

TD8: Learning II

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https://github.com/Elieoriol/2021_UlmM2_ThNeuro/tree/master/TD8

Following on our series of tutorials on learning, this one is devoted to unsupervised learning.

1 Inputs

We consider a neuron receiving two inputs, for example visual input from the left eye I_L and visual input from the right eye I_R . Each input is drawn from a random distribution of mean 0 and variance v . Moreover, the two inputs are correlated according to: $\langle I_L I_R \rangle = c$.

1. For $v = 1$ and $c = -1, 0, 1$, sketch a distribution where each input varies between -1 and 1. Explain why for visual input we should have $c \geq 0$.
2. Show that $-v \leq c \leq v$.
3. What are the symmetry axes \vec{e}_1, \vec{e}_2 of the distribution? Write them as a function of the basis vectors \vec{e}_L, \vec{e}_R . For any vector $\vec{X} = x_L \vec{e}_L + x_R \vec{e}_R$, what are the corresponding coordinates in the new basis $\vec{X} = x_1 \vec{e}_1 + x_2 \vec{e}_2$?
4. Calculate the correlations $\langle I_1^2 \rangle, \langle I_2^2 \rangle, \langle I_1 I_2 \rangle$.

2 Hebbian learning algorithm

The activity of the neuron is given by $V = \vec{W} \cdot \vec{I}$. We consider a Hebbian learning rule in which, every time an input $\vec{I}(t)$ is presented, the neuron weights are updated according to:

$$\vec{W}(t+1) = \vec{W}(t) + \epsilon V(t) \vec{I}(t) \quad (1)$$

5. We denote α the angle between $\vec{I}(t)$ and $\vec{W}(t)$. Supposing that $\|\vec{I}\| = 1$, sketch the update as a function of α . How does $\|\vec{W}\|$ evolve?

To make more precise statements, we remove ϵ to simplify and study the mean dynamics:

$$\frac{d\vec{W}}{dt} = \langle V(t) \vec{I}(t) \rangle \quad (2)$$

where the average $\langle \cdot \rangle$ is taken over the distribution of the inputs \vec{I} .

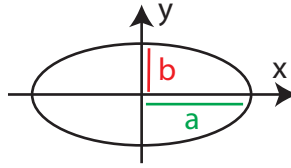
6. For each of the symmetry axes \vec{e}_1, \vec{e}_2 of the input distribution, suppose that initially \vec{W} is along one of this direction, then in which direction is $\frac{d\vec{W}}{dt}$? Along which of these two directions will $\frac{d\vec{W}}{dt}$ have the largest magnitude?
7. Obtain a linear differential equation on \vec{W} . What are the eigenvectors and associated eigenvalues? Comment on the dynamics.
8. We add a "homeostatic" term to the dynamics so as to prevent the weights from growing exponentially:

$$\frac{d\vec{W}}{dt} = \langle V(t)\vec{I}(t) \rangle - \langle V(t)^2 \rangle \vec{W}(t) \quad (3)$$

Can we obtain a **linear** differential equation on \vec{W} ? Obtain a differential equation on \vec{W} in the basis (\vec{e}_1, \vec{e}_2) .

9. Draw the nullclines. For this you will need the equation of an ellipse, given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (4)$$



What are the equilibrium points for \vec{W} ? Are they stable?

10. We would now like to study competitive Hebbian learning. We add a term to the dynamics so as to introduce competition between the left and right inputs. In the (\vec{e}_L, \vec{e}_R) , the dynamics are now given by:

$$\frac{d\vec{W}}{dt} = \langle V(t)\vec{I}(t) \rangle - \langle V(t) \left(\frac{I_L + I_R}{2} \right) \rangle \quad (5)$$

Obtain a linear differential equation on \vec{W} in the basis (\vec{e}_1, \vec{e}_2) . Comment on the dynamics.

11. We enforce the weights to remain positive. Show that, depending on the initial conditions, only one of the two weights will be non-zero.