Exercise 8.

A block of information is transmitted repeated over a noisy channel until an error-free block is received. Let $M \geq 1$ be the number of blocks required for a transmission. Define the following sets

(i) $A = \{M \text{ is even}\}$

(ii) $B = \{M \text{ is a multiple of } 5\}$

(iii) $C = \{M \text{ is less than or equal to } 7\}$

Assume that the probability of requiring one additional block is half of the probability without the additional block. That is:

$$\mathbb{P}[M = k] = \left(\frac{1}{2}\right)^k, \qquad k = 1, 2, \ldots.$$

Determine the following probabilities

(a) $\mathbb{P}[A]$, $\mathbb{P}[B]$, $\mathbb{P}[C]$, $\mathbb{P}[C^c]$

(b) $\mathbb{P}[\overset{10,20}{A \cap B}]$, $\mathbb{P}[\overset{2,4,6,8,n}{A \setminus B}]$, $\mathbb{P}[A \cap B \cap C]$

(c) $\mathbb{P}[A \mid B]$, $\mathbb{P}[B \mid A]$

(d) $\mathbb{P}[A \mid B \cap C]$, $\mathbb{P}[A \cap B \mid C]$

Hint: $\mathbb{P}[A] \neq \frac{1}{2}$. $\mathbb{P}[A] = \sum_{k=\text{even}} \mathbb{P}[M = k]$.

$\mathbb{P}[A] = (\frac{1}{2})^2 + (\frac{1}{2})^4 + \ldots (\frac{1}{2})^{2n}$

$\mathbb{P}(B) = (\frac{1}{2})^5 + (\frac{1}{2})^{10} + \ldots (\frac{1}{2})^{5n}$

$\mathbb{P}[C] = (\frac{1}{2})^1 + (\frac{1}{2})^2 + \ldots (\frac{1}{2})^7$

A. $\mathbb{P}[A] = \sum_{n=1}^{\infty} (\frac{1}{2})^{2n}$, $\mathbb{P}[B] = \sum_{n=1}^{\infty} (\frac{1}{2})^{5n}$, $\mathbb{P}[C] = \sum_{n=1}^{7} (\frac{1}{2})^n$, $\mathbb{P}[C^c] = \sum_{n=8}^{\infty} (\frac{1}{2})^n$

B. $\mathbb{P}[A \cap B] = \sum_{n=1}^{\infty} (\frac{1}{2})^{10n}$, $\mathbb{P}[A \setminus B] =$, $\mathbb{P}[A \cap B \cap C] = 0$

C. $\mathbb{P}[A \mid B] = \dfrac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \dfrac{\sum_{n=1}^{\infty} (\frac{1}{2})^{10n}}{\sum_{n=1}^{\infty} (\frac{1}{2})^{2n}} = \sum_{n=1}^{\infty} (\frac{1}{2})^{8n}$, $\mathbb{P}[B \mid A] = \sum_{n=1}^{\infty} (\frac{1}{2})^{5n}$

D. $\mathbb{P}[A \mid B \cap C] = \dfrac{\mathbb{P}[A \cap B \cap C]}{\mathbb{P}[B \cap C]} = 0$, $\mathbb{P}[A \cap B \mid C] = \dfrac{\mathbb{P}[A \cap B \cap C]}{\mathbb{P}[C]} = 0$