LØSNINGSFORSLAG

Oppgave 1 Valg i kommune

$P(KBF) = 0.7$

$P(KSF) = 0.3$

Definer

$X \in \{KBF, KSF\} = \{0, 1\}$

Bernouilli forsøk:

$$X_1, \ldots, X_n \text{ uif } p(x) = \begin{cases} x = 0 & \text{prob } 0.7 \\ x = 1 & \text{prob } 0.3 \end{cases}$$

$$Y_n = \sum_{i=1}^{n} X_i \Rightarrow bin(y; n, 0.3) = \binom{n}{y} 0.3^y \, 0.7^{n-y}$$

a)

- $P(X_1 = 0, X_2 = 0, X_3 = 1) = P(X_1 = 0) P(X_2 = 0) P(X_3 = 1)$

$$= 0.7 \times 0.7 \times 0.3 = \underline{\underline{0.147}}$$

- KSF 40% ved $n = 5 \Rightarrow Y_5 = 2$

$$P(Y_5 = 2) = \binom{5}{2} 0.3^2 \, 0.7^3 = \underline{\underline{0.309}}$$

- KSF største parti $n = 9 \Rightarrow Y_9 \geq 5$

$$P(Y_9 \geq 5) = 1 - P(Y_9 \leq 4) = \underline{\underline{0.099}}$$

$P(FBF/KBF) = 0.80$ $\quad$ $P(FBF/KSF) = 0.10$

$P(FCL/KBF) = 0.15$ $\quad$ $P(FCL/KSF) = 0.20$

$P(FSF/KBF) = 0.05$ $\quad$ $P(FSF/KSF) = 0.70$

b)
$$P(KSF/FCL) = \frac{P(KSF, FCL)}{P(KSF, FCL) + P(KBF, FCL)}$$

$$= \frac{P(FCL/KSF)\,P(KSF)}{P(FCL/KSF)P(KSF) + P(FCL/KBF)\,P(KBF)}$$

$$= \frac{0.2 \times 0.3}{0.2 \times 0.3 + 0.15 \cdot 0.7} = \underline{\underline{0.364}}$$

$$P(KSF/FBF) = \cdots = \underline{0.051}$$

Definer: OFCL – oppgeri: stemt FCL

$P(FCL / OFCL) = 0.7$

$P(FBF / OFCL) = 0.3$

$P(FSF / OFCL) = 0.0$

$$P(KSF / OFCL) =$$

$$\begin{aligned}
= \; & P(KSF, FBF / OFCL) && = P(KSF / FBF, OFCL)\,P(FBF/OFCL) \\
+ \; & P(KSF, FCL / OFCL) && = P(KSF / FCL, OFCL)\,P(FCL/OFCL) \\
+ \; & P(KSF, FSF / OFCL) && = P(KSF / FSF, OFCL)\,P(FSF/OFCL)
\end{aligned}$$

$$= 0.051 \cdot 0.3 + 0.364 \cdot 0.7 + \cdot \cdot 0.0 = \underline{\underline{0.2701}}$$

c)

$$P(FBF) = P(FBF, KBF) + P(FBF, KSF)$$
$$= P(FBF|KBF)\,P(KBF) + P(FBF|KSF)\,P(KSF)$$
$$= 0.80 \quad 0.70 + 0.10 \quad 0.3$$
$$\underline{= 0.590}$$

$$P(FCL) = 0.15 \quad 0.70 + 0.20 \quad 0.3$$
$$\underline{= 0.165}$$

$$P(FSF) = 0.05 \quad 0.70 + 0.70 \quad 0.3$$
$$\underline{= 0.245}$$

Definer:

$X = 10 \cdot$ andel FBF $\Rightarrow$ $p_x = 0.590$

$Y = 10 \cdot$ andel FCL $\quad p_y = 0.165$

$Z = 10 \cdot$ andel FSF $\quad p_z = 0.245$

$$f(x,y,z) = \text{Multinom}((x,y,z); 10, p_x, p_y, p_z)$$
$$= \begin{cases} \dfrac{10!}{x!\,y!\,z!} p_x^x\, p_y^y\, p_z^z & x+y+z = 10 \\ 0 & \text{ellers} \end{cases}$$

$$f(5,2,3) = \frac{10!}{5!\,2!\,3!}\, 0.59^5 \cdot 0.165^2 \cdot 0.245^3$$

$$=$$

$$f(x,y|z) = \frac{f(x,y,z)}{f(z)}$$

Merk

$$f(z) - \text{binom}(z;10,p_z) = \frac{10!}{(10-z)!z!} p_z^z (1-p_z)^{10-z} \quad z$$

