PURDUE UNIVERSITY

Homework 4

Fall 2020
(Due: October 12, 2020, Monday)

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Homework is due at 11:59pm (midnight) Eastern Time. Please print this homework, write your solution, and scan the solution. Submit your homework through Gradescope. No late homework will be accepted.

Exercise 1.
Two dice are tossed. Let X be the absolute difference in the number of dots facing up. Let

$$g(X) = \begin{cases} X^2, & \text{if } X > 2 \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad h(X) = -|X-2|.$$

(a) Find $\mathbb{E}[g(X)]$.

(b) Find $\mathbb{E}[h(X)]$.

$X=0$ | (1,1), (2,2), (3,3), (4,4), (5,5), (6,6) $= 6/36 = 1/6$
$X=1$ | (1,2), (2,3), (3,4), (4,5), (5,6) x2 $= 10/36 = 5/18$
$X=2$ | (1,3), (2,4), (3,5), (4,6) x2 $= 8/36 = 2/9$
$X=3$ | (1,4), (2,5), (3,6) x2 $= 6/36 = 1/6$
$X=4$ | (1,5), (2,6) x2 $= 4/36 = 1/9$
$X=5$ | (1,6) x2 $= 2/36 = 1/18$

A) $E[g(X)] = 3^2(1/6) + 4^2(1/9) + 5^2(1/18)$

$E[g(X)] = 14/3$

B) $E[h(X)] = -|-2|(1/6) - |-1|(5/18) - |0|(2/9) - |1|(1/6) - |2|(1/9) - |3|(1/18)$

$E[h(X)] = -7/6$

Exercise 2.

A modem transmits a +2 voltage signal into a channel. The channel adds to this signal a noise term that is drawn from the set $\{0, -1, -2, -3\}$ with respective probabilities $\{1/10, 2/10, 3/10, 4/10\}$.

(a) Find and sketch the PMF and the CDF of the output Y of the channel.

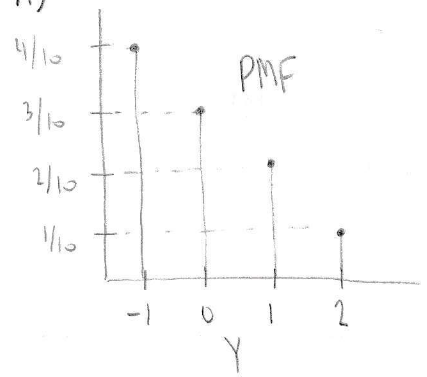
(b) What is the probability that the output of the channel is equal to the input of the channel?

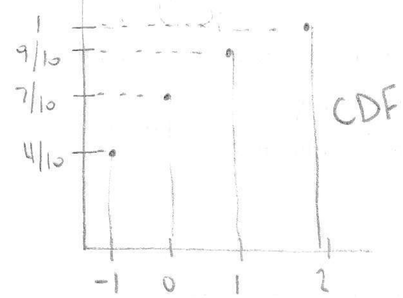
(c) What is the probability that the output of the channel is positive?

(d) Find the expected value and variance of Y.

$\mathbb{P}[Y=2] = 1/10$, $\mathbb{P}[Y=1] = 2/10$, $\mathbb{P}[Y=0] = 3/10$, $\mathbb{P}[Y=-1] = 4/10$

A)





B) $\mathbb{P}[\text{input} = \text{output}] = \mathbb{P}[Y=2] = 1/10$

C) $\mathbb{P}[Y>0] = \mathbb{P}[Y=2] + \mathbb{P}[Y=1] = 3/10$

D) $\mathbb{E}[Y] = 2(1/10) + 1(2/10) + 0(3/10) - 1(4/10)$

$\mathbb{E}[Y] = 0$

$\text{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$

$\mathbb{E}[Y^2] = 2^2(1/10) + 1(2/10) + 0(3/10) + 1(4/10) = 1$, $(\mathbb{E}[Y])^2 = 0$

$\text{Var}[Y] = 1 - 0 = 1$

Exercise 3.
A voltage X is uniformly distributed in the set $\{0, 1, 2, 3\}$.

(a) Find the mean and variance of X.

(b) Find the mean and variance of $Y = X^2 - 2$.

(c) Find the mean of $W = \sin(\pi X/4)$.

(d) Find the mean of $Z = \sin^2(\pi X/4)$.

