Differentiation in the polar spherical coordinate

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In the last seminar, we learned how to express velocity $\vec{v}$ in the polar spherical coordinate system. In order to extend it to acceleration $\vec{a} = \dfrac{d\vec{v}}{dt}$, however, it is better to develop a more systematic way of dealing derivative of polar spherical coordinates. Here is what you can do!

問題 1 Unit vectors in the polar spherical coordinate-1

Let's denote the polar spherical coordinates with (r, θ, φ), and their unit vectors with $(\hat{r}, \hat{\theta}, \hat{\varphi})$. See Fig.1(a). Those unit vectors have important differences as compared with the unit vectors of Cartesian coordinates $(\hat{x}, \hat{y}, \hat{z})$. Namely, they change their directions when the point of consideration changes. In another words, they are functions of (r, θ, φ); *i.e.* $\hat{r}(r, \theta, \varphi)$, $\hat{\theta}(r, \theta, \varphi)$, $\hat{\varphi}(r, \theta, \varphi)$

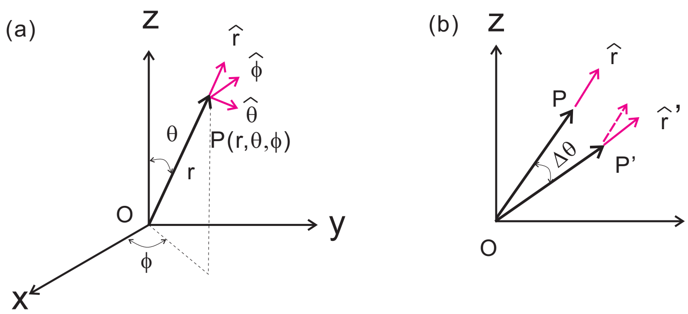

図 1: (a) Polar spherical coordinate and its unit vectors. (b) Calculation of $\frac{\partial \hat{r}}{\partial \theta}$. The dashed arrow is $\hat{r}$, which is parallel-displaced to P'.

	$\hat{r}$	$\hat{\theta}$	$\hat{\varphi}$
$\partial/\partial r$	?	?	?
$\partial/\partial \theta$	$\hat{\theta}$	?	?
$\partial/\partial \varphi$	?	?	$-(\hat{r}\sin\theta + \hat{\theta}\cos\theta)$

表 1: Derivatives of the unit vectors in the spherical coordinate system

For example, if $P(r, \theta, \varphi)$ moves to $P'(r, \theta + \Delta\theta, \varphi)$, as shown in Fig.1(b), then the unit vector $\hat{r}$ changes to $\hat{r}'$. The difference between the two is given by

$$\hat{r}' - \hat{r} \simeq \Delta\theta \ \ \hat{\theta}.$$

Note that the direction of $\hat{r}' - \hat{r}$ is in the direction of $\hat{\theta}$. Thus it implies that

$$\frac{\partial \hat{r}}{\partial \theta} \equiv \lim_{\Delta\theta \to 0} \frac{\hat{r}' - \hat{r}}{\Delta\theta} = \hat{\theta}. \tag{1}$$

Let's consider another example. This time we want to calculate $\dfrac{d\hat{\varphi}}{d\varphi}$. As shown in Fig.2(a), we consider $\hat{\varphi}$ at P and $\hat{\varphi}'$ at P'. The difference vector $\hat{\varphi}' - \hat{\varphi}$ lies in the $x-y$ plane and directs from P to Q; its magnitude is $\Delta\varphi$. See Fig.2(b). The vector $\overrightarrow{PQ}$ can be represented by $-(\hat{r}\sin\theta + \hat{\theta}\cos\theta)$, which is normalized to the unit length.. See Fig.2(c). We thus conclude

$$\frac{\partial\hat{\varphi}}{\partial\varphi} \equiv \lim_{\Delta\varphi\to 0}\frac{\hat{\varphi}'-\hat{\varphi}}{\Delta\varphi} = -(\hat{r}\sin\theta + \hat{\theta}\cos\theta). \tag{2}$$

Now your task is to complete Table 1.

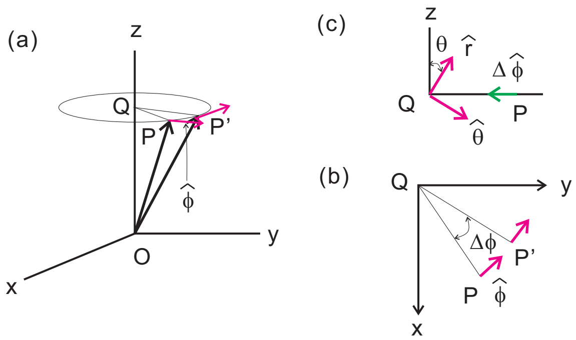

図 2: Calculation of $\frac{\partial\hat{\varphi}}{\partial\varphi}$. (a) The two nearby points P and P' to be considered. (b) Change in $\hat{\varphi}$ in xy palne. (c) the vector from P to Q.

