Linear Algebra & Differential Equations

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7 Week 7

Consider the set of three vectors in $\mathbb{R}^2$: $\mathbf{v}_1 = (1,0)$, $\mathbf{v}_2 = (0,1)$, and $\mathbf{v}_3 = (1,1)$.

These vectors span $\mathbb{R}^2$ because you can combine them (using scalar multiplication and addition) to get any vector in $\mathbb{R}^2$. However, one of these vectors is not necessary for spanning $\mathbb{R}^2$.

$$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$$

This shows that $\mathbf{v}_3$ can be obtained by adding $\mathbf{v}_1$ and $\mathbf{v}_2$, making $\mathbf{v}_3$ the "extra" or dependent vector.

If you remove $\mathbf{v}_3$, the remaining set $\{\mathbf{v}_1, \mathbf{v}_2\}$ still spans $\mathbb{R}^2$, and neither can be obtained from the other. This illustrates the concept of linear independence: a set of vectors is linearly independent if no vector in the set can be written as a linear combination of the others. In this example, after removing $\mathbf{v}_3$, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent because you cannot multiply one by a scalar to get the other. Their linear independence is critical for them to span $\mathbb{R}^2$ because it ensures that they cover the entire plane without redundancy.

This example motivates the definitions of dependency and independency in vector spaces: a set of vectors is dependent if at least one vector can be expressed as a combination of the others, reducing the set's "efficiency" in spanning a space. Conversely, a set is independent if no such combination is possible, maximizing the set's utility in spanning and defining dimensions within vector spaces.

7.1 Linear Dependence & Independence

Definition 7.1. *A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V over a field F is said to be **linearly dependent** if there exist scalars $a_1, a_2, \ldots, a_n$ in F, not all zero, such that the linear combination of these vectors equals the zero vector:*

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$$

*where $\mathbf{0}$ denotes the zero vector in V. If no such non-trivial combination exists (i.e., if the equation above holds only when $a_1 = a_2 = \cdots = a_n = 0$), then the vectors are said to be **linearly independent**.*

Remark. The definition implies a trivial result: The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if they are NOT linearly dependent.

Example 1. (The Zero Vector) The set containing only the zero vector in any vector space, such as $\{\mathbf{0}\}$ in $\mathbb{R}^n$, is linearly dependent. For any scalar $a \neq 0$,

$$a \cdot \mathbf{0} = \mathbf{0}.$$

Example 2. (A Singleton Set of a Nonzero Vector) A singleton set containing one nonzero vector, such as $\{\mathbf{v}\}$ where $\mathbf{v} \neq \mathbf{0}$, is linearly independent because for any nonzero scalar a, the vector $a \cdot v \neq \mathbf{0}$.

Example 3. Consider the vectors $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (2, 4)$ in $\mathbb{R}^2$. These vectors are linearly dependent because there exist scalars a, b such that $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{0}$, specifically $a = -2, b = 1$, and we have

$$-2(1, 2) + (2, 4) = (0, 0).$$

Example 4. Consider the vectors $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, and $\mathbf{v}_3 = (1, 1, 0)$ in $\mathbb{R}^3$. These vectors are linearly dependent because $\mathbf{v}_3$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2.$$

So we have $-\mathbf{v_1} - \mathbf{v_2} + \mathbf{v_3} = \mathbf{0}$.

Note that any two of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. For example, suppose we have $a\mathbf{v}_1 + b\mathbf{v}_3 = (0, 0, 0)$, then it means $(a, b, 0) = (0, 0, 0)$ so $a = b = 0$. Similarly, you can show $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent sets.

Example 5. We will prove that the vectors $\mathbf{v}_1 = (3, 1, 2)$, $\mathbf{v}_2 = (1, 0, 2)$, and $\mathbf{v}_3 = (2, 1, 1)$ are linearly independent, so we need to show that the only solution to the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$$

is $a_1 = a_2 = a_3 = 0$.

