

[https://github.com/Elieoriol/2021-UlmM2\\_ThNeuro/tree/master/TD9](https://github.com/Elieoriol/2021-UlmM2_ThNeuro/tree/master/TD9)

This supplementary tutorial tackles some topics of coding in Neuroscience, namely mutual information, Fisher information, and Bayesian inference.

## 1 Mutual Information

### 1.1 Characterizing the distribution of a discrete random variable

Suppose  $X$  is a random variable which can take the discrete values  $\{X_1, \dots, X_n\}$  with probability  $p(X_i)$ . The entropy of the distribution is defined as:

$$H(X) = - \sum_{i=1}^n p(X_i) \log(p(X_i))$$

This is also called the "information" of the distribution, and we shall see why. Suppose that the  $X_i$  correspond to the possible colors of a ball picked randomly from an urn. I observe the color of the ball, and you can ask me questions which I can only answer with yes or no. Calculate the entropy and the average number of questions needed to find out the color of the ball in the following cases:

1. If there is only one possible color.
2. If there are two possible colors.
3. If half the balls are red, one fourth are green and one fourth are blue.

In general the entropy of  $X$  is an upper bound on the average number of questions needed to find out  $X$ , with the unit: 1 question = 1 bit of information =  $\log(2)$ .

### 1.2 Mutual information between two discrete random variables

The brain is typically viewed as gathering information about the outside world through its sensors. One way to model this is to consider that the events  $s$  (for stimulus) in the outside world are stochastic and that the activity  $r$  (for response) of the neurons is correlated with the exterior events. The mutual information between stimulus and response quantifies how much observing one tells you about the other:

$$I(s, r) = H(s) - H(s|r) = H(r) - H(r|s) = H(r) + H(s) - H(s, r) = \sum_{s, r} p(s, r) \log \left[ \frac{p(s, r)}{p(s)p(r)} \right]$$

Answer the following questions without calculation (give the answer in bits):

1. What is the mutual information between stimulus and response if they are uncorrelated?
2. Consider a binary stimulus ( $s = A, B$ ), and a single binary neuron with activity  $r = 0$  if  $s = A$  and  $r = 1$  if  $s = B$ . If both stimuli have the same probability, what is the mutual information between  $s$  and the neural activity?
3. Consider now two neurons, with activities  $r_i(s)$ ,  $i = 1, 2$ . Suppose that, if  $s = A$ ,  $r_1(A) = r_2(A) = 1$ , and if  $s = B$ ,  $r_1(B) = r_2(B) = 0$ . Obviously the neural code is redundant. We want to quantify this redundancy.

What is the mutual information  $I(s, \{r_1, r_2\})$ ? Between the stimulus and only neuron 1,  $I(s, r_1)$ ?

One way to define the redundancy is  $R \equiv I(s, r_1) + I(s, r_2) - I(s, \{r_1, r_2\})$ . What is the redundancy in this particular case?

### 1.3 Mutual information for continuous random variables

Entropy and mutual information can be extended to the case of continuous random variables (in which the sum becomes an integral), according to:

$$H(s) = \int ds p(s) \log [p(s)] \quad (1)$$

$$I(s, r) = \int ds dr p(s, r) \log \left[ \frac{p(s, r)}{p(s)p(r)} \right] \quad (2)$$

1. Suppose the neural response is then processed:  $s \rightarrow r \rightarrow x$ . Show that processing cannot increase information:  $I(s, x) \leq I(s, r)$  (hint: use the concavity of the logarithm). **TODO**
2. Compute the entropy of the Gaussian distribution:

$$P(r) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{(r-r_0)^2}{2\sigma^2}}$$

3. Consider the simple linear system,

$$r = WS + z$$

where  $S$  is a scalar stimulus,  $W$  is a (positive) weight,  $z$  is a Gaussian noise with zero mean and variance  $\sigma^2$ :

$$P(z) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{z^2}{2\sigma^2}}$$

and the stimulus has also a Gaussian distribution,

$$\rho(S) = \frac{1}{(2\pi c^2)^{1/2}} e^{-\frac{S^2}{2c^2}}$$

Compute the mutual information  $I(r, S)$  between the stimulus  $S$  and the neural response  $r$ .

