Mathematical Olympiads Discord Server

POTD Solutions

The Big Document

Main Contributors

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Discord Server Link: `https://discord.gg/m22vNrX`
Problem Spreadsheet: `http://bit.ly/potd-history`

1 Introduction

This document is an set of problems and solutions which have been listed in the `#problem-of-the-day` channel of the Mathematical Olympiads Discord Server. Problems are selected from past contests and range from very easy to IMO3/6 level and beyond.

Although the document has been compiled by staff, solutions will be typically member-submitted. As these are problems "of the day", the set of problems is continually growing and thus solutions are welcome. If you wish to submit a solution, please send a direct message to the bot `Staff Mail` via the command `m.submit`. Alternatively, you may submit a pull request on Github. You will be credited if you wish.

The staff team are indebted to `www.imomath.org` for making available English translations of many national and international Mathematics Olympiads.

Tag system Each problem is tagged according to its genre and difficulty. Genre is indicated using initials[1] and difficulty is indicated on a scale with 1 being very easy, 6 being the typical difficulty of an IMO1/4 problem, and 10 being the typical difficulty of an IMO3/6 problem. For example, a problem assigned the [NCg2] tag is an easy problem in number theory involving some combinatorial geometry.

Note that this document is a compilation of all the problems ever to be submitted. Monthly releases will also be available.

2 Problems

These begin on the next page.

[1] **A**lgebra, **C**ombinatorics, **G**eometry, **N**umber theory and **C**ombinatorial **g**eometry.

March 26 [N4] 2015 Romanian MoM, Q1 Day 1

Does there exist an infinite sequence of positive integers $a_1, a_2, a_3, \ldots$ such that a_m and a_n are coprime if and only if $|m-n| = 1$?

Solution 1 submitted by SharkyKesa 268970368524484609

Suppose the primes are p_1, p_2, p_3, ... Set

$$a_1 = p_2 \cdot p_3$$

and

$$a_n = p_{n+2} \cdot p_{n-1} \cdot p_{n-3} \cdot p_{n-5} \cdots$$

Then it is trivial to show consecutive a_i are co-prime, but the rest are not co-prime.[2]

[2]It would be great if someone were to provide a solution which explicitly showed how this sequence fulfilled the conditions of the problem.

March 27 [C5] 2018 IMO Shortlist, C1 Day 2

A rectangle R with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of R are either all odd or all even.

Solution 1 submitted by SharkyKesa 268970368524484609

Colour in chessboard fashion with the corners as black, so the number of blacks is 1 greater than whites. Then there exists an internal rectangle with more blacks than whites, so it must have all corners as blacks, which means it satisfies the property that the distance to the sides is all odd or even.

March 28 [A4] 2014 BMO2, Q2 Day 3

Prove that it is impossible to have a cuboid for which the volume, the surface area and the perimeter are numerically equal. (The perimeter of a cuboid is the sum of the lengths of all its twelve edges.)

Solution 1 submitted by Tony Wang `541318134699786272`

Let the sides of the cuboid be a, b and c. Furthermore let $X = abc$, $Y = 2(ab + bc + ca)$, and $Z = 4(a + b + c)$.
Now suppose that $XZ = Y^2$, from $X = Y = Z$. Then this means that

$$4(a^2bc+b^2ca+c^2ab) = 4(a^2b^2+ab^2c+a^2bc+ab^2c+b^2c^2+abc^2+a^2bc+abc^2+a^2c^2)$$

So
$$a^2b^2 + ab^2c + b^2c^2 + a^2bc + a^2c^2 = 0$$

Since a, b, $c > 0$ this is impossible, as required.

March 29 [Cg4] 2015 APMO, Q4 Day 4

Let n be a positive integer. Consider $2n$ distinct lines on the plane, no two of which are parallel. Of the $2n$ lines, n are colored blue, the other n are colored red. Let $\mathcal{B}$ be the set of all points on the plane that lie on at least one blue line, and $\mathcal{R}$ the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects $\mathcal{B}$ in exactly $2n-1$ points, and also intersects $\mathcal{R}$ in exactly $2n-1$ points.

Solution 1 submitted by SharkyKesa 268970368524484609

Consider the pair of red-blue lines with maximal angle between them, and consider a circle of increasing radius tangent to them through this angle. Trivial angle chasing yields that this circle must eventually intersect every other line (else you get a bigger angle)[3]

[3]Not rigorous yet. Additions are welcome.

March 30 [N5] 2005 IMO, Q4 Day 5

Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

Solution 1 submitted by SharkyKesa 268970368524484609

Note that $2|a_1 = 10$, $3|a_2 = 48$. Now suppose $p > 3$ and p is prime. Then,

$$\begin{aligned} a_{p-2} &= (3^{p-2}+1)(2^{p-2}+1) - 2 \\ &= (1/3+1)(1/2+1) - 2 \\ &= 4/3 \times 3/2 - 2 = 2 - 2 \\ &= 0 \mod p \end{aligned}$$

So $p|a_{p-2}$. Thus all primes p eventually divide a_n, so only 1 satisfies.

