# Løsningsforslag Eksamen i Statistikk Aug 2000

**Oppgave 1)**

**a)**

$f(x) = F'(x) = \frac{x}{\alpha} exp(-\frac{x^2}{2\alpha})$

$f'(x) = \frac{1}{\alpha} exp(-\frac{x^2}{2\alpha}) + \frac{x}{\alpha}(-\frac{x}{\alpha}) exp(-\frac{x^2}{2\alpha}) = (\frac{1}{\alpha} - \frac{x^2}{\alpha^2}) exp(-\frac{x^2}{2\alpha})$
Setter deriverte lik null, og løser ut mhp $\alpha$.
$\frac{1}{\alpha} - \frac{x^2}{\alpha^2} = 0 \qquad x = \sqrt{\alpha}$

**b)**

Hendelsen $D$ er gitt ved at $A$ og minst en av $B$ eller $C$ fungerer. Dette betyr $D = A \cap (B \cup C)$. Denne delmengden kan deles i tre biter som vi kan finne sannsynligheten for.

$p(D) = p(A \cap B) + p(A \cap C) - p(A \cap B \cap C) = p(A)p(B) + p(A)p(C) - p(A)p(B)p(C)$

Vi har: $p(A) = p(B) = p(C) = 1 - F(2) = exp(-\frac{2^2}{2 \cdot 1}) = 0.135$.

$p(D) = 0.135^2 + 0.135^2 - 0.135^3 = 0.034$

**Oppgave 2)**
$E(\hat{\mu}) = \mu$
$Var(\hat{\mu}) = \frac{\tau_0^4 Var(X) + \sigma_0^4 Var(Y)}{(\tau_0^2 + \sigma_0^2)^2} = \frac{\tau_0^4 \sigma_0^2 + \sigma_0^4 \tau_0^2}{(\tau_0^2 + \sigma_0^2)^2} = \frac{\sigma_0^2 \tau_0^2}{\tau_0^2 + \sigma_0^2}$

$\hat{\mu}$ er en lineærkombinasjon av normalfordelte variable, og er dermed normalfordelt.
$Z = \frac{\hat{\mu} - \mu}{\sqrt{\frac{\sigma_0^2 \tau_0^2}{\tau_0^2 + \sigma_0^2}}} \sim N(0,1)$

$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$
$P(\hat{\mu} - z_{\alpha/2}\sqrt{\frac{\sigma_0^2 \tau_0^2}{\tau_0^2 + \sigma_0^2}} < \mu < \hat{\mu} + z_{\alpha/2}\sqrt{\frac{\sigma_0^2 \tau_0^2}{\tau_0^2 + \sigma_0^2}}) = 1 - \alpha$

Ett $1 - \alpha$ konfidensintervall for $\mu$ er da :
$(\hat{\mu} - z_{\alpha/2}\sqrt{\frac{\sigma_0^2 \tau_0^2}{\tau_0^2 + \sigma_0^2}}, \hat{\mu} + z_{\alpha/2}\sqrt{\frac{\sigma_0^2 \tau_0^2}{\tau_0^2 + \sigma_0^2}})$

**Oppgave 3)**
$E(X) = \lambda \nu$

**a)**

$\lambda$ : forventet (gjennonsnitlig) antall bakterier pr. liter vann.

$P(X=0) = \frac{(\lambda\nu)^0}{0!}e^{-\lambda\nu} = e^{-3(0.5)} = \underline{0.223}$

$P(X>3) = 1 - P(X \leq 3) = 1 - \sum_{x=0}^{3} \frac{(1.5)^x}{x!}e^{-1.5} = 1 - 0.934 = \underline{0.066}$

**b)**

$P(X > 3 \mid x > 0) = \frac{P(X>3 \cap x>0)}{P(X>0)} = \frac{P(X>3)}{1-P(X=0)} = \frac{0.066}{1-0.223} = \underline{0.0849}$

$Y = X_1 + X_2 \sim Po(\lambda\nu_1 + \lambda\nu_2)$; med $\lambda\nu_1 + \lambda\nu_2 = 3(3) = 9$

$P(X_1 + X_2 > 3) = 1 - P(X_1 + X_2 \leq 3) = 1 - \sum_{x=0}^{3} \frac{9^x}{x!}e^{-9} = \underline{0.97877}$

