Homework 9 and Study Problems - MATH 225

In this document, you will find two types of problems: homework and study problems. You are required to submit **only the homework problems** to Gradescope. The study problems are intended to help you grasp the topics thoroughly and prepare for exams. It is strongly advised to attempt all study problems for a comprehensive understanding.

Please submit your homework to Gradescope until **April 10, 11pm**.

Homework problems

1. Let A and B be $n \times n$ matrices, and assume that $v \in \mathbb{R}^n$ is an eigenvector of A corresponding to the eigenvalue λ and also an eigenvector of B corresponding to the eigenvalue μ.

   (a) Prove that v is an eigenvector of the matrix AB. What is the corresponding eigenvalue?

   (b) Prove that v is an eigenvector of the matrix $A+B$. What is the corresponding eigenvalue?

2. Determine whether the given matrix is defective or nondefective.

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 6 & 5 \\ -5 & -4 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}.$$

3. Consider the characteristic polynomial of a 3×3 matrix A; namely,

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

   which can be written in either of the following equivalent forms:

$$p(\lambda) = -\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3,$$

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda),$$

   where $\lambda_1, \lambda_2, \lambda_3$ (not necessarily distinct) are the eigenvalues of A.

   (a) Use the given equations to show that

$$b_1 = -(a_{11} + a_{22} + a_{33}),$$

$$b_3 = \det(A).$$

   Recall that the quantity $a_{11} + a_{22} + a_{33}$ is called the *trace* of the matrix A, denoted $\operatorname{tr}(A)$.

(b) Use the given equations to show that

$$b_1 = -(\lambda_1 + \lambda_2 + \lambda_3),$$

$$b_3 = \lambda_1 \lambda_2 \lambda_3.$$

(c) Use your results from (a) and (b) to show that

$$\det(A) = \text{product of the eigenvalues of } A,$$

$$\text{tr}(A) = \text{sum of the eigenvalues of } A.$$

Remark. We ask for the proof of 3×3 case, but this is true for any $n \times n$ matrix.

4. Prove the following properties for similar matrices:

   (a) A matrix A is always similar to itself.

   (b) If A is similar to B, then B is similar to A.

   (c) If A is similar to B and B is similar to C, then A is similar to C.

   Hint. For each part, you should find an invertible matrix S that satisfies the similarity as in the definition.

5. Determine a complete set of eigenvectors for the given matrix A. Construct a matrix S that diagonalizes A and explicitly verify that $S^{-1}AS = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

$$A = \begin{bmatrix} 1 & -3 & 3 \\ -2 & -4 & 6 \\ -2 & -6 & 8 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -2 & 3 & -2 \\ -2 & 3 & -2 & 3 \\ 3 & -2 & 3 & -2 \\ -2 & 3 & -2 & 3 \end{bmatrix}.$$

Study problems

1. True-False Reviews on Page 443, 451, 459.

2. Problems 7.1.12-32

3. Problems 7.2.1-28

4. Problems 7.3.1-15