Real Mathematical Analysis

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The following lecture notes are intended to be used as a supplemental resource for studying Real Analysis. The main sources used are *Real Mathematical Analysis* [2], *MAT357 Lecture Notes* [1], and *Fourier Analysis* [3]. I hope these notes will assist you in your studies!

Contents

1 Real Numbers

1.1 Cuts

Although we have used them since primary school, how are real numbers even defined? We were given a set of axioms about real numbers to solve problems involving them, but what if no such set satisfies these axioms? We would have been solving problems on the empty set! The following is a basic construction of the real numbers using Dedekind cuts.

Definition 1.1. A **cut** in $\mathbb{Q}$ is a pair of subsets A, B of $\mathbb{Q}$ such that

- $A \cup B = \mathbb{Q}$, $A, B \neq \emptyset$, and $A \cap B = \emptyset$
- If $a \in A$ and $b \in B$ Then $a < b$
- A contains no largest element

We denote the cut as $x = A|B$.

Definition 1.2. If $x = A|B$ and $y = C|D$ such that $A \subset C$ then x is **less than or equal** to y and we write $x \leq y$. If in addition, $A \neq C$ then x is **less than** y and we write $x < y$.

The most important property that distinguishes $\mathbb{Q}$ from $\mathbb{R}$ involves bounds.

Definition 1.3. $M \in \mathbb{R}$ is an **upper bound** for a set $S \subset \mathbb{R}$ if each $s \in S$ satisfies

$$s \leq M$$

We could also say that S is **bounded above** by M.

Definition 1.4. An upper bound for S that is less than all other upper bounds for S is a **least upper bound** for S. Denoted by $l.u.b.(S)$

Note that a least upper bound for S may or may not belong to S. This is why you should say "for S" rather than "of S"

Theorem 1.5. The set $\mathbb{R}$ is complete in that is satisfies the **Least Upper Bound Property**: If S is a nonempty subset of $\mathbb{R}$ and is bounded above then there exists a least upper bound $S \in \mathbb{R}$

Proof. Let $\mathcal{C} \subseteq \mathbb{R}$ be bounded above. Define

$$C = \{a \in \mathbb{Q} : \text{ for some } A|B \in \mathcal{C} \text{ we have } a \in A\} \text{ and } D = \mathbb{Q} \setminus C$$

with $z = C|D$ which is clearly an upper bound for $\mathcal{C}$. Let $z' = C'|D'$ be any upper bound for $\mathcal{C}$. By the assumption that $A|B \leq C'|D'$ for all $A|B \in \mathcal{C}$, we see that the A for every member of $\mathcal{C}$ is contained in C'. Hence $C \subset C'$, so $z \leq z'$. That is, among all upper bounds for $\mathcal{C}$, z is least. □

Arithmetic on cuts is defined to match out intuition from arithmetic on $\mathbb{R}$. The ordering defined by cuts allows for the following to be true:

Theorem 1.6. The set $\mathbb{R}$ of all cuts in $\mathbb{Q}$ is a complete ordered field that contains $\mathbb{Q}$ as an ordered subfield.

Proposition 1.7. Triangle Inequality: For all $x, y \in \mathbb{R}$ we have $|x + y| \leq |x| + |y|$

This definition of cuts in $\mathbb{Q}$ continues to be robust to further cutting. Defining further cuts on $\mathbb{R}$ does not result in a more "refined" set since the cuts would be at a real number by the $l.u.b.$ property.

1.2 Cauchy Sequences

$\mathbb{R}$ is also **complete** in the sense that every Cauchy sequence also converges.

Theorem 1.8. A Cauchy sequence $(a_n) \subseteq \mathbb{R}$ converges

Theorem 1.9. Every interval (a, b), no matter how small, contains both infinitely many rational and irrational numbers

Proof. (Sketch): Since $a < b$ are cuts, there exists $r < s \in \mathbb{Q}$ s.t. $r, s \in (a, b)$. Now, consider the transformation $T : [0, 1] \to (a, b)$ defined by $T(t) = r + (r - s)t$. Since $[0, 1]$ has infinitely many rationals and irrationals, (a, b) does as well. □

1.3 Euclidean Space

Given two sets A, B the **Cartesian Product** of A and B is the product $A \times B = \{(a, b) : a \in A \text{and} b \in B\}$

$\mathbb{R}^m$ is the m-th cartesian product of $\mathbb{R}$ on itself. Given $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, the dot product is defined to be,

$$\langle x, y \rangle = x_1 y_1 + ... + x_m y_m$$

Which is bilinear, symmetric, and positive definite. That is, for $x, y, z \in \mathbb{R}^m$ and $c \in \mathbb{R}$

1. $\langle x, y + cz \rangle = \langle x, y \rangle + c\langle x, z \rangle$
2. $\langle x, y \rangle = \langle y, x \rangle$
3. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = \vec{0}$

One defines the **length** to be $|x| = \sqrt{\langle x, x \rangle}$. With this, we have one of the most well known inequalities.

Proposition 1.10. Cauchy-Schwarz Inequality For all $x, y \in \mathbb{R}^m$ we have $\langle x, y \rangle \leq |x||y|$

This inequality easily implies that of the general **Triangle Inequality** in Euclidean space. With this background, we now move on to more general spaces.

2 Topology

2.1 Metric Spaces

Metric spaces are a specific example of a topological space. These notes will focus solely on them, when discussing topology.

Definition 2.1. A **Metric Space** is a set M, the elements of which are referred to as points of M, together with a metric d s.t. $\forall x, y, z \in M$

- **Positive Definiteness:** $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$
- **Symmetry:** $d(x, y) = d(y, x)$
- **Triangle Inequality:** $d(x, z) \leq d(x, y) + d(y, z)$

Remember that for a sequence of points in a metric space, $(p_n)_{n\in\mathbb{N}}$ is different than $\{p_n : n \in \mathbb{N}\}$. The former having an ordering for the points and the latter, a set where they are all jumbled. Formally, a sequence in M is a function $f : \mathbb{N} \to M$. The nth term in the sequence is $f(n) = p_n$.

