

# Mathematical Olympiads Discord Server

## POTD Solutions

for April 2019

### CONTRIBUTORS

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Discord Server Link: <https://discord.gg/m22vNrX>

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## 1 Introduction

This document is an set of problems and solutions which have been listed in the **#problem-of-the-day** channel of the Mathematical Olympiads Discord Server. Problems are selected from past contests and range from very easy to IMO3/6 level and beyond.

Although the document has been compiled by staff, solutions will be typically member-submitted. As these are problems “of the day”, the set of problems is continually growing and thus solutions are welcome. If you wish to submit a solution, please send a direct message to the bot **Staff Mail** via the command **m.submit**. Alternatively, you may submit a pull request on Github. You will be credited if you wish.

The staff team are indebted to [www.imomath.org](http://www.imomath.org) for making available English translations of many national and international Mathematics Olympiads.

**Tag system** Each problem is tagged according to its genre and difficulty. Genre is indicated using initials<sup>1</sup> and difficulty is indicated on a scale with 1 being very easy, 6 being the typical difficulty of an IMO1/4 problem, and 10 being the typical difficulty of an IMO3/6 problem. For example, a problem assigned the [NCg2] tag is an easy problem in number theory involving some combinatorial geometry.

## 2 Problems

These begin on the next page.

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<sup>1</sup>Algebra, Combinatorics, Geometry, Number theory and Combinatorial geometry.

**April 1 [C0] 2019 AFMO, Q3****Day 7**

Suppose there are a line of prisoners, each of whom is wearing either a green or red hat. Any individual prisoner can see all the infinitely many prisoners and hats in front of them but none of the finitely many prisoners or hats behind them. They also can't see their own hat. In these circumstances, each prisoner then guesses the colour of their hat by writing it down, and the prison warden sets free any prisoner who correctly guesses the colour of their own hat. Assuming that the prisoners use the best strategy possible, what is the maximum guaranteed density of prisoners set free?

**Solution 1**

submitted by Tony Wang

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This problem was an April Fool's joke, with AFMO being an acronym for April Fool's Mathematical Olympiad. It would not appear on a mathematical competition for it's "abuse of axiom of choice". That being said, you can find a document explaining the question and solution here:  
<https://bit.ly/prisoner-problems-solution>.

**April 2 [G2] 2017 BMO1, Q3****Day 8**

The triangle  $ABC$  has  $AB = CA$  and  $BC$  is its longest side. The point  $N$  is on the side  $BC$  and  $BN = AB$ . The line perpendicular to  $AB$  which passes through  $N$  meets  $AB$  at  $M$ . Prove that the line  $MN$  divides both the area and the perimeter of triangle  $ABC$  into equal parts.

**April 3 [A5] 2017 Canadian MO, Q2****Day 9**

Define a function  $f(n)$  from the positive integers to the positive integers such that  $f(f(n))$  is the number of positive integer divisors of  $n$ . Prove that if  $p$  is prime, then  $f(p)$  is prime.

**April 4 [Cg6] 2015 IMO, Q1****Day 10**

We say that a finite set  $S$  of points in the plane is *balanced* if, for any two different points  $A$  and  $B$  in  $S$ , there is a point  $C$  in  $S$  such that  $AC = BC$ .

We say that  $S$  is *centre-free* if for any three different points  $A, B$  and  $C$  in  $S$ , there is no point  $P$  in  $S$  such that  $PA = PB = PC$ .

1. Show that for all integers  $n \geq 3$ , there exists a balanced set consisting of  $n$  points.
2. Determine all integers  $n \geq 3$  for which there exists a balanced centre-free set consisting of  $n$  points.

**April 5 [A9] 2015 IMO, Q2****Day 11**

Let  $\mathbb{R}$  be the set of real numbers. Determine all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that, for all real numbers  $x$  and  $y$ ,

$$f(f(x)f(y)) + f(x + y) = f(xy).$$



**April 6 [NG5] 2013 BMO2, Q4****Day 12**

Suppose that  $ABCD$  is a square and that  $P$  is a point which is on the circle inscribed in the square. Determine whether or not it is possible that  $PA$ ,  $PB$ ,  $PC$ ,  $PD$  and  $AB$  are all integers.