Solution 2 submitted by SharkyKesa 268970368524484609

We proceed same as before for $p = 2, 3$.
Then, note that

$$\begin{aligned} 6a_{p-2} &= 6^{p-1} + 3 \times 2^{p-1} + 2 \times 3^{p-1} - 6 \\ &= 1 + 3 + 2 - 6 \\ &= 0 \mod p \end{aligned}$$

so we're done again

March 31 [Cg3] 2007 Romanian Final MO, F9, Q3 Day 6

The plane is partitioned into unit-width parallel bands, each colored white or black. Show that one can always place an equilateral triangle of side length 100 in the plane such that its vertices lie on the same color.

April 1 [C0] 2019 AFMO, Q3 Day 7

Suppose there are a line of prisoners, each of whom is wearing either a green or red hat. Any individual prisoner can see all the infinitely many prisoners and hats in front of them but none of the finitely many prisoners or hats behind them. They also can't see their own hat. In these circumstances, each prisoner then guesses the colour of their hat by writing it down, and the prison warden sets free any prisoner who correctly guesses the colour of their own hat. Assuming that the prisoners use the best strategy possible, what is the maximum guaranteed density of prisoners set free?

Solution 1 submitted by Tony Wang `541318134699786272`

This problem was an April Fool's joke, with AFMO being an acronym for April Fool's Mathematical Olympiad. It would not appear on a mathematical competition for it's "abuse of axiom of choice". That being said, you can find a document explaining the question and solution here:
`https://bit.ly/prisoner-problems-solution`.

April 2 [G2] 2017 BMO1, Q3 Day 8

The triangle ABC has $AB = CA$ and BC is its longest side. The point N is on the side BC and $BN = AB$. The line perpendicular to AB which passes through N meets AB at M. Prove that the line MN divides both the area and the perimeter of triangle ABC into equal parts.

Solution 1 submitted by Daniel `118831126239248397`

Let A' be the reflection of A in MN. Since MN is by definition perpendicular to AB, A' lies on AB. Thus

$$AM = MA',$$

$$AN = A'N,$$

and

$$\angle NAA' = \angle AA'N.$$

It is given that $BN = AB$, and so

$$\angle BNA = \angle NAB = \angle NAA' = \angle AA'N.$$

Thus also

$$\angle NA'B = \angle ANC.$$

It is also given that $AB = CA$, so

$$\angle ABC = \angle BCA,$$

and so

$$\angle A'BN = \angle NCA.$$

Thus triangles $A'BN$ and NCA are congruent. Note by the definition of A' that triangles AMN and $A'MN$ are also congruent. Thus

$$\text{area } AMN + \text{area } NCA = \text{area } A'MN + \text{area } A'BN,$$

so MN divides the area of triangle ABC into equal parts. Also

$$BA' + A'M + BN = AM + CN + AC,$$

so MN divides the perimeter of triangle ABC into equal parts.

Solution 2 submitted by Daniel 118831126239248397

Note that since $AB = CA$, $\angle B = \angle C$ and hence $\angle A = 180° - 2\angle B$. Thus by the sine rule,

$$\frac{BC}{\sin(180° - 2\angle B)} = \frac{AB}{\sin(\angle B)}.$$

Since $\sin(180° - x) \equiv \sin(x)$ and $\sin(2x) \equiv 2\sin(x)\cos(x)$,

$$BC = 2\cos(\angle B)AB.$$

Note also since $BN = AB$,

$$BM = \cos(\angle B)AB,$$

and

$$MN = \sin(\angle B)AB.$$

Hence $NB + BM = (1 + \cos(\angle B))AB = \frac{1}{2}(2 + 2\cos(\angle B))AB = \frac{1}{2}$ perimeter ABC, so MN divides the perimeter of triangle ABC into equal parts. Also area $NMB = \frac{1}{2}\cos(\angle B)\sin(\angle B)AB^2 = \frac{1}{4}\sin(2\angle B)AB^2 = \frac{1}{2}$ area ABC, so MN also divides the area of triangle ABC into equal parts.

April 3 [A5] 2017 Canadian MO, Q2 Day 9

Define a function $f(n)$ from the positive integers to the positive integers such that $f(f(n))$ is the number of positive integer divisors of n. Prove that if p is prime, then $f(p)$ is prime.

