Mathematical Olympiads Discord Server

2019 October Beginner Contest

Time: 4 hours *Each problem is worth 7 points*

Calculators and protractors are not allowed. Do not write your name on your working. After your timeslot finishes, please read the instructions in `#how-to-submit-scripts`*. Do not discuss the contents of this paper outside the text channel* `#finished-contestants` *and the voice channel* `Post-Contest Banter` *until notified by staff.*

Problem 1. A positive integer is called *square-free* if it is not a multiple of any square other than 1. George and his n friends sit around a table. George thinks of a positive integer $k > 1$ and writes it on the blackboard. The person to his left then divides the number on the blackboard by a square-free number to obtain another positive integer $k_1 < k$, and replaces k with k_1 on the blackboard. The process repeats with each person in succession, going clockwise around the table, generating positive integers $k_1 > k_2 > k_3 > \cdots$ and so on. The first person to write 1 on the blackboard wins.

Prove that for any value of n, George can always think of a positive integer k such that he is guaranteed to win.

Problem 2. Let $\mathbb{Q}$ denote the set of rational numbers. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for all rational a and b,

$$f(a)f(b) = f(a + b).$$

Problem 3. Do there exist points $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y$, and Z in the Euclidean plane, not all the same, such that $ABCD, EFGH, IJKL, MNOP, QRST, UVWX, YZAB, CDEF, GHIJ, KLMN, OPQR, STUV$, and $WXYZ$ are all squares?

(Note that the vertices of a square do not necessarily have to be in order, so that if $ABCD$ is a square then so is $ACBD$.)

Problem 4. Let ABC be a triangle and denote by M the midpoint of BC. Suppose X is the point on the perimeter of ABC such that MX bisects the perimeter of ABC. Show that MX is parallel to the internal angle bisector of $\angle BAC$.