Real Mathematical Analysis

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Contents

1 Real Numbers

1.1 Cuts

Although we have used them since primary school, how are real numbers even defined? We were given a set of axioms about real numbers to solve problems involving them, but what if no such set satisfies these axioms? We would have been solving problems on the empty set! The following is a basic construction of the real numbers using Dedekind cuts.

Definition 1.1. A **cut** in $\mathbb{Q}$ is a pair of subsets A, B of $\mathbb{Q}$ such that

- $A \cup B = \mathbb{Q}$, $A, B \neq$, and $A \cap B =$
- If $a \in A$ and $b \in B$ Then $a < b$
- A contains no largest element

We denote the cut as $x = A|B$.

Definition 1.2. If $x = A|B$ and $y = C|D$ such that $A \subset C$ then x is **less than or equal** to y and we write $x \leq y$. If in addition, $A \neq C$ then x is **less than** y and we write $x < y$.

The most important property that distinguishes $\mathbb{Q}$ from $\mathbb{R}$ involves bounds.

Definition 1.3. $M \in \mathbb{R}$ is an **upper bound** for a set $S \subset \mathbb{R}$ if each $s \in S$ satisfies

$$s \leq M$$

We could also say that S is **bounded above** by M.

Definition 1.4. An upper bound for S that is less than all other upper bounds for S is a **least upper bound** for S. Denoted by $l.u.b.(S)$

Note that a least upper bound for S may or may not belong to S. This is why you should say "for S" rather than "of S"

Theorem 1.5. The set $\mathbb{R}$ is complete in that is satisfies the
Least Upper Bound Property: If S is a nonempty subset of $\mathbb{R}$ and is bounded above then there exists a least upper bound $S \in \mathbb{R}$

Proof. Let $\mathcal{C} \subseteq \mathbb{R}$ be bounded above. Define

$$C = \{a \in \mathbb{Q} : \text{ for some } A|B \in \mathcal{C} \text{ we have } a \in A\} \text{ and } D = \mathbb{Q} \setminus C$$

with $z = C|D$ which is clearly an upper bound for $\mathcal{C}$. Let $z' = C'|D'$ be any upper bound for $\mathcal{C}$. By the assumption that $A|B \leq C'|D'$ for all $A|B \in \mathcal{C}$, we see that the A for every member of $\mathcal{C}$ is contained in C'. Hence $C \subset C'$, so $z \leq z'$. That is, among all upper bounds for $\mathcal{C}$, z is least. $\square$

Arithmetic on cuts is defined to match out intuition from arithmetic on $\mathbb{R}$. The ordering defined by cuts allows for the following to be true:

Theorem 1.6. The set $\mathbb{R}$ of all cuts in $\mathbb{Q}$ is a complete ordered field that contains $\mathbb{Q}$ as an ordered subfield.

Proposition 1.7. Triangle Inequality: For all $x, y \in \mathbb{R}$ we have $|x + y| \leq |x| + |y|$

This definition of cuts in $\mathbb{Q}$ continues to be robust to further cutting. Defining further cuts on $\mathbb{R}$ does not result in a more "refined" set since the cuts would be at a real number by the $l.u.b.$ property.

1.2 Cauchy Sequences

$\mathbb{R}$ is also **complete** in the sense that every Cauchy sequence also converges.

Theorem 1.8. A Cauchy sequence $(a_n) \subseteq \mathbb{R}$ converges

Theorem 1.9. Every interval (a, b), no matter how small, contains both infinitely many rational and irrational numbers

Proof. (Sketch): Since $a < b$ are cuts, there exists $r < s \in \mathbb{Q}$ s.t. $r, s \in (a, b)$. Now, consider the transformation $T : [0, 1] \to (a, b)$ defined by $T(t) = r + (r - s)t$. Since $[0, 1]$ has infinitely many rationals and irrationals, (a, b) does as well. $\square$

1.3 Euclidean Space

Given two sets A, B the **Cartesian Product** of A and B is the product $A \times B = \{(a, b) : a \in A \text{and} b \in B\}$

$\mathbb{R}^m$ is the m-th cartesian product of $\mathbb{R}$ on itself. Given $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, the dot product is defined to be,

$$\langle x, y \rangle = x_1 y_1 + ... + x_m y_m$$

Which is bilinear, symmetric, and positive definite. That is,

2 Topology

2.1 Metric Spaces

Metric spaces are a specific example of a topological space. These notes will focus solely on them, when discussing topology.

Definition 2.1. A **Metric Space** is a set M, the elements of which are referred to as points of M, together with a metric d s.t. $\forall x, y, z \in M$

- **Positive Definiteness:** $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$
- **Symmetry:** $d(x, y) = d(y, x)$
- **Triangle Inequality:** $d(x, z) \leq d(x, y) + d(y, z)$

Remember that for a sequence of points in a metric space, $(p_n)_{n\in\mathbb{N}}$ is different than $\{p_n : n \in \mathbb{N}\}$. The former having an ordering for the points and the latter, a set where they are all jumbled. Formally, a sequence in M is a function $f : \mathbb{N} \to M$. The nth term in the sequence is $f(n) = p_n$.

