Mathematical Olympiads Discord Server

2019 September Advanced Contest

Time: 4 hours *Each problem is worth 7 points*

Calculators and protractors are not allowed. Do not write your name on your working. After your timeslot finishes, please read the instructions in `#how-to-submit-scripts`*. Do not discuss the contents of this paper outside the text channel* `#finished-contestants` *and the voice channel* `Post-Contest Banter` *until notified by staff.*

Problem 1. In a 2019×2019 grid, the middle square is initially blue, and all other squares are white. In every move, three things happen in order:

1. Steve chooses some squares to colour black.
2. If any connected black region had a square that was initially blue before step 1., then all white squares adjacent to any black square in this region becomes blue.
3. All black squares become white.

(Two squares are *adjacent* if and only if they share an edge, and two squares A and B are in the same *connected region* if and only if there exists a sequence of squares $A, S_1, S_2, \ldots, S_n, B$ of the same colour such that any two consecutive squares in the sequence are adjacent.)

(a) What is the maximum number of blue squares that can exist on the grid at once?

(b) What is the minimum number of moves required to achieve this number of squares?

Problem 2. Find all n such that there exists a set of n consecutive nonnegative integers whose squares can be partitioned into two subsets of equal sum.

Problem 3. ABC is a triangle with incentre I. The feet of the altitudes from I to BC, AC, AB are D, E, F respectively, and the line through D parallel to AI intersects AB and AC at X and Y respectively. Prove that the circles with diameters XF and YE have a common point on the circumcircle of ABC.

Problem 4. Is it true that for any n there exists a geometric progression of positive integers, with ratio not equal to any power of 10, such that the base-10 representation of the first n terms contain no 9s?