Substituting the given vectors into the equation, we obtain:

$$a_1(3, 1, 2) + a_2(1, 0, 2) + a_3(2, 1, 1) = (0, 0, 0)$$

This leads to the following system of linear equations:

$$\begin{aligned} 3a_1 + a_2 + 2a_3 &= 0 \\ a_1 + a_3 &= 0 \\ 2a_1 + 2a_2 + a_3 &= 0 \end{aligned}$$

Solving this system of equations, we seek to find values of a_1, a_2, and a_3 that satisfy all three equations simultaneously. However, the system has only the trivial solution since the determinant of the coefficient matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ is nonzero ($\det(A) = -1$). So we have

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0$$

This solution indicates that the only way the linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ results in the zero vector is if all coefficients a_1, a_2, and a_3 are zero.

Therefore, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

Example of Linearly Independent 2×2 Matrices

Consider the following two 2×2 matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To determine if these matrices are linearly independent, we need to see if the only solution to the equation

$$c_1 A + c_2 B = \mathbf{0}$$

is $c_1 = c_2 = 0$, where **0** is the 2×2 zero matrix.

Expanding the equation, we get:

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This leads to a system of equations:

$$\begin{cases} c_1 + 0 = 0 \\ 0 + c_2 = 0 \\ 0 + c_2 = 0 \\ c_1 + 0 = 0 \end{cases}$$

The only solution is $c_1 = 0$ and $c_2 = 0$, proving that matrices A and B are linearly independent.

Example of Linearly Dependent 2×2 Matrices

Consider the two 2×2 matrices:

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

To check if these matrices are linearly dependent, we look for scalars c_1 and c_2 not both zero, such that:

$$c_1 C + c_2 D = \mathbf{0}$$

Notice that matrix D is exactly 2 times matrix C. Thus, we can choose $c_1 = 2$ and $c_2 = -1$ to satisfy:

$$2\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} - 1\begin{pmatrix}2 & 4\\ 6 & 8\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

This demonstrates that matrices C and D are linearly dependent since we found c_1 and c_2 that are not both zero and satisfy the equation.

Example with polynomials To determine that the polynomials $p_1(x) = 1-4x^2$, $p_2(x) = x+1$, and $p_3(x) = 2+2x+x^2$ are linearly independent or not, we consider a linear combination of these polynomials equal to the zero polynomial:

$$a_1(1-4x^2) + a_2(x+1) + a_3(2+2x+x^2) = 0$$

Expanding and rearranging terms, we get:

$$(a_1 + a_2 + 2a_3) + (a_2 + 2a_3)x + (-4a_1 + a_3)x^2 = 0$$

For the above polynomial to be the zero polynomial, the coefficients of x^0, x^1, and x^2 must all be zero:

$$\begin{aligned} a_1 + a_2 + 2a_3 &= 0 \\ a_2 + 2a_3 &= 0 \\ -4a_1 + a_3 &= 0 \end{aligned}$$

Since $\det\left(\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & 2\\ -4 & 0 & 1\end{bmatrix}\right) = 1$, the system has only the trivial solution.

Since $a_1 = a_2 = a_3 = 0$ is the only solution, the polynomials $1-4x^2$, $x+1$, and $2+2x+x^2$ are linearly independent.

Let's continue with the theory. We have a simple but useful observation:

Theorem 7.1. *A set of k vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space V over a field F is linearly dependent if and only if at least one of the vectors in the set can be written as a linear combination of the remaining vectors in the set.*

Proof. **(⇒)** Assume the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly dependent. By definition, this means there exist scalars $a_1, a_2, \dots, a_k$, not all zero, such that:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0} \tag{1}$$

Without loss of generality, assume $a_1 \neq 0$. We can solve for $\mathbf{v}_1$ as follows:

$$\mathbf{v}_1 = -\frac{a_2}{a_1}\mathbf{v}_2 - \frac{a_3}{a_1}\mathbf{v}_3 - \cdots - \frac{a_k}{a_1}\mathbf{v}_k \tag{2}$$

This shows that $\mathbf{v}_1$ is a linear combination of the remaining vectors, proving the sufficiency condition.

