Dirac delta-function

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1 Introduction– what is $\delta(x)$?–

Dirac's delta-function, $\delta(x)$, is a "function" introduced by Dirac. It is 0 everywhere except at $x = 0$ where it is infinitely high, and its area underneath is 1. It is a spike-like function which satisfies the conditions below:

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x) = 1 \tag{1}$$

For any arbitrary function $f(x)$[1], the relation

$$\int_{-\infty}^{-\infty} \delta(x) f(x)\, dx = f(0) \tag{2}$$

is satisfied.

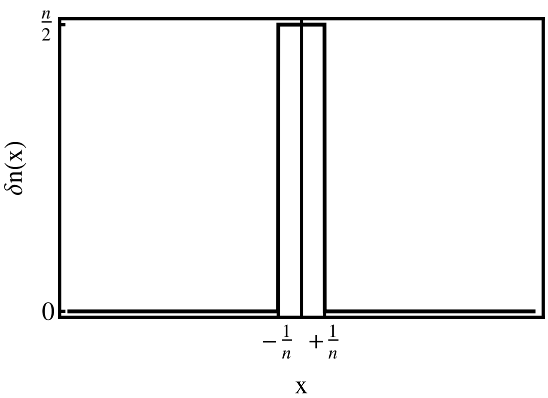

図 1: delta series function ($\delta_n(x)$)

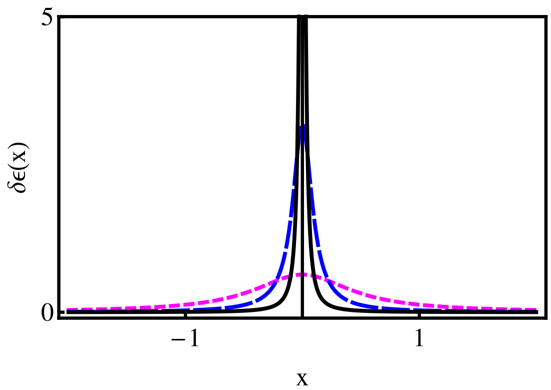

図 2: $\delta_\epsilon(x) = \dfrac{1}{\pi}\dfrac{\epsilon}{x^2+\epsilon^2}$。Parameters: $\epsilon = 0.5$(dotted)、$\epsilon = 0.1$(dashed), and $\epsilon = 0.02$(solid)。

From the view point of "mathematicians", the delta-function cannot be identified as a genuine function, and a better definition is necessary. Here is one way.
Suppose there is a series of functions $\delta_n(x)$, which are all good functions even for mathematicians, satisfying

$$\lim_{n\to\infty} \int_{-\infty}^{-\infty} \delta_n(x) f(x)\, dx = f(0), \tag{3}$$

then we identify $\lim_{n\to\infty} \delta_n(x)$ as $\delta(x)$, and we denote

$$\lim_{n\to\infty} \delta_n(x) \equiv \delta(x). \tag{4}$$

[1]For any functions we use in physics.

Note that in Eq (3) we first perform integration and then take limits. Let's give an example. Consider a series of functions defined by

$$\delta_n(x) = \begin{cases} 0 & (|x| > 1/n) \\ \dfrac{n}{2} & (|x| < 1/n) \end{cases}, \tag{5}$$

where $n(n > 0)$ is a positive parameter. Obviously their areas underneath are always 1 for any n. As $n \to \infty$, the seiries has all necessary properties indicated in Eq (1). In particular, the left side of Eq (3) is given by

$$\int_{-\infty}^{+\infty} \delta_n(x) f(x) dx = \int_{-1/n}^{+1/n} \frac{n}{2} f(x) dx. \tag{6}$$

Replacing $n = 1/h$, in the limit of $h \to 0$, it becomes

$$\frac{1}{2h} \int_{-h}^{h} f(x) dx = \frac{F(h) - F(-h)}{2h}, \quad \longrightarrow \quad f(0) \tag{7}$$

where $F(x)$ is the primitive function of $f(x)$. Thus we conclude that $\delta_n(x)$ approaches to $\delta(x)$ in the limit of $n \to \infty$. From the example above, any series of functions whose area is 1, and which approaches to infinity at $x = 0$, and zero at $x \neq 0$, then the series becomes $\delta(x)$. We denote such a series of functions as the δ-series in this note.

