MO Server*PoTD Solutions

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April 25, 2019

1 Introduction

This document is an set of problems and solutions which have been listed in the problems-of-the-day channel of the Mathematical Olympiads Discord Server. Problems are Olympiad-style and range from very easy to IMO P3/6 level, with difficulty indicated on a scale of 1-10, where 10 is IMO P3/6 and 5-6 is IMO P1/4 level.
Although the document has been compiled by a moderator, solutions will be typically member-submitted. As these are problems "of the day", the set of problems is continually growing and thus solutions will be welcome. If you wish to submit a solution please either PM a moderator of the Discord server (these will be clearly marked when you join the server) or submit a pull request. You will be credited if you wish.

2 Problems

Day 1 — Mar 26, 2019 (Number Theory, D4)
Does there exist an infinite sequence of positive integers $a_1, a_2, a_3, \ldots$ such that a_m and a_n are coprime if and only if $|m - n| = 1$?
Source: 2015 Romanian MoM, Q1

Solution to Day 1:
Submitted by (userID = *134837275161788416*).
Construct a sequence

$$a_n = p_{2n} \times p_{2n-1} \times \prod_{i=1}^{n-2} p_{2i-1+(n \mod 2)}$$

where p_n is nth prime. This can be shown to produce the require co-primality conditions.

*`https://discord.gg/NPnGZYH`
†Individual solution authors are listed beside each problem

Day 2 — Mar 27, 2019 (Combinatorics, D5)
A rectangle R with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of R are either all odd or all even.
Source: 2018 IMO Shortlist, C1

Day 3 — Mar 28, 2019 (Algebra (Inequalities), D4)
Prove that it is impossible to have a cuboid for which the volume, the surface area and the perimeter are numerically equal. (The perimeter of a cuboid is the sum of the lengths of all its twelve edges.)
Source: 2014 BMO2, Q2

Day 4 — Mar 29, 2019 (Combinatorial Geometry, D4)
Let n be a positive integer. Consider $2n$ distinct lines on the plane, no two of which are parallel. Of the $2n$ lines, n are colored blue, the other n are colored red. Let $\mathcal{B}$ be the set of all points on the plane that lie on at least one blue line, and $\mathcal{R}$ the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects $\mathcal{B}$ in exactly $2n-1$ points, and also intersects $\mathcal{R}$ in exactly $2n-1$ points.
Source: 2015 APMO, Q4

Day 5 — Mar 30, 2019 (Number Theory, D5)
Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1,\ n \geq 1.$$

Source: 2005 IMO, Q4

Day 6 — Mar 31, 2019 (Combinatorial Geometry, D3)
The plane is partitioned into unit-width parallel bands, each colored white or black. Show that one can always place an equilateral triangle of side length 100 in the plane such that its vertices lie on the same color.
Source: 2007 Romanian Final MO, F9, Q3

Day 7 — April 1, 2019 (Combinatorics, DT)
Suppose there are a line of prisoners, each of whom is wearing either a green or red hat. Any individual prisoner can see all the infinitely many prisoners and hats in front of them but none of the finitely many prisoners or hats behind them. They also can't see their own hat. In these circumstances, each prisoner then guesses the colour of their hat by writing it down, and the prison warden sets free any prisoner who correctly guesses the colour of their own hat. Assuming that the prisoners use the best strategy possible, what is the maximum guaranteed density of prisoners set free?
Source: 2019 AFMO, Q3

Solution to Day 7:
Submitted by *Eliclax/Tony Wang* (userID = *541318134699786272*).
Note that this problem was an 'April Fools Joke', and was intended to be more challenging than a usual Problem of the Day; and will likely not appear on a mathematical competition. That being said, the answers are here: `https://bit.ly/prisoner-problems-solution`

Day 8 — April 2, 2019 (Geometry, D2)
The triangle ABC has $AB = CA$ and BC is its longest side. The point N is on the side BC and $BN = AB$. The line perpendicular to AB which passes through N meets AB at M. Prove that the line MN divides both the area and the perimeter of triangle ABC into equal parts.
Source: 2017 BMO1, Q3

Day 9 — April 3, 2019 (Algebra, D5)
Define a function $f(n)$ from the positive integers to the positive integers such that $f(f(n))$ is the number of positive integer divisors of n. Prove that if p is prime, then $f(p)$ is prime.
Source: 2017 Canadian MO, Q2

Day 10 — April 4, 2019 (Combinatorial Geometry, D6)
We say that a finite set S of points in the plane is *balanced* if, for any two different points A and B in S, there is a point C in S such that $AC = BC$. We say that S is *centre-free* if for any three different points A, B and C in S, there is no point P in S such that $PA = PB = PC$.
(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.
(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.
Source: 2015 IMO, Q1

Day 11 — April 5, 2019 (Algebra (Functional Equations), D9)
Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

Source: 2015 IMO, Q2

Day 12 — April 6, 2019 (Number Theory/Geometry, D4-5)
Suppose that $ABCD$ is a square and that P is a point which is on the circle inscribed in the square. Determine whether or not it is possible that PA, PB, PC, PD and AB are all integers.
Source: 2013 BMO2, Q4

Day 13 — April 7, 2019 (Number Theory, D5)
Consider the set

$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, \dots \right\}.$$

1. Prove that every integer $x \geq 2$ can be written as the product of one or more elements of A, which are not necessarily different.

2. For every integer $x \geq 2$, let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different.

Prove that there exist infinitely many pairs (x, y) of integers with $x \geq 2, y \geq 2$, and

$$f(xy) < f(x) + f(y).$$

(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \neq x_2$ or $y_1 \neq y_2$.)
Source: 2018 EGMO, Q2