The sequence (p_n) **converges to the limit** p in M if

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ n \geq N \Rightarrow d(p_n, p) < \epsilon$$

2.2 Continuity

Linear Algebra focuses on "Linear" objects. The main focus of the topic being that of linear transformations. In Analysis, the main focus is on "smooth" objects, in particular that of continuous functions.

Definition 2.2. A function $f : M \to N$ is **continuous** if it **preserves sequential convergence:** (p_n) converges in M to $p \Rightarrow (f(p_n))$ converges in N to $f(p)$

In Linear Algebra, spaces that look and behave the same are called isomorphic, that is, a linear bijection exists between them. In metric spaces, a bijection $f : M \to N$ is called a **Homeomorphism** if f and f^{-1} are continuous. M and N are then called **Homeomorphic** which is an equivalence relation and is denoted as $M \cong N$.

Note that the above definition of

Theorem 2.3. $f : M \to N$ is continuous if and only if it satisfies the **(ϵ, δ)-condition**:

$$\forall \epsilon > 0 \text{ and } p \in M \ \exists \delta > 0 \text{ such that } x \in M \text{ and } d_M(x, p) < \delta \Rightarrow d_N(f(x), f(p)) < \epsilon$$

Proof. Forwards via contradiction. Backwards direct. □

2.3 The Topology of a Metric Space

We now move onto two fundamental concepts in Topology. Let M be a metric space and let $S \subseteq M$. A point $p \in M$ is a **limit** of S if there exists a sequence (p_n) in S that converges to it.

Definition 2.4. S is a **closed set** if it contains all its limits

Definition 2.5. S is an open set if for each $p \in S$ there exists an $r > 0$ such that

$$d(p, q) < r \Rightarrow q \in S$$

Theorem 2.6. The complement of an open set is a closed set and the complement of a closed set is an open set

The **Topology** of M is the collection $\mathcal{T}$ of all open subsets of M.

Theorem 2.7. $\mathcal{T}$ is closed under arbitrary union, finite intersection, and it contains $\emptyset$, M.

Two sets, uniquely defined for a set S and point p are

$$limS = \{p \in M : p \text{ is a limit of } S\}$$

$$M_r p = \{q \in M : d(p, q) < r\}$$

The former being the **limit set** of S and the latter being the **r-neighborhood** of p

Theorem 2.8. $limS$ is a closed set and $M_r p$ is an open set

Proof. Suppose that $p_n \to p$ and each $p_n \in limS$. Then, $\exists (p_{n,k})_{k\in\mathbb{N}}$ in S that converges to p_n as $k \to \infty$. Then, consider the sequence $q_n = p_{n,k_n}$ such that

$$d(p_n, q_n) < \frac{1}{n}$$

Then as $n \to \infty$ we have

$$d(p, q_n) \leq d(p, p_n) + d(p_n, q_n) \to 0$$

□

Corollary 2.9. The interval (a, b) is open in $\mathbb{R}$ and so are $(-\infty, b)$, (a, ∞), and $(-\infty, \infty)$.

Proof. (a, b) is the r-neighborhood of its midpoint $m = \frac{a+b}{2}$ where $r = \frac{b-a}{2}$. Therefore (a, b) is open in $\mathbb{R}$ □

Note that $limS$ is the "smallest" closed set that contains S in the sense that it is a subset of all closed sets that contain S. Also, since limits of limits are limits, $\lim(\lim S) = \lim S$ (lim operator is idempotent). $limS$ will be referred to as the **closure** of S and will be denoted as $\bar{S}$.

Note that in Topology, a property of a metric space or of a mapping connecting them that can be solely described with respect to open sets is called a **topological property**. Let $f : M \to N$ be given. The **preimage** of a set $V \subset N$ is defined to be

3 Lebesgue Theory

Measuring simple sets is easy, for instance, the length of (a, b) is $b - a$. How do we measure more complicated sets?

3.1 Outer Measure

Definition 3.1. The **length** of the interval I is $|I| = b - a$. The **Lebesgue outer measure** of $A \subset \mathbb{R}$ is

$$m^*A = \inf\{\Sigma_k |I_k| : \{I_k\} \text{ is a covering of A by open intervals}\}$$

If for every covering of A, $\{I_k\}$, $\Sigma_k |I_k|$ diverges then $m^*A := \infty$. There are three immediate consequences of the above definition.

