Solutions

to the 2019 October Beginner Contest

version 1.0 (23 Sep 2019)

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Server invite link: https://discord.gg/94UnnAG

Contents

This solution booklet was compiled by the Mathematical Olympiads Discord Server (MODS) at `https://discord.gg/94UnnAG`. The problems on the contest were original problems authored by the MODS Contest Creator team, and the solutions are taken from both contest creators and submitted solutions.

The contest was hosted by nya10, brainysmurfs, trillian, and Tony Wang in the Mathematical Olympiads Discord Server on the **12th and 13th of October.** Throughout the document the following names correspond to the following users on Discord:

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The Problems

Time: 4 hours — *Each problem is worth 7 points*

Calculators and protractors are not allowed. Do not write your name on your working. After your timeslot finishes, please read the instructions in `#how-to-submit-scripts`*. Do not discuss the contents of this paper outside the text channel* `#finished-contestants` *and the voice channel* `Post-Contest Banter` *until notified by staff.*

Problem 1. A positive integer is called *square-free* if it is not a multiple of any square other than 1. George and his n friends sit around a table. George thinks of a positive integer $k > 1$ and writes it on the blackboard. The person to his left then divides the number on the blackboard by a square-free number to obtain another positive integer $k_1 < k$, and replaces k with k_1 on the blackboard. The process repeats with each person in succession, going clockwise around the table, generating positive integers $k_1 > k_2 > k_3 > \cdots$ and so on. The first person to write 1 on the blackboard wins.

Prove that for any value of n, George can always think of a positive integer k such that he is guaranteed to win.

Problem 2. Let $\mathbb{Q}$ denote the set of rational numbers. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for all rational a and b,

$$f(a)f(b) = f(a+b).$$

Problem 3. Do there exist points $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X,$ Y, and Z in the Euclidean plane, not all the same, such that $ABCD, EFGH, IJKL, MNOP, QRST,$ $UVWX, YZAB, CDEF, GHIJ, KLMN, OPQR, STUV$, and $WXYZ$ are all squares?

(Note that the vertices of a square do not necessarily have to be in order, so that if $ABCD$ is a square then so is $ACBD$.)

Problem 4. Let ABC be a triangle and denote by M the midpoint of BC. Suppose X is the point on the perimeter of ABC such that MX bisects the perimeter of ABC. Show that MX is parallel to the internal angle bisector of $\angle BAC$.

Problem 1

A positive integer is called *square-free* if it is not a multiple of any square other than 1. George and his n friends sit around a table. George thinks of a positive integer $k > 1$ and writes it on the blackboard. The person to his left then divides the number on the blackboard by a square-free number to obtain another positive integer $k_1 < k$, and replaces k with k_1 on the blackboard. The process repeats with each person in succession, going clockwise around the table, generating positive integers $k_1 > k_2 > k_3 > \cdots$ and so on. The first person to write 1 on the blackboard wins.

Prove that for any value of n, George can always think of a positive integer k such that he is guaranteed to win.

(Proposed by Tony Wang)

Solution 1

(by Tony Wang)

We will show using induction that if George picks $k = k_0 = p^{n+1}$ for some prime p, then George is guaranteed to win.

Our induction hypothesis is that $k_i = p^{n+1-i}$ for $i \in \{0, 1, 2, \ldots, n+1\}$. We verify that in the base case where $i = 0$, $k_0 = k = p^{n+1}$.

Now assume that $k_j = p^{n+1-j}$. Because $j < n+1$, we have $n+1-j > 0$ so $p^{n+1-j} \geq p^1$ which is divisible by p. However, the factors of p^{n+1-j} are $1, p, p^2, \ldots, p^{n+1-j}$, as p is prime. We note that the j-th person cannot divide k_j by 1, as then $k_{j+1} = k_j$. However he also cannot divide k_j by $p^2, p^3, \ldots, p^{n+1-j}$, as these are all divisible by p^2, and are therefore not square-free. Hence the j-th person can only divide k_j by p, and this means that $k_{j+1} = p^{n-j}$, as desired.

