

## TD4: Balanced Networks

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TD material is available at:

[https://github.com/Elieoriol/2122\\_UlmM2\\_ThNeuro/tree/master/TD4](https://github.com/Elieoriol/2122_UlmM2_ThNeuro/tree/master/TD4)

In this tutorial, we will try to model the stochasticity neurons are subject to and its consequences on balanced networks.

## 1 Noisy Leaky Integrate-and-Fire neurons

Neurons are connected to a multitude of other neurons through synapses. Therefore, the current which was supposed constant in our early LIF model, is in fact stochastic in biological networks. We propose to model this stochastic current with a white noise of mean  $\mu$  and variance  $\tau_m \sigma^2$ :

$$\tau_m \frac{dV}{dt} = E_l - V + I(t)/g_l \quad I(t) = \mu + \sigma \sqrt{\tau_m} \eta(t) \quad (1)$$

with  $\eta(t)$  a reduced centered white noise defined by  $\langle \eta(t) \rangle = 0$  and  $\langle \eta(t) \eta(t') \rangle = \delta(t - t')$ .

To simulate such a stochastic system, one can pick a small step  $\Delta t$  as in the deterministic case. Increments  $\Delta V$  in membrane potential therefore write:

$$\Delta V = \frac{E_l - V + \mu/g_l}{\tau_m} \Delta t + \frac{\sigma/g_l}{\sqrt{\tau_m}} \Delta \eta_t \quad (2)$$

where  $\Delta \eta_t = \eta(t) \Delta t$  is called a Wiener process; it represents the integral of the white noise, in other words the white noise summed over the interval  $\Delta t$ . An important property to remember of a Wiener process  $W$  is the following:

$$W(t) - W(s) \sim \mathcal{N}(0, t - s) \quad (t > s) \quad (3)$$

Therefore,  $\Delta \eta_t \sim \sqrt{\Delta t} \mathcal{N}(0, 1)$  and:

$$\Delta V = \frac{E_l - V + \mu/g_l}{\tau_m} \Delta t + \frac{\sigma}{g_l} \sqrt{\frac{\Delta t}{\tau_m}} \mathcal{N}(0, 1) \quad (4)$$

*Note: One can see why the variance was defined as  $\tau_m \sigma^2$  instead of just  $\sigma^2$ ; as we have seen that  $\eta(t)$  has the dimension of the inverse square root of a time,  $\sigma$  then has the dimension of a current.*

1. **num** Can you describe the effect of the white noise term, particularly in the sub-threshold regime? What is the influence of  $\sigma$  on the f-I curve? Show numerically that the new f-I curve is given by:

$$\nu = \frac{1}{\tau_m \sqrt{\pi} \int_{\frac{V_T - m}{\sigma_0}}^{\frac{V_T h - m}{\sigma_0}} e^{s^2} (1 + \operatorname{erf}(s)) ds} \quad (5)$$

with  $m = E_l + \mu/g_l$  and  $\sigma_0 = \frac{\sigma}{g_l \sqrt{\tau_m}}$ . The same parameters as in the first tutorial can be used.

*Note: The analytical resolution for the equations with noise are much harder than the classical ones. For more information see Dynamics of the Firing Probability of Noisy Integrate-and-Fire Neurons, N. Fourcaud and N. Brunel, 2002 Neural Computation.*

2. **num** Plot the histogram of interspike intervals under different conditions of noise in the sub-threshold and supra-threshold regimes. How does it look like in the sub-threshold regime?

## 2 Poisson Process

An experimenter can record the times at which a neuron emits spikes in response to a given stimulus. However, repeating the same stimulus, one observes that the spikes don't occur at the same time; the number of spikes is not even conserved. One way to describe these spike trains is therefore to use a random process with a given rate of spike occurrence  $R$ .

The simplest random process is the Poisson process that describes well the statistics of firing of single neurons subject to stochastic inputs as seen in the previous exercise. In the Poisson process, all events (here spikes) are independent, such that the mean number of spikes in an interval of duration  $T$  is  $RT$ , and the actual number of spikes on a given trial in one interval is independent of the number of spikes in any other interval.

