M2 Biology **18/11/21**

TD7: Learning II

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TD material is available at:

`https://github.com/Elieoriol/2122_UlmM2_ThNeuro/tree/master/TD7`

Following on our series of tutorials on learning, this one is devoted to unsupervised learning.

1 Inputs

We consider a neuron receiving two inputs, for example visual input from the left eye I_L and visual input from the right eye I_R. Each input is drawn from a random distribution of mean 0 and variance v. Moreover, the two inputs are correlated according to: $\langle I_L I_R \rangle = c$.

1. For $v = 1$ and $c = -1, 0, 1$, sketch a distribution where each input varies between -1 and 1. Explain why for visual input we should have $c \geq 0$.
2. Show that $-v \leq c \leq v$.
3. What are the correlation and anti-correlation axes $\vec{e}_1, \vec{e}_2$ of the distribution? Write them as a function of the basis vectors $\vec{e}_L, \vec{e}_R$. For any vector $\vec{X} = x_L \vec{e}_L + x_R \vec{e}_R$, what are the corresponding coordinates in the new basis $\vec{X} = x_1 \vec{e}_1 + x_2 \vec{e}_2$?
4. Calculate the correlations $\langle I_1^2 \rangle$, $\langle I_2^2 \rangle$, $\langle I_1 I_2 \rangle$.

2 Hebbian learning algorithm

The activity of the neuron is given by $V = \vec{W} \cdot \vec{I}$. We consider a Hebbian learning rule in which, every time an input $\vec{I}(t)$ is presented, the neuron weights are updated according to:

$$\vec{W}(t+1) = \vec{W}(t) + \epsilon V(t) \vec{I}(t) \tag{1}$$

5. We denote α the angle between $\vec{I}(t)$ and $\vec{W}(t)$. Supposing that $\|\vec{I}\| = 1$, sketch the update as a function of α. How does $\|\vec{W}\|$ evolve?

To make more precise statements, we remove ϵ to simplify and study the mean dynamics:

$$\frac{\mathrm{d}\vec{W}}{\mathrm{d}t} = \langle V(t) \vec{I}(t) \rangle \tag{2}$$

where the average $\langle \cdot \rangle$ is taken over the distribution of the inputs $\vec{I}$.

6. For each of the axes $\vec{e}_1$, $\vec{e}_2$ of the input distribution, suppose that initially $\vec{W}$ is along one of this direction, then in which direction is $\frac{\mathrm{d}\vec{W}}{\mathrm{d}t}$? Along which of these two directions will $\frac{\mathrm{d}\vec{W}}{\mathrm{d}t}$ have the largest magnitude?

7. Obtain a linear differential equation on $\vec{W}$. What are the eigenvectors and associated eigenvalues? Comment on the dynamics.

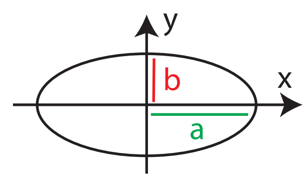
8. We add a "homeostatic" term to the dynamics so as to prevent the weights from growing exponentially:

$$\frac{\mathrm{d}\vec{W}}{\mathrm{d}t} = \langle V(t)\vec{I}(t)\rangle - \langle V(t)^2\rangle\vec{W}(t) \tag{3}$$

Can we obtain a **linear** differential equation on $\vec{W}$? Obtain a differential equation on $\vec{W}$ in the basis $(\vec{e_1}, \vec{e_2})$.

9. Draw the nullclines. For this you will need the equation of an ellipse, given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4}$$



What are the equilibrium points for $\vec{W}$? Are they stable?

10. We would now like to study competitive Hebbian learning. We add a term to the dynamics so as to introduce competition between the left and right inputs. In the $(\vec{e_L}, \vec{e_R})$, the dynamics are now given by:

$$\frac{\mathrm{d}\vec{W}}{\mathrm{d}t} = \langle V(t)\vec{I}(t)\rangle - \langle V(t)\begin{pmatrix} \frac{I_L+I_R}{2} \\ \frac{I_L+I_R}{2} \end{pmatrix}\rangle \tag{5}$$

Obtain a linear differential equation on $\vec{W}$ in the basis $(\vec{e_1}, \vec{e_2})$. Comment on the dynamics.

11. We enforce the weights to remain positive. Show that, depending on the initial conditions, only one of the two weights will be non-zero.