**c)**

$P(X_1 + X_2 > 3 \mid X_1 > 0 \cap X_2 > 0) = 1 - P(X_1 + X_2 \leq 3 \mid X_1 > 0 \cap X_2 > 0)$
$= 1 - \frac{P(X_1+X_2 \leq 3 \cap X_1>0 \cap X_2>0)}{P(X_1>0 \cap X_2>0)}$
$= 1 - \frac{P(X_1=1 \cap X_2=1)+P(X_1=1 \cap X_2=2)+P(X_1=2 \cap X_2=1)}{P(X_1>0).P(X_2>0)}$
$= 1 - \frac{P(X_1=1)P(X_2=1)+P(X_1=1)P(X_2=2)+P(X_1=2)P(X_2=1)}{(1-P(X_1=0))(1-P(X_2=0))}$
$= 1 - \frac{(3.6+3.6^2/2+3^2.6/2).e^{-3}.e^{-6}}{(1-e^{-3})(1-e^{-6})}$
$= \underline{0.987}$

Og

$L(\lambda) = \pi_{i=1}^{n} \frac{(\lambda\nu_i)^{x_i}}{x_i!}e^{-\lambda\nu_i}$

$l(\lambda) = \sum_{i=1}^{n} (x_i ln(\lambda\nu_i) + ln(x_i!) - \lambda\nu_i)$

$l'(\lambda) = \sum_{i=1}^{n} \left(x_i \frac{\nu_i}{\lambda\nu_i} + 0 - \nu_i\right) = \frac{1}{\lambda}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \nu_i$

$l'(\lambda) = 0 \Rightarrow \lambda = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} \nu_i}$
dvs. $\hat{\lambda} = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} \nu_i}$

$$E(\hat{\lambda}) = \frac{E\left(\sum_{i=1}^n X_i\right)}{\sum_{i=1}^n \nu_i} = \frac{\sum_{i=1}^n E(X_i)}{\sum_{i=1}^n \nu_i} = \frac{\sum_{i=1}^n \lambda\nu_i}{\sum_{i=1}^n \nu_i} = \lambda\frac{\sum_{i=1}^n \nu_i}{\sum_{i=1}^n \nu_i} = \lambda$$

$$Var(\hat{\lambda}) = \frac{Var\left(\sum_{i=1}^n X_i\right)}{\left(\sum_{i=1}^n \nu_i\right)^2} = \frac{\sum_{i=1}^n \lambda\nu_i}{\left(\sum_{i=1}^n \nu_i\right)^2} = \lambda\frac{\sum_{i=1}^n \nu_i}{\left(\sum_{i=1}^n \nu_i\right)^2} = \frac{\lambda}{\sum_{i=1}^n \nu_i}$$

**d)**

$H_0 : \lambda = \lambda_0 = 3$ mot $H_1 : \lambda > 3$

Test obs.

$U = \frac{\hat{\lambda}-\lambda_0}{\sqrt{\frac{\lambda_0}{\sum_{i=1}^n \nu_i}}} \approx N(0,1)$ under $H_0$.

Forkaster dersom $U > k$ der $k$ bestemmer fra $P(U > k$ når $H_0$ er riktig $) = \alpha$

dvs. $k = z_\alpha$

dvs. Forkaster dersom $U > z_\alpha$.

Innsatt tall: $z_{0.025} = 1.96$, og $\hat{\lambda} = 78/(20) = 3.9$

$u = \frac{3.9-3}{\sqrt{3/20}} = 2.32 > 1.96$

dvs. Forkaster $H_0$.

**e)**

Under $H_0 : Z \sim \text{bin}(n = 10, p_0)$
der $p_0 = P(X > 6 \mid \lambda = \lambda_0) = 1 - P(X \leq 5 \mid \lambda = \lambda_0) = 1 - 0.4457 = \underline{0.5543}$

Forkaster $H_0$ hvis $Z \geq k$ der $k$ bestemmes fra kravet:
$P(Z \geq k$ hvis $H_0$ er riktig$) \leq 0.025$.

For ulike verdier for $k$,

| Z | $P(Z = z$ hvis $H_0$ er riktig) | $P(Z \geq k$ når $H_0$ er riktig) |
|---|---|---|
| 10 | 0.002738 | 0.02738 |
| 9 | 0.022016 | 0.024754 |
| 8 | 0.07966 | 0.104414 |

Ser at en må ha $\underline{k = 9}$.