($\Leftarrow$) Now, assume that at least one of the vectors, say $\mathbf{v}_1$, can be written as a linear combination of the remaining vectors. Then, there exist scalars $c_2, c_3, \ldots, c_k$ such that:

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k$$

Rearranging, we get:

$$\mathbf{v}_1 - c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - \cdots - c_k\mathbf{v}_k = \mathbf{0}$$

This is a non-trivial linear combination that equals the zero vector, implying that the set of vectors is linearly dependent, which proves the necessity condition.

Hence, we have shown that a set of k vectors is linearly dependent if and only if at least one of the vectors can be written as a linear combination of the others. $\square$

Example 1. Consider the vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (2, 1, 3), \quad \mathbf{v}_3 = (3, 1, 4).$$

We can express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{v}_3 = 1 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2.$$

This shows that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent according to the theorem.

Example 2. Consider the vectors in $\mathbb{R}^2$:

$$\mathbf{v}_1 = (1, 2), \quad \mathbf{v}_2 = (2, 3).$$

There are no scalars a and b (other than $a = b = 0$) such that

$$a \cdot \mathbf{v}_1 + b \cdot \mathbf{v}_2 = \mathbf{0}.$$

Therefore, the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, which indirectly supports the theorem by showing a case where no vector in the set is a linear combination of the others, hence the set is not linearly dependent.

Remark. Another useful observation is that if the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ contains the zero vector, the set is indeed linearly dependent because its coefficient in the expansion $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$ can be nonzero.

7.2 Linear Dependence & Independence in $\mathbb{R}^n$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. To determine their linear dependency, we need to solve the coefficients in

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

However, this is the same as solving the system $A\mathbf{c} = \mathbf{0}$ where $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_k]$ and $\mathbf{c} = [c_1 \ c_2 \ \ldots \ c_n]^T$. So we have the following important result.

Theorem 7.2. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$ and $A = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_k]$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent if and only if the system $A\mathbf{c} = \mathbf{0}$ has a nontrivial solution for $\mathbf{c}$.*

The theorem has the following practical corollaries:

Corollary 7.2.1. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$ and $A = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_k]$.*

1. *If $k > n$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent.*
2. *If $k = n$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent if and only if det$(A) = 0$.*

Remark !! If $k < n$, then we need further investigation to determine the dependency.

Examples.

1. Consider the vectors in $\mathbb{R}^2$: $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (0, 1)$, and $\mathbf{v}_3 = (1, 1)$. Here, $k = 3 > n = 2$, so by the corollary, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.
2. Consider the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^3$ defined as follows:

$$\mathbf{v}_1 = (1, 2, 3), \quad \mathbf{v}_2 = (4, 5, 6), \quad \mathbf{v}_3 = (7, 8, 9).$$

The matrix A formed by these vectors as columns is

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

The determinant of A is calculated as follows:

$$\begin{aligned} \det(A) &= 1(5 \cdot 9 - 6 \cdot 8) - 4(2 \cdot 9 - 3 \cdot 8) + 7(2 \cdot 6 - 3 \cdot 5) \\ &= 1(-3) - 4(-6) + 7(-3) \\ &= -3 - 24 - 21 \\ &= -48. \end{aligned}$$

Since det$(A) \neq 0$, according to the corollary, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

3. Given the vectors $(1, 0, 2, 3)$, $(0, 1, 2, 1)$, $(4, 6, 2, 0)$, and $(3, 4, 5, 4)$ in $\mathbb{R}^4$, we form the matrix A as follows:

$$A = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & 6 & 4 \\ 2 & 2 & 2 & 5 \\ 3 & 1 & 0 & 4 \end{bmatrix}$$

We want to determine if these vectors are linearly dependent or independent. According to the corollary, if $k = n$ and the determinant of A is 0, then the vectors are linearly dependent. Since the determinant of A is 0, it follows that the vectors $(1, 0, 2, 3)$, $(0, 1, 2, 1)$, $(4, 6, 2, 0)$, and $(3, 4, 5, 4)$ are linearly dependent.

7.3 Linear Dependency in Function Spaces

Definition 7.2. *The set of functions $\{f_1, f_2, \ldots, f_k\}$ is linearly independent on an interval I if and only if the only values of the scalars $c_1, c_2, \ldots, c_k$ such that*

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) = 0 \quad \text{for all } x \in I$$

are $c_1 = c_2 = \cdots = c_k = 0$.