Solution 1

Let $\tau(x)$ be the divisor function so that by the problem condition,

$$f(f(n)) = \tau(n) \tag{1}$$

for all $n \in \mathbb{Z}^+$. If we substitute $n \to f(n)$ in (1), we find that both $f(f(f(n))) = \tau(f(n))$ and $f(f(f(n))) = f(\tau(n))$ depending on what order we evaluate the fs in $f(f(f(n)))$. Hence

$$\tau(f(n)) = f(\tau(n)) \tag{2}$$

If we substitute into (2) an arbitrary prime p, then since $\tau(p) = 2$, $\tau(f(p)) = f(2)$ for all such p. In particular, $\tau(f(2)) = f(2)$, and it is easily seen that the only fixed points of τ are 1 and 2. Hence $f(2) = 1$ or 2.
We now show that $f(2) \neq 1$. Suppose for sake of contradiction that $f(2) = 1$. Then $f(1) = f(f(2)) = 2$ and

$$f(f(3)) = 2 \implies f(f(f(3))) = f(2) = 1 \implies \tau(f(3)) = 1 \implies f(3) = 1$$

Now $f(f(9)) = 3$ since there are 3 positive divisors of 9, but this implies

$$f(f(f(9))) = f(3) = 1 \implies \tau(f(9)) = 1 \implies f(9) = 1 \implies f(f(9)) = f(1) = 2$$

which is a contradiction. Hence $f(2) = 2$. Then by (2), for all prime p we have $\tau(f(p)) = 2$ which means for any prime p, $f(p)$ is prime as desired.

April 4 [Cg6] 2015 IMO, Q1 Day 10

We say that a finite set S of points in the plane is *balanced* if, for any two different points A and B in S, there is a point C in S such that $AC = BC$. We say that S is *centre-free* if for any three different points A, B and C in S, there is no point P in S such that $PA = PB = PC$.

1. Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

2. Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

April 5 [(FE)A9] 2015 IMO, Q2 Day 11

Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

April 6 [NG5] 2013 BMO2, Q4 Day 12

Suppose that $ABCD$ is a square and that P is a point which is on the circle inscribed in the square. Determine whether or not it is possible that PA, PB, PC, PD and AB are all integers.

April 7 [N5] 2018 EGMO, Q2 Day 13

Consider the set

$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, \dots \right\}.$$

1. Prove that every integer $x \geq 2$ can be written as the product of one or more elements of A, which are not necessarily different.
2. For every integer $x \geq 2$, let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different.

Prove that there exist infinitely many pairs (x, y) of integers with $x \geq 2, y \geq 2$, and

$$f(xy) < f(x) + f(y).$$

(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \neq x_2$ or $y_1 \neq y_2$.)

April 8 [Cg4] 2017 NZ Squad Selection Test, Q5 Day 14

Let A and B be two distinct points in the plane. Find all points C in the plane such that there does not exist a point X in the plane with the property that X is closer to both A and B than C.

April 9 [C2] 2009 Russian MO (29th), Grade 11, Q1 Day 15

Some cities in a country are linked by roads, none of which intersect outside a city. Each city displays the shortest length of a trip (chain of roads) beginning in that city and passing through each of the other cities at least once. Prove that any two displayed lengths a and b satisfies $a \leq 1.5b$ and $b \leq 1.5a$.

April 10 [N6-8] 2005 Korean MO (18th), Final Round, Q5 Day 16

Find all positive integers m and n such that both $3^m + 1$ and $3^n + 1$ are divisible by mn.

April 11 [G4] 1999 Balkan MO (16th), Q1 Day 17

Let D be the midpoint of the shorter arc BC of the circumcircle of an acute-angled triangle ABC. The points symmetric to D with respect to BC and the circumcenter are denoted by E and F, respectively. Let K be the midpoint of EA.

(a) Prove that the circle passing through the midpoints of the sides of $\triangle ABC$ also passes through K.

(b) The line through K and the midpoint of BC is perpendicular to AF.

April 12 [N5] 2015/16 BMO1, Q6 Day 18

A positive integer is called *charming* if it is equal to 2 or is of the form 3^i5^j where i and j are non-negative integers. Prove that every positive integer can be written as a sum of different charming integers.

April 13 [(FE)A3] 2004 Swedish MO (44th), Final Round, Q3 Day 19

Find all functions f satisfying $f(x) + xf(1-x) = x^2$ for all real x.

