PURDUE UNIVERSITY

Homework 3

Fall 2020
(Due: September 28, 2020, Monday)

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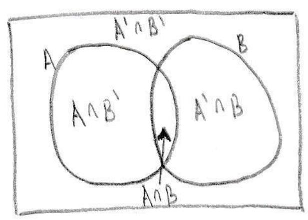
Homework is due at 11:59pm (midnight) Eastern Time. Please print this homework, write your solution, and scan the solution. Submit your homework through Gradescope. No late homework will be accepted.

Exercise 1.
Show that if A and B are independent events, then the pairs A and B^c, A^c and B, and A^c and B^c are also independent.

For A and B independent: $P(A \cap B) = P(A)P(B)$



1. $P(A \cap B') = P(A) - P(A \cap B)$
$= P(A) - P(A)(P(B)) = P(A)(1-P(B))$
$P(A \cap B') = P(A)P(B')$ ∴ independent

2. $P(A' \cap B) = P(B) - P(A \cap B)$
$= P(B) - P(A)(P(B)) = P(B)(1-P(A))$
$P(A' \cap B) = P(A')P(B)$ ∴ independent

3. $P(A' \cap B') = 1 - P(A \cup B)$
$= 1 - P(A) - P(B) + P(A \cap B)$
$= 1 - P(A) - P(B) + P(A)P(B)$
$= (1-P(A))(1-P(B))$
$P(A' \cap B') = P(A')P(B')$ ∴ independent

Exercise 2.

A binary communication system transmits a signal X that is either a +2 voltage signal or a -2 voltage signal. A malicious channel reduces the magnitude of the received signal by the number of heads it counts in two tosses of a coin. Let Y be the resulting signal. Possible values of Y are listed below.

	2 Heads	1 Head	No Head
$X = -2$	$Y = 0$	$Y = -1$	$Y = -2$
$X = +2$	$Y = 0$	$Y = +1$	$Y = +2$

Assume that the probability of having $X = +2$ and $X = -2$ is equal.

(a) Find the sample space of Y, and hence the probability of each value of Y.

(b) What are the probabilities $\mathbb{P}[X = +2 \,|\, Y = 1]$ and $\mathbb{P}[Y = 1 \,|\, X = -2]$?

A) $\Omega(Y) = \{-2, -1, 0, 1, 2\}$

$P(Y=-2) = 1/6$ $\quad P(Y=-1) = 1/6$ $\quad P(Y=0) = 1/3$

$P(Y=1) = 1/6$ $\quad P(Y=2) = 1/6$

B) $P[X=2 \,|\, Y=1] = 1$ (if $Y=1$, X must be 2)

$P[Y=1 \,|\, X=-2] = 0$ (if $X=2$, Y cannot be 1)

Exercise 3.
A computer manufacturer uses chips from three sources. Chips from source A, B and C are defective with probabilities 0.01, 0.003 and 0.008, respectively. The proportions of chips from A, B, C are 0.2, 0.3, 0.5 respectively. If a randomly selected chip is found to be defective, find

(a) the probability that the chips are from A

(b) the probability that the chips are from B

(c) the probability that the chips are from C

A) $P(A \mid \text{defective}) = \dfrac{P(\text{defective} \mid A)\,P(A)}{P(\text{defective})}$

$P(\text{defective}) = P(\text{def} \mid A)P(A) + P(\text{def} \mid B)P(B) + P(\text{def} \mid C)P(C)$

$P(\text{defective}) = 0.01(0.2) + 0.003(0.3) + 0.008(0.5) = 0.0069$

$P(A \mid \text{defective}) = \dfrac{0.01(0.2)}{0.0069} = 0.2899$

B) $P(B \mid \text{defective}) = \dfrac{P(\text{defective} \mid B)\,P(B)}{P(\text{defective}} = \dfrac{0.003(0.3)}{0.0069}$

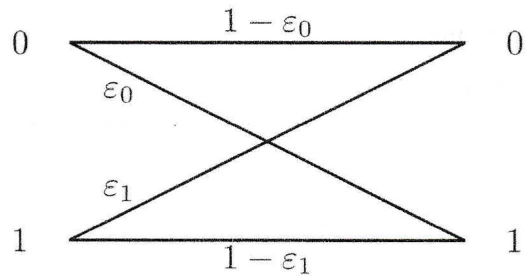
$P(B \mid \text{defective}) = 0.1304$

C) $P(C \mid \text{defective}) = \dfrac{P(\text{defective} \mid C)\,P(C)}{P(\text{defective})} = \dfrac{0.008(0.5)}{0.0069}$

$P(C \mid \text{defective}) = 0.5797$

Exercise 4.