The sequence (p_n) **converges to the limit** p in M if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; n \geq N \Rightarrow d(p_n, p) < \epsilon$$

2.2 Continuity

Linear Algebra focuses on "Linear" objects. The main focus of the topic being that of linear transformations. In Analysis, the main focus is on "smooth" objects, in particular that of continuous functions.

Definition 2.2. A function $f : M \to N$ is **continuous** if it **preserves sequential convergence:** (p_n) converges in M to $p \Rightarrow (f(p_n))$ converges in N to $f(p)$

In Linear Algebra, spaces that look and behave the same are called isomorphic, that is, a linear bijection exists between them. In metric spaces, a bijection $f : M \to N$ is called a **Homeomorphism** if f and f^{-1} are continuous. M and N are then called **Homeomorphic** which is an equivalence relation and is denoted as $M \cong N$.

Note that the above definition of

Theorem 2.3. $f : M \to N$ is continuous if and only if it satisfies the **(ϵ, δ)-condition**:

$$\forall \epsilon > 0 \text{ and } p \in M \; \exists \delta > 0 \text{ such that } x \in M \text{ and } d_M(x, p) < \delta \Rightarrow d_N(f(x), f(p)) < \epsilon$$

Proof. Forwards via contradiction. Backwards direct. □

2.3 The Topology of a Metric Space

We now move onto two fundamental concepts in Topology. Let M be a metric space and let $S \subseteq M$. A point $p \in M$ is a **limit** of S if there exists a sequence (p_n) in S that converges to it.

Definition 2.4. S is a **closed set** if it contains all its limits

Definition 2.5. S is an **open set** if for each $p \in S$ there exists an $r > 0$ such that

$$d(p, q) < r \Rightarrow q \in S$$

Theorem 2.6. The complement of an open set is a closed set and the complement of a closed set is an open set

The **Topology** of M is the collection $\mathcal{T}$ of all open subsets of M.

Theorem 2.7. $\mathcal{T}$ is closed under arbitrary union, finite intersection, and it contains $\emptyset$, M.

Two sets, uniquely defined for a set S and point p are

$$limS = \{p \in M : p \text{ is a limit of } S\}$$

$$M_r p = \{q \in M : d(p, q) < r\}$$

The former being the **limit set** of S and the latter being the **r-neighborhood** of p

Theorem 2.8. $limS$ is a closed set and $M_r p$ is an open set

Proof. Suppose that $p_n \to p$ and each $p_n \in limS$. Then, $\exists (p_{n,k})_{k\in\mathbb{N}}$ in S that converges to p_n as $k \to \infty$. Then, consider the sequence $q_n = p_{n,k_n}$ such that

$$d(p_n, q_n) < \frac{1}{n}$$

Then as $n \to \infty$ we have

$$d(p, q_n) \leq d(p, p_n) + d(p_n, q_n) \to 0$$

□

Corollary 2.9. The interval (a, b) is open in $\mathbb{R}$ and so are $(-\infty, b)$, (a, ∞), and $(-\infty, \infty)$.

Proof. (a, b) is the r-neighborhood of its midpoint $m = \frac{a+b}{2}$ where $r = \frac{b-a}{2}$. Therefore (a, b) is open in $\mathbb{R}$ □

Note that $limS$ is the "smallest" closed set that contains S in the sense that it is a subset of all closed sets that contain S. Also, since limits of limits are limits, $\lim(\lim S) = \lim S$ (lim operator is idempotent). $limS$ will be referred to as the **closure** of S and will be denoted as $\bar{S}$.

Note that in Topology, a property of a metric space or of a mapping connecting them that can be solely described with respect to open sets is called a **topological property**. Let $f : M \to N$ be given. The **preimage** of a set $V \subset N$ is defined to be

$$f^{-1}(V) = \{p \in M : f(p) \in V\}$$

Theorem 2.10. The following are equivalent for continuity of $f : M \to N$

1. The (ϵ, δ)-condition
2. The sequential convergence preservation condition
3. The **closed set condition**: The preimage of each closed set in N is closed in M
4. The **open set condition**: The preimage of each open set in N is open in M

Note that the set $\{x \in \mathbb{Q} : -\pi < x < \pi\}$ is open and closed in $\mathbb{Q}$, but is neither open nor closed in $\mathbb{R}$.

Proposition 2.11. Every metric subspace N of M inherits its topology from M in the sense that each subset $V \subset N$ which is open in N is actually the intersection $V = N \cap U$ for some $U \subset M$ that is open in M, and vice versa

The same can be said for the closed sets of N. With this, it is easy to see why $S = \{x \in \mathbb{Q} : -\pi < x < \pi\}$ is both open and closed. $S = \mathbb{Q} \cap K$ and $S = \mathbb{Q} \cap U$ where $U = (-\pi, \pi)$ and $K = [-\pi, \pi]$.

Corollary 2.12. Assume N is a metric subspace of M and is also a closed subset of M. A set $L \subset N$ is closed in N if and only if it is closed in M. Similarly if N is a metric subspace of M and is also an open subset of M then $U \subset N$ is open in N if and only if it is open in M

2.4 Product Metrics

We now define a metric on the cartesian product of two metric spaces $M = X \times Y$. There are three natural choices with $p = (x, y)$ and $p' = (x', y')$

$$d_E(p, p') = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}$$
$$d_{max}(p, p') = \max d_X(x, x'), d_Y(y, y')$$
$$d_{sum}(p, p') = d_X(x, x') + d_Y(y, y')$$

Proposition 2.13. $d_{max} \leq d_E \leq d_{sum} \leq 2d_{max}$

Proof. Trivial □

This result about the above metrics imply that in a topological sense they are equivalent. That is,

Corollary 2.14. Let $p_n = (p_{1n}, p_{2n})$ be a sequence in $M = M_1 \times M_2$:

a) (p_n) converges with respect to d_{max}
b) (p_n) converges with respect to d_E
c) (p_n) converges with respect to d_{sum}
d) (p_{1n}) and p_{2n} converge in M_1 and M_2 respectively

Corollary 2.15. If $f : M \to N$ and $g : X \to Y$ are continuous then so is their cartesian product $f \times g$ which sends $(p, x) \in M \times X$ to $(f(p), g(x)) \in N \times Y$.