Innsatt data: $z = 6 < k = 9$

dvs. Forkaster ikke $H_0$.

**f)**

La $\lambda = 3.5$ :

$$P(U > 1.96) = P\left(\frac{\hat{\lambda}-\lambda}{\sqrt{\frac{\lambda_0}{\sum \nu_i}}} > 1.96\right) = P(\hat{\lambda} > 1.96\sqrt{\frac{\lambda_0}{\sum \nu_i}} + \lambda_0)$$
$$= P\left(\frac{\hat{\lambda}-\lambda}{\sqrt{\frac{\lambda_0}{\sum \nu_i}}} > \frac{1.96\sqrt{\frac{\lambda_0}{\sum \nu_i}}+\lambda_0-\lambda}{\sqrt{\frac{\lambda}{\sum \nu_i}}}\right)$$
$$= P\left(\frac{\hat{\lambda}-\lambda}{\sqrt{\frac{\lambda_0}{\sum \nu_i}}} > 0.619\right) = 1 - \Phi(0.619) = 1 - 0.7324 = \underline{0.2676}.$$

$$P(Z \geq 9) = \sum_{z=9}^{10} \frac{10!}{z!(10-z)!} p^z (1-p)^{10-z}$$

der $p = P(X > 6 \mid \lambda = 3.5) = 1 - P(X \leq 5 \mid \lambda = 3.5) = 1 - 0.3007 = 0.6993$

$\Rightarrow P(Z \geq 9) = 0.02796 + 0.12025 = \underline{0.1482}$

**Oppgave 4)**

Definer $V = X - Y$

For $v \geq 0$:

$$
\begin{aligned}
F_V(v) &= P(V \leq v) = P(X - Y \leq v) \\
&= 1 - P(X - Y > v) \\
&= 1 - \int_v^\infty \int_0^{x-v} f_X(x) f_Y(y)\, dy\, dx = 1 - \int_v^\infty f_X(x) F_Y(x-v) dx \\
&= 1 - \int_v^\infty \lambda e^{-\lambda x}(1 - e^{-\lambda(x-v)}) dx \\
&= 1 - \int_v^\infty \lambda e^{-\lambda x} dx + \int_v^\infty \lambda e^{-2\lambda x} e^{\lambda v} dx \\
&= 1 - [-e^{-\lambda x}]_v^\infty + e^{\lambda v}[-\frac{1}{2} e^{-2\lambda x}]_v^\infty \\
&= 1 - (0 + e^{-\lambda v}) + e^{\lambda v}(0 + \frac{1}{2} e^{-2\lambda v}) \\
&= 1 - e^{-\lambda v} + \frac{1}{2} e^{-\lambda v} = 1 - \frac{1}{2} e^{-\lambda v}
\end{aligned}
$$

For $v \geq 0 : \; F_V(v) = 1 - \frac{1}{2} e^{-\lambda v}$

P.g.a. $X$ og $Y$ har samme fordeling må $f_V(v)$ være symmetrisk om 0, dvs.
For $v < 0$:

$$
\begin{aligned}
F_V(v) &= P(V \leq v) = 1 - P(V \geq v) \\
&= 1 - P(V \leq -v) = 1 - (1 - \frac{1}{2} e^{\lambda v}) = \frac{1}{2} e^{\lambda v}
\end{aligned}
$$

$Z = |V|$
$\underline{Z > 0}$:

$$
\begin{aligned}
F_Z(z) &= P(Z \leq z) = P(|V| \leq z) = 1 - P(|V| > z) \\
&= 1 - P(V > z \cup V < -z) \\
&= 1 - (P(V > z) + P(V < -z)) \\
&= 1 - (1 - F_V(z)) - F_V(-z) \\
&= F_V(z) - F_V(-z) \\
&= 1 - \frac{1}{2} e^{-\lambda z} - \frac{1}{2} e^{-\lambda z} = 1 - e^{-\lambda z}
\end{aligned}
$$

$$
\underline{\underline{f_Z(z) = \lambda e^{-\lambda z}}} \quad \text{dvs. eksponensial fordelt}
$$

Vet da at $\underline{\underline{\mathrm{E}[z] = \frac{1}{\lambda}}}$