Consider the following communication channel. A source transmits a string of binary symbols through a noisy communication channel. Each symbol is 0 or 1 with probability p and $1-p$, respectively, and is received incorrectly with probability ε_0 and ε_1, respectively. Errors in different symbols transmissions are independent.



Denote S as the source and R as the receiver.

(a) What is the probability that a symbol is correctly received? Hint: Find $\mathbb{P}[R=1 \cap S=1]$ and $\mathbb{P}[R=0 \cap S=0]$.

(b) Find the probability of receiving 0110 conditioned on that 0110 was sent, i.e., $\mathbb{P}[R=0110 \mid S=0110]$.

(c) In an effort to improve reliability, each symbol is transmitted three times and the received string is decoded by majority rule. In other words, a 0 (or 1) is transmitted as 000 (or 111, respectively), and it is decoded at the receiver as a 0 (or 1) if and only if the received three-symbol string contains at least two 0s (or 1s, respectively). What is the probability that the symbol is correctly decoded, given that we send a 0?

(d) Suppose that the scheme of part (c) is used. What is the probability that a 0 was sent conditioned on that the string 100 was received?

(e) Suppose the scheme of part (c) is used, and given that a 0 was sent. For what value of ε_0 is there an improvement in the probability of correct decoding? Assume that $\varepsilon_0 \neq 0$.

A) $P(\text{correct}) = \mathbb{P}[R=1 \cap S=1] + \mathbb{P}[R=0 \cap S=0] = (1-p)(1-\varepsilon_1) + p(1-\varepsilon_0)$

$P(\text{correct}) = (1-p)(1-\varepsilon_1) + p(1-\varepsilon_0)$

B) $P(\text{all correct}) = \mathbb{P}(0 \text{ correct})^2\, \mathbb{P}(1 \text{ correct})^2$

$P(\text{all correct}) = (1-\varepsilon_0)^2(1-\varepsilon_1)^2$

C) $\mathbb{P}(2/3 \text{ or } 3/3 \text{ correct}) = \mathbb{P}(000) + \mathbb{P}(001) + \mathbb{P}(010) + \mathbb{P}(100)$

$= \mathbb{P}(3 \text{ correct}) + 3\,\mathbb{P}(2 \text{ correct})$

$= (1-\varepsilon_0)^3 + 3\varepsilon_0(1-\varepsilon_0)^2$

D) $\mathbb{P}(0 \text{ sent} \mid 100 \text{ received}) = \dfrac{\mathbb{P}(100 \text{ received} \mid 0 \text{ sent})\,\mathbb{P}(0 \text{ sent})}{\mathbb{P}(100 \text{ received} \mid 0 \text{ sent})\,\mathbb{P}(0 \text{ sent}) \ldots \ldots + \mathbb{P}(100 \text{ received} \mid 1 \text{ sent})\,\mathbb{P}(1 \text{ sent})}$

$$= \frac{p(\varepsilon_0(1-\varepsilon_0)^2)}{p(\varepsilon_0(1-\varepsilon_0)^2) + (1-p)(\varepsilon_1^2(1-\varepsilon_1))}$$

E) $(1-\varepsilon_0)^3 + 3\varepsilon_0(1-\varepsilon_0)^2 > 1-\varepsilon_0$

$2\varepsilon_0^3 - 3\varepsilon_0^2 + 1 > 1 - \varepsilon_0$

$2\varepsilon_0^3 - 3\varepsilon_0^2 + \varepsilon_0 > 0$

$\varepsilon_0(2\varepsilon_0 - 1)(\varepsilon_0 - 1) > 0$

Improvement for $0 < \varepsilon_0 < 1/2$

Exercise 5.
Two dice are tossed. Let X be the absolute difference in the number of dots facing up.