Proof. Follows immediately from 2.14 d) ☐

Note that proposition 2.13 can be generalized to an m-cartesian product of space implying that $d_{max} \leq d_E \leq d_{sum} \leq md_{max}$. In particular this implies that a sequence in $\mathbb{R}^m$ converges if and only if each of its components does.

Further, the metric $d : M \times M \to \mathbb{R}$ is continuous.

Theorem 2.16. d is continuous

Proof. Checking (ϵ, δ)-continuity with respect to the metric d_{sum} on $M \times M$. Given an $\epsilon > 0$ take $\delta = \epsilon$. If $d_{sum}((p, q), (p', q')) < \delta$ the triangle inequality gives that

$$d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < d(p', q') + \epsilon$$
$$d(p', q') \leq d(p', p) + d(p, q) + d(q, q') < d(p, q) + \epsilon$$

which gives that

$$d(p, q) - \epsilon < d(p', q') < d(p, q) + \epsilon$$

Thus $|d(p', q') - d(p, q)| < \epsilon$ and we get continuity with respect to the metric d_{sum}. By theorem 17 it does not matter which metric we use on $M \times M$. ☐

Figure 1: The unit disk in the max metric and sum (taxi-cab metric)

2.5 Completeness

There is a natural extension to cauchy sequences in $\mathbb{R}$ to metric spaces. The sequence (p_n) in M satisfies the **Cauchy Criterion** provided that for each $\epsilon > 0$ there is an integer N such that for all $k, n \geq N$ we have $d(p_k, p_n) < \epsilon$, and (p_n) is said to be a Cauchy sequence. It follows immediately by use of the triangle inequality that convergence $\to$ Cauchy.

Note that in $\mathbb{R}$, cauchiness implies convergence, however, in a general metric space, this need not be true. Consider $\mathbb{Q}$ with the inherited metric from $\mathbb{R}$ and the sequence $(p_n) = (3, 3.1, 3.14, \dots)$. This sequence is cauchy, but does not converge to a point in $\mathbb{Q}$.

Definition 2.17. A metric space M is **complete** if each Cauchy sequence in M converges to a limit in M

Theorem 2.18. $\mathbb{R}^n$ is complete

Proof. This follows immediately as a result of Corrollary 2.14 and the fact that $\mathbb{R}$ is complete ☐

Note that completeness is not a topological property as $(-1, 1)$ is homeomorphic to $\mathbb{R}$, but $(-1, 1)$ is not complete.

2.6 Compactness

Compactness is one of the most important concepts in real analysis, it reduces the infinite to the finite.

Definition 2.19. A subset A of a metric space M is (sequentially) **compact** if every sequence (a_n) in A has a subsequence (a_{n_k}) that converges to a limit in A

Note that it immediately follows that all finite sets and the empty set are compact.

Theorem 2.20. Every compact set is closed and bounded

Proof. Every convergent subsequence most converge to the same point as the convergent mother sequence and convergent sequences are bounded (a contradiction would occur if a compact set was unbounded as a unbounded sequence would exist). ☐

Theorem 2.21. The closed interval $[a, b] \subset \mathbb{R}$ is compact

Proof. Let (x_n) be a sequence in $[a, b]$ and set

$$C = \{x \in [a, b] : x_n < x \text{ only finitely often}\}$$

$a \in C$ and b is an upper bound so the L.U.B propery implies $\exists c \in [a, b]$ supremum for C. We claim a subsequence (a_{n_k}) converges to c. Suppose not, that no subsequence does. Then, for some $r > 0$, x_n lies in $(c-r, c+r)$ only finitely many times. But then since there are finitely many points smaller than c, there are finitely many points smaller than $c + r$ implying that c is not the least upper bound of C, thus some subsequence of (x_n) converges to c and therefore $[a, b]$ is compact. □

Theorem 2.22. Every closed subset of a compact set is compact

Theorem 2.23. Heine-Borel Theorem: Every closed and bounded subset of $\mathbb{R}^n$ is compact

Proof. Let $A \subset \mathbb{R}^m$ be closed and bounded. Boundedness implies that A is contained in some box, which is compact. Since A is closed, A is compact by the above. □

Note that this is not true for general metric spaces M. Consider $\mathbb{N}$ equipped with the discrete metric. Then $\mathbb{N}$ is complete, closed, and bounded, but the sequence $(1, 2, 3, 4, \dots)$ does not have any convergent subsequence. Further, consider the metric space $C([0, 1], \mathbb{R})$ whose metric is $d(f, g) = \max\{|f(x) - g(x)|\}$. The space is complete but its closed unit ball is not compact. For example, consider the sequence of functions $f_n = x^n$ has no subsequence that converges with respect to the metric d. In fact, every closed ball is noncompact...

Theorem 2.24. The intersection of a nested sequence of compact nonempty sets is compact and nonempty

The **diameter** of a nonempty set $S \subset M$ is the supremum of the distance $d(x, y)$ between points of S

Theorem 2.25. If in addition to being nested, nonempty, and compact, the sets A_n have diameter that tends to 0 as $n \to \infty$ then $A = \cap A_n$ is a singleton

2.6.1 Continuity and Compactness

Theorem 2.26. If $f : M \to N$ is continuous and A is a compact subset of M then $f(A)$ is a compact subset of N. That is, the continuous image of compact is compact.

