Mathematical Olympiads Discord Server

POTD Solutions

The Big Document

Main Contributors

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Discord Server Link: `https://discord.gg/m22vNrX`
Problem Spreadsheet: `http://bit.ly/potd-history`

1 Introduction

This document is an set of problems and solutions which have been listed in the `#problem-of-the-day` channel of the Mathematical Olympiads Discord Server. Problems are selected from past contests and range from very easy to IMO3/6 level and beyond.

Although the document has been compiled by staff, solutions will be typically member-submitted. As these are problems "of the day", the set of problems is continually growing and thus solutions are welcome. If you wish to submit a solution, please send a direct message to the bot `Staff Mail` via the command `m.submit`. Alternatively, you may submit a pull request on Github. You will be credited if you wish.

The staff team are indebted to `www.imomath.org` for making available English translations of many national and international Mathematics Olympiads.

Tag system Each problem is tagged according to its genre and difficulty. Genre is indicated using initials[1] and difficulty is indicated on a scale with 1 being very easy, 6 being the typical difficulty of an IMO1/4 problem, and 10 being the typical difficulty of an IMO3/6 problem. For example, a problem assigned the [NCg2] tag is an easy problem in number theory involving some combinatorial geometry.

Note that this document is a compilation of all the problems ever to be submitted. Monthly releases will also be available.

2 Problems

These begin on the next page.

[1] **A**lgebra, **C**ombinatorics, **G**eometry, **N**umber theory and **C**ombinatorial **g**eometry.

March 26 [N4] 2015 Romanian MoM, Q1 Day 1

Does there exist an infinite sequence of positive integers $a_1, a_2, a_3, \ldots$ such that a_m and a_n are coprime if and only if $|m-n| = 1$?

Solution 1 submitted by SharkyKesa 268970368524484609

Suppose the primes are p_1, p_2, p_3, ... Set

$$a_1 = p_2 \cdot p_3$$

and

$$a_n = p_{n+2} \cdot p_{n-1} \cdot p_{n-3} \cdot p_{n-5} \cdots$$

Then it is trivial to show consecutive a_i are co-prime, but the rest are not co-prime.[2]

[2] It would be great if someone were to provide a solution which explicitly showed how this sequence fulfilled the conditions of the problem.

March 27 [C5] 2018 IMO Shortlist, C1 Day 2

A rectangle R with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of R are either all odd or all even.

Solution 1 submitted by SharkyKesa 268970368524484609

Colour in chessboard fashion with the corners as black, so the number of blacks is 1 greater than whites. Then there exists an internal rectangle with more blacks than whites, so it must have all corners as blacks, which means it satisfies the property that the distance to the sides is all odd or even.

March 28 [A4] 2014 BMO2, Q2 Day 3

Prove that it is impossible to have a cuboid for which the volume, the surface area and the perimeter are numerically equal. (The perimeter of a cuboid is the sum of the lengths of all its twelve edges.)

Solution 1 submitted by Tony Wang `541318134699786272`

Let the sides of the cuboid be a, b and c. Furthermore let $X = abc$, $Y = 2(ab + bc + ca)$, and $Z = 4(a + b + c)$.
Now suppose that $XZ = Y^2$, from $X = Y = Z$. Then this means that

$$4(a^2bc+b^2ca+c^2ab) = 4(a^2b^2+ab^2c+a^2bc+ab^2c+b^2c^2+abc^2+a^2bc+abc^2+a^2c^2)$$

So

$$a^2b^2 + ab^2c + b^2c^2 + a^2bc + a^2c^2 = 0$$

Since a, b, $c > 0$ this is impossible, as required.

March 29 [Cg4] 2015 APMO, Q4 Day 4

Let n be a positive integer. Consider $2n$ distinct lines on the plane, no two of which are parallel. Of the $2n$ lines, n are colored blue, the other n are colored red. Let $\mathcal{B}$ be the set of all points on the plane that lie on at least one blue line, and $\mathcal{R}$ the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects $\mathcal{B}$ in exactly $2n-1$ points, and also intersects $\mathcal{R}$ in exactly $2n-1$ points.

Solution 1 submitted by SharkyKesa 268970368524484609

Consider the pair of red-blue lines with maximal angle between them, and consider a circle of increasing radius tangent to them through this angle. Trivial angle chasing yields that this circle must eventually intersect every other line (else you get a bigger angle)[3]

[3]Not rigorous yet. Additions are welcome.

