TDT4136 - Artificial Intelligence Methods
Assignment 2 - Probabilistic Reasoning over Time

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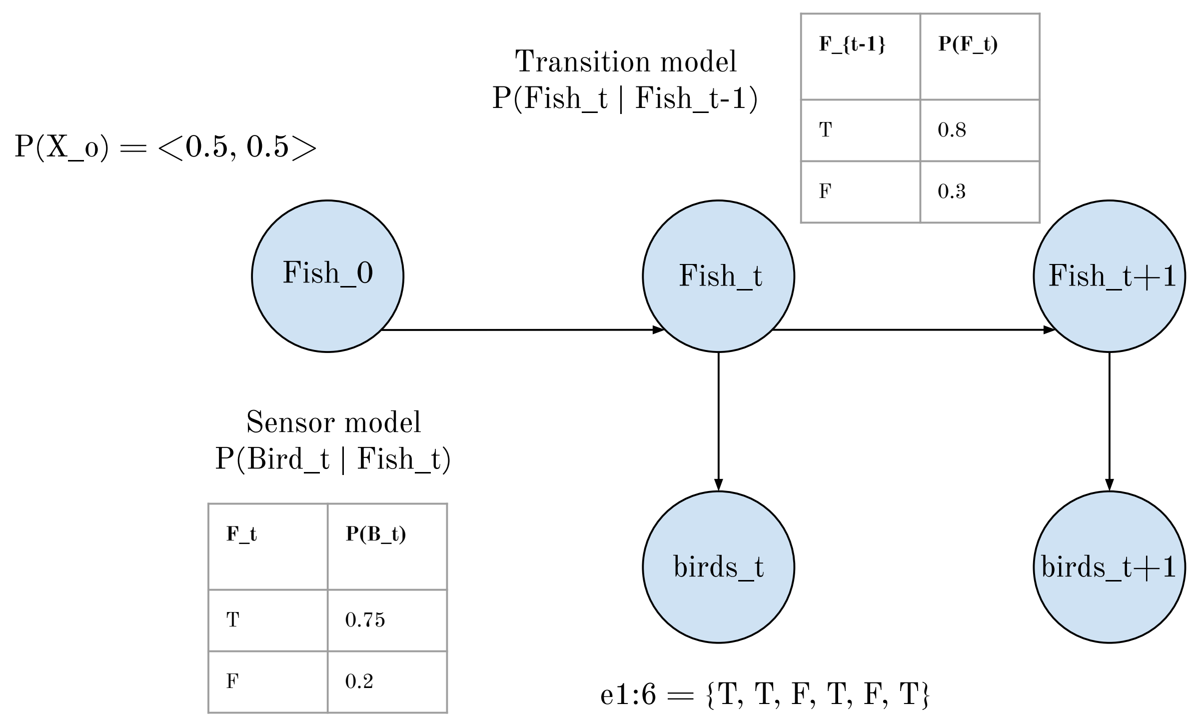
Problem 1

1a

Hidden Markov Model with probability tables:



1b

The following operation for this problem is *filtering* which provides us with an updated probability of fish nearby given the observations of birds for each of the previous days.

Output from code implementation:

```
/Users/eliassovikgunnarsson/TDT4171/assignment_2/hidden_markov.py
Problem 1b:

P(X_1|e_1:1):
[[0.82089552]
 [0.17910448]]

P(X_2|e_1:2):
[[0.90197069]
 [0.09802931]]

P(X_3|e_1:3):
[[0.48518523]
 [0.51481477]]

P(X_4|e_1:4):
[[0.81645924]
 [0.18354076]]

P(X_5|e_1:5):
[[0.43134895]
 [0.56865105]]

P(X_6|e_1:6):
[[0.79970863]
 [0.20029137]]
```

1c

The following operation for this problem is *prediction* which provides us a predicted probability of fish nearby in the future, given the observations of birds until present time.