Proof. Let $(b_n) \subseteq N$. For each b_n pick a single $a_n \in M$ such that $f(a_n) = b_n$. Using the compactness of A and continuity of f, the result follows. □

Corollary 2.27. A continuous real-valued function deifned on a compact set is bounded; it assumes maximum and minimum values.

Note that the above theorems imply that compactness is a topological property. Thus compactness can be used to prove spaces are not homeomorphic to each other.

Theorem 2.28. If M is compact then a continuous bijection $f : M \to N$ is a homeomorphism - its inverse bijection $f^{-1} : N \to M$ is automatically continuous

Proof. Suppose that $q_n \to q$ in N. $p_n = f^{-1}(q_n)$ and $p = f^{-1}(q)$ are well defined. Assume (p_n) does not converge to p. Then there is a subsequence (p_{n_k}) such that for all k we have $d(p_{n_k}, p) > \delta > 0$. Compactness implies that there exists a sub-subsequence (p_{k_l}) that converges to a point $p^* \in M$ as $l \to \infty$. Thus $p \neq p^*$. Since f is continuous we have

$$f(p_{n_{k_l}}) \to f(p^*)$$

. Then, $f(p^*) = f(p)$ and so by the bijectivity of f, $p_n \to p$ and therefore f^{-1} is continuous. □

Note that if M is not compact, this fact does not hold. Consider the continuous bijection from $[0, 2\pi)$ to S^1 mentioned previously.

One says that $h : M \to N$ **embeds** M into N if h is a homeomorphism from M onto $h(M)$. Topologically, M and $h(M)$ are then equivalent, a property of M that holds for every embedded copy of M is an **absolute** or **intrinsic** property of M.

Proposition 2.29. *A compact is absolutely closed and absolutely bounded.*

2.6.2 Uniform Continuity and Compactness

We can extend the definition of uniform continuity to metric spaces with an analogous definition.

Definition 2.30. A function $f : M \to N$ is **uniformly continuous** if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$p, q \in M \text{ and } d_M(p, q) < \delta \Rightarrow d_N(f(p), f(q)) < \epsilon$$

The following is one of the most important results in analysis:

Theorem 2.31. Every continuous function defined on a compact set is uniformly continuous.

Proof. Suppose not that $\exists f : M \to N$ continuous M compact which is not uniformly continuous. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$ $\exists p, q \in M$ with $d(p, q) < \delta$ but $d(f(p), f(q)) \geq \epsilon$. By taking $\delta = \frac{1}{n}$ and letting (p_n) and (q_n) being the sequence of points for such a delta. Compactness implies that there exists a subsequence p_{n_k} which converges to some $p \in M$. By their definition, (q_n) also converges to p. Continuity thus implies that $f(p_{n_k}) \to f(p)$ and $f(q_{n_k}) \to f(p)$. But then, if k is large enough,

$$d(f(p_{n_k}), f(q_{n_k})) \leq d(f(p_{n_k}), f(p)) + d(f(p), f(q_{n_k})) < \epsilon$$

contrary to how p_n and q_n were defined. Thus, f is uniformly continuous on M. $\square$

2.7 Connectedness

Let A be a subset of a metric space. If A is neither the empty set nor M then A is said to be a **proper** subset of M.

Definition 2.32. If M has a proper clopen subset A then M is **disconnected**. This can be seen since there exists a **separation** of M into proper disjoint clopen subsets

$$M = A \cup A^c$$

M is said to be **connected** if it is not disconnected

Theorem 2.33. If M is connected, $f : M \to N$ is continuous and onto then N is connected. That is, the continuous image of a connected set is connected.

Connectedness is then a topological property.

Theorem 2.34. (**Generalized Intermediate Value Theorem**) Every continuous real-valued function defined on a connected domain has the intermediate value property.

This is because if you assumed otherwise, a disconnection in the original space would exist.

Theorem 2.35. $\mathbb{R}$ is connected

Proof. If $U \subset \mathbb{R}$ is nonempty and clopen we claum that $U = \mathbb{R}$. Choose some $p \in U$ and consider the set

$$X = \{x \in U : (p, x) \text{is contained in U}\}$$

Let s be the supremum of X. If $s < \infty$ then s is a limit point of U and so is in U but then some open interval would exist around s as U is open. A contradiction to s being the supremum of the set. Thus, s is infinite and so $(p, \infty) \subset U$. The same reasoning on the other end gives that $U = \mathbb{R}$ $\square$

Corollary 2.36. (**Intermediate Value Theorem for $\mathbb{R}$**) Every continuous function $f : \mathbb{R} \to \mathbb{R}$ has the intermediate value property.

R being compact has a further corollary. Indeed the sets (a, b), $[a, b]$, and S^1. Connected can also be used to show that no interval $[a, b]$ is homeomorphic to S^1 as removing a point in either of them would make $[a, b]$ disconnected, whilst S^1 would remain connected.

Theorem 2.37. The closure of a connected set is connected. More generally, if $S \subset M$ is connected and $S \subset T \subset \bar{S}$ then T is connected

Theorem 2.38. The union of connected sets sharing a common point P is connected

Proof. Let $S = \cup S_\alpha$ where $P \in \cap S_\alpha$. If S was disconnected then there exists a proper clopen A, A^c such that $S = A^c \cup A$. One of them contains p say A. Then $A \cap S_\alpha$ is a nonempty clopen subset of S_α. Since S_α is connected, $A \cap S_\alpha = S_\alpha$ for each α. And so $A = S$. This implies that $A^c = \emptyset$, a contradiction. Therefore S is connected. $\square$

The fact that S^2 is connected comes from the fact that it is the union of spinning rings around the north south pole. Furthermore, every convex set $C \subset \mathbb{R}^m$ is connected.

