ECE 302: Probabilistic Methods in Electrical and Computer Engineering
Fall 2020
Instructor: Prof. Stanley H. Chan

PURDUE UNIVERSITY

Homework 8

Fall 2020
(Due: Dec 4, 2020, Friday)

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Homework is due at 11:59pm (midnight) Eastern Time. Please print this homework, write your solution, and scan the solution. Submit your homework through Gradescope. No late homework will be accepted.

Exercise 1.
Let $Y(t) = X(t) - X(t-d)$.

(a) Find $R_{X,Y}(\tau)$ and $S_{X,Y}(\omega)$.

(b) Find $R_Y(\tau)$.

(c) Find $S_Y(\omega)$.

A) $R_{X,Y}(\tau) = \mathbb{E}[X(t+\tau)Y(t)] = \mathbb{E}[X(t+\tau)X(t) - X(t+\tau)X(t-d)]$

$= R_X(\tau) - R_X(\tau+d)$

$S_{X,Y}(\omega) = \mathcal{F}\{R_{X,Y}(\tau)\} = \mathcal{F}\{R_X(\tau)\} - \mathcal{F}\{R_X(\tau+d)\}$

$= S_X(\omega) - S_X(\omega)e^{j\omega d}$

B) $R_Y(\tau) = \mathbb{E}[Y(t+\tau)Y(t)] = \mathbb{E}[(X(t+\tau) - X(t+\tau-d))(X(\tau) - X(\tau-d))]$

$= \mathbb{E}[X(t+\tau)X(t)] - \mathbb{E}[X(t+\tau-d)X(t)] - \mathbb{E}[X(t+\tau)X(t-d)] - \mathbb{E}[X(t+\tau-d)X(t-d)]$

$= R_X(\tau) - R_X(\tau-d) - R_X(\tau+d) + R_X(\tau) = 2R_X(\tau) - R_X(\tau-d) - R_X(\tau+d)$

C) $S_Y(\omega) = 2S_X(\omega) - S_X(\omega)e^{-j\omega d} - S_X(\omega)e^{j\omega d}$

$= 2S_X(\omega) - 2S_X(\omega)\left[\frac{e^{j\omega d} + e^{-j\omega d}}{2}\right]$

$= 2S_X(\omega)[1 - \cos(\omega d)]$

Exercise 2.
Let $X(t)$ be a zero-mean WSS process with autocorrelation function $R_X(\tau)$. Let $Y(t) = X(t)\cos(\omega t + \Theta)$, where $\Theta \sim \text{uniform}(-\pi, \pi)$ and Θ is independent of the process $X(t)$.

(a) Find the autocorrelation function $R_Y(\tau)$.

(b) Find the cross-correlation function of $X(t)$ and $Y(t)$.

(c) Is $Y(t)$ WSS? Why?

A) $R_Y(\tau) = E[Y(t+\tau)Y(t)] = E[X(t+\tau)\cos(\omega(t+\tau)+\theta)X(t)\cos(\omega t+\theta)]$

$= E[X(t+\tau)X(t)]E[\cos(\omega(t+\tau)+\theta)\cos(\omega t+\theta)]$

$= R_X(\tau)E\left[\frac{\cos(2\omega t + \omega\tau + 2\theta) + \cos(\omega\tau)}{2}\right] = \frac{1}{2}R_X(\tau)\cos(\omega\tau)$

B) $R_{X,Y}(\tau) = E[X(t+\tau)Y(t)] = E[X(t+\tau)X(t)\cos(\omega t+\theta)]$

$= E[X(t+\tau)X(t)]E[\cos(\omega t+\theta)] = 0$

c) $E[Y(t)] = E[X(t)\cos(\omega t+\theta)] = E[X(t)]E[\cos(\omega t+\theta)]$

$= 0$

$R_Y(\tau) = \frac{1}{2}R_X(\tau)\cos(\omega\tau)$

Constant mean, autocorrelation depends only on τ

$\therefore$ $Y(t)$ is WSS

Exercise 3.
Consider the system

$$Y(t) = e^{-t} \int_{-\infty}^{t} e^{\tau} X(\tau) d\tau.$$

Assume that $X(t)$ is zero mean white noise with power spectral density $S_X(\omega) = N_0/2$. Find

(a) $S_{XY}(\omega)$

(b) $R_{XY}(\tau)$

(c) $S_Y(\omega)$

(d) $R_Y(\tau)$

A) $S_{X,Y}(\omega) = F\{R_{X,Y}(\tau)\} =$

Exercise 4.
Consider the random process

$$X(t) = 2A\cos(t) + (B-1)\sin(t),$$

where A and B are two independent random variables with $\mathbb{E}[A] = \mathbb{E}[B] = 0$, and $\mathbb{E}[A^2] = \mathbb{E}[B^2] = 1$.