Output from code implementation:

```
/Users/eliassovikgunnarsson/TDT4171/assignment_2/hidden_markov.py
Problem 1c:

P(X_7|e_1:6):
[[0.69985432]
 [0.30014568]]

P(X_8|e_1:6):
[[0.64992716]
 [0.35007284]]

P(X_9|e_1:6):
[[0.62496358]
 [0.37503642]]

P(X_10|e_1:6):
[[0.61248179]
 [0.38751821]]
```

```
P(X_11|e_1:6):
[[0.60624089]
 [0.39375911]]

P(X_12|e_1:6):
[[0.60312045]
 [0.39687955]]

P(X_13|e_1:6):
[[0.60156022]
 [0.39843978]]

P(X_14|e_1:6):
[[0.60078011]
 [0.39921989]]

P(X_15|e_1:6):
[[0.60039006]
 [0.39960994]]

P(X_16|e_1:6):
[[0.60019503]
 [0.39980497]]

P(X_17|e_1:6):
[[0.60009751]
 [0.39990249]]

P(X_18|e_1:6):
[[0.60004876]
 [0.39995124]]

P(X_19|e_1:6):
[[0.60002438]
 [0.39997562]]

P(X_20|e_1:6):
[[0.60001219]
 [0.39998781]]

P(X_21|e_1:6):
[[0.60000609]
 [0.39999391]]

P(X_22|e_1:6):
[[0.60000305]
 [0.39999695]]

P(X_23|e_1:6):
[[0.60000152]
 [0.39999848]]

P(X_24|e_1:6):
[[0.60000076]
```

```
 [0.39999924]]

P(X_25|e_1:6):
[[0.60000038]
 [0.39999962]]

P(X_26|e_1:6):
[[0.60000019]
 [0.39999981]]

P(X_27|e_1:6):
[[0.6000001]
 [0.3999999]]

P(X_28|e_1:6):
[[0.60000005]
 [0.39999995]]

P(X_29|e_1:6):
[[0.60000002]
 [0.39999998]]

P(X_30|e_1:6):
[[0.60000001]
 [0.39999999]]
```

1d

The following operation for this problem is *smoothing* which provides us an updated probability of fish nearby in earlier days by taking account for the probability of fish nearby the following days.

Output from code implementation:

```
/Users/eliassovikgunnarsson/TDT4171/assignment_2/hidden_markov.py
Problem 1d:

P(X_0|e1:6)
[[0.66485218]
 [0.33514782]]

P(X_1|e1:6)
[[0.88699926]
 [0.11300074]]

P(X_2|e1:6)
[[0.86984716]
 [0.13015284]]

P(X_3|e1:6)
[[0.59991343]
 [0.40008657]]
```

```
P(X_4|e1:6)
[[0.76738105]
 [0.23261895]]

P(X_5|e1:6)
[[0.57119509]
 [0.42880491]]
```

1e

the following operation for this problem *most likely sequence* of hidden states i.e. the most probable path of states which in this case, observing *fish nearby* every day out of the six days is the most probable path.
Output from code implementation:

```
/Users/eliassovikgunnarsson/TDT4171/assignment_2/hidden_markov.py
Problem 1e:
Most likely hidden state sequence:
 [True, True, True, True, True, True]
```

Problem 2

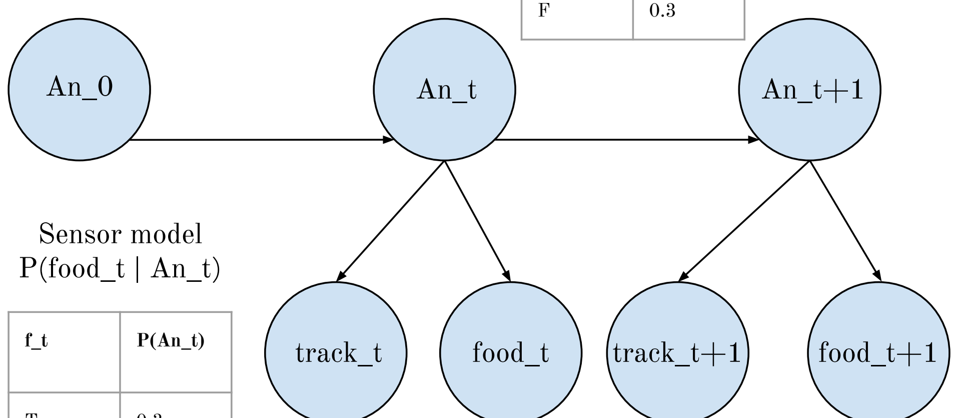
2a

Dynamic Bayesian Model with probability tables:

Transition model
P(An_t | An_t-1)

A_{t-1}	P(An_t)
T	0.8
F	0.3

P(X_o) = <0,7, 0.3>



Sensor model
P(track_t | An_t)

t_t	P(An_t)
T	0.7
F	0.2

Sensor model
P(food_t | An_t)

f_t	P(An_t)
T	0.3
F	0.1

e1:4 = {{T, T}, {F, T}, {F, F}, {T, F}}(track, food)