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Example 1 Prove that the functions below is a δ-series.

$$\delta_\epsilon(x) = \lim_{\epsilon \to +0} \frac{1}{\pi} \left(\frac{\epsilon}{x^2 + \epsilon^2} \right) \tag{8}$$

Solution We first show Eq (8) has unit area. Changing the independent variable from x to θ with $x = \epsilon \tan\theta$, the integration becomes

$$\int_{-\infty}^{+\infty} \frac{1}{\pi} \left(\frac{\epsilon}{x^2 + \epsilon^2} \right) dx = \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} \frac{1}{1 + \tan^2\theta} \frac{d\theta}{\cos^2\theta} = \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} d\theta = 1. \tag{9}$$

For $x \neq 0$, $\delta_\epsilon(x)$ approaches to 0 in the limit of $\epsilon \to 0$, while at $x = 0$, it approaches to ∞ in the limit of $\epsilon \to 0$. Thus Eq (8) is the δ-series.

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Problem 1 Sketch the shape of the series functions below and prove that they are the δ-series.

$$\begin{aligned} &(1) \quad \delta_s(x) = \frac{a}{2} e^{-a|x|}, \qquad a \to +\infty \\ &(2) \quad \delta_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2} \right), \qquad \sigma \to +0 \end{aligned} \tag{10}$$

2 Properties of $\delta(x)$

Let's study properties of $\delta(x)$. We assume that $f(x)$ and/or $g(x)$ are square-integrable functions: $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$ and/or $\int_{-\infty}^{+\infty} |g(x)|^2 dx < \infty$. It is easy to show

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x)\delta(x-a)dx &= f(a) \\ \int_{-\infty}^{+\infty} f(x)\delta(kx)dx &= \frac{1}{|k|}f(0) \end{aligned} \tag{11}$$

where a or k are constants. Note that it must be $|k|$, not k, in the denominator of the second formula. Next let's consider $\displaystyle\int_{-\infty}^{+\infty} f(x)\delta\left(g(x)\right) dx$. You will notice that the zeros of $g(x)$ would be important because $\delta(\cdots)$ becomes infinity when $(\cdots)$ becomes zero. We assume $g(x_i) = 0$ for $i = 1, 2, \cdots$, and we expand $g(x)$ around x_i by Taylor series as

$$g(x) = (x - x_i)g'(x_i) + \frac{1}{2}(x - x_i)^2 g''(x_i) + \cdots .$$

setting $a = x_i$ and $k = g'(x_i)$ in Eq (11), we obtain

$$\int_{-\infty}^{+\infty} f(x)\delta\left(g(x)\right) dx = \sum_i \frac{1}{|g'(x)|_{x=x_i}} f(x_i), \qquad\qquad g(x_i) = 0$$

where the summation runs over all zeros.

Problem 2 Calculate the following integrals.

$$\begin{aligned} (i) &: \quad \int_{-\infty}^{+\infty} (1+x^2)\delta(2x-1)dx \\ (ii) &: \quad \int_{-\infty}^{+\infty} (1+x^3)\delta(x^2-1)dx \\ (iii) &: \quad \int_{1}^{+\infty} \frac{1}{x(x+\pi)}\delta(\sin x)dx \end{aligned}$$

Problem 3 Determine the function $f(y)$ below.

$$\begin{aligned} (i) &: \quad f(y) = \int_{-\infty}^{+\infty} x^2\delta(y^2-x)dx \\ (ii) &: \quad f(y) = \int_{-\infty}^{+\infty} x^2\delta(y-x^2)dx \end{aligned}$$

Problem 4 Prove (i) and calculate (ii).

$$\begin{aligned} (i) &: \quad \int_{-\infty}^{+\infty} f(x)\frac{d\delta(x)}{dx}dx = -f'(0) \\ (ii) &: \quad \int_{-\infty}^{+\infty} (1+x^2+x^3)\frac{d\delta(x-1)}{dx}dx \end{aligned}$$

3 δ function and Fourier transformation

We are now in the position to study very important connection between δ function and Fourier transformation. We are going to prove an important formula of

δ function expressed by Fourier transform

$$2\pi\delta(x) = \int_{-\infty}^{+\infty} e^{-ikx} dk. \tag{12}$$

Le's reconsider the δ-series defined in Eq (5). The Fourier transform of $\delta_n(x)$ is given by

$$\widetilde{\delta}_n(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta_n(x) e^{ikx} dx = \frac{n}{\sqrt{2\pi}} \frac{e^{ik/n} - e^{-ik/n}}{2ik} = \frac{1}{\sqrt{2\pi}} \frac{\sin(k/n)}{k/n} \tag{13}$$

where the symbol $\widetilde{\cdots}$ denotes the Fourier transform of the function underneath $(\cdots)$. In the limit of $k/n \to 0$, $\widetilde{\delta}_n(k)$ converges to the value of $1/\sqrt{2\pi}$. We now consider the inverse transform of $\widetilde{\delta}_n(k)$

$$\delta_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{\delta}_n(k) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\sin(k/n)}{k/n} e^{-ikx} dx \tag{14}$$

and we take the limit of $n \to \infty$, leading to

$$\lim_{n\to\infty} \delta_n(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lim_{n\to\infty} \left(\frac{\sin(k/n)}{k/n} \right) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} dx \tag{15}$$

