{E}[Y]$ and $\mathbb{E}[Y^2]$ for the following random variables:

(a) $Y = A\sin(\omega t + \theta)$, where $A \sim \mathcal{N}(\mu, \sigma^2)$.

(b) $Y = a\sin(\omega t + \Theta)$, where $\Theta \sim \text{Uniform}(0, \pi)$.

(c) $Y = a\sin(\omega T + \theta)$, where $T \sim \text{Uniform}\left(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right)$.

A) $\mathbb{E}[A\sin(wt+\theta)] = \sin(wt+\theta)\,\mathbb{E}[A] \rightarrow \mathbb{E}[Y] = \mu\sin(wt+\theta)$

$\text{Var}[A\sin(wt+\theta)] = \sin^2(wt+\theta)\,\text{Var}[A] \rightarrow \text{Var}[Y] = \sigma^2\sin^2(wt+\theta)$

$\mathbb{E}[Y^2] = \text{Var}[Y] + \mathbb{E}[Y]^2 \rightarrow \mathbb{E}[Y^2] = (\mu^2+\sigma^2)\sin(wt+\theta)$

B) $\mathbb{E}[Y] = a\,\mathbb{E}[\sin(wt+\theta)] = a\int_0^{2\pi} \frac{\sin(wt+\theta)}{2\pi}\,d\theta \rightarrow \mathbb{E}[Y] = 0$

$\mathbb{E}[Y^2] = \mathbb{E}[a^2\sin^2(wt+\theta)] = a^2\int_0^{2\pi} \frac{\sin^2(wt+\theta)}{2\pi}\,d\theta$

$= \frac{a^2}{2\pi}\int_0^{2\pi}\left(0.5 - 0.5\,\cancelto{0}{\cos(2wt+2\theta)}\right)d\theta$

$= \frac{a^2}{2\pi}\left[0.5\theta\right]_0^{2\pi} \rightarrow \mathbb{E}[Y^2] = \frac{a^2}{2}$

C) $\mathbb{E}[Y] = a\,\mathbb{E}[\sin(wT+\theta)] = a\int_{-\pi/2w}^{\pi/2w} \frac{\sin(wT+\theta)}{\pi/w}\,dT \rightarrow \mathbb{E}[Y] = 0$

$\mathbb{E}[Y^2] = a^2\int_{-\pi/2w}^{\pi/2w} \frac{\sin^2(wT+\theta)}{\pi/w}\,dT$

$= \frac{a^2 w}{\pi}\int_{-\pi/2w}^{\pi/2w}\left(0.5 - 0.5\,\cancelto{0}{\cos(2wT+\theta)}\right)dT = \frac{a^2 w}{\pi}\left[0.5T\right]_{-\pi/2w}^{\pi/2w}$

$= \frac{a^2 w}{2\pi}\left(\frac{\pi}{2w} - -\frac{\pi}{2w}\right) = \frac{a^2 w}{2\cancel{\pi}}\left(\frac{\cancel{\pi}}{\cancel{w}}\right) \rightarrow \mathbb{E}[Y^2] = \frac{a^2}{2}$

Exercise 5.

A continuous random variable X is uniformly distributed in $[-2, 2]$. $\rightarrow f_X(x) = \frac{1}{4} \cdot \mathbb{1}_{[-2,2]}(x)$

(a) Let $Y = \sin(\pi X/8)$. Find $f_Y(y)$.

(b) Let $Z = 2X^2 + 1$. Find $f_Z(z)$.

Hint: Compute $F_Y(y)$ from $F_X(x)$, and use $\frac{d}{dy}\sin^{-1} y = \frac{1}{\sqrt{1-y^2}}$.