Solution 1

Notice that if we substitute $x \to 1 - x$ in our original equation, we find that

$$f(1-x) + (1-x)f(x) = (1-x)^2 \tag{1}$$

If we let $a = x$ and $b = 1 - x$, upon comparing (1) with the original equation we have a simultaneous equation in a and b:

$$\begin{cases} f(a) + af(b) = a^2 \\ f(b) + bf(a) = b^2 \end{cases} \implies \begin{cases} f(a) + af(b) = a^2 \\ baf(a) + af(b) = ab^2 \end{cases}$$

From the last pair of simultaneous equations, if we subtract the first equation from the second, we find

$$\begin{aligned} (ba-1)f(a) = a(b^2 - a) &\implies f(a) = \frac{a(b^2-a)}{ab-1} \\ &\implies f(x) = \frac{x((1-x)^2 - x)}{x(1-x) - 1} \\ &\qquad\quad\;\; = \frac{-x^3 + 3x^2 - x}{x^2 - x + 1} \end{aligned}$$

Because this satisfies the original equation (the reader is invited to verify if so inclined), $f(x) = \frac{-x^3+3x^2-x}{x^2-x+1}$ is the only solution to the original equation.

April 14 [C5] 2018 IMO Shortlist, C2 Day 20

Let n be a positive integer. Define a *chameleon* to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters a, b, and c. Define a *swap* to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X, there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

April 15 [N3] 2009 Japanese MO, Final Round, Q2 Day 21

Let N be a positive integer. Prove that if the sum of the elements in $1, 2, \ldots, N$ is even, then it is possible to paint each each element red or green so that the sum of the red numbers is equal to the sum of the green numbers.

April 16 [Cg6] 2004 Swedish MO (44th), Final Round, Q4 Day 22

A square with integer side length $n \geq 3$ is divided into n^2 unit squares, and $n-1$ lines are drawn so that each square's interior is cut by at least one line.

(a) Give an example of such a configuration for some n.

(b) Show that some two of the lines must meet inside the square

April 17 [A5-A6] 2018 Euclid Contest, Q10 (adapted) Day 23

In an infinite grid with two rows, each row continues to the left and right without bound. Each cell contains a positive real number. Prove that if each cell is the average of its three neighbours, then all the numbers in the grid are equal.

April 18 [G2] 2015/16 BMO1, Q2 Day 24

Let $ABCD$ be a cyclic quadrilateral and let the lines CD and BA meet at E. The line through D which is tangent to the circle ADE meets the line CB at F. Prove that the triangle CDF is isosceles.

April 19 [(Poly)A5] 1993 IMO (34th), Q1 Day 25

Let $n > 1$ be an integer and let $f(x) = x^n + 5x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that $f(x) = g(x)h(x)$.

April 20 [C4] 2000 Dutch MO, Second Round, Q5 of 5 Day 26

Consider an infinite strip of unit squares numbered $1, 2, 3, \ldots$. A pawn starting on one of these squares can, at each step, move between squares numbered n, $2n$, and $3n + 1$. Show that the pawn will be able to reach the square 1 after finitely many steps.

April 21 [A4/N4] 2005 Canadian MO (37th), Q2 of 5 Day 27

Let ((a,b,c)) be a Pythagorean triple, i.e. a triplet of positive integers with (a^2 + $b^2 = c^2$).

(a) Prove that $\left(\frac{c}{a} + \frac{c}{b}\right)^2 > 8$).

(b) Prove that there are no integers n and Pythagorean triples (a, b, c) satisfying $\left(\frac{c}{a} + \frac{b}{a}\right)^2 = n$.

Solution 1

(a) First note that

$$\left(\frac{c}{a} + \frac{c}{b}\right)^2 = \left(\frac{c(a+b)}{ab}\right)^2 = \frac{c^2(a+b)^2}{(ab)^2} = \frac{(a^2+b^2)(a+b)^2}{ab^2} = \frac{a^2+b^2}{ab} \times \frac{(a+b)^2}{ab}$$

and that $\frac{a^2+b^2}{ab} \geq 2 \iff a^2 + b^2 \geq 2ab \iff (a-b)^2 \geq 0$ which is true, as well as $\frac{(a+b)^2}{ab} \geq 4 \iff a^2 + 2ab + b^2 \geq 4ab \iff (a-b)^2 \geq 0$ which is true. Then by multiplying these two inequalities,

$$\left(\frac{c}{a} + \frac{c}{b}\right)^2 = \frac{a^2+b^2}{ab} \times \frac{(a+b)^2}{ab} \geq 2 \times 4 = 8$$

but equality holds only if $a = b$, which is not possible in a right triangle with integer lengths. Hence $\left(\frac{c(a+b)}{ab}\right)^2 > 8$, as desired[4].

(b) Because $\frac{c}{a} + \frac{c}{b}$ is rational[5], it suffices to prove that there are no integers n such that $\frac{c}{a} + \frac{c}{b} = n$. If we multiply both sides by abn we find $c(a+b) = abn$. Now consider the equation

$$a^2 + b^2 = c^2$$

By dividing out $d = \gcd(a, b, c)$, we can assume that a, b and c are pairwise coprime. Since all squares are either 0 or 1 (mod 4), if a, b are both odd, then $a^2 \equiv b^2 \equiv 1 \pmod 4 \implies c^2 \equiv 2 \pmod 4$ which is a contradiction. a, b cannot be both even as they are coprime. Hence exactly one of a and b is odd. Then $a + b$ is odd, so $c(a+b)$ is odd, but abn is even. Therefore $c(a+b) \neq abn$ and so there are no integers n satisfying $\frac{c}{a} + \frac{c}{b} = n^2$.