Example. Let $f_1(x) = \cos x$ and $f_2(x) = \sin x$ defined on $[0, 2\pi]$. If $a, b \in \mathbb{R}$ are such that $a(\cos x) + b(\sin x) = 0$ for all $x \in [0, 2\pi]$. Then if $x = 0$, then we get $a = 0$. If $x = \frac{\pi}{2}$, then $b = 0$. Since $a = b = 0$, we get $\{\cos x, \sin x\}$ is linearly independent.

Although the example seems easy, if we have lots of functions to check, evaluating at right points may not be easy. The main point to notice is that the condition must hold for all x in I. A key tool in deciding whether or not a collection of functions is linearly independent on an interval I is the Wronskian.

Definition 7.3. *Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$, namely, functions that can differentiable $k-1$ times. The Wronskian of these functions is the order k determinant defined by*

$$W[f_1, f_2, \ldots, f_k](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{vmatrix}$$

Remark. Notice that the Wronskian is a function defined on I. Also note that this function depends on the order of the functions in the Wronskian. For example, using properties of determinants, $W[f_2, f_1, \ldots, f_k](x) = -W[f_1, f_2, \ldots, f_k](x)$.

We use Wronskian to determine linear dependency of functions via the following theorem:

Theorem 7.3. *Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$. If the Wronskian $W[f_1, f_2, \ldots, f_k]$ is nonzero at some point x_0 in I, then the set $\{f_1, f_2, \ldots, f_k\}$ is linearly independent on I.*

Proof. In the textbook. □

Examples.

- Consider the functions $f_1(x) = e^x$ and $f_2(x) = e^{2x}$. To compute the Wronskian of these functions, we first find their derivatives:

$$f_1'(x) = e^x,$$
$$f_2'(x) = 2e^{2x}.$$

Then, the Wronskian $W[f_1, f_2](x)$ is given by the determinant:

$$W[f_1, f_2](x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x(2e^{2x}) - e^{2x}(e^x) = e^{3x}.$$

Since $W[f_1, f_2](x) = e^{3x} \neq 0$ for all x, the functions $f_1(x) = e^x$ and $f_2(x) = e^{2x}$ are linearly independent.

- Consider the functions $g_1(x) = x$ and $g_2(x) = x^2$. To compute the Wronskian of these functions, we first find their derivatives:

$$g_1'(x) = 1,$$
$$g_2'(x) = 2x.$$

Then, the Wronskian $W[g_1, g_2](x)$ is given by the determinant:

$$W[g_1, g_2](x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x(2x) - x^2(1) = 2x^2 - x^2 = x^2.$$

Since $W[g_1, g_2](x) = x^2 \neq 0$ for all x except $x = 0$, the functions $g_1(x) = x$ and $g_2(x) = x^2$ are linearly independent on any interval that does not include $x = 0$.

Remarks:

1. Notice that it is only necessary for the Wronskian $W[f_1, f_2, \ldots, f_k](x)$ to be nonzero at one point in I for $\{f_1, f_2, \ldots, f_k\}$ to be linearly independent on I.

2. Theorem 7.3 does not say that if $W[f_1, f_2, \ldots, f_k](x) = 0$ for every x in I, then $\{f_1, f_2, \ldots, f_k\}$ is linearly dependent on I. Namely, the theorem is not an "if and only if" statement. Instead, the logical equivalent of the preceding theorem is: If $\{f_1, f_2, \ldots, f_k\}$ is linearly dependent on I, then $W[f_1, f_2, \ldots, f_k](x) = 0$ at every point x of I.

3. If $W[f_1, f_2, \ldots, f_k](x) = 0$ for all x in I, Theorem 7.3 gives no information as to the linear dependence or independence of $\{f_1, f_2, \ldots, f_k\}$ on I.

The **Wronskian** is a fundamental tool in mathematics, particularly in the domains of differential equations and linear algebra. It plays a crucial role in determining solutions to differential equations. We will establish that if we have n functions that are solutions of an equation of the form

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = 0$$

on an interval I, then if the Wronskian of these functions is identically zero on I. However, these topics will be the subject of the lectures in the upcoming weeks :)