1. This property of independence is enough to obtain the full probability distribution of the number of spikes in any given interval. A way to obtain it is to divide the interval into  $M$  bins of length  $\Delta T \ll 1/R$  such that there is at most one spike per bin (for instance  $\Delta T$  can correspond to the absolute refractory period of the neuron). What is the probability of observing  $n$  spikes in the total interval?
2. What is the distribution of interspike intervals?
3. Let  $X$  be a random variable with probability density  $P(X)$ . We define:

$$G_X(\alpha) = \int e^{\alpha X} P(x) dX = \langle e^{\alpha X} \rangle_X \quad (6)$$

Show that:

$$\left. \frac{d^n G_X}{d\alpha^n} \right|_{\alpha=0} = \langle X^n \rangle \quad (7)$$

$G_X$  is called the **moment-generating function** of the random variable  $X$ . We have a similar definition with sums instead of integrals in the case of a discrete probability distribution.

4. Use the moment-generating function to compute the mean and variance of the number of spikes generated by a homogeneous Poisson process of rate  $R$  in a window of size  $T$ . Compute the Fano factor  $\Delta n^2 / \langle n \rangle$ .

### 3 Poisson inputs in a balanced network

Consider a neuron that receives  $C_E$  excitatory synapses and  $C_I$  inhibitory synapses. We will represent incoming synaptic currents by delta pulses: a presynaptic spike in neuron  $k$  at time  $t_0$  elicits a postsynaptic current  $i_k(t)$  given by:

$$i_k(t) = \tau_m J_k \delta(t - t_0) \quad (8)$$

where  $\tau_m$  is the membrane time-scale and  $J_k$  is the strength of synapse.

1. Assuming that all excitatory synapses have the same strength  $J$  and all inhibitory synapses have the same strength  $-gJ$ , compute the mean and the auto-correlation function of the total synaptic input  $I(t)$  for Poisson presynaptic spike trains of rate  $r$ . Compare with a white noise process of mean  $\mu$  and variance  $\tau_m \sigma^2$ .
2. The membrane potential dynamics of an integrate-and-fire neuron is given by:

$$\tau_m \frac{dV}{dt} = -V + I(t) \quad (9)$$

where  $\tau_m$  is the membrane time constant and  $I(t)$  is an arbitrary input current.

Not considering the threshold, compute the mean  $\langle V(t) \rangle$  and auto-correlation  $\langle V(t)V(t+\tau) \rangle - \langle V(t) \rangle \langle V(t+\tau) \rangle$  for a white noise input  $I(t) = \mu + \sqrt{\tau_m} \sigma \cdot \eta(t)$ , with  $\langle \eta(t) \rangle = 0$  and  $\langle \eta(t)\eta(t') \rangle = \delta(t - t')$ .  $\langle \cdot \rangle$  indicates here an average across the realizations of the white noise.

*Hint: Consider the function  $f_t = V(t)e^{t/\tau_m}$  and apply Itô's lemma to it. This is the standard way to get the properties of the Ornstein-Uhlenbeck process that we are in fact studying here. We recall the Itô formula when working with 1-D stochastic variables  $X_t$  subject to a white noise of variance  $\sigma_t^2$  and  $f(X_t, t)$  a sufficiently regular function:*

$$d[f(X_t, t)] = \frac{\partial f}{\partial t}(X_t, t) dt + \frac{\partial f}{\partial x}(X_t, t) dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(X_t, t) \sigma_t^2 dt \quad (10)$$

3. Suppose the integrate-and-fire neuron receives  $C_E$  excitatory synapses and  $C_I$  inhibitory synapse of respective strengths  $J$  and  $-gJ$ . What are the mean and auto-correlation function of the membrane-potential dynamics (forgetting the threshold) if presynaptic spike trains are Poisson with rate  $r$ ?