[4] I wish to leave it as a challenge to find the best minimum bound. I've proven that $\left(\frac{c}{a} + \frac{c}{b}\right)^2 > 4\sqrt{3}$ and would be interested to know if there was a better bound, possibly with equality.

[5] It is well known that $\sqrt{x} \in \mathbb{R} \setminus \mathbb{Q}$ if x is not a perfect square.

April 22 [G6] 2016 IMO (57th), Q1 Day 28

Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that BD, FX and ME are concurrent.

April 23 [N3] 2018 Putnam, B3 Day 29

Find all positive integers $n < 10^{100}$ for which simultaneously n divides $2^n, n-1$ divides $2^n - 1$ and $n-2$ divides $2^n - 2$.

April 24 [C4] 2008 Polish MO, Second Round, Q4 Day 30

An integer is written in every square of an $n \times n$ board such that the sum of all the integers in the board is 0. A move consists of choosing a square and decreasing the number in it by the number of neighbouring squares (by side), while increasing the numbers in each of the neighbouring squares by 1. Determine if there is an $n \geq 2$ for which it is always possible to turn all the integers into zeros in finitely many moves.

Solution 1 submitted by Denial 118831126239248397

Consider all the squares on the main diagonal of the board (shown in yellow and red), as shown in the figure below. Let S represent the sum of all the numbers of the squares on the main diagonal.

Lemma 1: S is invariant mod 2.
Note that any 'move' performed will only affect the integers on the main diagonal if it is performed on either a square on the main diagonal (a yellow or red square in the diagram) or on a square adjacent to the main diagonal. We handle these cases separately.

1. The move is performed on one of the squares of the main diagonal. Then if it is a yellow square, its number increases by 2; whereas if it is a red square, its number increases by 4. So S will be invariant mod 2 in this case. Note that since squares on the main diagonal are not adjacent to each other, performing a move on one square will not affect any of the other squares.

2. The move is performed on one of the squares adjacent to the main diagonal. Each of these squares is adjacent to exactly two squares on the main diagonal, and so the S decreases by 2 and is thus invariant mod 2.

Note that having all 0's on the main diagonal means $S = 0 \mod 2$. However, since S is invariant mod 2, then if the original configuration had $S = 1 \mod 2$, then it would be impossible to achieve the configuration with all the numbers being 0.

April 25 [A4] 2005 Serbia and Montenegro TST, Test 1, Q3 Day 31

Find all polynomials $P(x)$ that satisfy $P\left(x^2+1\right) = P(x)^2 + 1$ for all x.

April 26 [C5] 2008 BMO2, Q3 Day 32

Adrian has drawn a circle in the $xy-$plane whose radius is a positive integer at most 2008. The origin lies somewhere inside the circle. You are allowed to ask him questions of the form ”Is the point (x, y) inside your circle?” After each question he will answer truthfully ”yes” or ”no”. Show that it is always possible to deduce the radius of the circle after at most sixty questions. [Note: Any point which lies exactly on the circle may be considered to lie inside the circle.]

April 27 [N6] 2013 IMO Shortlist, N3 Day 33

Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

April 28 [G5] 2013 Australian TST, Q2 Day 34

Let ABC be a triangle with orthocentre H. Let D be the point such that $AHCD$ is a parallelogram. Let M be the midpoint of BC, and the perpendicular from M to AB meet it at E. Let the line parallel to BD through A intersect ME at G. Suppose F is the midpoint of ME. Show that A, M, C and F are concyclic if, and only if, BF bisects CG.

April 29 [N4/Cg4] 2002 Taiwan MO (11th), Day 1, Q2 Day 35

A lattice point X in the plane is said to be *visible* from the origin O if the line segment OX does not contain any other lattice points. Show that for any positive integer n, there is a square $ABCD$ of area n^2 such that none of the lattice points inside the square is visible from the origin.

April 30 [(FE)A5] 1997 Balkan MO (14th), Q4 Day 36

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(xf(x) + f(y)) = (f(x))^2 + y$ for all $x, y \in \mathbb{R}$.

