PURDUE UNIVERSITY

Homework 1

Fall 2020
(Due: 09/04/2020)

Name: Elias Talcott

Email: etalcott@purdue.edu

Homework is due at 11:59pm (midnight) Eastern Daylight Time. Please print this homework, write your solution, and scan the solution. Or you can use a tablet. Submit your homework through Gradescope. No late homework will be accepted.

Exercise 1.
Calculate the infinite series

$$\sum_{k=0}^{\infty} k \cdot \left(\frac{2}{3}\right)^{k+1}$$

$= 0\left(\frac{2}{3}\right) + 1\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^5 + \dots \qquad r = 2/3$

$= r^2 + 2r^3 + 3r^4 + 4r^5 + \dots$

$= r^2(1 + 2r + 3r^2 + 4r^3 + \dots)$

$= r^2\left(\frac{1}{(1-r)^2}\right)$

$= \frac{r^2}{(1-r)^2} = \frac{4/9}{(1-2/3)^2} = \frac{4/9}{(1/3)^2} = \frac{4/9}{1/9}$

$= 4$

Exercise 2.
Evaluate the integrals

(a)
$$\int_a^b \frac{1}{b-a}\left(x-\frac{a+b}{2}\right)^2 dx$$

(b)
$$\int_0^\infty \lambda x e^{-\lambda x} dx$$

(c)
$$\int_{-\log y}^{\log y} \frac{\lambda x}{4} e^{-\lambda|x|} dx,$$

where $y > 1$.

a) $= \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx$

$= \frac{1}{b-a} \int_a^b x^2 - \frac{x(a+b)}{2} - \frac{x(a+b)}{2} + \frac{(a+b)^2}{4} dx$

$= \frac{1}{b-a} \int_a^b x^2 - ax - bx + \frac{(a+b)^2}{4} dx$

$= \frac{1}{b-a} \left[\frac{1}{3}x^3 - \frac{1}{2}ax^2 - \frac{1}{2}bx^2 + \frac{(a+b)^2}{4}x\right]_a^b$

$= \frac{1}{b-a}\left[\frac{b^3}{3} - \frac{a^3}{3} - \frac{ab^2}{2} + \frac{a^3}{2} - \frac{b^3}{2} + \frac{a^2b}{2} + \frac{(a+b)^2 b}{4} - \frac{(a+b)^2 a}{4}\right]$

$= \frac{(b-a)^2}{12}$

b) $= \lambda \int_0^\infty x e^{-\lambda x} dx$

$u = x \quad dv = e^{-\lambda x} dx$

$du = dx \quad v = \frac{-1}{\lambda} e^{-\lambda x}$

$= \cancel{\lambda}\left[\frac{-x}{\cancel{\lambda}} e^{-\lambda x} + \frac{1}{\cancel{\lambda}} \int e^{-\lambda x} dx\right]_0^\infty$

$= \left[-x e^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda}\right]_0^\infty = (0+0) - \left(0 - \frac{1}{\lambda}\right) = \frac{1}{\lambda}$

c) $= \frac{\lambda}{4} \int_{-\ln y}^{\ln y} x e^{-\lambda|x|} dx \rightarrow$ odd

$= 0$

Exercise 3. .
Evaluate the infinite series

$$\sum_{k=0}^{\infty}(k-\lambda)^2\frac{\lambda^k e^{-\lambda}}{k!} \tag{1}$$

$$\sum_{k=0}^{\infty}(k-\lambda)^2\frac{\lambda^k e^{-\lambda}}{k!}$$

$$= \underbrace{\sum_{k=0}^{\infty}\frac{k^2\lambda^k e^{-\lambda}}{k!}}_{\lambda(\lambda+1)} - 2\lambda\underbrace{\sum_{k=0}^{\infty}\frac{ke^{-\lambda}\lambda^k}{k!}}_{\lambda} + \lambda^2\underbrace{\sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}}_{1}$$