2b

$\boxed{\text{Compute } P(X_t \mid e_{1:t}) \text{ for } t = 1,2,3,4}$

$$P(X_1) = \sum_{x_0} P(X_1 \mid x_0) \cdot P(x_0)$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.7 + \langle 0.3, 0.7 \rangle \cdot 0.3$$

$$= \langle 0.65, 0.35 \rangle$$

$$P(X_1 \mid e_{1:1}) = P(X_1 \mid t_1, f_1) \qquad (e_{t_1} = T,\ e_{f_1} = T)$$

$$= \alpha\ P(t_1 \mid X_1) \cdot P(f_1 \mid X_1) \cdot P(X_1)$$

$$= \alpha \langle 0.65 \cdot 0.7 \cdot 0.3,\ 0.35 \cdot 0.2 \cdot 0.1 \rangle$$

$$= \alpha \langle 0.1365,\ 0.007 \rangle = \underline{\underline{\langle 0.951,\ 0.049 \rangle}}$$

$$P(X_2) = \sum_{x_1} P(X_2 \mid x_1) \cdot P(x_1)$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.951 + \langle 0.3, 0.7 \rangle \cdot 0.049$$

$$= \langle 0.776, 0.224 \rangle$$

$$P(X_2 \mid e_{1:2}) = P(X_2 \mid t_2, f_2) \qquad (e_{t_2} = F,\ e_{f_2} = T)$$

$$= \alpha\ P(t_2 \mid X_2) \cdot P(f_2 \mid X_2) \cdot P(X_2)$$

$$= \alpha \langle 0.776 \cdot 0.3 \cdot 0.3,\ 0.224 \cdot 0.8 \cdot 0.1 \rangle$$

$$= \alpha \langle 0.06984,\ 0.01792 \rangle = \underline{\underline{\langle 0.795,\ 0.205 \rangle}}$$

$$P(X_3) = \sum_{x_2} P(X_3 \mid x_2) \cdot P(x_1)$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.795 + \langle 0.3, 0.7 \rangle \cdot 0.205$$

$$= \langle 0.698, 0.302 \rangle$$

$$P(X_3 \mid e_{1:3}) = P(X_3 \mid t_3, f_3) \qquad (E_{t_3} = F,\ E_{f_3} = F)$$

$$= \alpha\ P(t_3 \mid X_3) \cdot P(f_3 \mid X_3) \cdot P(X_3)$$

$$= \alpha \langle 0.698 \cdot 0.3 \cdot 0.7,\ 0.302 \cdot 0.8 \cdot 0.9 \rangle$$

$$= \alpha \langle 0.14658, 0.21744 \rangle = \underline{\underline{\langle 0.402, 0.598 \rangle}}$$

$$P(X_4) = \sum_{x_3} P(X_4 \mid x_3) \cdot P(x_3)$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.402 + \langle 0.3, 0.7 \rangle \cdot 0.598$$

$$= \langle 0.501, 0.499 \rangle$$

$$P(X_4 \mid e_{1:4}) = P(X_4 \mid t_4, f_4) \qquad (E_{t4} = T, E_{f4} = F)$$

$$= \alpha P(t_4 \mid X_4) \cdot P(f_4 \mid X_4) \cdot P(X_4)$$

$$= \alpha \langle 0.7 \cdot 0.7 \cdot 0.501, 0.2 \cdot 0.9 \cdot 0.499 \rangle$$

$$= \alpha \langle 0.24549, 0.08982 \rangle = \underline{\underline{\langle 0.732, 0.268 \rangle}}$$

2c

$$\boxed{\text{Compute} \quad P(X_t | e_{1:4}) \text{ for } t = 5, 6, 7, 8}$$

$$P(X_5 | e_{1:4}) = \sum_{x_4} P(X_5 | x_4) \cdot P(x_5 | e_{1:4})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.732 + \langle 0.3, 0.7 \rangle \cdot 0.268$$

$$= \underline{\underline{\langle 0.666, 0.334 \rangle}}$$

$$P(X_6 | e_{1:4}) = \sum_{x_5} P(X_6 | x_5) \cdot P(x_6 | e_{1:4})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.666 + \langle 0.3, 0.7 \rangle \cdot 0.334$$

$$= \underline{\underline{\langle 0.633, 0.367 \rangle}}$$

$$P(X_7 | e_{1:4}) = \sum_{x_6} P(X_7 | x_6) \cdot P(X_7 | e_{1:4})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.633 + \langle 0.3, 0.7 \rangle \cdot 0.367$$

$$= \underline{\underline{\langle 0.617, 0.383 \rangle}}$$

$$P(X_8 | e_{1:4}) = \sum_{x_7} P(X_8 | x_7) \cdot P(X_8 | e_{1:4})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.617 + \langle 0.3, 0.7 \rangle \cdot 0.383$$

$$= \underline{\underline{\langle 0.608, 0.392 \rangle}}$$

2d

$\boxed{\text{Verify that } \lim_{t \to \infty} P(X_t | e_{1:4}) = \langle 0.6, 0.4 \rangle}$

We have that $P(X_t | e_{1:4}) = \langle 0.6, 0.4 \rangle$

if we then calculate

$$P(X_{t+1} | e_{1:4}) = \sum_{x_t} P(X_{t+1} | x_t) \cdot P(X_{t+1} | e_{1:4})$$

$$= \langle 0.8, 0.2 \rangle \cdot 0.6 + \langle 0.3, 0.7 \rangle \cdot 0.4$$

$$= \langle 0.6, 0.4 \rangle$$

thus the probability has converged q.e.d.