Theorem 3.2.
1. $m^*\emptyset$=0
2. If $A \subset B$ then $m^*A \leq m^*B$
3. If $A = \cup_{n=1}^{\infty} A_n$ then $m^*A \leq \sum_{n=1}^{\infty} m^*A_n$

Proof. For $\epsilon > 0$ there exists for each n a covering $\{I_{k,n} : k \in \mathbb{N}\}$ such that

$$\sum_{n=1}^{\infty} |I_{k,n}| < m^*A_n + \frac{\epsilon}{2^n}$$

The collection $\{I_{k,n} : k, n \in \mathbb{N}\}$ covers A and

$$\sum_{k,n} |I_{k,n}| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |I_{k,n}| \leq \sum_{k=1}^{\infty} (m^*A_n + \frac{\epsilon}{2^n}) = \sum_{k=1}^{\infty} m^*A_n + \epsilon$$

□

To assert 3., it suffices by definition of the infimum (think reverse of the epsilon definition of the supremum), to find a covering of A which can in fact be made smaller than the series presented plus an additional arbitrary epsilon. To do this we can use the definition of infimums on each one of the A_n and essentially squeeze the covering onto A.

Definition 3.3. If $Z \subset \mathbb{R}^n$ has outer measure zero, then it is a **zero set**

Proposition 3.4. Every subset of a zero set is a zero set. The countable union of zero sets is a zero set. Each plane $P_i(a)$ us a zero set in $\mathbb{R}^n$

This follows immediately from the above theorem. It seems that this theorem will be quite important.

Theorem 3.5. The linear outer measure of a closed interval is its length. The n-dimensional outer measure of a closed box is its volume.

Proof. Let [a,b] be a closed interval. Note that $(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$ is a cover for [a,b] and so $m^*([a, b]) \leq b - a + \epsilon$ It follows by the epsilon definition of the infimum that $m^*([a, b]) \leq b - a$.

To get the reverse inequality, we need to show that for every covering of [a,b] its total length is greater than or equal to $b - a$. Firstly, since **[a,b] is compact it suffices to only consider finite open coverings of [a,b]**. We then reason inductively.
The base case follows immediately as the endpoints of the open interval that contains [a,b] must be further apart than a and b themselves. Now, assume that the claim holds for an arbitrary n. Let $I_1, ..., I_{n+1}$ be an open covering of [a,b] with $I_i = (a_i, b_i)$. We claim that $\sum_{i=1}^{n+1} |I_i| > b - a$. One of the intervals contains a, say (a_1, b_1)
If $b_1 \geq b$ then $\sum_{i=1}^{n} |I_i| \geq |I_1| = b_1 - a_1 > b - a$
If $b_1 < b$ then

$$[a, b] = [a, b_1) \cup [b_1, b]$$

and $|I_1| > b_1 - a$. $[b_1, b]$ is a closed interval covered by n many open intervals, $I_2, ..., I_{n+1}$. It follows by the induction hypothesis that $\sum_{i=2}^{n+1} |I_i| > b - b_1$. Thus,

$$\sum_{i=1}^{n+1} |I_i| = |I_1| + \sum_{i=2}^{n+1} |I_i| > (b_1 - a) + (b - b_1) = b - a$$

$\square$

An interesting proof. Try and remember all the tools you have in $\mathbb{R}$, in particular that of compactness and the results that follow from its formal definition.

Proof. Let $R = [a, b] \times [c, d] \subset \mathbb{R}^2$. Using a similar covering as above, it is easy to see that $m^*R \leq (b-a)*(d-c)$. R has a Lebesgue number λ. Break R up into a finite set of rectangles $S_j \subset R$ where each S_j has a diameter less than λ. By their construction, $\sum |S_j| = (b-a)*(d-c)$. It follows that

$$\sum_j |S_j| \leq \sum_i \sum_{S_j \subset R_i} |S_j| \leq \sum |R_i|$$

which implies that $(b-a)*(d-c) \leq \sum |R_i|$. Thus, $(b-a)*(d-c) = m^*R$ $\square$

3.2 Measurability

If A and B are disjoint intervals in $\mathbb{R}$

$$m^*(A \sqcup B) = m^*A + m^*B$$

It seems like this should follow to arbirary disjoint sets. However, this is not the case in general and we need an added condition.

Definition 3.6. A set $E \subset \mathbb{R}$ is **(Lebesgue) measurable** if the division $E|E^c$ of $\mathbb{R}$ is such that for each $X \subset \mathbb{R}$ (test set) we have

$$m^*X = m^*(X \cap E) + m^*(X \cap E^c)$$

Next, we denote $\mathcal{M} = \mathcal{M}(\mathbb{R}^n)$ as the collection of all Lebesgue measurable subsets of $\mathbb{R}^n$. If E is measurable, its **Lebesgue Measure** $mA = m^*A$.

We now move out of $\mathbb{R}^n$ and into more abstract sets.

Definition 3.7. Let M be a set. An **Outer Measure** on M is a function $\omega : 2^M \to [0, \infty]$ such that

1. $\omega(\emptyset) = 0$
2. $A \subseteq B \Rightarrow \omega(A) \leq \omega(B)$

References

[1] Charles C. Pugh. Real Mathematical Analysis. Springer, 2015. isbn: 978-3-319-17770-0