Definition 2.39. A **path** joining p and q in a metric space M is a continuous function $f; [a, b] \to M$ such that $f(a) = p$ and $f(b) = q$. If each pair of points in M can be joined by a path in M then M is said to be **path-connected.**

Theorem 2.40. Path-connected implies connected

Proof. Trivial $\square$

2.8 Other Metric Space Concepts

If $S \subset M$ then its **closure** $(\bar{A})$ is the smallest closed subset of M that contains S, its **interior**(int$S$) is the largest open set contained in S, and its **boundary** (∂S) is the difference between its closure and its interior.

2.8.1 Clustering and Condensing

The set S clusters at p (p is a **cluster point** of S) if each $M_r(p)$ contains infinitely many points of S. The set S condenses at p (and p is a **condensation point** of S) if each $M_r(p)$ contains uncountably many points in S. Thus, S limits at p, clusters at p, or condenses at p according to whether each $M_r(p)$ contains some, infinitely many, or uncountably many points of S.

capture2.png

Figure 2: limiting, clustering, and condensing behaviour)

Note that S clusters at p if there is a sequence of distinct points in S that converges to p or if each neighborhood of p contains a point of S other than p.

$S^{'}$ is frequently used to denote the set of clusters points of S

2.8.2 Perfect Metric Spaces

Definition 2.41. A space M is **perfect** if $M' = M$ that is, each $p \in M$ is a cluster point of M. Recall that M clusters at p if each $M_r(p)$ is an infinite set.

Theorem 2.42. Every nonempty, perfect, complete metric space is uncountable

Proof. (sketch) Reason inductively to find a point that does not belong in a enumeration of the set $\square$

2.8.3 Boundedness

Definition 2.43. S a subset of a metric space M is **bounded** if for some $p \in M$ and some $r > 0$

$$S \subset M_r(p)$$

a set which is not bounded is **unbounded**

Note that boundedness is not a topological as $(0, 1)$ is homeomorphic to $\mathbb{R}$ but $\mathbb{R}$ is not bounded.

Further, a function $f : M \to N$ is **bounded** if its range is a bounded subset of N. That is, if $\exists q \in N$ and $r > 0$ such that $f(M) \subset N_r(q)$

2.9 Coverings

The concept of coverings can be quite challenging, but it is in fact central to much of analysis (especially measure theory).

Definition 2.44. A collection $\mathfrak{U}$ of subsets of M **covers** $A \subset M$ if A is contained in the unin of the set belonging to $\mathfrak{U}$.

The collection $\mathcal{U}$ is called a **covering** of A.

Definition 2.45. If $\mathcal{U}$ and $\mathcal{V}$ both cover A and if $\mathcal{V} \subset \mathcal{U}$ then we say $\mathcal{U}$ reduces to $\mathcal{V}$ and that $\mathcal{V}$ is **subcovering** of A

The formal topological definition of compactness will now be presented.

Definition 2.46. If every open covering of A reduces to a finite subcovering of A then we say that A is **covering compact**

Theorem 2.47. For a subset A of a metric space M the following are equivalent:

- A is covering compact
- A is sequentially compact

To complete this proof, we need the following definition:

Definition 2.48. A **Lebesgue number** for a covering $\mathcal{U}$ of A is a $\lambda > 0$ such that for each $a \in A$ there is some $U \in \mathcal{U}$ with $M_\lambda(a) \subset U$

The next lemma is necessary for proving Theorem 2.47

Definition 2.49. Lebesgue Number Lemma Every open covering of a sequentially compact set has a lebesgue number $\lambda > 0$

Definition 2.50. A set $A \subset M$ is **totally bounded** if for each $\epsilon > 0$ there exists a finite covering of A with ϵ-neighborhoods.

With this new definition, we can provide a generaliztion of the Heine-Borel theorem in $\mathbb{R}^n$

Theorem 2.51. Generalized Heine-Borel Theorem A subset of a complete metric space is compact if and only if it is closed and totally bounded.

3 Function Spaces

3.1 Uniform Convergence

Points converge to a limit if they get physically closer and closer to it. What about a sequence of functions? The simplest notion of convergence for functions is that of pointwise convergence.

Definition 3.1. A sequence of functions $f_n : [a, b] \to \mathbb{R}$ **converges pointwise** to a limit function $f : [a, b] \to \mathbb{R}$ if for each $x \in [a, b]$ we have

$$\lim_{n\to\infty} f_n(x) = f(x)$$

This is denoted as $f_n \to f$. Uniform convergence is stronger than that of pointwise.

Definition 3.2. $f_n : [a, b] \to \mathbb{R}$ converges uniformly to the limit functions $f : [a, b] \to \mathbb{R}$ if $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $x \in [a, b]$ we have

$$|f_n(x) - f(x)| < \epsilon$$

This is denoted by $f_n \rightrightarrows f$. This can be visualized by all functions in the sequence eventually in an ϵ band of f. Note that pointwise convergence does not imply uniform. This can be seen by the sequence of functions $f_n(x) = x^n$ on $[0, 1]$. Which converges to $\begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$ but note that this convergence is not uniform because of the discontinuity in the limit at $x = 1$. We are now concerned with finding out which properties of functions are preserved under uniform convergence.

Theorem 3.3. If $f_n \rightrightarrows f$ and each f_n is continuous at x_0 then f is continuous at x_0

Proof. Let $\epsilon > 0$ and $x_0 \in [a, b]$ be given. Then there is a N such that $\forall n \geq N$ and all $x \in [a, b]$ we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

Note that the function f_N is continuous at x_0 and so $\exists \delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$$

With this δ,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Completing the proof that f is continuous at x_0 □

Note that a continuous pointwise limit does not result in uniform convergence. This can be seen by the case of x^n on $(0, 1)$. It still doesn't hold on compact domains either.