4. Compute the mutual information  $I(r, S)$  for the model

$$r = \sum_{j=1}^N W_j S_j + z$$

with Gaussian inputs of zero mean,  $\langle S_j \rangle = 0$  for every  $j$ , and covariance matrix  $C$ :

$$\langle S_j S_{j'} \rangle = C_{j,j'}$$

Discuss the maximization of mutual information with respect to the choice of the weights  $\{W_j, j = 1, \dots, N\}$ .

## 2 Fisher Information

Shannon's information is a measure of how much information the response of a neuron provides about the whole stimulus space. However, individual neurons in the brain appear to be "tuned" to certain regions of the stimulus space. It can therefore be useful to introduce a local measure of information, such as the Fisher information, which can be characterized in two equivalent ways.

### 2.1 Distance between probability distributions

The 'precision' of the information that a neuron provides about a stimulus  $s_0$  corresponds to how easy it is, given the response of that neuron, to tell  $s_0$  apart from nearby values  $s_0 + \delta s$ . The more the distributions  $p(r|s_0)$  and  $p(r|s_0 + \delta s)$  overlap, the less 'precise' this information is. To quantify the distance between these two distributions, we introduce the Kullback-Leibler divergence:

$$KL(p||q) = \int dx p(x) \log \left[ \frac{p(x)}{q(x)} \right] \quad (3)$$

$$KL(p||q) \geq 0 \quad (4)$$

$$KL(p||q) = 0 \Leftrightarrow p = q \quad (5)$$

Note that this distance has already been used to define the mutual information between  $s$  and  $r$  as the distance between  $p(s, r)$  and  $p(s)p(r)$ .

1. Sketch  $KL(p(r|s_0)||p(r|s))$  as a function of  $s$ .
2. Explain why the following gives a measure of the information that  $r$  locally provides about the stimulus:

$$F(s_0) = \frac{d^2 KL(p(r|s_0)||p(r|s))}{ds^2}(s_0) = - \int dr p(r|s_0) \frac{\partial}{\partial S} \log(p(r|s))(s_0)$$

### 2.2 Variance of the locally optimal estimator

Another approach to quantify the 'precision' of the information that the neural response provides about the stimulus is to try to estimate the stimulus from the response, using an estimator  $\hat{s}(r)$ . We only consider locally unbiased estimators; estimators which are accurate on average for values of the stimulus close to  $s_0$ :

$$\frac{\partial}{\partial S} \langle \hat{s} \rangle(s_0) = 1$$

where  $\langle \hat{s} \rangle(s) = \int \hat{s}(r) p(r|s) dr$ .

1. Such an estimator, although accurate on average, will generally not provide an exact estimate of the stimulus on each trial. How could we measure the 'precision' of such an estimator?
2. Using the fact that the estimator is unbiased, and the Cauchy-Schwarz inequality:

$$\int f(x)g(x) \, dx \leq \int f^2(x) \, dx \int g^2(x) \, dx$$

with equality if and only if  $f(x) = ag(x)$ , show that:

$$\int dr p(r|s_0)(\hat{s}(r) - s_0)^2 \int dr p(r|s_0) \left( \frac{\partial}{\partial S} \log(p(r|s))(s_0) \right)^2 \geq 1$$

Using the fact that the probability distribution is normalized, it is possible to show that this limit which we obtain on the 'precision' of the information is equivalent to the Fisher Information defined in the previous section:

$$F(s) = \int dr p(r|s) \left[ \frac{\partial}{\partial S} \log p(r|s) \right]^2 = - \int dr p(r|s) \frac{\partial^2}{\partial S^2} \log p(r|s)$$

3. Using the case of equality in the Cauchy-Schwarz inequality, find a locally unbiased estimator whose variance is equal to the inverse of the Fisher Information.

## 2.3 Some examples

1. Suppose that the mean response of a neuron to a stimulus  $s$  is  $f(s)$  and the variance of the neuron's response is  $\sigma(s)^2$ . How do you expect the Fisher Information to depend on  $f(s)$  and  $\sigma(s)$ ?
2. Give the Fisher Information for a neuron with a Poisson firing rate:

$$P(r|s) = \frac{f(s)^r}{r!} e^{-f(s)}$$

What is the optimal estimator?

3. What is the Fisher Information for a neuron with Gaussian noise:

$$P(r|s) = \frac{1}{(2\pi)^{1/2}\sigma(s)} \exp\left(-\frac{(r - f(s))^2}{2\sigma(s)^2}\right)$$

4. What is the Fisher Information for two independent neurons?

What is the optimal estimator? For simplicity, consider that the variance is constant:  $\sigma'(s) = 0$