**April 7 [N5] 2018 EGMO, Q2****Day 13**

Consider the set

$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, \dots \right\}.$$

1. Prove that every integer  $x \geq 2$  can be written as the product of one or more elements of  $A$ , which are not necessarily different.
2. For every integer  $x \geq 2$ , let  $f(x)$  denote the minimum integer such that  $x$  can be written as the product of  $f(x)$  elements of  $A$ , which are not necessarily different.

Prove that there exist infinitely many pairs  $(x, y)$  of integers with  $x \geq 2, y \geq 2$ , and

$$f(xy) < f(x) + f(y).$$

(Pairs  $(x_1, y_1)$  and  $(x_2, y_2)$  are different if  $x_1 \neq x_2$  or  $y_1 \neq y_2$ .)

**April 8 [Cg4] 2017 NZ Squad Selection Test, Q5 Day 14**

Let  $A$  and  $B$  be two distinct points in the plane. Find all points  $C$  in the plane such that there does not exist a point  $X$  in the plane with the property that  $X$  is closer to both  $A$  and  $B$  than  $C$ .

**April 9 [C2] 2009 Russian MO (29th), Grade 11, Q1 Day 15**

Some cities in a country are linked by roads, none of which intersect outside a city. Each city displays the shortest length of a trip (chain of roads) beginning in that city and passing through each of the other cities at least once. Prove that any two displayed lengths  $a$  and  $b$  satisfies  $a \leq 1.5b$  and  $b \leq 1.5a$ .

**April 10 [N6-8] 2005 Korean MO (18th), Final Round,  
Q5 Day 16**

Find all positive integers  $m$  and  $n$  such that both  $3^m + 1$  and  $3^n + 1$  are divisible by  $mn$ .

**April 11 [G4] 1999 Balkan MO (16th), Q1 Day 17**

Let  $D$  be the midpoint of the shorter arc  $BC$  of the circumcircle of an acute-angled triangle  $ABC$ . The points symmetric to  $D$  with respect to  $BC$  and the circumcenter are denoted by  $E$  and  $F$ , respectively. Let  $K$  be the midpoint of  $EA$ .

- (a) Prove that the circle passing through the midpoints of the sides of  $\triangle ABC$  also passes through  $K$ .
- (b) The line through  $K$  and the midpoint of  $BC$  is perpendicular to  $AF$ .

**April 12 [N5] 2015/16 BMO1, Q6****Day 18**

A positive integer is called *charming* if it is equal to 2 or is of the form  $3^i 5^j$  where  $i$  and  $j$  are non-negative integers. Prove that every positive integer can be written as a sum of different charming integers.

**April 13 [A(FE)3] 2004 Swedish MO (44th), Final Round,  
Q3 Day 19**

Find all functions  $f$  satisfying  $f(x) + xf(1-x) = x^2$  for all real  $x$ .



**April 14 [C5] 2018 IMO Shortlist, C2****Day 20**

Let  $n$  be a positive integer. Define a *chameleon* to be any sequence of  $3n$  letters, with exactly  $n$  occurrences of each of the letters  $a$ ,  $b$ , and  $c$ . Define a *swap* to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon  $X$ , there exists a chameleon  $Y$  such that  $X$  cannot be changed to  $Y$  using fewer than  $3n^2/2$  swaps.

**April 15 [N3] 2009 Japanese MO, Final Round, Q2 Day 21**

Let  $N$  be a positive integer. Prove that if the sum of the elements in  $1, 2, \dots, N$  is even, then it is possible to paint each element red or green so that the sum of the red numbers is equal to the sum of the green numbers.