March 30 [N5] 2005 IMO, Q4 Day 5

Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

Solution 1 submitted by SharkyKesa 268970368524484609

Note that $2|a_1 = 10$, $3|a_2 = 48$. Now suppose $p > 3$ and p is prime. Then,

$$\begin{aligned} a_{p-2} &= (3^{p-2}+1)(2^{p-2}+1) - 2 \\ &= (1/3+1)(1/2+1) - 2 \\ &= 4/3 \times 3/2 - 2 = 2 - 2 \\ &= 0 \mod p \end{aligned}$$

So $p|a_{p-2}$. Thus all primes p eventually divide a_n, so only 1 satisfies.

Solution 2 submitted by SharkyKesa 268970368524484609

We proceed same as before for $p = 2, 3$.
Then, note that

$$\begin{aligned} 6a_{p-2} &= 6^{p-1} + 3 \times 2^{p-1} + 2 \times 3^{p-1} - 6 \\ &= 1 + 3 + 2 - 6 \\ &= 0 \mod p \end{aligned}$$

so we're done again

March 31 [Cg3] 2007 Romanian Final MO, F9, Q3 Day 6

The plane is partitioned into unit-width parallel bands, each colored white or black. Show that one can always place an equilateral triangle of side length 100 in the plane such that its vertices lie on the same color.

April 1 [C0] 2019 AFMO, Q3 Day 7

Suppose there are a line of prisoners, each of whom is wearing either a green or red hat. Any individual prisoner can see all the infinitely many prisoners and hats in front of them but none of the finitely many prisoners or hats behind them. They also can't see their own hat. In these circumstances, each prisoner then guesses the colour of their hat by writing it down, and the prison warden sets free any prisoner who correctly guesses the colour of their own hat. Assuming that the prisoners use the best strategy possible, what is the maximum guaranteed density of prisoners set free?

Solution 1 submitted by Tony Wang `541318134699786272`

This problem was an April Fool's joke, with AFMO being an acronym for April Fool's Mathematical Olympiad. It would not appear on a mathematical competition for it's "abuse of axiom of choice". That being said, you can find a document explaining the question and solution here: `https://bit.ly/prisoner-problems-solution`.

April 2 [G2] 2017 BMO1, Q3 Day 8

The triangle ABC has $AB = CA$ and BC is its longest side. The point N is on the side BC and $BN = AB$. The line perpendicular to AB which passes through N meets AB at M. Prove that the line MN divides both the area and the perimeter of triangle ABC into equal parts.

April 3 [A5] 2017 Canadian MO, Q2 Day 9

Define a function $f(n)$ from the positive integers to the positive integers such that $f(f(n))$ is the number of positive integer divisors of n. Prove that if p is prime, then $f(p)$ is prime.

April 4 [Cg6] 2015 IMO, Q1 Day 10

We say that a finite set S of points in the plane is *balanced* if, for any two different points A and B in S, there is a point C in S such that $AC = BC$. We say that S is *centre-free* if for any three different points A, B and C in S, there is no point P in S such that $PA = PB = PC$.

1. Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.
2. Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

April 5 [A9] 2015 IMO, Q2 Day 11

Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

April 6 [NG5] 2013 BMO2, Q4 Day 12

Suppose that $ABCD$ is a square and that P is a point which is on the circle inscribed in the square. Determine whether or not it is possible that PA, PB, PC, PD and AB are all integers.

April 7 [N5] 2018 EGMO, Q2 Day 13

Consider the set
$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, \ldots \right\}.$$

1. Prove that every integer $x \geq 2$ can be written as the product of one or more elements of A, which are not necessarily different.

2. For every integer $x \geq 2$, let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different.

Prove that there exist infinitely many pairs (x, y) of integers with $x \geq 2, y \geq 2$, and
$$f(xy) < f(x) + f(y).$$
(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \neq x_2$ or $y_1 \neq y_2$.)

April 8 [Cg4] 2017 NZ Squad Selection Test, Q5 Day 14

Let A and B be two distinct points in the plane. Find all points C in the plane such that there does not exist a point X in the plane with the property that X is closer to both A and B than C.

April 9 [C2] 2009 Russian MO (29th), Grade 11, Q1 Day 15

Some cities in a country are linked by roads, none of which intersect outside a city. Each city displays the shortest length of a trip (chain of roads) beginning in that city and passing through each of the other cities at least once. Prove that any two displayed lengths a and b satisfies $a \leq 1.5b$ and $b \leq 1.5a$.