We already know that $\lim_{n\to\infty} \delta_n(x) = \delta(x)$, and thus

$$2\pi\delta(x) = \int_{-\infty}^{+\infty} e^{-ikx} dk. \tag{16}$$

There is another way of proving Eq (12). To this end, let's define a series of function by

$$\delta_T(\omega) \equiv \frac{\sin(\omega T)}{\pi\omega} = \frac{1}{2\pi} \int_{-T}^{+T} e^{i\omega T} dt, \qquad (T > 0). \tag{17}$$

First we prove the following formula:

$$\lim_{T\to\infty} \int_{-\infty}^{+\infty} f(\omega) \frac{\sin(\omega T)}{\pi\omega} d\omega = f(0) \tag{18}$$

If we change the integration variable from ω to $z = \omega T$, then the left side of the above equation becomes

$$\lim_{T\to\infty} \int_{-\infty}^{+\infty} f(\omega) \frac{\sin(\omega T)}{\pi\omega} d\omega = \int_{-\infty}^{+\infty} \lim_{T\to\infty} f(z/T) \frac{\sin(z)}{\pi z} dz = f(0) \int_{-\infty}^{+\infty} \frac{\sin(z)}{\pi z} dz \tag{19}$$

We know that the last integration becomes $f(0)$ (see Eq (21) in the next page). It means that $\lim_{T\to\infty} \delta_T(\omega) = \delta(\omega)$. The formula

$$2\pi\delta(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} dt \tag{20}$$

is often used to prove the "energy conservation law" in quantum mechanics.

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Example 2 Prove the following equality.

$$\int_{-\infty}^{+\infty} \mathrm{sinc}(z)dz = \pi, \qquad \mathrm{sinc}(z) \equiv \frac{\sin(z)}{z} \tag{21}$$

Solution Instead of $\frac{\sin(z)}{z}$, we integrate $g(z) = e^{iz}/z$, and take the imaginary part of the result at the end. To find the integration, we first integrate $g(z)$ along the contour C, as shown in Fig. (3) on the complex plane. Since $g(z)$ is an analytic function inside C, the integration along C should be zero as a whole:

$$\oint_C g(z)dz = \left\{ \int_{\text{Semi-cir } R} + \int_{-R}^{-r} + \int_r^R + \int_{\text{Semi-cir } r} \right\} g(z)dz = 0 \tag{22}$$

Let's investigate each contribution above. The first term becomes zero as $R \to +\infty$. This is because $|g(z)| = |\frac{e^{ix-y}}{R}| = \frac{e^{-y}}{R}$ on the semi-circle C, with $z = x + iy$ $(y > 0)$. Thus as $R, y \to \infty$, the integration goes to zero. The second and the third terms are what we want. In order to calculate the 4th term, we introduce $z = re^{i\theta}$ and $dz = izd\theta$;

$$\int_{\text{Semi-cir } r} \frac{\exp(iz)}{z} dz = i\int_\pi^0 \exp(ire^{i\theta})d\theta \qquad \xrightarrow{r\to 0} \qquad i\int_\pi^0 d\theta = -i\pi \tag{23}$$

Combining these results we arrive at Eq (21).

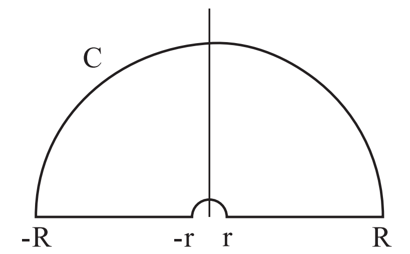

図 3: Integration contour C for $\mathrm{sinc}(z)$. It consists of a semi-circle of R, a segment $[-R, -r]$ on the real axis, a semi-circle of r, and a segment $[r, R]$.

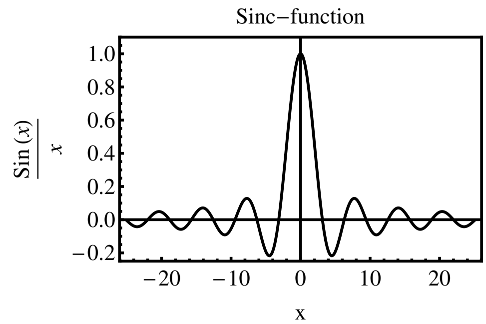

図 4: $\mathrm{sinc}(z) \equiv \dfrac{\sin(z)}{z}$

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Problem 5 Prove the integration below.

$$\int_{-\infty}^{+\infty} \frac{\sin^2(z)}{z^2} dz = \pi \tag{24}$$

(Hint) Rewrite the numerator of the kernel to $\sin^2(z) = \dfrac{1-\cos(2z)}{2}$, and use the same contour C as in Fig. (3). At around the origin, the numerator of the kernel can be Taylor expanded.