問題 2 Unit vectors in the polar spherical coordinate-2

If you don't like the method above, there is another way. In terms of the Cartesian components (x, y, z), the unit vectors $(\hat{r}, \hat{\theta}, \hat{\varphi})$ are expressed by

$$\begin{aligned}
\hat{r} &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
\hat{\theta} &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\
\hat{\varphi} &= (-\sin\varphi, \cos\varphi, 0),
\end{aligned} \tag{3}$$

where $(\cdots, \cdots, \cdots)$ denotes the Cartesian components. Prove the followings:

$$\begin{aligned}
&\hat{r}\cdot\hat{\theta} = 0, \qquad \hat{\theta}\cdot\hat{\varphi} = 0, \qquad \hat{\varphi}\cdot\hat{r} = 0 \\
&\hat{r}\times\hat{\theta} = \hat{\varphi}, \qquad \hat{\theta}\times\hat{\varphi} = \hat{r}, \qquad \hat{\varphi}\times\hat{r} = \hat{\theta}
\end{aligned} \tag{4}$$

You can differentiate each of Eq.(3) with respect to r, θ, φ. For example,

$$\begin{aligned}
\frac{\partial\hat{r}}{\partial\theta} &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) = \hat{\theta} \\
\frac{\partial\hat{\varphi}}{\partial\varphi} &= -(\cos\varphi, \sin\varphi, 0) = -(\hat{r}\sin\theta + \hat{\theta}\cos\theta).
\end{aligned} \tag{5}$$

問題　3　Velocity and acceleration in the spherical coordinate system

Now we are ready to calculate $\vec{v}$ and $\vec{a}$ in spherical coordinate system. We start from $\vec{r} = r\hat{r}$. Then

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\left(\frac{d\hat{r}}{dr}\dot{r} + \frac{d\hat{r}}{d\theta}\dot{\theta} + \frac{d\hat{r}}{d\varphi}\dot{\varphi}\right). \tag{6}$$

With help of Table 1, we are able to calculate $\vec{v}$ in terms of (r, θ, φ) and/or $(\dot{r}, \dot{\theta}, \dot{\varphi})$. Give the expression for $\vec{v}$. The you can differentiate $\vec{v}$ with respect to t to calculate $\vec{a}$. Prove that $\vec{a} = (a_r, a_\theta, a_\varphi)$ is given by

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 - r(\sin\theta\dot{\varphi})^2, \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta \\ a_\varphi &= r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta \end{aligned} \tag{7}$$

If we consider a particle (mass $m = 1$) moving in a central force potential, then we know its motion is confined in a plane; thus we can set $\varphi = 0$ without loss of generality. Prove that the angular momentum ($\vec{L} = \vec{r} \times \vec{p}$) of the particle is conserved in this case using Eq.(7).

問題　4　Gradient operator ∇

Suppose that we have a scalar function $f(r, \theta, \varphi)$, which depends on the polar spherical coordinate (r, θ, φ). We consider the values of f at two nearby points $P(r, \theta, \varphi)$ and $P'(r + \Delta r, \theta + \Delta\theta, \varphi + \Delta\varphi)$. First of all, the vector from P to P', denoted by $\Delta\vec{r}$, is given by

$$\Delta\vec{r} \equiv \overrightarrow{PP'} = \hat{r}\,\Delta r + \hat{\theta}\; r\Delta\theta + \hat{\varphi}\; r\sin\theta\Delta\varphi$$

The gradient of a scalar function, $\nabla f(r, \theta, \varphi)$, is defined by

$$\nabla f \cdot \Delta\vec{r} \simeq f(r + \Delta r, \theta + \Delta\theta, \varphi + \Delta\varphi) - f(r, \theta, \varphi). \tag{8}$$

The right hand-side can be expanded to

$$f(r + \Delta r, \theta + \Delta\theta, \varphi + \Delta\varphi) - f(r, \theta, \varphi) = \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial\theta}\Delta\theta + \frac{\partial f}{\partial\varphi}\Delta\varphi \tag{9}$$

Comparing Eq.(9) and Eq.(8), we obtain

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}. \tag{10}$$

Laplace operator Δ is defined to by $\Delta \equiv \vec{\nabla} \cdot \vec{\nabla}$. Prove the following:

$$\begin{aligned} \Delta &= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) \cdot \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \end{aligned}$$