As George is the $(n+1)$-th person around the table, he writes $k_{n+1} = (p^{(n+1)-(n+1)}) = 1$ on the blackboard and wins. $\square$

Problem 2

Let $\mathbb{Q}$ denote the set of rational numbers. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for all rational a and b,

$$f(a)f(b) = f(a+b).$$

(Proposed by Tony Wang)

Solution 1

(by Tony Wang)

Firstly, making the substitution $(x, y) = (0, 0)$ yields the equation $f(0)^2 = f(0) \implies f(0) = 0$ or 1. If $f(0) = 0$, substituting $(x, y) = (x, 0)$ gives $f(x)f(0) = f(x) = 0$ for all x, so assume that $f(0) = 1$.

Suppose that $f(c) = a$, where $c \in \mathbb{Q}$. Note that $f\left(\frac{c}{2}\right) f\left(\frac{c}{2}\right) = f(c)$ so $f\left(\frac{c}{2}\right) = \sqrt{a}$. However, we may repeat this process to obtain $f\left(\frac{c}{4}\right) = \sqrt[4]{a}$, $f\left(\frac{c}{8}\right) = \sqrt[8]{a}$, and so on. However, it is well known that the RHS will eventually become irrational or complex unless $f(c) = 0$ or 1. If $f(c) = 0$ for any $c \in \mathbb{Q}$ then $f(c)f(-c) = f(0) = 0$, which is a contradiction. Thus $f(c) = 1$ for all c, and thus $f(x) = 1$ for all rational x is the only possible other solution.

It is easy to check that both of these solutions works, as either $f(x)f(y) = 1 = f(x+y)$ or $f(x)f(y) = 0 = f(x+y)$ for all rational x and y. $\square$

Solution 2

(by brainysmurfs)

$f\left(\frac{x}{2}\right)^2 = f(x)$ so $f(x) \geq 0$ for all x. Assume that for some x, we have $f(x) = 0$. Then

$$0 = f(x)f(b) = f(x+b)$$

for all b so $f \equiv 0$.

Thus $f(x) > 0$ for all x. Therefore we can consider $g(x) = \log f(x)$, which gives us

$$g(a) + g(b) = g(a+b),$$

and veteran problem solvers will recognise that this is Cauchy's functional equation. Thus the solutions to g over rationals are $g(x) = cx$ so $f(x) = c^x$, and $f(1)$ gives $c > 0$ is rational. Finally, we notice that $c = 1$: assume p^n is the power of p which divides c, for $n \neq 0$ and p prime (yes, fundamental theorem of arithmetic for positive rationals holds). Suppose the power of p that divides $c^{\frac{1}{n}}$ is p^k. Then we have $(n+1)k|n$ which is a blatant contradiction.

Finally, it is easy to see that these functions work, as proved in solution 1. $\square$

Solution 3

(by brainysmurfs)

As in solutions 1 and 2 we get that $f(x) \geq 0$ and that either $f = 0$ or $f(0) = 1$.

Note that $f(x+1) = f(x)f(1) = cf(x)$, and so by induction $f(x) = c^x \quad \forall x \in \mathbb{Z}$. We see that $f(mx) = f(\overbrace{x + x + x + \cdots x}^{m})$ for $m \in \mathbb{N}$, so

$$f(mx) = f(\overbrace{x + x + x \cdots x}^{m}) = f(\overbrace{x + x + x \cdots x}^{m-1})f(x) = \cdots = \overbrace{f(x)f(x)\cdots f(x)}^{m} = f(x)^m.$$

Thus $f(\frac{m}{n}) = f(m)^{\frac{1}{n}} = c^{\frac{m}{n}}$ for $m, n \in \mathbb{Z}$, i.e. $f(x) = c^x \quad \forall x \in \mathbb{Q}$. However note that if $c \neq 1$, then, as proven in solution 2, we can find x such that $c^x \notin \mathbb{Q}$. So $c = 1$ and thus the only solutions are $f = 1$ or $f = 0$.