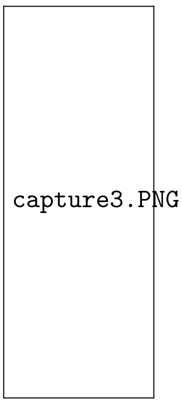

Figure 3: A sequence of functions whose pointwise limit is continuous, but fails to converge uniformly to it

Then $\lim_{n\to\infty} f_n(x) = 0$ for each x and f_n converges pointwise to 0, the functions have compact domains of definition, uniformly continuous, but still do not have uniform convergence.

We now move to an equivalent formulation of uniform convergence. Let $C_b = C_b([a, b], \mathbb{R})$ denote the set of all bounded functions.

Definition 3.4. The **sup norm** on C_b is defined to be
$$||f|| = \sup\{|f(x)| : x \in [a,b]\}$$
with this a metric can be defined as
$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]\}$$

Proposition 3.5. Convergence with respect to the sup metric d is equivalent to uniform convergence

The metric space defined above has nice properties. For instance

Theorem 3.6. C_b is a complete metric space

Proof. The pointwise limit of f_n exists by the completeness of $\mathbb{R}$. □

Let $C^0([a,b],\mathbb{R})$ denote the set of continuous functions $[a,b] \to \mathbb{R}$. Note that $C^0 \subset C_b$.
Just as it is reasonable to consider the convergence of a sequence of functions, we can also consider the convergence of a series of functions $\sum f_k$. Consider the n^{th} partial sum
$$F_n(x) = \sum_{k=0}^{n} f_k(x)$$
It is merely a function. We can say that the sequence (F_n) converges to a limit F and write
$$F(x) = \sum_{k=0}^{\infty} f_k(x)$$
If the sequence of partial sums converges uniformly then we say the series **converges uniformly**. If $\sum |f_k(x)|$ converges then we say that $\sum f_k$ converges **absolutely**.

Proposition 3.7. **(Weierstrass M-test)** If $\sum M_k$ is a convergent series of constants and if $f_k \in C_b$ satisfies $||f_k|| \le M_k$ for all k then $\sum f_k$ converges uniformly and absolutely

Theorem 3.8. The uniform limit of Riemann integrable functions is Riemann integrable, and the limit of the integrals is the integral of the limit. That is,
$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b unif \lim_{n\to\infty} f_n(x)$$

In other words, $\mathcal{R}$, the set of Riemann integrable functions defined on $[a,b]$ is a closed subset of C_b and so $f \to \int_a^b f(x)dx$ is a continuous map from $\mathcal{R}$ to $\mathbb{R}$. Thus,
$$C_b \supset \mathcal{R} \supset C^0 \supset C^1 \supset ... \supset C^\infty \supset C^\omega$$

Proof. Since C_b is a complete metric space, f is bounded and the Riemann-Lebesgue theorem implies that $f \in \mathcal{R}$ since it is discontinuous on a zero set and so
$$\begin{aligned}\left|\int_a^b f(x)dx - \int_a^b f_n(x)dx\right| &= \left|\int_a^b f(x) - f_n(x)dx\right| \\ &\le \int_a^b |f(x) - f_n(x)|dx \\ &\le d(f,f_n)(b-a) \\ &\to 0\end{aligned}$$
□

Theorem 3.9. The uniform limit of a sequence of differentiable functions is differentiable provided that the sequence of derivatives also converges uniformly

Proof.
If we first assume that f_n' are all continuous then
$$f_n(x) = f_n(x) + \int_a^x f_n'(t)dt \rightrightarrows f(a) + \int_a^x g(t)dt$$
so then $f(x) = f(a) + \int_a^x g(t)dt$ and so $f' = g$ by the FTC.
For the general case, $x \in [a,b]$ and define
$$\phi_n(t) = \begin{cases} \frac{f_n(t) - f_n(x)}{t-x} \text{ if } t \neq x \\ f_n'(x) \text{ if } t = x \end{cases}$$

and

$$\phi(t) = \begin{cases} \frac{f(t)-f(x)}{t-x} \text{ if } t \neq x \\ g(x) \text{ if } t = x \end{cases}$$

Each ϕ_n is continuous since $\phi_n(t)$ converges to $f'_n(x)$ as $t \to x$. Also, it is clear that ϕ_n converges pointwise to ϕ. For any m, n the MVT applied to $f_m - f_n$ gives that

$$\phi_m(t) - \phi_n(t) = \frac{(f_m(t) - f_n(t)) - (f_m(x) - f_n(x))}{t - x} = f'_m(\theta) - f'_n(\theta)$$

for some θ in between t and x. Since $f'_n \rightrightarrows g$ the difference $f'_m - f'_n$ tends uniformly to 0 as $m, n \to \infty$. Thus (ϕ_n) is cauchy in C^0. By completeness, ϕ_n converges uniformly to a limit function ψ and ψ is continuous. As already remarked, the pointwise limit of ϕ_n is ϕ and so $\psi = \phi$. Continuity of $\psi = \phi$ implies that $g(x) = f'(x)$.

□

3.2 Compactness and Equicontinuity

The Heine-Borel theorem says that a closed and bounded set in R^m is compact. Closed and bounded sets in C^0 are rarely compact. Consider the closed unit ball

$$\mathcal{B} = \{f \in C^0([a,b], \mathbb{R}) : ||f|| \leq 1\}$$

To see that $\mathcal{B}$ is not compact, consider the sequence $f_n(x) = x^n$. It lies entirely in $\mathcal{B}$, however any subsequence would converge to a function that is 0 everywhere except for at 1 which does not belong to C^0. The problem comes from the fact that C^0 is infinite dimensional.