**April 16 [Cg6] 2004 Swedish MO (44th), Final Round,  
Q4 Day 22**

A square with integer side length  $n \geq 3$  is divided into  $n^2$  unit squares, and  $n - 1$  lines are drawn so that each square's interior is cut by at least one line.

- (a) Give an example of such a configuration for some  $n$ .
- (b) Show that some two of the lines must meet inside the square

**April 17 [A5-6] 2018 Euclid Contest, Q10 (adapted) Day 23**

In an infinite grid with two rows, each row continues to the left and right without bound. Each cell contains a positive real number. Prove that if each cell is the average of its three neighbours, then all the numbers in the grid are equal.

**April 18 [G2] 2015/16 BMO1, Q2****Day 24**

Let  $ABCD$  be a cyclic quadrilateral and let the lines  $CD$  and  $BA$  meet at  $E$ . The line through  $D$  which is tangent to the circle  $ADE$  meets the line  $CB$  at  $F$ . Prove that the triangle  $CDF$  is isosceles.

**April 19 [A(Poly)5] 1993 IMO (34th), Q1 Day 25**

Let  $n > 1$  be an integer and let  $f(x) = x^n + 5x^{n-1} + 3$ . Prove that there do not exist polynomials  $g(x), h(x)$ , each having integer coefficients and degree at least one, such that  $f(x) = g(x)h(x)$ .

**April 20 [C4] 2000 Dutch MO, Second Round, Q5 of 5  
Day 26**

Consider an infinite strip of unit squares numbered  $1, 2, 3, \dots$ . A pawn starting on one of these squares can, at each step, move between squares numbered  $n$ ,  $2n$ , and  $3n + 1$ . Show that the pawn will be able to reach the square 1 after finitely many steps.

**April 21 [A/N4] 2005 Canadian MO (37th), Q2 of 5 Day 27**

Let  $((a,b,c))$  be a Pythagorean triple, i.e. a triplet of positive integers with  $(a^2 + b^2 = c^2)$ .

- (a) Prove that  $\left(\frac{c}{a} + \frac{b}{a}\right)^2 > 8$ .
- (b) Prove that there are no integers  $n$  and Pythagorean triples  $(a, b, c)$  satisfying  $\left(\frac{c}{a} + \frac{b}{a}\right)^2 = n$ .



**April 22 [G6] 2016 IMO (57th), Q1****Day 28**

Triangle  $BCF$  has a right angle at  $B$ . Let  $A$  be the point on line  $CF$  such that  $FA = FB$  and  $F$  lies between  $A$  and  $C$ . Point  $D$  is chosen so that  $DA = DC$  and  $AC$  is the bisector of  $\angle DAB$ . Point  $E$  is chosen so that  $EA = ED$  and  $AD$  is the bisector of  $\angle EAC$ . Let  $M$  be the midpoint of  $CF$ . Let  $X$  be the point such that  $AMXE$  is a parallelogram (where  $AM \parallel EX$  and  $AE \parallel MX$ ). Prove that  $BD, FX$  and  $ME$  are concurrent.

**April 23 [N3] 2018 Putnam, B3****Day 29**

Find all positive integers  $n < 10^{100}$  for which simultaneously  $n$  divides  $2^n$ ,  $n - 1$  divides  $2^n - 1$  and  $n - 2$  divides  $2^n - 2$ .

**April 24 [C4] 2008 Polish MO, Second Round, Q4 Day 30**

An integer is written in every square of an  $n \times n$  board such that the sum of all the integers in the board is 0. A move consists of choosing a square and decreasing the number in it by the number of neighbouring squares (by side), while increasing the numbers in each of the neighbouring squares by 1. Determine if there is an  $n \geq 2$  for which it is always possible to turn all the integers into zeros in finitely many moves.

**April 25 [A4] 2005 Serbia and Montenegro TST, Test 1,  
Q3 Day 31**

Find all polynomials  $P(x)$  that satisfy  $P(x^2 + 1) = P(x)^2 + 1$  for all  $x$ .

### 3 Appendix