Problem 6 Using Eq (24), derive the following formula which is useful in Fermi's Golden rule:

$$\delta(\Delta\omega) = \lim_{T\to+\infty} \frac{\sin^2(T\Delta\omega)}{\pi T(\Delta\omega)^2} \tag{25}$$

Problem 7 Find Fourier transforms of $f_\epsilon(t)$ below, and sketch their shapes.

$$f_\epsilon(t) = \frac{1}{\sqrt{2\pi}} e^{-\epsilon|t|} \qquad (\epsilon > 0), \qquad\qquad \widetilde{f}_\epsilon(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{i\omega t} dt \tag{26}$$

Show that $\widetilde{f}_\epsilon(\omega)$ is a δ-series in the limit of $\epsilon \to +0$, and that Eq (12) holds.

4 Extension and application of the δ-function

Extension to the 3-dimensional δ-function Up to now we have dealt with 1-dimensional δ-function. It is easy to extend it to 3-dimensional δ-function. For example, a δ-function which is infinite at $\vec{r} = 0$ and zero elsewhere may be expressed by

$$\delta(x)\delta(y)\delta(z) \equiv \delta^{(3)}(\vec{r}).$$

Using the formula above, it is evident that the following relation holds:

$$\int_V \delta^{(3)}(\vec{r} - \vec{a}) f(\vec{r}) d^3\vec{r} = f(\vec{a})$$

where the region V must contain the point $\vec{a}$.

Application to electromagnetism Next let's prove one of the most familiar formulae used in electromagnetism;

$$\nabla^2 \frac{1}{r} \equiv \nabla \cdot \nabla \frac{1}{r} = -4\pi\delta^{(3)}(\vec{r}) \qquad (r = \sqrt{x^2 + y^2 + z^2}) \tag{27}$$

It is easy to show that the above function on the left is infinite at $\vec{r} = 0$ and zero elsewhere (see the problem below). Next we integrate it over the volume V including the origin. With the Gauss theorem, the volume integral becomes the surface integral:

$$\int_V \nabla \cdot \nabla \frac{1}{r} dv = \int_S \left(\nabla \frac{1}{r} \right) \cdot d\vec{S}$$

Noting that $\nabla \frac{1}{r} = -\frac{\hat{r}}{r^2}$ and $\hat{r} \cdot d\vec{S} = dS$ (take a shpere for V), the right-hand side becomes just solid angle of S. Thus the right-hand side is -4π times the δ-function.

Problem 8 Find $\nabla^2 \frac{1}{\sqrt{r^2 + a^2}}$ for $a > 0$. Using the result, show that the left-hand side of Eq (27) is infinite at $\vec{r} = 0$ and zero elsewhere.

Problem 9 Express the charge density $\rho(\vec{r})$ using appropriate $\delta-$functions under the circumstances described below.
For example, the point like particle with charge Q is $\rho(x, y, z) = Q\delta(x)\delta(y)\delta(z)$.

(i) Charge Q distributed uniformly on the surface of a sphere of radius a. (polar spherical coordinate)

(ii) Charge Q distributed uniformly along a ring of radius a. (cylindrical coordinate)

(iii) Charge Q distributed uniformly on the surface of a disk of radius a. (cylindrical coordinate)

Problem 10 Perform the following integral. In this problem, we use $\vec{p} = (p_x, p_y, p_z)$ instead of $\vec{r} = (x, y, z)$.

(i) $g(E) = \dfrac{1}{h^3}\displaystyle\int_{-\infty}^{+\infty} \delta(E - \frac{p^2}{2m}) dp_x dp_y dp_z$

(Note) E is a given parameter while h, m are constants, and $p^2 = \vec{p} \cdot \vec{p}$.

(Hint) Use polar spherical coordinates.

This function gives "density of states" for electrons, used frequently in solid state physics. See "Introduction to solid state physics" by C. Kittel, for example.

(ii) $\Phi_2(M) = \displaystyle\int_{-\infty}^{+\infty} \frac{1}{4p\sqrt{p^2 + m^2}} \delta(M - p - \sqrt{p^2 + m^2}) dp_x dp_y dp_z$

(Note) M, m are constants.

This function gives "relativistic phase space", and will show up when calculating decay rates, or scattering cross sections, *etc.*