Solution 1

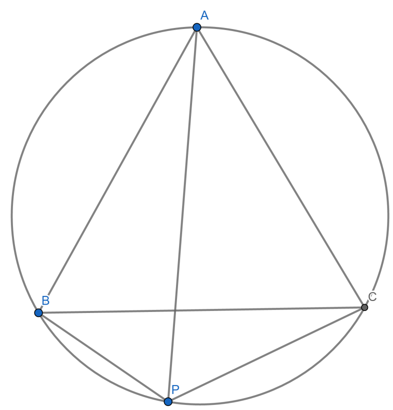
Since the right hand side of the original equation can attain any real value when x is fixed, f is clearly surjective. Hence their exists a constant c such that $f(c) = 0$. Substituting $x \to c$, we find that $f(f(y)) = y$ and hence f is also an involution. Then since $xf(x)$ remains the same whether you replace x with x or $f(x)$, we have

$$\begin{aligned} f(xf(x) + f(y)) &= (f(x))^2 + y \\ &= x^2 + y && \text{if we substitute } x \to f(x) \\ \implies (f(x))^2 + y &= x^2 + y \\ \implies (f(x))^2 &= x^2 \end{aligned}$$

Note that while $f(x) = \pm x$, we cannot conclude that $f(x) = x$ and $f(x) = -x$ are the only solutions. For example, $f(x) = \begin{cases} x \text{ if } x \in S \\ -x \text{ if } x \notin S \end{cases}$ for some arbitrary $S \subset \mathbb{R}$ satisfies $(f(x))^2 = x^2$. However, such a solution seems unlikely, and we now prove such a function that isn't $f(x) \equiv x$ or $f(x) \equiv -x$ cannot exist.
Suppose for sake of contradiction that there exist two positive reals a, b such that $f(a) = a$ and $f(b) = -b$. Then if we substitute (a, b) for (x, y) in the original equation, we find $f(a^2 - b) = a^2 + b$. But we know that $f(a^2 - b) = \pm(a^2 - b)$, and $a^2 + b \neq a^2 - b$ for positive reals a, b.
Hence the only functions satisfying the original equation are $f(x) = x$ and $f(x) = -x$, which clearly work.

April 31 [G3] 2018 NZ Camp Selection Test, Q2 of 9 Day 37

Let ABC be an equilateral triangle and let P be a point on the minor arc BC of the circumcircle of ABC. Prove that $PB + PC = PA$.



Solution 1 submitted by brainysmurfs 281300961312374785

Suppose $\angle BAP = a$. Then
$\angle CBP = 60 - a \implies \angle ACP = a \implies \angle BCP = 60 + a$.
By extended sine rule,

$$\begin{aligned}
PA &= 2R \sin \angle ABP \\
&= 2R \sin a \\
PB &= 2R \sin(60 + a) \\
&= 2R(\sin a \cos 60 + \cos a \sin 60) \\
&= 2R(\frac{1}{2} \sin a + \frac{\sqrt{3}}{2} \cos a) \\
PC &= 2R(\sin(60 - a)) \\
&= 2R(\frac{\sqrt{3}}{2} \cos a - \frac{1}{2} \sin a)
\end{aligned}$$

So,

$$\begin{aligned} PA + PC &= 2R(\sin a - \frac{1}{2}\sin a + \frac{\sqrt{3}2}{\cos} a) \\ &= 2R(\frac{\sqrt{3}}{2}\cos a + \frac{1}{2}\sin a) \\ &= PB \end{aligned}$$

as required. This finishes the proof.

Solution 2

By applying Ptolemy's theorem on cyclic quadrilateral $ABPC$, we find that

$$\begin{aligned} PA \cdot BC &= PB \cdot AC + PC \cdot AB \\ &= PB \cdot BC + PC \cdot BC \\ \iff PA &= PB + PC \end{aligned}$$

as desired, since $AB = AC = BC$.

May 1 [C4] 2007 Canadian MO (37th), Q2 of 5 Day 38

"For two real numbers a, b with $ab \neq 1$, define operation $\star$ by $a \star b = \frac{a+b-2ab}{1-ab}$. Start with a list of $n \geq 2$ real numbers that all satisfy $0 < x < 1$. Select any two numbers a and b in the list; remove them and put the number $a \star b$ at the end of the list, therefore reducing its length by 1. Repeat this procedure until a single number remains.
(a) Prove that this single number is the same regardless of the choice of pair at each stage.
(b) Suppose the condition on the numbers in the list is weakened to $0 < x \leq 1$. What happens if the list contains exactly one 1?"