A) $F_Y(y) = \mathbb{P}(\sin(\pi X/8) \leq y) = \mathbb{P}(X \leq \frac{8}{\pi}\sin^{-1}(y)) = F_X(\frac{8}{\pi}\sin^{-1}(y))$

$$f_Y(y) = \frac{d(F_X(\frac{8}{\pi}\sin^{-1}(y)))}{dy} = \frac{d(F_X(\frac{8}{\pi}\sin^{-1}(y)))}{d(\sin^{-1}(y))} \cdot \frac{d(\sin^{-1}(y))}{dy}$$

$$f_Y(y) = \frac{1}{4}\left(\frac{8}{\pi}\right)\left(\frac{1}{\sqrt{1-y^2}}\right) \rightarrow f_Y(y) = \frac{2}{\pi\sqrt{1-y^2}} \cdot \mathbb{1}_{[-1/\sqrt{2},\, 1/\sqrt{2}]}(y)$$

B) $F_Z(z) = \mathbb{P}(2X^2 + 1 \leq z) = \mathbb{P}(X \leq \sqrt{\frac{z-1}{2}}) = F_X(\sqrt{\frac{z-1}{2}})$

$$f_Z(z) = \frac{d(F_X(\sqrt{\frac{z-1}{2}}))}{dz} = \frac{d(F_X(\sqrt{\frac{z-1}{2}}))}{d(\sqrt{\frac{z-1}{2}})} \cdot \frac{d(\sqrt{\frac{z-1}{2}})}{dz}$$

$$f_Z(z) = f_X(x)\left(\frac{1}{4}\right)\left(\sqrt{\frac{z-1}{2}}\right)$$

$$f_Z(z) = \frac{1}{16}\sqrt{\frac{z-1}{2}} \cdot \mathbb{1}_{[4,9]}(z)$$

Exercise 6.

A few days ago my daughter's piano teacher asked me to scan a music sheet for my daughter. I have a IPEVO document camera, which is not the best in the market but quite descent at the \$100-\$200 range. I took a picture, and adjusted the lighting as much as I could. Here is the picture. The background is gray-ish, and the color did not look good. FYI, you can read an image and display in MATLAB by using these commands. (For Python, you can do `cv2.imread`, `cv2.cvtColor` and `cv2.imshow`.)

```
x = im2double(imread('some_file_name.png'));
x = rgb2gray(x);
imshow(x);
```

(a) Raw input (b) Binary quantization (c) What I want

Figure 1: (a) Raw input of a photo-scan of a document. The goal of this exercise is to design a nonlinear transformation that can correct the color of the document. (b) A simple binary thresholding. (c) The desired output. You are welcome to do something better than mine.

I was not happy about the undesired color of the raw image. So I tried some simple tricks. The thing I did was to convert the image into binary color. This is done using a simple command

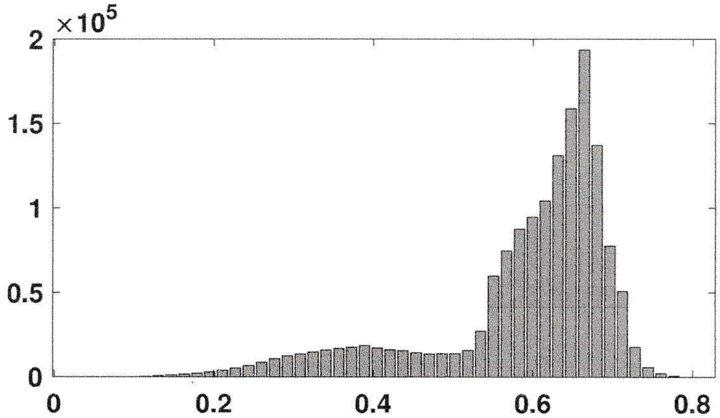
```
x(x>0.5) = 1;
x(x<0.5) = 0;
```

Now, the image looks much better as you can see in part (b) of the figure. So I brought this to my wife. And she complained. She said the hand written notes are gone, and some music notes are not clear. So I looked at the histogram of the data. It appeared to me that the histogram has a bi-modal distribution which is expected because there is foreground text and background color. The problem is that the two distributions are too close to each other. One way to address this is to come up with a nonlinear transformation $g(\cdot)$ such that

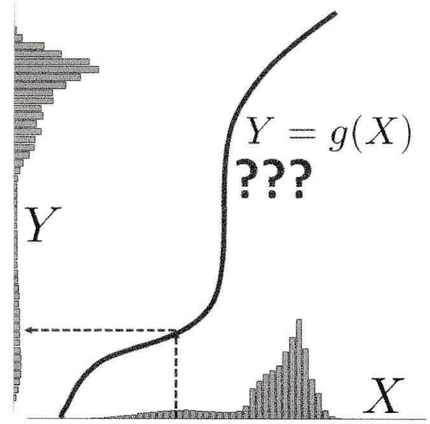
$$Y = g(X) \tag{1}$$

has the desired histogram, where X is the original random variable and Y is the transformed random variable. Here is your exercise. I want you to come up with a nonlinear transformation $g(\cdot)$. For example, you can try:

- Quadratic equation, but I do not think this will work. $g(X) = X^2$. In MATLAB, this would be `y = x.^2`, where `x` is the input image.