A) $E[X] = 0(1/4) + 1(1/4) + 2(1/4) + 3(1/4)$

$E[X] = 3/2$

$Var[X] = 0(1/4) + 1(1/4) + 4(1/4) + 9(1/4) - (3/2)^2$

$Var[X] = 5/4$

B) $Y = \{-2, -1, 2, 7\}$

$E[Y] = -2(1/4) - 1(1/4) + 2(1/4) + 7(1/4) \rightarrow E[Y] = 3/2$

$Var[Y] = 4(1/4) + 1(1/4) + 4(1/4) + 49(1/4) - (3/2)^2 \rightarrow Var[Y] = 49/4$

C) $W = \{0, \sqrt{2}/2, 1, -1\}$

$E[W] = 0(1/4) + \sqrt{2}/2(1/4) + 1(1/4) - 1(1/4) \rightarrow E[W] = \sqrt{2}/8$

$Var[W] = 0(1/4) + 1/2(1/4) + 1(1/4) + 1(1/4) - (\frac{\sqrt{2}}{8})^2 \rightarrow Var[W] = 19/32$

D) $Z = \{0, 1/2, 1, 1\}$

$E[Z] = 0(1/4) + 1/2(1/4) + 1(1/4) + 1(1/4) \rightarrow E[Z] = 5/8$

$Var[Z] = 0(1/4) + 1/4(1/4) + 1(1/4) + 1(1/4) - (5/8)^2 \rightarrow Var[Z] = 11/64$

Exercise 4.

(a) If X is Poisson(λ), compute $\mathbb{E}[3/(X+2)]$.

(b) If X is Bernoulli(p) and Y is Bernoulli(q), compute $\mathbb{E}[(2X+Y)^2]$ if X and Y are independent.

A) $\mathbb{P}_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 1, 2, \ldots$

$$\mathbb{E}\left[\frac{3}{X+2}\right] = \sum_{k=0}^{\infty} \frac{3}{k+2} \mathbb{P}_X(k) = \sum_{k=0}^{\infty} \frac{3\lambda^k e^{-\lambda}}{(k+2)\,k!} = \sum_{k=0}^{\infty} \frac{3\lambda^k e^{-\lambda}}{(k+2)!/k+1}$$

$$= \sum_{k=0}^{\infty}$$

Exercise 5.
Let X be the number of photons counted by a receiver in an optical communication system. It is known that X is a Poisson random variable with rate λ_1 when a signal is present and a Poisson random variable with rate $\lambda_0 < \lambda_1$ when a signal is absent. The probability that the signal is present is p. Suppose that we observe $X = k$ photons. We want to determine a threshold T such that if $k \geq T$ then we claim that the signal is present; and if $k < T$ then we claim that the signal is absent. What is this T?

For λ_1: $\mathbb{P}(X=k \mid \lambda=\lambda_1) = \dfrac{\lambda_1^k e^{-\lambda_1}}{k!}$ For λ_0: $\mathbb{P}(X=k \mid \lambda=\lambda_0) = \dfrac{\lambda_0^k e^{-\lambda_0}}{k!}$

Present if $k \geq T$

$$\mathbb{P}(\lambda=\lambda_1 \mid X=k) \geq \mathbb{P}(\lambda=\lambda_0 \mid X=k)$$

$$\frac{\mathbb{P}(X=k \mid \lambda=\lambda_1)\,\mathbb{P}(\lambda=\lambda_1)}{\cancel{\mathbb{P}(X=k)}} \geq \frac{\mathbb{P}(X=k \mid \lambda=\lambda_0)\,\mathbb{P}(\lambda=\lambda_0)}{\cancel{\mathbb{P}(X=k)}}$$

$$\frac{\lambda_1^k e^{-\lambda_1}}{\cancel{k!}}\, p \geq \frac{\lambda_0^k e^{-\lambda_0}}{\cancel{k!}}(1-p)$$

$$\frac{\lambda_1^k}{\lambda_0^k} \geq \frac{e^{-\lambda_0}(1-p)}{e^{-\lambda_1}(p)}$$

$$\left(\frac{\lambda_1}{\lambda_0}\right)^k \geq \left(\frac{1-p}{p}\right)e^{\lambda_1-\lambda_0}$$

$$k\ln\left(\frac{\lambda_1}{\lambda_0}\right) \geq \ln\left(\frac{1-p}{p}\right) + \lambda_1 - \lambda_0$$

$$k \geq \frac{\ln\left(\frac{1-p}{p}\right) + \lambda_1 - \lambda_0}{\ln\left(\frac{\lambda_1}{\lambda_0}\right)}$$

$$\therefore \quad T = \frac{\ln\left(\frac{1-p}{p}\right) + \lambda_1 - \lambda_0}{\ln\left(\frac{\lambda_1}{\lambda_0}\right)}$$