Problem 3

Do there exist points $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y,$ and Z in the Euclidean plane, not all the same, such that $ABCD, EFGH, IJKL, MNOP, QRST, UVWX, YZAB, CDEF, GHIJ, KLMN, OPQR, STUV,$ and $WXYZ$ are all squares?

(Note that the vertices of a square do not necessarily have to be in order, so that if $ABCD$ is a square then so is $ACBD$.)

(Proposed by Sharky Kesa)

Solution 1 *(by Tony Wang)*

We claim that the answer is no.

Rename the points $A, B, C, \ldots, Z$ as $P_1, P_2, \ldots, P_{26}, \ldots$, with indices taken mod 26. Let f be a function taking two points and returning the x-coordinate of the lower point, if it exists.

As not all points are the same, the sides and diagonals of any square is non-zero. Hence we can scale and rotate the segment P_1P_2 such that $P_1 = (0, 0)$ and $P_2 = (1, 1)$. As $P_1P_2P_3P_4$ is a square, then $\{P_3, P_4\}$ must be $\{(0, 1), (1, 0)\}$, $\{(-1, 1), (0, 2)\}$, or $\{(1, -1), (2, 0)\}$. In each case, $f(P_3, P_4)$ is odd. Since $P_1P_2 \cong P_3P_4$, we may repeat this argument on P_3P_4 to deduce that $f(P_5, P_6)$ must be even. Continuing in this way, we deduce that $f(P_{27}, P_{28})$ must be odd. But this is a contradiction as $P_1P_2 = P_{27}P_{28}$, and $f(P_1, P_2)$ is even. $\square$

Solution 2 *(by pianocat31)*

The answer is no.

As above, rename the points $A, B, C, \ldots, Z$ as $P_1, P_2, \ldots, P_{26}, \ldots$, with indices taken mod 26. Let $M_1, M_2, \ldots, M_{13}$ be the midpoints of P_1P_2, P_3P_4, and so on.

Rescale and shift the plane so that P_1P_2 is the line segment defined by $(-\frac{1}{2}, 0), (\frac{1}{2}, 0)$, then $M_1 = (0, 0)$. We note that there are three ways to place P_3P_4 in such a way that they form a square with AB. Two produce a parallel line segment distance 1 away, and the third produces a perpendicular segment with the same length and midpoint. Thus all line segments $P_1P_2, P_3P_4, \ldots$ must be parallel or perpendicular to P_1P_2, and hence $M_{i+1} - M_i$ must be either $\binom{0}{1}, \binom{0}{-1}, \binom{1}{0}, \binom{-1}{0}$, or $\binom{0}{0}$. We now note that in order for $P_{27}P_{28}$ to equal P_1P_2, we must have:

1. M_{14} equal to M_1, from which we can deduce there must be an equal number of $\binom{0}{1}$ and $\binom{0}{-1}$ moves, and an equal number of $\binom{1}{0}$ and $\binom{-1}{0}$ moves.
2. $P_{27}P_{28}$ parallel to P_1P_2, from which we can deduce there must be an even number of rotations, or $\binom{0}{0}$ moves.

Hence the total number of moves must be even, but this is a contradiction as there are 13 moves between P_1P_2 and $P_{27}P_{28}$. $\square$