(a) Original histogram



(b) Transformation of random variables

- Piecewise linear transformation, which you need to figure out all these parameters.

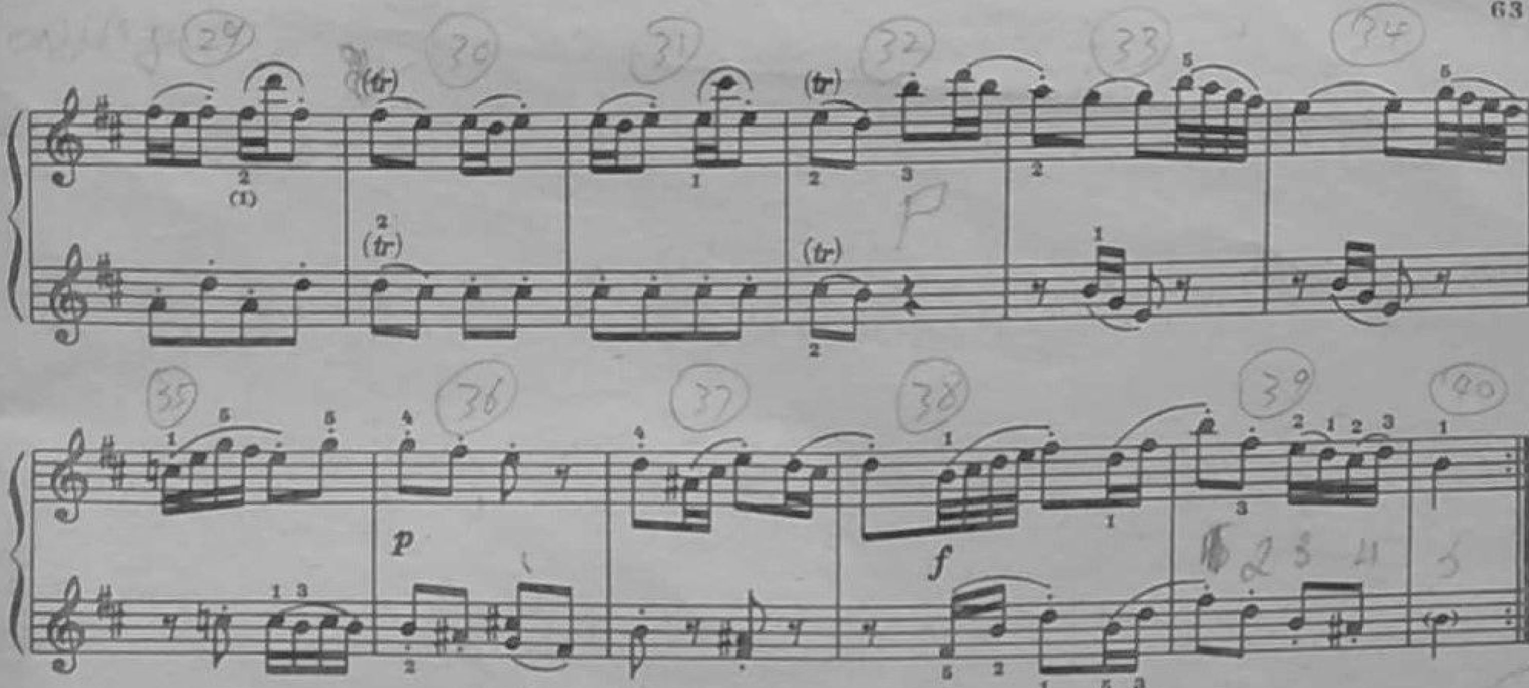
$$g(X) = \begin{cases} a_1 X + b_1, & \ell_1 \le X \le u_1 \\ a_2 X + b_2, & \ell_2 \le X \le u_2 \\ a_3 X + b_3, & \ell_3 \le X \le u_3 \end{cases}$$

- Exponential function, which I have not tried, $g(X) = e^X$. (Or some variants of that.)
- Logarithmic function, which I also have not tried, $g(X) = \log X$. (Or some variants of that.)
- Your creative solution!