$$= \lambda(\lambda+1) - 2\lambda(\lambda) + \lambda^2(1)$$

$$= \lambda^2 + \lambda - 2\lambda^2 + \lambda^2$$

$$= \lambda$$

Exercise 4.
Simplify the following sets with the domain of real numbers in mind

(a) $[2,5] \cap ([1,3] \cup \{0,3,4\})$

(b) $(1,2)^c \cup [4,6]$

(c) $\bigcap_{n=1}^{\infty}(2-1/n, 2+1/n)$

(d) $\bigcup_{n=1}^{\infty}[3, 6-\frac{1}{n}]$

a) $[2,5] \cap (\{3\}) \rightarrow \{3\}$

b) $(1,2)^c \cup [4,6] \rightarrow (-\infty,1] \cup [2,\infty) \cup [4,6]$

$\rightarrow (-\infty,1] \cup [2,\infty)$

c) $\bigcap_{n=1}^{\infty} (2-\frac{1}{n}, 2+\frac{1}{n}) \rightarrow \left.\begin{array}{l} A_1 = (1,3) \\ A_2 = (1.5, 2.5) \\ A_\infty \approx (2,2) \end{array}\right\} \rightarrow [2,2]$

$\rightarrow \{2\}$

d) $\bigcup_{n=1}^{\infty} [3, 6-\frac{1}{n}] \rightarrow \begin{array}{l} A_1 = [3,5) \\ A_2 = [3,5.5) \\ A_\infty \approx [3,6) \end{array}$

$\rightarrow [3,6)$

Exercise 5.
A space S and three of its subsets are given by $S = \{1, 3, 5, 7, 9, 11\}$, $A = \{1, 3, 5\}$, $B = \{7, 9, 11\}$, and $C = \{1, 3, 9, 11\}$. Find (a) $A \cap B \cap C$, (b) $A^c \cap B$, (c) $A \backslash C$, and (d) $(A \backslash B) \cup B$.

a) $A \cap B \cap C = \phi$

b) $A^c \cap B = \{7,9,11\} \cap \{7,9,11\} = \{7,9,11\}$

c) $A \backslash C = \{\cancel{1},\cancel{3},5\} = \{5\}$

d) $(A \backslash B) \cup B = A \cup B = \{1,3,5,7,9,11\} = S$

Exercise 6.
Prove the second part of DeMorgan's Law, i.e., show that $(A \cup B)^c = A^c \cap B^c$.

$X = (A \cup B)^c$, $Y = A^c \cap B^c$

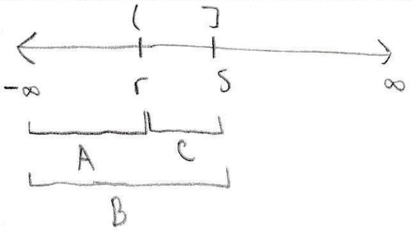
$x_1 \in X \rightarrow x_1 \in (A \cup B)^c \rightarrow x_1 \notin (A \cup B)$

$\rightarrow x_1 \notin A$ and $x_1 \notin B \rightarrow x_1 \in Y$

$y_1 \in Y \rightarrow y_1 \in A^c \cap B^c \rightarrow y_1 \notin A$ and $y_1 \notin B$

$\rightarrow y_1 \notin (A \cup B) \rightarrow y_1 \in (A \cup B)^c \rightarrow y_1 \in X$

Exercise 7.
Let $A = (-\infty, r]$ and $B = (-\infty, s]$ where $r \leq s$. (a) Find an expression for $C = (r, s]$ in terms of A and B. (b) Show that $B = A \cup C$, and $A \cap C = \emptyset$.



a) C = in B but not in A

$C = A^c \cap B$

b) $C = (r, s]$, $A = (-\infty, r] \rightarrow A \cup C = (-\infty, s] = B$

$C = (r, s]$, $A = (-\infty, r] \rightarrow A \cap C = \emptyset$

C starts where A stops. Both parts of B illustrated above.

Exercise 8.
Show that if $A \cup B = A$ and $A \cap B = A$, then $A = B$.

If $A \cup B = A$, there is nothing in B that is not in A.
If $A \cap B = A$, there is nothing in A that is not in B.

Therefore, $A == B$

Exercise 9.
This is a programming exercise. You can use either MATLAB or Python.