Problem 4

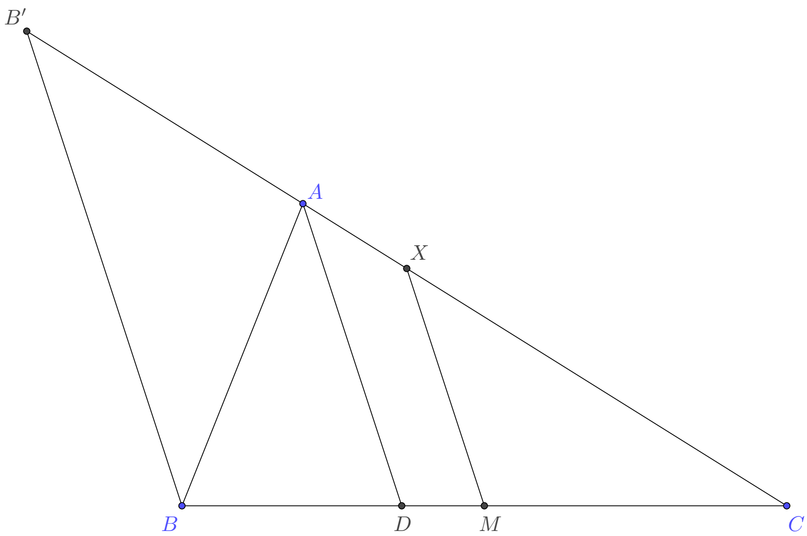
Let ABC be a triangle and denote by M the midpoint of BC. Suppose X is the point on the perimeter of ABC such that MX bisects the perimeter of ABC. Show that MX is parallel to the internal angle bisector of $\angle BAC$.

(Proposed by Sharky Kesa)

Solution 1

(by pianocat31 and Goldtiger997)

WLOG $AC \geq AB$. Let the angle bisector of $\angle BAC$ meet BC at D. Construct B' on CA such that $AB' = AB$ and A is contained in segment $B'C$.

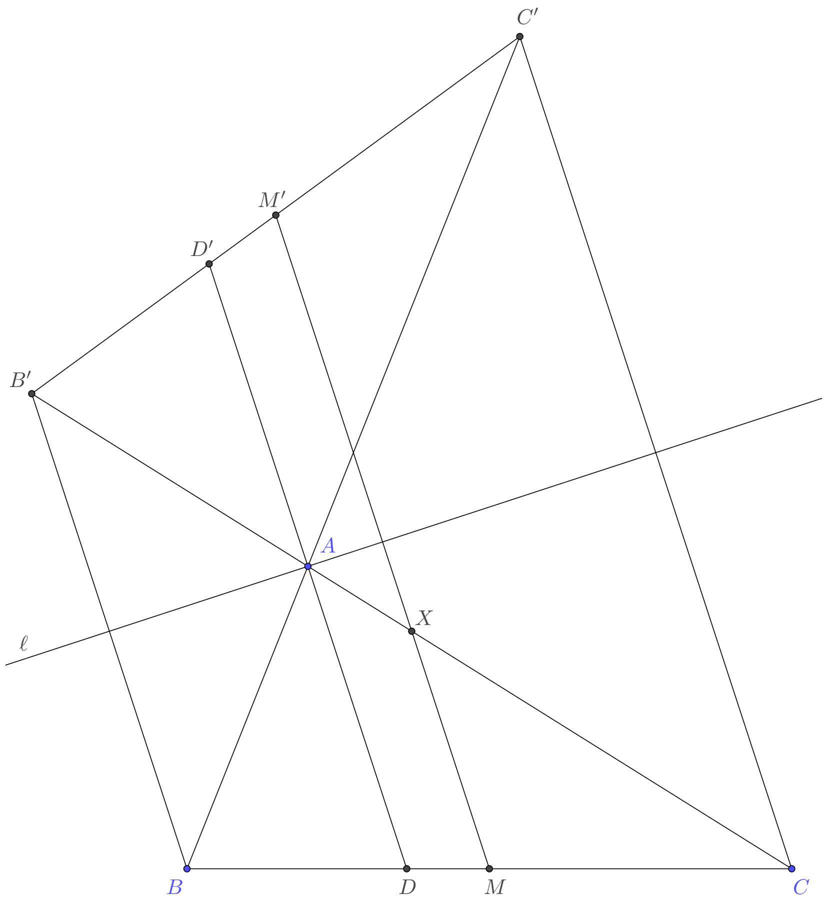


Note that as BAB' is an isosceles triangle with base BB', $\angle BB'A = \angle B'BA = \frac{1}{2}(180^\circ - \angle BAB') = \frac{1}{2}\angle BAC = \angle DAC$, and so $BB' \parallel DA$. Also note that $\frac{1}{2} = \frac{MC}{DB} = \frac{CX}{CA+AB} = \frac{CX}{CB'}$, and so by similar triangles $\triangle BCB'$ and $\triangle MCX$ we also have $MX \parallel BB'$. Thus $MX \parallel DA$ as desired. $\square$