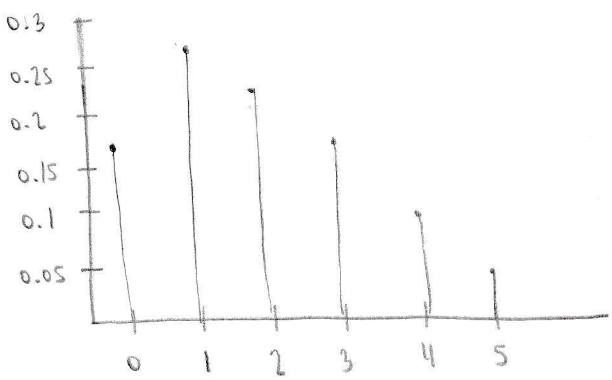
(a) Find and plot the PMF of X.

(b) Find the probability that $X \leq 2$.

(c) Find $\mathbb{E}[X]$ and $\text{Var}[X]$.

A) $\mathbb{P}_X(0) = 6/36$, $\mathbb{P}_X(1) = 10/36$, $\mathbb{P}_X(2) = 8/36$,
$\mathbb{P}_X(3) = 6/36$, $\mathbb{P}_X(4) = 4/36$, $\mathbb{P}_X(5) = 2/36$

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0



B) $\mathbb{P}[X \leq 2] = \mathbb{P}_X(0) + \mathbb{P}_X(1) + \mathbb{P}_X(2) = \dfrac{6+10+8}{36}$

$\mathbb{P}[X \leq 2] = 2/3$

C) $\mathbb{E}[X] = 0(6/36) + 1(10/36) + 2(8/36) + 3(6/36) + 4(4/36) + 5(2/36)$

$\mathbb{E}[X] = 35/18$

$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$, $\mathbb{E}[X^2] = 1(10/36) + 4(8/36) + 9(6/36) + 16(4/36) + 25(2/36)$
$= 35/6$

$\text{Var}[X] = 35/6 - (35/18)^2 \rightarrow \text{Var}[X] = 665/324$

Exercise 6.
Let X be a random variable with PMF $p_k = c/2^k$ for $k = 1, 2, \ldots$.

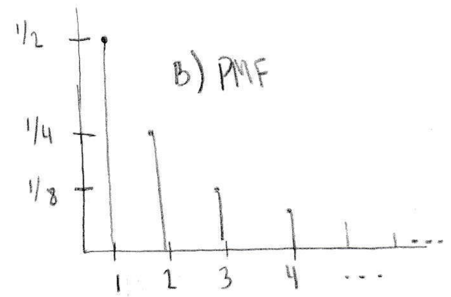
(a) Determine the value of c.

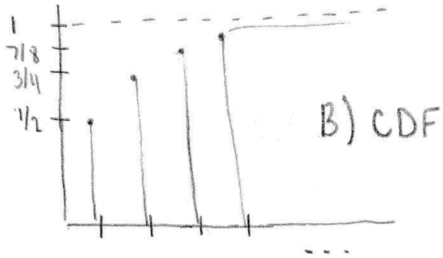
(b) Plot the PMF and the CDF.

(c) Find $\mathbb{P}(X > 3)$ and $\mathbb{P}(1 \le X \le 4)$.

(d) Find $\mathbb{E}[X]$ and $\text{Var}[X]$.

A) $\sum_{k=1}^{\infty} \frac{c}{2^k} = 1$, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$ $\therefore$ $c = 1$





C) $\mathbb{P}(X > 3) = 1 - \mathbb{P}(X \le 3) = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} \rightarrow \mathbb{P}(X > 3) = 1/8$

$\mathbb{P}(1 \le X \le 4) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \rightarrow \mathbb{P}(1 \le X \le 4) = 15/16$

D) $\mathbb{E} = \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k = \frac{1}{2} \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^{k-1} = \frac{1}{2}\left(\frac{1}{(1-\frac{1}{2})^2}\right) = \frac{1}{2}(4) \rightarrow \mathbb{E}[X] = 2$

$\mathbb{E}[X^2] = \sum_{k=1}^{\infty} k^2 \left(\frac{1}{2}\right)^k = \frac{1}{4} \sum_{k=1}^{\infty} k^2 \left(\frac{1}{2}\right)^{k-2} = \frac{1}{4}\left[\sum_{k=1}^{\infty} k(k-1)\left(\frac{1}{2}\right)^{k-2} + \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^{k-2}\right]$

$\mathbb{E}[X^2] = \frac{1}{4} \frac{d^2 \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k}{dt^2} + \frac{1}{2} \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^{k-1} = \frac{1}{4}\left(\frac{2}{(1-1/2)^3}\right) + 2 = 4 + 2 = 6$

$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 6 - 2^2 \rightarrow \text{Var}[X] = 2$