(a) Find $\mu_X(t)$

(b) Find $R_X(t_1, t_2)$

(c) Find $C_X(t_1, t_2)$

A) $\mu_X(t) = \mathbb{E}[2A\cos(t) + (B-1)\sin(t)] = 2\cos(t)\mathbb{E}[\overset{0}{A}] + \sin(t)\mathbb{E}[\overset{-1}{B-1}]$

$\mu_X(t) = -\sin(t)$

B) $R_X(t_1,t_2) = \mathbb{E}[X(t_1)X(t_2)]$

$= \mathbb{E}[(2A\cos(t_1) + (B-1)\sin(t_1))(2A\cos(t_2) + (B-1)\sin(t_2))]$

$= \cos(t_1)\cos(t_2)\underbrace{\mathbb{E}[4A^2]}_{4} + \cos(t_1)\sin(t_2)\underbrace{\mathbb{E}[2A(B-1)]}_{\mathbb{E}[\overset{0}{2A}]\mathbb{E}[B-1]=0} + \sin(t_1)\cos(t_2)\underbrace{\mathbb{E}[2A(B-1)]}_{0} + \sin(t_1)\sin(t_2)\underbrace{\mathbb{E}[(B-1)^2]}_{\mathbb{E}[B^2-2B+1]=2}$

$R_X(t_1,t_2) = 4\cos(t_1)\cos(t_2) + 2\sin(t_1)\sin(t_2)$

C) $C_X(t_1,t_2) = \mathbb{E}[X(t_1)X(t_2)] - \mathbb{E}[X(t_1)]\mathbb{E}[X(t_2)]$

$= 4\cos(t_1)\cos(t_2) + 2\sin(t_1)\sin(t_2) - \sin(t_1)\sin(t_2)$

$C_X(t_1,t_2) = 4\cos(t_1)\cos(t_2) + \sin(t_1)\sin(t_2)$

Exercise 5.
Find the autocorrelation function $R_X(\tau)$ corresponding to each of the following power spectral densities:

(a) $\delta(\omega - \omega_0) + \delta(\omega + \omega_0)$

(b) $e^{-\omega^2/2}$

(c) $e^{-|\omega|}$

A) $R_x(\tau) = F^{-1}\{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\}$

$R_x(\tau) = \frac{1}{2\pi}\left(e^{-j\omega_0\tau} + e^{j\omega_0\tau}\right) \rightarrow R_x(\tau) = \frac{1}{\pi}\cos(\omega\tau)$

B) $R_x(\tau) = F^{-1}\{e^{-\omega^2/2}\}$

$R_x(\tau) = \frac{1}{\sqrt{2\pi}} e^{-\tau^2/2}$

C) $R_x(\tau) = F^{-1}\{e^{-|\omega|}\}$

$R_x(\tau) = \frac{1}{\pi}\left(\frac{1}{1+\tau^2}\right)$

Exercise 6.
A WSS process $X(t)$ with autocorrelation function $R_X(\tau) = e^{-\tau^2/(2\sigma_T^2)}$ is passed through an LTI system with transfer function $H(\omega) = e^{-\omega^2/(2\sigma_H^2)}$. Denote the system output by $Y(t)$. Find

(a) $S_{XY}(\omega)$

(b) $R_{XY}(\tau)$

(c) $S_Y(\omega)$

(d) $R_Y(\tau)$

$$S_x(\omega) = \mathcal{F}\{R_x(\tau)\} = \sqrt{2\pi}\,\sigma_T\, e^{\frac{-\sigma_T^2\omega^2}{2}}$$

A) $S_{X,Y}(\omega) = H^*(\omega) S_X(\omega) = e^{\omega^2/2\sigma_H^2}\left(\sqrt{2\pi}\, e^{\frac{-\sigma_T^2\omega^2}{2}}\right)\sqrt{2\pi}\,\sigma_T\, e^{\frac{-\sigma_T^2\omega^2}{2}}$

$$S_{X,Y}(\omega) = \frac{\sigma_T}{\sqrt{\sigma_T^2+\sigma_H^{-2}}}\sqrt{2\pi}\cdot\sqrt{\sigma_T^2+\sigma_H^{-2}}\; e^{-\left(\frac{\sigma_T^2+\sigma_H^{-2}}{2}\right)\omega^2}$$

B) $R_{X,Y}(\tau) = \mathcal{F}^{-1}\{S_{X,Y}(\omega)\}$

$$\rightarrow R_{X,Y}(\tau) = \frac{\sigma_T}{\sqrt{\sigma_T^2+\sigma_H^{-2}}}\, e^{-\left(\frac{\tau^2}{2(\sigma_T^2+\sigma_H^{-2})}\right)}$$

C) $S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \left|e^{-\omega^2/2\sigma_H^2}\right|^2 \sqrt{2\pi}\,\sigma_T\, e^{\frac{-\sigma_T^2\omega^2}{2}}$

$$S_Y(\omega) = \sqrt{2\pi}\,\sigma_T\, e^{\frac{-\omega^2}{\sigma_H^2}}\, e^{\frac{-\sigma_T^2\omega^2}{2}}$$