Definition 3.10. A sequence of functions (f_n) in C^0 is **equicontinuous** (uniform) if

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } |s - t| < \delta \text{ and } n \in \mathbb{N} \Rightarrow |f_n(s) - f_n(t)| < \epsilon$$

This is in contrast to **pointwise equicontinuity**, which requires that

$$\forall \epsilon > 0 \text{ and } \forall x \in [a,b] \ \exists \delta > 0 \text{ such that } |x - t| < \delta \text{ and } n \in \mathbb{N} \Rightarrow |f_n(x) - f_n(t)| < \epsilon$$

Note that both of these definitions work equally well for any arbitrary collection of functions. Equicontinuity can be thought of as a fundamental property among spaces of functions. An important result stemming from the definition being

Theorem 3.11. Arzela-Ascoli Theorem Every bounded equicontinuous sequence of functions in $C^0([a,b], \mathbb{R})$ has a uniformly convergent subsequence.

Proof. $[a,b]$ has a countable dense subset $D = \{d_1, d_2, ...\}$. Boundedness of (f_n) implies that $(f_n(d_1))$ is a sequence in a compact set. So a subseqeuence $f_{1,k}(d_1)$ exists which converges to some y_1 as $k \to \infty$. Then, the subsequence $(f_{1,k})$ evaluated at d_2 has a convergent subsequence $(f_{2,k})$ which converges to y_2 at d_2 and y_1 at d_1. This continues forever where

$$(f_{m,k}) \text{ is a subsequence of } (f_{m-1,k})$$
$$j \leq m \Rightarrow f_{m,k}(d_j) \to y_j \text{ as } k \to \infty$$

Now, consider the diagonal subsequence $(g_m) = (f_{m,m})$ It clearly converges pointwise on D.
TO ensure that the subsequence converges uniformly on $[a,b]$ is suffices to show that (g_m) is a Cauchy sequence. Let $\epsilon > 0$ be given. Equicontinuity gives a $\delta > 0$ such that for all $s, t \in [a,b]$ we have

$$|s - t| < \delta \Rightarrow |g_m(s) - g_m(t)| < \frac{\epsilon}{3}$$

With this δ there is J such that $\cup_{i \leq J} M_\delta(d_i) = [a,b]$. Then since $\{d_1, ...d_J\}$ is a finite set and $g_m(d_j)$ converges for each d_j there is $N \in \mathbb{N}$ such that $\forall l, m \geq N$ and all $j \leq J$,

$$|g_m(d_j) - g_l(d_j)| < \frac{\epsilon}{3}$$

Now assume $l, m \geq N$ and let $x \in [a,b]$ then choose d_j with $|d_j - x| < \delta$ and $j \leq J$. Then

$$\begin{aligned} |g_m(x) - g_l(x)| &\leq |g_m(x) - g_m(d_j)| + |g_m(d_j) - g_l(d_j)| + |g_l(d_j) - g_l(x)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Hence, (g_m) is Cauchy in C^0 and so it converges in C^0. □

Corollary 3.12. Assume that $f_n : [a,b] \to \mathbb{R}$ is a sequence of differentiable functions whose derivatives are uniformly bounded. If for one point x_0, the sequence $(f_n(x_0))$ is bounded as $n \to \infty$ then the sequence (f_n) has a subsequence that converges uniformly on the whole interval $[a,b]$.

Proof. Let M be a bound for the derivatives $|f'_n(x)|$, valid for all $n \in \mathbb{N}$ and all $x \in [a,b]$. Equicontinuity of (f_n) follows from the Mean Value Theorem:

$$|s - t| < \delta \Rightarrow |f_n(s) - f_n(t)| = |f'_n(\theta)||s - t| \leq M\delta$$

choosing $\delta = \frac{\epsilon}{M+1}$ shows that (f_n) is equicontinuous. Then with C being a bound for $|f_n(x_0)|$,

$$|f_n(x)| \leq |f_n(x) - f_n(x_0)| + |f_n(x_0)| \leq M|b-a| + C$$

The Arezela-Ascoli theorem can then be applied to find that a uniformly convergent subsequence exists. □

Theorem 3.13. Heine-Borel Theorem in a Function Space A subset $\mathcal{E} \subset C^0$ is compact if and only if it is closed, bounded, and equicontinuous.

Proof.

□

3.3 Uniform Approximation

Theorem 3.14. Weierstrass Approximation Theorem The set of polynomials is dense in $C^0([a,b],\mathbb{R})$

Remember that in C^0 density means that for every $f \in C^0$ and each $\epsilon > 0$ there is a polynomial function $p(x)$ such $\forall x \in [a,b]$

$$|f(x) - p(x)| < \epsilon$$

The approximating function is built from f by sampling the values of f and recombining them in some clever way. WLOG assume $[a.b]$ is $[0,1]$

Proof. WLOG $[a,b] = [0,1]$ $f(0) = f(1) = 0$ and we can extend f to $\mathbb{R}$ by defining $f(x) = 0$ for all $x \in \mathbb{R} \setminus [0,1]$ then we consider the function

$$\beta_n(t) = b_n(1-t^2)^n \quad -1 \leq t \leq 1$$

Where b_n is chosen so that $\int_{-1}^{1} \beta_n(t)dt = 1$. Then, for $x \in [0,1]$ set

$$P_n(x) = \int_{-1}^{1} f(x+t)\beta_n(t)dt$$

This is the weighted average of the values of f using the weight function β_n.

$$P_n(x) = \int_{x-1}^{x+1} f(u)\beta_n(u-x)du = \int_0^1 f(u)b_n(1-(u-x)^2)^n du$$

To check that $P_n \rightrightarrows f$ as $n \to \infty$, we need to estimate $\beta_n(t)$. We claim that if $\delta > 0$ then

$$\beta_n(t) \rightrightarrows 0$$

as $n \to \infty$ and $\delta \leq |t| \leq 1$. Proceeding more rigorously and using the definition of β_n as $\beta_n(t) = b_n(1-t^2)^n$ then

$$1 = \int_{-1}^{1} \beta_n(t)dt \geq \int_{\frac{-1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} b_n(1-t^2)^n dt \geq b_n \frac{2}{\sqrt{n}}(1-\frac{1}{n})^n$$