Solution 1

We first show that for any $0 < x, y < 1$, $0 < x \star y < 1$. To prove $0 < x \star y$, note

$$\begin{aligned} 0 &< \frac{x+y-2xy}{1-xy} \\ \iff x+y-2xy &> 0 \\ \iff x+y &> 2xy \end{aligned}$$

but since x, y are positive reals, $x + y \geq 2\sqrt{xy}$. However, notice that for $0 < x, y < 1$, $\sqrt{xy} > xy$ because, in general for some positive constant $c < 1$, $c > c^2$. Here, equality cannot hold because $\sqrt{xy} = xy \implies xy = 1$ which is not possible. Hence $x + y \geq 2\sqrt{xy} > 2xy$, proving one direction.
Now to prove $x \star y < 1$, note that

$$\begin{aligned} \frac{x+y-2xy}{1-xy} &< 1 \\ \iff x+y-2xy &< 1-xy \\ \iff x+y-xy-1 &< 0 \\ \iff (x-1)(1-y) &< 0 \end{aligned}$$

but this last inequality is obviously true, since $x - 1$ is always negative and $1 - y$ is always positive. So the set of reals between 0 and 1 is closed over $\star$. Clearly, $\star$ is commutative due to symmetry; we now prove that it is assosciative. For general $0 < x, y, z < 1$ it is, while tedious, quite easy to compute that

$$x \star (y \star z) = \frac{x+y+z-2xy-2xz-2yz+3xyz}{1-xy-xz-yz+2xyz} = (x \star y) \star z$$

(the reader is invited to verify this if desired). Then any chain of $\star$ operations performed on a list of reals (between 0 and 1) is equivalent to any other rearrangement of the same chain of operations, so that the final result will always be the same. If the list contains exactly one 1, note that $x \star 1 = 1$ for any $x \neq 1$ and so the procedure will always end with the single number 1.

May 2 [2014 BMO1, Q3 of 6] C3 Day 39

A hotel has ten rooms along each side of a corridor. An olympiad team leader wishes to book seven rooms on the corridor so that no two reserved rooms on the same side of the corridor are adjacent. In how many ways can this be done?

May 3 [2002 IMO (43rd), Q4] N5 Day 40

The positive divisors of the integer $n > 1$ are $d_1 < d_2 < \cdots < d_k$, so that $d_1 = 1, d_k = n$. Let $d = d_1 d_2 + d_2 d3 + \cdots + dk - 1 d_k$. Show that $d < n^2$ and find all n for which d divides n^2.

May 4 [2006 Romanian MO, 9th Form, Q1 of 4] A2 Day 41

Find the maximum value of $\left(x^3+1\right)\left(y^3+1\right)$, where x and y are real numbers such that $x+y=1$.

May 5 [2008 Spanish MO, Q4 of 6] N3 Day 42

Let p and q be two different prime numbers. Prove that there are two positive integers, a and b, such that the arithmetic mean of the divisors of $n = p^a q^b$ is an integer.

May 6 [2019 NZ Squad Selection Test, Q1] G2 Day 43

The rectangle $ABCD$ has longest side AB. The point E lies on the line AD such that BE is perpendicular to AC, and the point F lies on the segment CD such that $AF = AB$.
Prove that the lines AF and EF are perpendicular.

May 7 [2006 Flanders MO, Q3 of 4] C3 Day 44

A total of 60 elves and trolls are seated around a table. Trolls always lie, and elves always speak the truth, except when they make a little mistake. Everybody claims to sit between an elf and a troll, but exactly two elves made a mistake! How many trolls are there at the table?

May 8 [2017 IMO Shortlist, N2] N6 Day 45

Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $0, 1, \ldots, p-1$ that was not chosen before by either of the two players and then chooses an element a_i of the set $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. Eduardo has the first move. The game ends after all the indices $i \in 0, 1, \ldots, p-1$ have been chosen. Then the following number is computed:

$$M = a_0 + 10 \cdot a1 + \cdots + 10^{p-1} \cdot ap - 1 = \sum_{j=0}^{p-1} a_j \cdot 10^j. \tag{4}$$

The goal of Eduardo is to make the number M divisible by p, and the goal of Fernando is to prevent this. Prove that Eduardo has a winning strategy.

May 9 [2006 Flanders MO, Q4 of 4] A4 Day 46

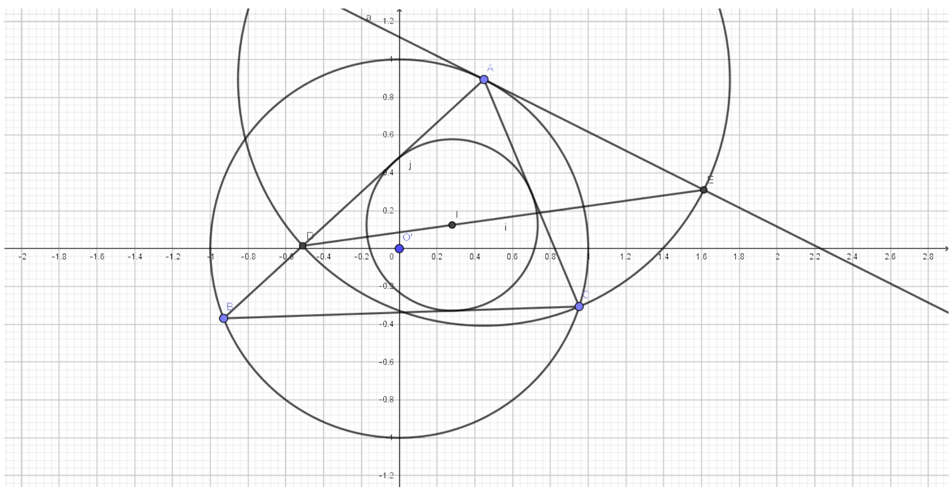
Find all functions $f : \mathbb{R} \setminus 0, 1 \to \mathbb{R}$ such that