(a) Compute the result of the following matrix vector multiplication.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(b) Plot a sine function on the interval $[-\pi, \pi]$ with 1000 data points using `matplotlib.pyplot.plot` in Python or `plot` in MATLAB.

(c) Generate 10,000 uniformly distributed random numbers on interval $[0, 1)$. Use `hist` in MATLAB or `matplotlib.pyplot.hist` Python to generate a histogram of all the random numbers.

Please insert your code / solution after this page.

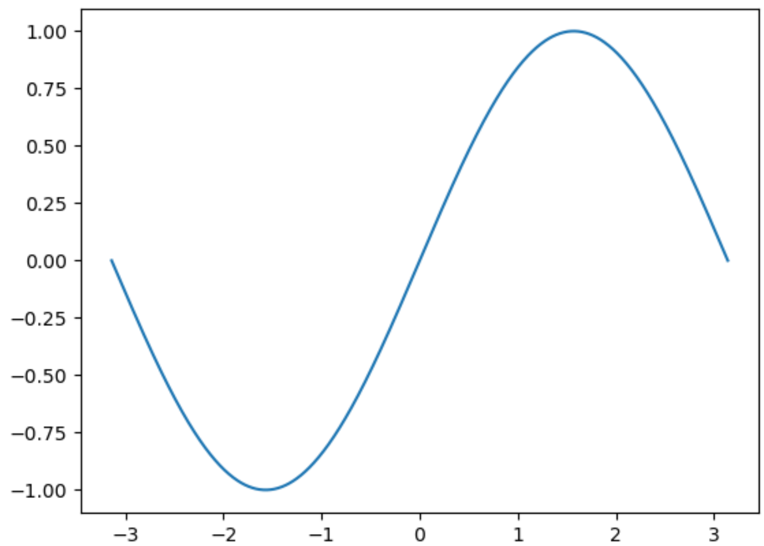
Exercise 9 Code

```
#!/usr/bin/env python

import numpy as np
import matplotlib.pyplot as plot

# Calculate the result of a matrix vector multiplication
def matrix_vector_multiplication():
    mat = np.array([[1, 2, 3], [4, 5, 6], [7, 8, 9]])
    vec = np.array([1, 2, 3])
    print("Result: {}".format(np.matmul(mat, vec)))


# Plot sine from -pi to pi with 1,000 data points
def plot_sine():
    x = np.linspace(-np.pi, np.pi, 1000)
    y = [np.sin(x_i) for x_i in x]
    plot.plot(x, y)
    plot.show()


# Plot 10,000 uniformly distributed random numbers on [0, 1)
def generate_histogram():
    x = np.random.uniform(0, 1, 10000)
    plot.hist(x, bins=20)
    plot.show()


if __name__ == "__main__":
    matrix_vector_multiplication()
    plot_sine()
    generate_histogram()
```

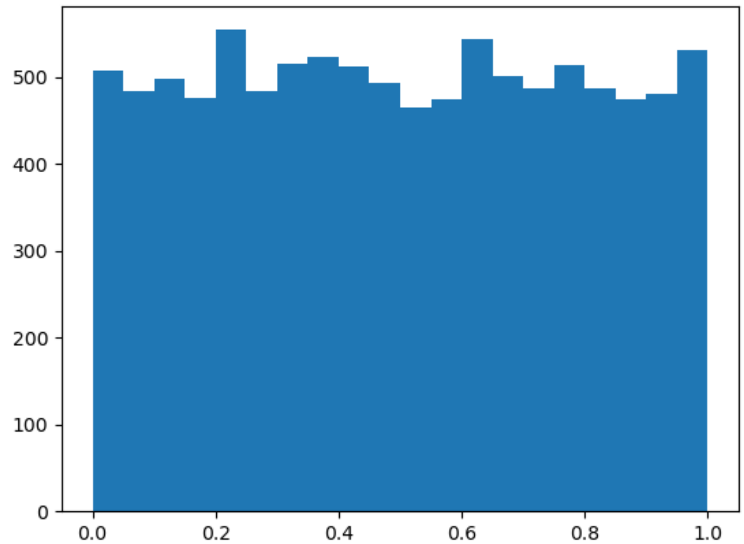
Exercise 9 Result

(a) Result: $[14, 32, 50]$

(b) sin(x) on [-π, π] for 1000 data points of x



(c) 10,000 uniformly distributed random numbers on [0, 1)



Exercise 10.
A collection of letters, a-z, is mixed in a jar. Two letters are drawn at random, one after the other.