Exercise 6.
A random variable X has PDF:

$$f_X(x) = \begin{cases} cx(4-x^2), & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find c

(b) Find $F_X(x)$

(c) Find $\mathbb{E}[X]$

A) $\int_0^2 cx(4-x^2)\,dx = c\int_0^2 4x - x^3\,dx$

$= c\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = c((8-4)-(0)) = 4c = 1 \rightarrow c = 1/4$

B) $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_0^x \frac{1}{4}x(4-x^2)\,dx$

$= \frac{1}{4}\int_0^x 4x - x^3\,dx = \frac{1}{4}\left[2x^2 - \frac{1}{4}x^4\right]_0^x = \frac{1}{4}\left(2x^2 - \frac{1}{4}x^4\right)$

$F_X(x) = \left(\frac{1}{2}x^2 - \frac{1}{16}x^4\right)\mathbb{1}_{[0,2]}(x)$

c) $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^2 \frac{1}{4}x^2(4-x^2)\,dx$

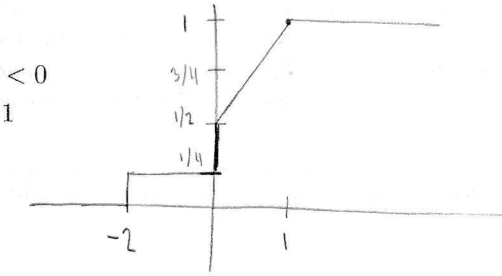
$= \frac{1}{4}\int_0^2 4x^2 - x^4\,dx = \frac{1}{4}\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$

$= \frac{1}{4}\left(\frac{32}{3} - \frac{32}{5}\right)$

$\mathbb{E}[X] = 16/15$

Exercise 7.
Consider a CDF

$$F_X(x) = \begin{cases} 0, & \text{if } x < -2 \\ 0.25, & \text{if } -2 \le x < 0 \\ (x+1)/2, & \text{if } 0 \le x < 1 \\ 1, & \text{otherwise.} \end{cases}$$



(a) Find $\mathbb{P}[X < -2]$, $\mathbb{P}[X = 0]$ and $\mathbb{P}[X > 0.5]$.

(b) Find $f_X(x)$.

A) $\mathbb{P}[X < -2] = 0$

$\mathbb{P}[X = 0] = \mathbb{P}[X \le 0] - \mathbb{P}[X < 0] = \frac{(0+1)}{2} - 0.25 = 1/4$

$\mathbb{P}[X = 0] = 1/4$

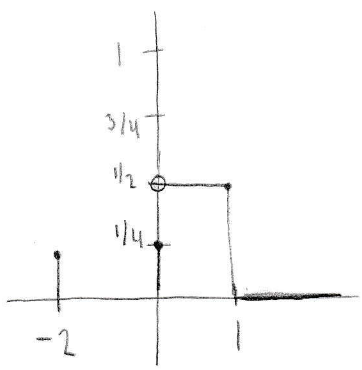
$\mathbb{P}[X > 0.5] = 1 - \mathbb{P}[X \le 0.5] = 1 - \frac{(0.5+1)}{2} = 1/4$

$\mathbb{P}[X > 0.5] = 1/4$

B)

$\frac{d}{dx}\left(\frac{x+1}{2}\right) = \frac{1}{2}$

$$f_X(x) = \begin{cases} 0 & , x < -2 \\ 0.25 & , x = -2 \\ 0 & , -2 < x < 0 \\ 0.25 & , x = 0 \\ 0.5 & , 0 < x < 1 \\ 0 & , x \ge 1 \end{cases}$$



Exercise 8.

Consider a discrete PMF $p_X(k) = \begin{bmatrix} 0.3 & 0.1 & 0.15 & 0.25 & 0.1 & 0.08 & 0.02 \end{bmatrix}$. Write a MATLAB / Python function that takes this PMF and generates $N = 100,000$ realizations of X. Your function can only use the uniform random number generator `rand` in MATLAB (or `numpy.random.rand` in Python) and no other random number generators. Submit your code and the empirical histogram of X.

Please attach your code and plot after this page.