M2 Biology **21/10/21**

TD5: Rate Models

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TD material is available at:

`https://github.com/Elieoriol/2122_UlmM2_ThNeuro/tree/master/TD5`

In this tutorial, we will consider a very standard example of rate model: the ring model.

1 Ring model

We consider a network of neurons responsive to a stimulus θ which spans the range $[0, 2\pi]$. Each neuron has a preferred stimulus and the connection strength between two neurons respectively preferring θ_1 and θ_2 is proportional to $J_0 + J_1 \cos(\theta_1 - \theta_2)$. Additionally, each neuron receives en external input which depends on the neuron's preferred stimulus $h(\theta)$.

In the limit where the number of neurons is large and their preferred stimuli uniformly span the range $[0, 2\pi]$, we can write the neural activity as a continuous function $m(\theta, t)$. The input current to a neuron preferring θ can then be written as:

$$I(\theta, t) = h(\theta) + \int_0^{2\pi} \frac{\mathrm{d}\theta'}{2\pi} \left[J_0 + J_1 \cos(\theta - \theta')\right] m(\theta', t)$$

The activity then evolves according to:

$$\frac{\mathrm{d}m(\theta, t)}{\mathrm{d}t} = -m(\theta, t) + f[I(\theta, t)] \tag{1}$$

$$f(x > 0) = x \tag{2}$$

$$f(x < 0) = 0 \tag{3}$$

1. Suppose that there is a constant, uniform and positive external current $h(\theta) = h_0$, which is sufficient for there to be positive, uniform network activity $m(\theta, t) = m_0$. What is the current received by each neuron? What is the network activity m_0? How does it depend on J_0?

We wish to study whether this uniform state is stable, we therefore consider small perturbations around it: $m(\theta, t) = m_0 + \delta m(\theta, t)$. We wish to see how these evolve with time.

To do so, we want to find a simple description of the dynamics. We introduce the order parameters:

$$M(t) = \int_0^{2\pi} \frac{\mathrm{d}\theta'}{2\pi} \delta m(\theta', t) \tag{4}$$

$$C(t) = \int_0^{2\pi} \frac{\mathrm{d}\theta'}{2\pi} \delta m(\theta', t) e^{i\theta'} \tag{5}$$

2. What does $M(t)$ correspond to?
3. Suppose that the perturbation is uniform:

$$\delta m(\theta, t) = \epsilon$$

   What are the values of M and C?

4. Suppose that the perturbation is a small bump centered around the angle ϕ:

$$\delta m(\theta, t) = \epsilon \cos(\theta - \phi) = \epsilon \, \frac{e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}}{2}$$

   What are the values of M and C?

5. What does $C(t)$ correspond to?
6. Now that we have understood what $M(t)$ and $C(t)$ characterize, we will try to obtain a description of the dynamics in terms of the evolution of these two order parameters. Linearize the dynamics of the activity around m_0 and express it as a function of $M(t)$ and $C(t)$.
7. Derive the differential equations governing the evolution of the order parameters.
8. Under what conditions is the uniform activity stable? What happens when either of these conditions is not met?
9. Consider $J_0 < 1$, $J_1 < 2$. Consider an external input with weak modulation, $h(\theta) = h_0 + \epsilon \cos(\theta)$, where $\epsilon \ll 1$. What is the profile of activity of the network induced by such an external input?
10. Suppose that the firing rate is given by $m(\theta, t) = m_0 + m_1 \cos(\theta)$, under what conditions does the network amplify the input, ie: $\frac{m_1}{m_0} > \frac{\epsilon}{h_0}$?

Trigonometry:

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \tag{6}$$

$$\int_0^{2\pi} \frac{d\theta'}{2\pi} \cos(\theta') = \int_0^{2\pi} \frac{d\theta'}{2\pi} \sin(\theta') = 0 \tag{7}$$

$$\int_0^{2\pi} \frac{d\theta'}{2\pi} \cos(\theta')^2 = \int_0^{2\pi} \frac{d\theta'}{2\pi} \sin(\theta')^2 = \frac{1}{2} \tag{8}$$

$$\int_0^{2\pi} \frac{d\theta'}{2\pi} \cos(\theta')\sin(\theta') = 0 \tag{9}$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{10}$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{11}$$