(a) What is the probability of drawing a vowel (a,e,i,o,u) and a consonant in either order?

(b) Write a MATLAB / Python program to verify your answer in part (a). That is, randomly draw two letters without replacement and check whether one is a vowel and the other is a consonant. Compute the probability by repeating the experiment for 10000 times.

Please write your hand-written solution here.

Vowel → consonant $= \frac{\cancel{5}}{26}\left(\frac{21}{\cancel{25}_{5}}\right) = \frac{21}{130} \approx 0.161$

Consonant → vowel $= \frac{21}{26}\left(\frac{\cancel{5}}{\cancel{25}_{5}}\right) = \frac{21}{130} \approx 0.161$

→ Consonant → vowel
OR
Vowel → consonant $= \frac{21}{65} \approx 0.323$

Please insert your code / solution after this page.

Exercise 10 Code

```
#!/usr/bin/env python

import numpy as np

# Do n random draws of letters without replacement and compute fraction
    that included one vowel and one consonant
def random_draw(n = 10000):
    matches = 0
    for _ in range(0, n):
        # Make list of all letters and of vowels
        letters = ['a', 'b', 'c', 'd', 'e', 'f', 'g', 'h', 'i',
                   'j', 'k', 'l', 'm', 'n', 'o', 'p', 'q', 'r',
                   's', 't', 'u', 'v', 'w', 'x', 'y', 'z']
        vowels = ['a', 'e', 'i', 'o', 'u']
        # Draw and remove first letter
        draw1 = letters[np.random.randint(0, 25)]
        letters.remove(draw1)
        # Draw second letter
        draw2 = letters[np.random.randint(0, 24)]
        # Return True if one is a vowel and one is a consonant
        if (draw1 in vowels and draw2 not in vowels) or (draw1 not in
            vowels and draw2 in vowels):
            matches += 1
    print("Fraction of draws with one vowel and one consonant: {}".
        format(matches / n))


if __name__ == "__main__":
    random_draw()
```

Exercise 10 Result

(a) Fraction of draws with one vowel and one consonant: 0.3366

Exercise 11.
There are 50 students in a classroom.

(a) What is the probability that there is at least one pair of students having the same birthday? Show your steps.

(b) Write a MATLAB / Python program to simulate the event, and verify your answer in (a). Hint: You probably need to repeat the simulation for many times to obtain a probability. Submit your code and result.

You may assume that a year only has 365 days. You may also assume that all days have equal likelihood to be taken.

Please write your hand-written solution here.

$$\bar{p}(n) = 1\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)(\cdots)\left(\frac{365-n+1}{365}\right)$$

$$\bar{p}(n) = \frac{365!}{365^n(365-n)!}$$

$$\bar{p}(50) = \frac{365!}{365^{50}(315)!} \approx 0.0296$$

$$\therefore \quad p(50) = 1 - 0.0296$$

$$p(50) = 0.9704$$

Please insert your code / solution after this page.

Exercise 11 Code

```python
#!/usr/bin/env python

import numpy as np

# Do m runs of n students with random birthdays to determine
    probability of collision
def birthday_problem(m = 10000, n = 50):
    matches = 0
    for _ in range(0, m):
        # Assign each student a random birthday between 0 and 364
        students = np.random.randint(0, 365, n)
        # Checks for birthday matches since set doesn't contain
            duplicates
        if len(students) != len(set(students)):
            matches += 1
    print("Fraction of simulations with at least one matching birthday:
       {}".format(matches / m))


if __name__ == "__main__":
    birthday_problem()
```

Exercise 11 Result

(a) Fraction of simulations with at least one matching birthday: 0.9676