

TD6: Rate Models

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https://github.com/Elieoriol/2021_UlmM2_ThNeuro/tree/master/TD6

In this tutorial, we will consider a very standard example of rate model: the ring model.

1 Ring model

We consider a network of neurons responsive to a stimulus θ which spans the range $[0, 2\pi]$. Each neuron has a preferred stimulus and the connection strength between two neurons respectively preferring θ_1 and θ_2 is proportional to $J_0 + J_1 \cos(\theta_1 - \theta_2)$. Additionally, each neuron receives an external input which depends on the neuron's preferred stimulus $h(\theta)$.

In the limit where the number of neurons is large and their preferred stimuli uniformly span the range $[0, 2\pi]$, we can write the neural activity as a continuous function $m(\theta, t)$. The input current to a neuron preferring θ can then be written as:

$$I(\theta, t) = h(\theta) + \int_0^{2\pi} \frac{d\theta'}{2\pi} [J_0 + J_1 \cos(\theta - \theta')] m(\theta', t)$$

The activity then evolves according to:

$$\frac{dm(\theta, t)}{dt} = -m(\theta, t) + f[I(\theta, t)] \quad (1)$$

$$f(x > 0) = x \quad (2)$$

$$f(x < 0) = 0 \quad (3)$$

1. Suppose that there is a constant, uniform and positive external current $h(\theta) = h_0$, which is sufficient for there to be positive, uniform network activity $m(\theta, t) = m_0$. What is the current received by each neuron? What is the network activity m_0 ? How does it depend on J_0 ?

Each neuron receives:

$$I(\theta) = h_0 + \int_0^{2\pi} \frac{d\theta'}{2\pi} (J_0 + J_1 \cos(\theta - \theta')) m_0 = h_0 + J_0 m_0$$

Therefore:

$$m_0 = f(h_0 + J_0 m_0) = h_0 + J_0 m_0 \quad \Rightarrow \quad m_0 = \frac{h_0}{1 - J_0}$$

The mean network activity is positive for $J_0 < 1$, goes to 0 for $J_0 \rightarrow -\infty$ and diverges to $+\infty$ for $J_0 \rightarrow 1$. It cannot exceed 1 here, otherwise m_0 would be negative and we assumed it to be positive.

We wish to study whether this uniform state is stable, we therefore consider small perturbations around it: $m(\theta, t) = m_0 + \delta m(\theta, t)$. We wish to see how these evolve with time.

To do so, we want to find a simple description of the dynamics. We introduce the order parameters:

$$M(t) = \int_0^{2\pi} \frac{d\theta'}{2\pi} \delta m(\theta', t) \quad (4)$$

$$C(t) = \int_0^{2\pi} \frac{d\theta'}{2\pi} \delta m(\theta', t) e^{i\theta'} \quad (5)$$

2. What does $M(t)$ correspond to?

$M(t)$ is the average firing rate across neurons, to which m_0 is subtracted. It is as such the deviation of the average firing rate from the uniform state.

3. Suppose that the perturbation is uniform:

$$\delta m(\theta, t) = \epsilon$$

What are the values of M and C ?

One directly computes the integrals to find $M = \epsilon$ and $C = 0$. The average deviation from the uniform state is logically equal to the uniform perturbation.

4. Suppose that the perturbation is a small bump centered around the angle ϕ :

$$\delta m(\theta, t) = \epsilon \cos(\theta - \phi) = \epsilon \frac{e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}}{2}$$

What are the values of M and C ?

We directly find $M = 0$, the average deviation is null. We compute C :

$$\begin{aligned} C &= \epsilon \int_0^{2\pi} \frac{d\theta'}{2\pi} \frac{e^{i(\theta'-\phi)} + e^{-i(\theta'-\phi)}}{2} e^{i\theta'} \\ &= \frac{\epsilon}{2} \left[\int_0^{2\pi} \frac{d\theta'}{2\pi} e^{i(2\theta'-\phi)} + \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{i\phi} \right] \\ &= \frac{\epsilon}{2} e^{i\phi} \end{aligned}$$

5. What does $C(t)$ correspond to?

C portrays how "bumpy" $\delta m(\theta, t)$ is: it is a complex number whose phase corresponds to the center of the bump and whose amplitude corresponds to the amplitude of the activity modulation.

6. Now that we have understood what $M(t)$ and $C(t)$ characterize, we will try to obtain a description of the dynamics in terms of the evolution of these two order parameters. Linearize the dynamics of the activity around m_0 and express it as a function of $M(t)$ and $C(t)$.