April 10 [N6-8] 2005 Korean MO (18th), Final Round, Q5 Day 16

Find all positive integers m and n such that both $3^m + 1$ and $3^n + 1$ are divisible by mn.

April 11 [G4] 1999 Balkan MO (16th), Q1 Day 17

Let D be the midpoint of the shorter arc BC of the circumcircle of an acute-angled triangle ABC. The points symmetric to D with respect to BC and the circumcenter are denoted by E and F, respectively. Let K be the midpoint of EA.

(a) Prove that the circle passing through the midpoints of the sides of $\triangle ABC$ also passes through K.

(b) The line through K and the midpoint of BC is perpendicular to AF.

April 12 [N5] 2015/16 BMO1, Q6 Day 18

A positive integer is called *charming* if it is equal to 2 or is of the form 3^i5^j where i and j are non-negative integers. Prove that every positive integer can be written as a sum of different charming integers.

April 13 [A(FE)3] 2004 Swedish MO (44th), Final Round, Q3 Day 19

Find all functions f satisfying $f(x) + xf(1-x) = x^2$ for all real x.

April 14 [C5] 2018 IMO Shortlist, C2 Day 20

Let n be a positive integer. Define a *chameleon* to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters a, b, and c. Define a *swap* to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X, there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

April 15 [N3] 2009 Japanese MO, Final Round, Q2 Day 21

Let N be a positive integer. Prove that if the sum of the elements in $1, 2, \ldots, N$ is even, then it is possible to paint each each element red or green so that the sum of the red numbers is equal to the sum of the green numbers.

April 16 [Cg6] 2004 Swedish MO (44th), Final Round, Q4 Day 22

A square with integer side length $n \geq 3$ is divided into n^2 unit squares, and $n-1$ lines are drawn so that each square's interior is cut by at least one line.

(a) Give an example of such a configuration for some n.

(b) Show that some two of the lines must meet inside the square

April 17 [A5-6] 2018 Euclid Contest, Q10 (adapted) Day 23

In an infinite grid with two rows, each row continues to the left and right without bound. Each cell contains a positive real number. Prove that if each cell is the average of its three neighbours, then all the numbers in the grid are equal.

April 18 [G2] 2015/16 BMO1, Q2 Day 24

Let $ABCD$ be a cyclic quadrilateral and let the lines CD and BA meet at E. The line through D which is tangent to the circle ADE meets the line CB at F. Prove that the triangle CDF is isosceles.

April 19 [A(Poly)5] 1993 IMO (34th), Q1 Day 25

Let $n > 1$ be an integer and let $f(x) = x^n + 5x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that $f(x) = g(x)h(x)$.

April 20 [C4] 2000 Dutch MO, Second Round, Q5 of 5 Day 26

Consider an infinite strip of unit squares numbered $1, 2, 3, \ldots$. A pawn starting on one of these squares can, at each step, move between squares numbered n, $2n$, and $3n + 1$. Show that the pawn will be able to reach the square 1 after finitely many steps.

April 21 [A/N4] 2005 Canadian MO (37th), Q2 of 5 Day 27

Let ((a,b,c)) be a Pythagorean triple, i.e. a triplet of positive integers with (a^2 + $b^2 = c^2$).

(a) Prove that ($\left(\frac{c}{a} + \frac{b}{a}\right)^2 > 8$).

(b) Prove that there are no integers n and Pythagorean triples (a, b, c) satisfying $\left(\frac{c}{a} + \frac{b}{a}\right)^2 = n$.

April 22 [G6] 2016 IMO (57th), Q1 Day 28

Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that BD, FX and ME are concurrent.

April 23 [N3] 2018 Putnam, B3 Day 29

Find all positive integers $n < 10^{100}$ for which simultaneously n divides $2^n, n-1$ divides $2^n - 1$ and $n-2$ divides $2^n - 2$.

April 24 [C4] 2008 Polish MO, Second Round, Q4 Day 30

An integer is written in every square of an $n \times n$ board such that the sum of all the integers in the board is 0. A move consists of choosing a square and decreasing the number in it by the number of neighbouring squares (by side), while increasing the numbers in each of the neighbouring squares by 1. Determine if there is an $n \geq 2$ for which it is always possible to turn all the integers into zeros in finitely many moves.

April 25 [A4] 2005 Serbia and Montenegro TST, Test 1, Q3 Day 31

Find all polynomials $P(x)$ that satisfy $P\left(x^2+1\right) = P(x)^2 + 1$ for all x.