samt

$$1-p_z = p_x+p_y$$

$$10-z = x+y$$

herav

$$f(x,y|z) = \begin{cases} \dfrac{\frac{10!}{x!y!z!} p_x^x p_y^y p_z^z}{\frac{10!}{(x+y)!z!} p_z^z (p_x+p_y)^{x+y}} & x+y+z=10 \\ 0 & \text{ellers} \end{cases}$$

$$= \begin{cases} \dfrac{(x+y)!}{x!\,y!} \left(\dfrac{p_x}{p_x+p_y}\right)^x \left(\dfrac{p_y}{p_x+p_y}\right)^y & x+y=10-z \\ 0 & \text{ellers} \end{cases}$$

$$f(x,y|3) = \begin{cases} \dfrac{7!}{x!\,y!} \left(\dfrac{p_x}{p_x+p_y}\right)^x \left(\dfrac{p_y}{p_x+p_y}\right)^y & x+y=7 \\ 0 & \text{ellers} \end{cases}$$

altså er x $\text{bin}\left(x; 7, \dfrac{p_x}{p_x+p_y}\right)$

Oppgave 2: Bilsikringer

$$T \rightsquigarrow f_T(t;\alpha) = \begin{cases} 0 & t \le 0 \\ 2\alpha t e^{-\alpha t^2} & t > 0 \end{cases}$$

herav

$$T \rightsquigarrow F_T(t;\alpha) = \int_0^t f_T(u;\alpha)\,du = 1 - e^{-\alpha t^2}$$

Tilfeldig utvalg:

$T_1, \dots, T_8$ uif $f_T(t;\alpha)$

$t_1, \dots, t_8$

a) SME basert på $T_1, \dots, T_8$:

$$l(\alpha) = \ln\left\{\prod_i^8 f_T(t_i;\alpha)\right\} = \ln\left\{\prod_i^8 2\alpha t_i e^{-\alpha t_i^2}\right\}$$

$$= 8\ln 2 + 8\ln\alpha + \sum_i \ln t_i - \alpha \sum_i t_i^2$$

$$\frac{dl(\alpha)}{d\alpha} = 0 \Rightarrow \frac{8}{\alpha} - \sum_i t_i^2 = 0 \Rightarrow$$

$$\hat{\alpha} = 8 \cdot \left[\sum_i^8 t_i^2\right]^{-1}$$

Sanns.max.estimator:

$$\underline{\underline{\hat{\alpha} = 8 \cdot \left[\sum_i^8 T_i^2\right]^{-1}}}$$

Forventning

$$E\{\hat{\alpha}\} = 8 \cdot E\left\{\left[\sum_i T_i^2\right]^{-1}\right\}$$ – vanskelig å regne ut, men ikke generelt lik α.

<u>Ikke forventningsrett</u>

b) Definér:

$S = \min\{T_1, \dots, T_5\} \rightsquigarrow f_S(s;\alpha)$

herav:

$$S \rightsquigarrow F_S(s;\alpha) = \text{Prob}\{S < s\} = 1 - \text{Prob}\{S > s\}$$

$$= 1 - \text{Prob}\{T_i > s \,;\, i = 1, \dots, 5\}$$

$$= 1 - \left[1 - F_T(s;\alpha)\right]^5$$

samt

$$f_S(s;\alpha) = \frac{d\,F_S(s;\alpha)}{ds} = 5\left[1 - F_T(s;\alpha)\right]^4 f_T(s;\alpha)$$

$$= \begin{cases} 5\left[e^{-\alpha s^2}\right]^4 2\alpha s e^{-\alpha s^2} & s > 0 \\ 0 & s \le 0 \end{cases}$$

$$= \begin{cases} 10\alpha s e^{-5\alpha s^2} & s > 0 \\ 0 & s \le 0 \end{cases}$$

Observasjoner:

$$\begin{array}{ccc} S_1, & S_2, & S_3 \\ s_1 & s_2 & s_3 \end{array} \quad \text{uif} \quad f_S(s;\alpha)$$