For this exercise, please just submit: (1) Your nonlinear transformation, basically the equation $g(X)$. (2) Your histograms, before and after. (3) Your final output image.

Please do not use photoshop or any off-the-shelf commercial software because this will defeat the educational purpose of this exercise. You can also search on Wikipedia. A few ideas that were relevant are: (a) Balanced histogram thresholding. This idea does not really work for us because it outputs a binary image. I want a gray-scale image, and I want to preserve the hand written note as much as I can. (b) Histogram equalization. I tried this, but it did not work. You can try it too. The bottom line is that you can try whatever you want, but I am only looking for a very simple transformation $g(X)$. It should be one single equation. As a reference, the solution I came up with is just a single line of MATLAB code.

Please write your transformation here
$g(X) =$
Please attached your histograms and the final image after this page. No need to submit your code.



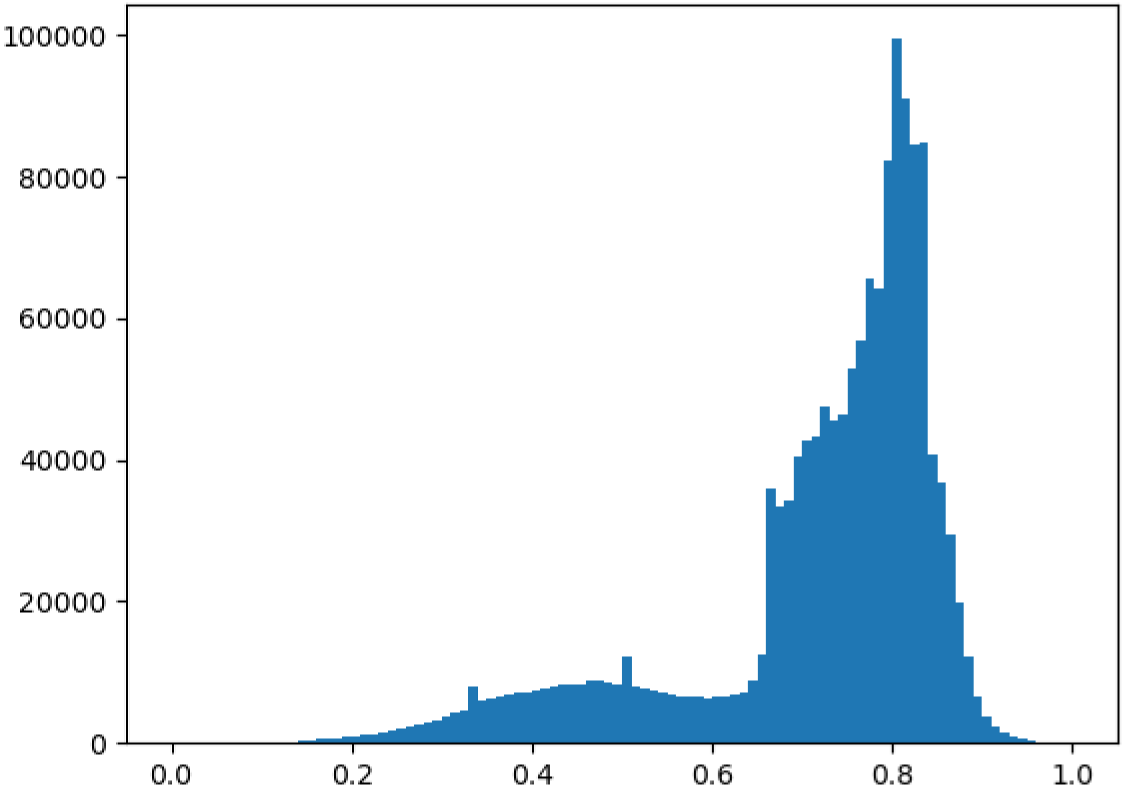
ТАНЕЦ БАЛЕРИНЫ из балета «Петрушка»

ПЕРВАЯ ПАРТИЯ

Переложение Э. Денисова

И. СТРАВИНСКИЙ
(1882—1971)

Allegro [Скоро]


100000
80000
60000
40000
20000
0
0.0
0.2
0.4
0.6
0.8
1.0