D) $R_Y(\tau) = \mathcal{F}^{-1}\{S_Y(\omega)\} = \mathcal{F}^{-1}\left\{\sqrt{2\pi}\,\sigma_T\, e^{\frac{-\sigma_T^2\omega^2}{2}}\, e^{\frac{-\omega^2}{\sigma_H^2}}\right\}$

$$R_Y(\tau) = e^{-\left(\frac{\tau^2}{2\sigma_T^2}\right)}$$

Exercise 7.
A WSS process $X(t)$ with autocorrelation function

$$R_X(\tau) = 1/(1+\tau^2)$$

is passed through an LTI system with impulse response

$$h(t) = 3\sin(\pi t)/(\pi t).$$

Let $Y(t)$ be the system output. Find $S_Y(\omega)$. Sketch $S_Y(\omega)$

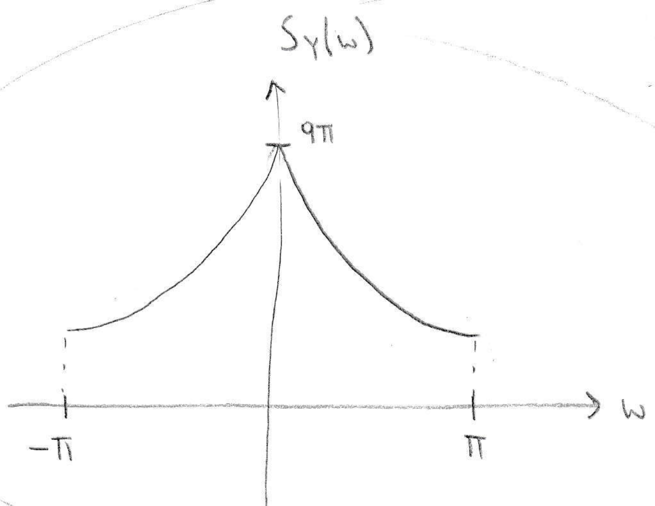
$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$$

$$S_X(\omega) = \mathcal{F}\left\{\frac{1}{1+\tau^2}\right\} = \pi e^{-|\omega|}$$

$$H(\omega) = \mathcal{F}\left\{\frac{3\sin(\pi t)}{\pi t}\right\} = \mathcal{F}\{3\,\mathrm{sinc}(\pi t)\} = 3\,\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$

$$S_Y(\omega) = \left|3\,\mathrm{rect}\left(\frac{\omega}{2\pi}\right)\right|^2 \pi e^{-|\omega|}$$

$$S_Y(\omega) = 9\pi e^{-|\omega|} \cdot \mathbb{1}_{[-\pi,\pi]}(\omega)$$



Exercise 8.
Consider a WSS process $X(t)$ with autocorrelation function

$$R_X(\tau) = \text{sinc}(\pi\tau).$$

The process is sent to an LTI system, with input-output relationship

$$2\frac{d^2}{dt^2}Y(t) + 2\frac{d}{dt}Y(t) + 4Y(t) = 3\frac{d^2}{dt^2}X(t) - 3\frac{d}{dt}X(t) + 6X(t).$$

Find the autocorrelation function $R_Y(\tau)$.

$S_X(\omega) = \text{rect}\left(\frac{\omega}{2\pi}\right)$

$H(\omega) = \dfrac{3(j\omega)^2 - 3(j\omega) + 6}{2(j\omega)^2 + 2(j\omega) + 4} = \dfrac{3(2-\omega^2 - j\omega)}{2(2-\omega^2 + j\omega)}$

$|H(\omega)| = \frac{3}{2}$

$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \frac{9}{4}\,\text{rect}\left(\frac{\omega}{2\pi}\right)$

$\rightarrow R_Y(\tau) = \frac{9}{4}\,\text{sinc}(\pi\tau)$

Exercise 9.
Let $X(t)$ be a WSS process with correlation function

$$R_X(\tau) = \begin{cases} 1 - |\tau|, & \text{if } -1 \le \tau \le 1 \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

It is known that when $X(t)$ is input to a system with transfer function $H(\omega)$, the system output $Y(t)$ has a correlation function

$$R_Y(\tau) = \frac{\sin \pi\tau}{\pi\tau}. \quad (2)$$

Find the transfer function $H(\omega)$.

$$S_X(\omega) = \text{sinc}^2\left(\frac{\omega}{2}\right)$$

$$S_Y(\omega) = \text{rect}\left(\frac{\omega}{2\pi}\right)$$

$$|H(\omega)|^2 = \frac{S_Y(\omega)}{S_X(\omega)} = \frac{1}{\text{sinc}^2\left(\frac{\omega}{2}\right)} \cdot \mathbb{1}_{[-\pi,\pi]}(\omega)$$

$$H(\omega) = \frac{1}{\text{sinc}\left(\frac{\omega}{2}\right)} \cdot \mathbb{1}_{[-\pi,\pi]}(\omega)$$

$$H(\omega) = \frac{\omega}{2\sin\left(\frac{\omega}{2}\right)} \cdot \mathbb{1}_{[-\pi,\pi]}(\omega)$$