Solution 2

(by Tony Wang)

WLOG $AC \geq AB$. Let the angle bisector of $\angle BAC$ meet BC at D. Let ℓ be the external angle bisector of $\angle ABC$ and reflect $BDMC$ over ℓ to get points $B'D'M'C'$.

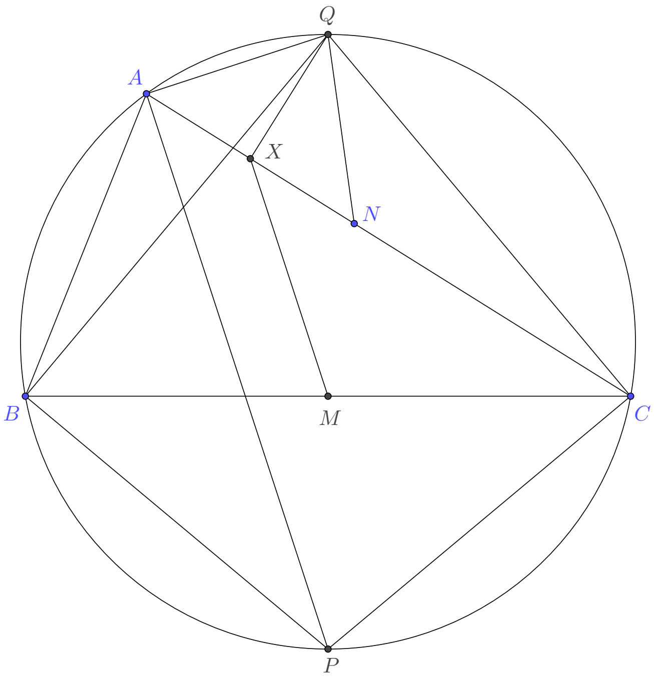


Note that we have $BB' \parallel DD' \parallel MM' \parallel CC' \perp \ell$, so it remains to prove that X lies on MM'.

To do this, we first show that $\angle BAB' = 2\left(\frac{1}{2}(180 - \angle BAC)\right) = 180 - \angle BAC$, so B', A, and C are collinear. Because $B'M' = M'C' = CM = MB$, MM' lies exactly halfway between BB' and CC'. However, we also know that $CX = XA + AB = XA + AB' = XB'$, and so X also lies halfway between BB' and CC'. Hence it must lie on MM', as desired. $\square$

Solution 3 *(by InvisibleRabbit)*

Note that if $AC = AB$, then ABC is isosceles, and in this case DA and MX are both perpendiculars from A to BC, so they are the same line and are therefore parallel. Now WLOG assume $AC > AB$. Denote by Ω the circumcircle of ABC. Let the internal angle bisector of $\angle BAC$ intersect Ω at $P \neq A$, external angle of $\angle BAC$ intersect Ω at $Q \neq A$. Let N be the point on segment AC such that $CN = AB$.