$$f(x) + f(\frac{1}{1-x}) = 1 + \frac{1}{x(1-x)} \tag{5}$$

for all real x.

May 10 [2009 Croatian TST, Q3 of 4] G4 Day 47

Let ABC be a triangle such that $AB > AC$. Let l be a tangent at A to the circumcircle of ABC. A circle with centre A and radius AC intersects AB at D and the line l at E and F (in such a way that C and E are on the same side of AB). Prove that the line DE passes through the incentre of ABC.



Solution 1 submitted by brainysmurfs 281300961312374785

Suppose $\angle BAC = 2a$, $\angle ABC = 2b$, $\angle ACB = 2c$. By Alternate Segment Theorem, $\angle EAC = 2b \implies \angle AEC = 90 - b$ by base angles in isosceles triangle. But it is well known that $\angle AIC = 90 + (\angle ABC)/2 = 90 + b$. So $AECI$ is cyclic. This means that $\angle IEC = \angle IAC = a$. By angles at center is twice angles on the circumference, we get that $\angle DEC = (\angle DAC)/2 = a$. Since $\angle IEC = \angle DEC$ we get that DE passes through I.

May 11 [2009 Canada MO, Q5 of 5] CG3 Day 48

A set of points is marked on the plane, with the property that any three marked points can be covered with a disk of radius 1. Prove that the set of all marked points can be covered with a disk of radius 1.

May 12 [2015 IMO Shortlist, C1] C5 Day 49

"In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a right and a left bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, the bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B being to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly, B can sweep A away if the left bulldozer of B can move to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town which cannot be swept away by any other one."

May 13 [2014 BMO2, Q3 of 4] N3 Day 50

"Let $a_0 = 4$ and define a sequence of terms using the formula $an = an - 1^2 - a_{n-1}$ for each positive integer n. a) Prove that there are infinitely many prime numbers which are factors of at least one term of the sequence. b) Are there infinitely many prime numbers which are factors of no term in the sequence?"

May 14 [2008 Polish MO Round 1, Q9 of 12] A3 Day 51

"Determine the smallest real number a having the following property: For any real numbers $x, y, z \geq a$ satisfying $x + y + z = 3$, it holds that $x^3 + y^3 + z^3 \geq 3$.

May 15 [2000 Mexico MO, Q6 of 6] G3 Day 52

Let ABC be a triangle with $\angle B > 90^\circ$ such that there is a point H on side AC with $AH = BH$ and BH perpendicular to BC. Let D and E be the midpoints of AB and BC respectively. A line through H parallel to AB cuts DE at F. Prove that $\angle BCF = \angle ACD$.

May 16 [2019 NZ Squad Selection Test, Q7 of 9] C4 Day 53

A set S of positive integers is *self-indulgent* if $\gcd(a, b) = |a - b|$ for any two distinct $a, b \in S$.

(a) Prove that any self-indulgent set is finite.

(b) Prove that for any positive integer n, there exists a self-indulgent set with at least n elements.

Solution 1

(a) Let a_1 be the smallest element of the set. Let a_n be the largest element of the set. Since $a_n - a_0 = gcd(a_0, a_n) \leq a_0$ we have that $a_n \leq 2a_0$. So each self-indulgent set is bounded above by twice its smallest element, so is necessarily finite.

(b) Note that the following sets are self-indulgent for the various values of n:

n=1: 1

n=2: 5,10

n=3: 6,8,9

n=4: 8,9,10,12

We prove this result for all $n \geq 4$ by induction on n.

Base case ($n = 4$) is taken care of earlier.

Inductive step: Let $\{a_1, a_2, a_3, \ldots, a_n\}$ be a self-indulgent set with n elements. We claim that the set $\{b_0, b_1, b_2, \ldots, b_n\}$ is also self-indulgent with $n + 1$ elements, where $b_0 = \text{lcm}(a_1, a_2, \ldots a_n)$ and $b_i = a_i + b_0$ $\forall$ $0 < i \leq n$. Then it is trivial to note this new set is also self-indulgent.

May 17 [2018 Putnam, Q1 of 12] A2 Day 54

Find all ordered pairs of integers (a, b) for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}$$