SME basert på $T_1, \dots, T_8, S_1, \dots, S_3$

$$h(\alpha) = \ln\left\{\prod_i^8 f_T(t_i;\alpha) \prod_j^3 f_S(s_i;\alpha)\right\}$$

$$= \ln\left\{\prod_i^8 2\alpha t_i e^{-\alpha t_i^2} \prod_j^3 10\alpha s_i e^{-5\alpha s_i^2}\right\}$$

$$= \left[8\ln 2 + 3\ln 10\right] + 11\ln\alpha + \left[\sum_i \ln t_i + \sum_j \ln s_j\right] - \alpha\left[\sum_i t_i^2 + 5\sum_j s_j^2\right]$$

$$\frac{dL(\alpha)}{d\alpha} = 0 \Rightarrow \frac{11}{\alpha} - \left[\sum_i t_i^2 + 5\sum_j s_j^2\right] = 0 \Rightarrow$$

$$\hat{\alpha} = 11 \cdot \left[\sum_i^8 t_i^2 + 5\sum_j^3 s_j^2\right]^{-1}$$

Sann. max. estimator:

$$\underline{\underline{\hat{\alpha} = 11 \left[\sum_i^8 T_i^2 + 5\sum_j^3 S_j^2\right]^{-1}}}$$

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Oppgave 1

a) La X være kraften som er nødvendig for å trekke korken.

$$P(300 < X < 310) = P(\frac{300-310}{36} < Z < \frac{310-310}{36}) = P(-0.28 < Z < 0) = 0.1103$$

$$\begin{aligned} P(X > 360 | X > 330) &= \frac{P(X > 360 \cup X > 330)}{P(X > 330)} = \frac{P(X > 360)}{P(X > 330)} = \frac{P(Z > \frac{360-310}{36})}{P(Z > \frac{330-310}{36})} \\ &= \frac{P(Z > 1.39)}{P(Z > 0.56)} = \frac{0.0824}{0.2893} = 0.28 \end{aligned}$$

La $\bar{X} = 1/8 \sum_{i=1}^{8} X_i$.

$$P(\bar{X} > 320 = P(\frac{\bar{X}-310}{36/\sqrt{8}} > \frac{320-310}{36/\sqrt{8}}) = P(Z > 0.79) = 0.2160$$

b) $H_0 : \mu = 310$ mot $H_1 : \mu \neq 310$
Under H_0 er testobservator

$$Y = \frac{\bar{X} - 310}{\sigma/\sqrt{n}}$$

standard normalfordelt.
Akseptområde blir $A = (-z_{\alpha/2}, z_{\alpha/2}) = (-z_{0.005}, z_{0.005}) = (-2.58, 2.58)$

$$y = \frac{259.64 - 310}{36/\sqrt{8}} = -3.96 \notin A$$

og nullhypotesen forkastes.

$$
\begin{aligned}
P(Y \in A|\mu = 250) &= P(Y + \frac{310-250}{36/\sqrt{8}} \in (-2.58 + \frac{310-250}{36/\sqrt{8}}, 2.58 + \frac{310-250}{36/\sqrt{8}}|\mu = 250) \\
&= P(Z \in (-2.58 + 4.71, 2.58 + 4.71)) \\
&= P(Z \in (2.13, 7.29)) = 0.016
\end{aligned}
$$

Sannsynlighet for forkastning med $\mu = 250$ var dermed $1 - 0.016 = 0.984$.
Alternativt:
Testobservator $\bar{X}$ med akseptområde

$$
\begin{aligned}
A &= (310 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, 310 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) \\
&= (310 - 2.58\frac{36}{\sqrt{8}}, 310 + 2.58\frac{36}{\sqrt{8}}) \\
&= (277.16, 342.84)
\end{aligned}
$$

$\bar{x} = 259.64 \notin A$ og H_0 forkastes.

$$
\begin{aligned}
P(\bar{X} \in A|\mu = 250) &= P(Z \in (\frac{277.16-250}{36/\sqrt{(8)}}, \frac{342.84-250}{36/\sqrt{(8)}})) \\
&= P(Z \in (2.13, 7.29)) = 0.016
\end{aligned}
$$

osv.

c) $H_0 : \sigma = 36$ mot $H_1 : \sigma > 36$
Under H_0 er testobservatoren

$$V = \frac{7S^2}{36^2}$$

χ^2-fordelt med 7 frihetsgrader.
Kritisk område blir $C = (\chi^2_{7,0.05}, \infty) = (14.07, \infty)$. Med dataene i oppgaven får en

$$v = \frac{7 \cdot 1091.2}{36^2} = 20.278$$

og H_0 forkastes.
p-verdien er definert som minste signifikansnivå som gir forkastning av nullhypotesen. Dette kan ses på som sannsynligheten for å få en like ekstrem eller mer ekstrem indikasjon mot H_0 gitt at H_0 er sann.

$$p = P(V > 20.278) = 0.005$$