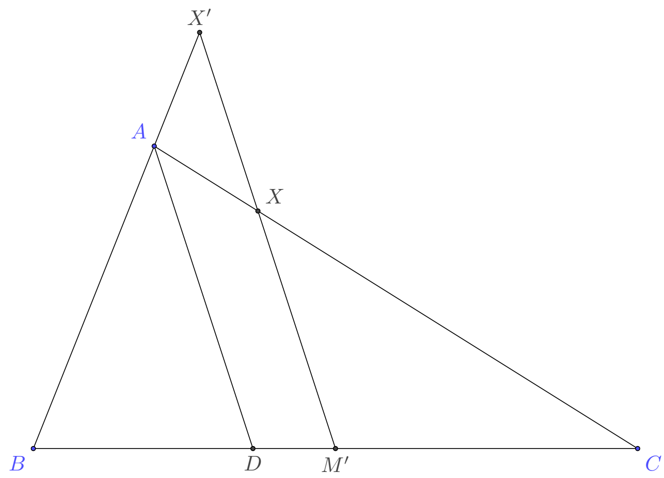


It is well known that both $\triangle QBC$ and $\triangle PBC$ are isosceles triangles with altitudes QM and PM respectively. Then we note that $BA + AX = XC$, so $AX = CX - BA = CX - CN = NX$, so X is the midpoint of AN. Since we have $AB = NC$, $QB = QC$, and $\angle ABQ = \angle NCQ$, we can deduce that $\triangle ABQ \cong \triangle NCQ$ and hence that $QA = QN$. As X is the midpoint of the base of isosceles triangle QAN, $\angle QXC = 90° = \angle QMC$, and so $QXMC$ is a cyclic quad. Then, as Q, M, and P are collinear, $\angle MXC = \angle MQC = \angle PQC = \angle PAC$, as required. $\square$

Solution 4

(by 9271kain)

Note that if $AC = AB$, then ABC is isosceles, and in this case DA and MX are both perpendiculars from A to BC, so they are the same line and are therefore parallel. Now WLOG assume $AC > AB$. Let X' be a point on ray BA past A such that $AX = AX'$, and let N be the intersection between the internal angle bisector of $\angle BAC$ and BC.



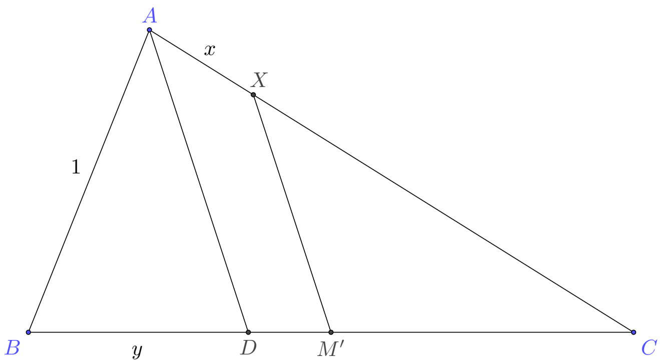
Note that as XAX' is an isosceles triangle with base XX', we have $\angle AX'X = \angle AXX' = \frac{1}{2}(180° - \angle XAX') = \frac{1}{2}\angle BAC = \angle BAD$. Hence $X'M \parallel AN$, and so it suffices to prove that X lies on $X'M$. However, by Menalaus' Theorem, we know that $X'XM$ are collinear if and only if

$$\frac{BM}{MC}\frac{CX}{XA}\frac{AX'}{X'B} = 1.$$

Note that as $BM = MC$, $XA = AX'$, and $CX = BA + AX = BX'$, everything on the LHS cancels and so indeed equals 1. Therefore $X'XM$ are collinear and so $XM \parallel X'M \parallel AD$, as required. $\square$

Solution 5 *(by chfrn)*

WLOG assume $AC > AB$. Let D be the intersection between the internal angle bisector of$\angle BAC$ and BC. Let $BA = 1$, $AX = x$, and $BD = y$. Let the line through X parallel to AD intersect BC at M'. by reverse reconstruction, it suffices to show that $BM' = M'C$, as P is the midpoint of $BC \iff CP = PB$.



As we have $XC = BA + AX = 1 + x$, the angle bisector theorem gives us $DC = \dfrac{BD \cdot AC}{AB} = 2xy + y$, which implies $BC = 2(xy + y)$. By equal angles, $\triangle ACD \cong \triangle XCM'$, and so we have

$$CM' = \frac{CD \cdot CX}{CA} = \frac{(2xy + y)(x + 1)}{2x + 1} = yx + y.$$

Hence $CM' = \frac{1}{2}CB$. As X lies on segment AC, M' lies on segment DC. Thus M' bisects line segment CB, as desired. $\square$