M2 Biology

TD9b: Coding II

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https://github.com/Elieoriol/2021_UlmM2_ThNeuro/tree/master/TD9b

This last tutorial focuses on a model of unsupervised learning.

1 Bayesian inference

We consider a population of N neurons with various tuning curves $f_i(s)$: when a stimulus s is presented, the neuron i will fire on average $f_i(s)$ spikes in a given timebin. However, on each trial, the number of spikes emitted by neuron i is drawn from a Poisson process of mean $f_i(s)$. This variability is independent across neurons.

1. Suppose that a given stimulus s is presented, what is the probability of observing a given pattern of spikes $\{n_i\}$?

$$P(\{n_i\}|s) = \prod_{i=1}^{N} \frac{f_i(s)^{n_i}}{n_i!} \exp(-f_i(s))$$

2. Supposing that a given pattern of spikes $\{n_i\}$ was observed, can you tell what stimulus was presented?

   The same pattern can be caused by various stimuli, therefore by observing the pattern you can't tell exactly what the stimulus was.

3. Suppose that the prior on the stimulus $p(s)$ is uniform. Suppose also that the tuning curves all have the same shape, a bell, with the preferred stimuli s_i of the various neurons evenly distributed across the stimulus range. Suppose that a given pattern of spikes $\{n_i\}$ was observed, give an estimate of the stimulus that was presented.

   Each neuron votes for its preferred stimulus, and our estimate is a weighted average

$$\frac{\sum_{i=1}^{N} n_i s_i}{\sum_{i=1}^{N} n_i}$$

4. How does the accuracy of this estimate depend on the height f_0 and width σ of the tuning curves and on the number of neurons?

   Using the result of the previous exercice, the Fisher information is

$$\sum_{i=1}^{N} \frac{f_i'(s)^2}{f_i(s)} \approx N \frac{(f_0/\sigma)^2}{f_0} = \frac{N f_0}{\sigma^2}$$

5. We will now be more quantitative: find the probability distribution of the stimulus using Baye's rule

$$p(s|\{n_i\}) = \frac{p(\{n_i\}|s)p(s)}{p(\{n_i\})}$$

Include all the terms which don't depend on s in a function $\Phi(\{n_i\})$.

$$p(s|\{n_i\}) = \frac{p_s}{p(\{n_i\})} \prod_{i=1}^{N} \frac{1}{n_i!} \exp(-\sum_{i=1}^{N} f_i(s)) \exp(\sum_{i=1}^{N} n_i \log(f_i(s))) \tag{1}$$

Where $p_s = p(s) \forall s$.

The tuning curves being evenly distributed along the stimulus range, we can consider that $\sum_{i=1}^{N} f_i(s)$ doesn't depend on s.

$$p(s|\{n_i\}) = \Phi(\{n_i\}) \exp(\sum_{i=1}^{N} n_i \log(f_i(s))) \tag{2}$$

6. Consider that the tuning curves are Gaussian:

$$f_i(s) = f_0 \exp(-\frac{(s-s_i)^2}{2\sigma^2})$$

Show that $p(s|\{n_i\})$ is also a Gaussian. What is its mean and variance? How does the variance depend on the various parameters? Under what conditions may the variance become infinitely small?

- *This is the maximum of* $\log(p(\{n_i\}|s))$*:*

$$\frac{\partial}{\partial s} \log(p(\{n_i\}|s)) = \sum_{i=1}^{N} n_i(-2\frac{(s-s_i)}{2\sigma^2}) = 0 \Leftrightarrow \sum_{i=1}^{N} n_i s = \sum_{i=1}^{N} n_i s_i \tag{3}$$

$$\Leftrightarrow s = \frac{\sum_{i=1}^{N} n_i s_i}{\sum_{i=1}^{N} n_i} \tag{4}$$

- $$p(s|\{n_i\}) = \Phi(\{n_i\}) \exp(\sum_{i=1}^{N} -n_i \frac{(s-s_i)^2}{2\sigma^2}) \tag{5}$$

- $$\sum_{i=1}^{N} n_i(s-s_i)^2 = (\sum_{i=1}^{N} n_i)s^2 - 2(\sum_{i=1}^{N} n_i s_i)s + (\sum_{i=1}^{N} n_i s_i^2) \tag{6}$$

$$= (\sum_{i=1}^{N} n_i)(s - \frac{\sum_{i=1}^{N} n_i s_i}{\sum_{i=1}^{N} n_i})^2 + C \tag{7}$$

The mean is

$$\frac{\sum_{i=1}^{N} n_i s_i}{\sum_{i=1}^{N} n_i}$$

The variance is

$$\frac{\sigma^2}{\sum_{i=1}^{N} n_i}$$

- *The variance of the posterior depends on the sum of the* n_i, *which is proportional to the number of neurons times the height of the tuning curves, therefore the variance is proportional to*

$$\frac{\sigma^2}{N f_0}$$

- *The variance becomes infinitely small if the number of neurons becomes infinitely large or if the mean firing rate becomes infinitely large.*

7. Suppose now that instead of responding to s, neurons respond to a jittered version of s, namely $\hat{s}$, where $p(s|\hat{s})$ is a Gaussian of variance σ_j^2. The number of spikes they fire is drawn from a Poisson distribution of mean $f_i(\hat{s})$.

   For a given stimulus s, is the variability still independent across neurons? Does the variance of $p(s|\{n_i\})$ still become infinitely small in the conditions considered previously?

   The variability is now correlated across neurons

$$p(s|\{n_i\}) = p(s|\hat{s})p(\hat{s}|\{n_i\})$$

   It is now a product of Gaussians: the inverse variances add. In the previous conditions, $p(\hat{s}|\{n_i\})$ *becomes infinitely narrow, its variance becomes infinitely small. However the variance of* $p(s|\{n_i\})$ *is necessarily larger than the variance of* $p(s|\hat{s})$ *which doesn't depend on the number of neurons or on their mean firing rate.*