2e

Compute $P(X_t \mid e_{1:4})$ for $t = 0, 1, 2, 3$

$b_{5:4} = \langle 1, 1 \rangle$

$b_{4:4} = P(e_{4:4} \mid X_3)$

$= \sum_{x_4} P(e_4 \mid x_4) \cdot b_{5:4}(x_4) \cdot P(x_4 \mid X_3)$

$= \sum_{x_4} P(t_4 \mid x_4) \cdot (f_4 \mid x_4)\ b_{5:4}(x_4) \cdot P(x_4 \mid X_3)$ (T, F)

$= 0.7 \cdot 0.7 \cdot 1 \langle 0.8, 0.2 \rangle + 0.2 \cdot 0.9 \langle 0.3, 0.7 \rangle$

$= \langle 0.446, 0.224 \rangle$

$b_{3:4} = P(e_{3:4} \mid X_2)$

$= \sum_{x_3} P(t_3 \mid x_3) \cdot P(f_3 \mid x_3) \cdot b_{4:4}(x_3) \cdot P(x_3 \mid X_2)$ (F, F)

$= 0.3 \cdot 0.7 \cdot 0.446 \langle 0.8, 0.2 \rangle + 0.8 \cdot 0.9 \cdot 0.224 \langle 0.3, 0.7 \rangle$

$= \langle 0.123, 0.132 \rangle$

$b_{2:4} = P(e_{2:4} \mid X_1)$

$= \sum_{x_2} P(t_2 \mid x_2) \cdot P(f_2 \mid x_2) \cdot b_{3:4} \cdot P(x_2 \mid X_1)$ (F, T)

$= 0.3 \cdot 0.3 \cdot 0.123 \langle 0.8, 0.2 \rangle + 0.8 \cdot 0.1 \cdot 0.132 \langle 0.3, 0.7 \rangle$

$= \langle 0.012, 0.0096 \rangle$

$b_{1:4} = P(e_{1:4} \mid X_0)$

$= \sum_{x_1} P(t_1 \mid x_1) \cdot P(f_1 \mid x_1) \cdot b_{2:4} \cdot P(x_1 \mid X_0)$ (T, T)

$= 0.7 \cdot 0.3 \cdot 0.0096 \langle 0.8, 0.2 \rangle \cdot 0.2 \cdot 0.1 \cdot 0.0096 \langle 0.3, 0.7 \rangle$

$= \langle 0.0017, 0.0005 \rangle$

$$P(X_0 | e_{1:4}) = \alpha\, f \cdot b_{1:4}$$

$$= \alpha \langle 0.7, 0.3 \rangle \langle 0.0017, 0.0005 \rangle$$

$$= \alpha \langle 0.0019, 0.00015 \rangle = \underline{\underline{\langle 0.937, 0.063 \rangle}}$$

$$P(X_1 | e_{1:4}) = \alpha\, f_{1:1} \cdot b_{2:4}$$

$$= \alpha \langle 0.951, 0.049 \rangle \cdot \langle 0.012, 0.0096 \rangle$$

$$= \alpha \langle 0.11412, 0.00047 \rangle$$

$$= \underline{\underline{\langle 0.995, 0.005 \rangle}}$$

$$P(X_2 | e_{1:4}) = \alpha\, f_{1:2} \cdot b_{3:4}$$

$$= \alpha \langle 0.795, 0.205 \rangle \cdot \langle 0.123, 0.132 \rangle$$

$$= \alpha \langle 0.978, 0.027 \rangle$$

$$= \underline{\underline{\langle 0.973, 0.027 \rangle}}$$

$$P(X_3 | e_{1:4}) = \alpha\, f_{1:3} \cdot b_{4:4}$$

$$= \alpha \langle 0.402, 0.598 \rangle \langle 0.446, 0.224 \rangle$$

$$= \alpha \langle 0.179, 0.134 \rangle$$

$$= \underline{\underline{\langle 0.572, 0.428 \rangle}}$$