Since we linearize around m_0 that corresponds to a positive current $I > 0$, we have $f(I) = I$. Then:

$$\begin{aligned} \frac{d\delta m(\theta, t)}{dt} &= -\delta m(\theta, t) + J_0 \int_0^{2\pi} \frac{d\theta'}{2\pi} \delta m(\theta', t) + J_1 \int_0^{2\pi} \frac{d\theta'}{2\pi} \cos(\theta - \theta') \delta m(\theta', t) \\ &= -\delta m(\theta, t) + J_0 M + \frac{J_1}{2} \left[\int_0^{2\pi} \frac{d\theta'}{2\pi} e^{i(\theta-\theta')} \delta m(\theta', t) + \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{-i(\theta-\theta')} \delta m(\theta', t) \right] \\ &= -\delta m(\theta, t) + J_0 M + J_1 \frac{e^{i\theta} \bar{C} + e^{-i\theta} C}{2} \end{aligned}$$

7. Derive the differential equations governing the evolution of the order parameters.

$$\begin{aligned}
\frac{dM}{dt} &= \int_0^{2\pi} \frac{d\theta'}{2\pi} \frac{d\delta_m(\theta', t)}{dt} \\
&= - \int_0^{2\pi} \frac{d\theta'}{2\pi} \delta_m(\theta', t) + J_0 M \int_0^{2\pi} \frac{d\theta'}{2\pi} + \frac{J_1}{2} \left[\bar{C} \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{i\theta'} + C \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{-i\theta'} \right] \\
&= (J_0 - 1) M
\end{aligned}$$

$$\begin{aligned}
\frac{dC}{dt} &= \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{i\theta} \frac{d\delta_m(\theta', t)}{dt} \\
&= - \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{i\theta} \delta_m(\theta', t) + J_0 M \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{i\theta} + J_1 \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{i\theta} \frac{e^{i\theta} \bar{C} + e^{-i\theta} C}{2} \\
&= -C + \frac{J_1}{2} \left[\bar{C} \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{2i\theta} + C \int_0^{2\pi} \frac{d\theta'}{2\pi} \right] \\
&= \left(\frac{J_1}{2} - 1 \right) C
\end{aligned}$$

8. Under what conditions is the uniform activity stable? What happens when either of these conditions is not met?

There are two conditions for the network activity to be stable:

- $J_0 < 1$ otherwise any deviation of the mean activity below m_0 leads to an activity of 0, and any deviation of the mean activity above m_0 leads to exponentially growing activity.
- $J_1 < 2$ otherwise any non-uniform activity is expanded into a bump whose amplitude grows exponentially.

9. Consider $J_0 < 1$, $J_1 < 2$. Consider an external input with weak modulation, $h(\theta) = h_0 + \epsilon \cos(\theta)$, where $\epsilon \ll 1$. What is the profile of activity of the network induced by such an external input?

Using the same method:

$$\begin{aligned}
\frac{d\delta_m(\theta, t)}{dt} &= -\delta_m(\theta, t) + \epsilon \cos(\theta) + J_0 M + \frac{J_1}{2} \left[e^{-i\theta} C + e^{i\theta} \bar{C} \right] \\
\frac{dM}{dt} &= (J_0 - 1) M \\
\frac{dC}{dt} &= -C + \epsilon \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{i\theta'} \frac{e^{i\theta'} + e^{-i\theta'}}{2} + \frac{J_1}{2} C \\
&= \frac{\epsilon}{2} + \left(\frac{J_1}{2} - 1 \right) C
\end{aligned}$$

Now C converges to $\frac{\epsilon/2}{1-J_1/2}$. The profile of activity converges to a bump centered on 0.

10. Suppose that the firing rate is given by $m(\theta, t) = m_0 + m_1 \cos(\theta)$, under what conditions does the network amplify the input, ie: $\frac{m_1}{m_0} > \frac{\epsilon}{h_0}$?

$$C = \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{i\theta'} m_1 \frac{e^{i\theta'} + e^{-i\theta'}}{2} = \frac{m_1}{2} = \frac{\epsilon/2}{1 - J_1/2}$$

Therefore:

$$\frac{m_1}{m_0} = \frac{\epsilon}{1 - J_1/2} \frac{1 - J_0}{h_0} = \frac{\epsilon}{h_0} \frac{1 - J_0}{1 - J_1/2}$$

The network amplifies the input if and only if $J_1 > 2J_0$.

Trigonometry:

$$\cos(\theta + \phi) = \cos(\theta) \cos(\phi) - \sin(\theta) \sin(\phi) \quad (6)$$

$$\int_0^{2\pi} \frac{d\theta'}{2\pi} \cos(\theta') = \int_0^{2\pi} \frac{d\theta'}{2\pi} \sin(\theta') = 0 \quad (7)$$

$$\int_0^{2\pi} \frac{d\theta'}{2\pi} \cos(\theta')^2 = \int_0^{2\pi} \frac{d\theta'}{2\pi} \sin(\theta')^2 = \frac{1}{2} \quad (8)$$

$$\int_0^{2\pi} \frac{d\theta'}{2\pi} \cos(\theta') \sin(\theta') = 0 \quad (9)$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (10)$$

$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \quad (11)$$