Then since $(1-\frac{1}{n})^n \to \frac{1}{e}$ we see that for some constant c and for all n,

$$b_n \leq c\sqrt{n}$$

It follows that $\beta_n(t) = b_n(1-t^2)^n \leq c\sqrt{n}(1-\delta^2)^n \to 0$ as $n \to \infty$. Further, uniform continuity of f gives $\delta > 0$ such that $|t| < \delta$ implies $|f(x+t) - f(x)| < \frac{\epsilon}{2}$. Since β_n has integral 1 on $[-1,1]$ we have

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} (f(x+t) - f(x))\beta_n(t)dt \right| \\
&\leq \int_{-1}^{1} |f(x+t) - f(x)|\beta_n(t)dt \\
&= \int_{|t|<\delta} |f(x+t) - f(x)|\beta_n(t)dt + \int_{|t|\geq\delta} |f(x+t) - f(x)|\beta_n(t)dt \\
&< \frac{\epsilon}{2} + 2M \int_{|t|\geq\delta} \beta_n(t)dt \\
&< \epsilon
\end{aligned}$$

Since $\beta_n \rightrightarrows 0$ the integral converges to 0 as well

□

We now move on to see how this result can be extended to functions on compact metric spaces.

Definition 3.15. A subset $\mathcal{A}$ of $C^0M = C^0(M,\mathbb{R})$ is a **function algebra** if it closed under addition, scalar multiplication, and function multiplication. That is for $f, g \in \mathcal{A}$ and $c \in \mathbb{R}$ we have

$$f + g, cf, f \cdot g \in \mathcal{A}$$

The function algebra $\mathcal{A}$ is said to **vanish at a point** if $f(p) = 0$ for all $f \in \mathcal{A}$. For example the function algebra of all polynomials with zero constant term vanishes at $x = 0$.

Definition 3.16. $\mathcal{A}$ **separates points** if for each pair of distinct points $p_1, p_2 \in M$ there is a function $f \in \mathcal{A}$ such that

$$f(p_1) \neq f(p_2)$$

Theorem 3.17. Stone-Weierstrass Theorem If M is a compact metric space and $\mathcal{A}$ is a function algebra in C^0M that vanishes nowhere and separates points then $\mathcal{A}$ is dense in C^0M

This proof requires the former and two lemmas.

Proposition 3.18. If $\mathcal{A}$ vanishes nowhere and separates points then there exists $f \in \mathcal{A}$ with specified values at any pair of distinct points.

3.4 Contractions and ODEs

If $f : M \to M$ and for some $p \in M$ then p is a **fixed-point** of f if $f(p) = p$. A particular type of function must have a fixed point because it strictly "shrinks" space.

Definition 3.19. A **contraction** of M is a mapping $f : M \to M$ such that for some constant $k < 1$ and all $x, y \in M$ we have

$$d(f(x), f(y)) \leq kd(x, y)$$

Theorem 3.20. Banach Contraction Principle Suppose that $f : M \to M$ is a contaction and the metric space M is complete. Then f has a unique fixed-point p and for any $x \in M$, the iterate $f^n(x) = f \circ f \circ ... \circ f(x)$ converges to p as $n \to \infty$

A **vector ODE** on U is given as M simultaneous scalar equations

$$\begin{aligned} x_1' &= f_1(x_1, x_2, ..., x_m) \\ x_2' &= f_2(x_1, x_2, ..., x_m) \\ &... \\ x_m' &= f_m(x_1, x_2, ..., x_m) \end{aligned}$$

We seek m real-valued functions $x_1(t), ..., x_m(t)$ that solve the ODE. If we define $F(x) = f_1(x), ..., f_m(x))$ then F is a vector field on U and we seek a **trajectory** of F, that is, a curve $\gamma : (a, b) \to U$ such that $a < 0 < b$ and for all $t \in (a, b)$ we have

$$\gamma'(t) = F(\gamma(t)) \text{ and } \gamma(0) = p$$

ODEs.png

Figure 4: γ is always tangent to F

Firstly assume that F is **Lipschitz**. That is, $\exists L$ such that for all points $x, y \in U$ we have

$$|F(x) - F(y)| \leq L|x - y|$$

Theorem 3.21. Picard's Theorem Given $p \in U$ there exists an F-trajectory $\gamma(t)$ in U through p. This means that $\gamma : (a, b) \to U$ solves the equations. Locally, γ is unique.

Firstly note that the integral version of the system is

$$\gamma(t) = p + \int_0^t F(\gamma(s))ds$$

Proof. Since F is continuous, there exists a compact neighborhood $N = \bar{N}_r(p) \subset U$ and a constant M such that $|F(x)| \leq M$ for all $x \in N$. Choosing $\tau > 0$ such that

$$\tau M \leq r \text{ and } \tau L < 1$$

Consider the set $\mathcal{C}$ of all continuous functions from $[-\tau, \tau]$ to N with respect to the sup metric. Then define the operator

$$\Phi(\gamma)(t) = p + \int_0^t F(\gamma(s))ds$$

Solving the integral equation is the same thing as solving for a fixed point of Φ that is a function γ such that $\Phi(\gamma) = \gamma$. We just need to show that Φ is a contraction on $\mathcal{C}$. We firstly see that $\Phi(\gamma)(t)$ is a continuous vector-valued function of t and that by

$$\Phi(\gamma)(t) - p = \left| \int_0^t F(\gamma(s))ds \right| \leq \tau M \leq r$$

$$\begin{aligned} d(\Phi(\gamma), \Phi(\sigma)) &= \sup_t \left| \int_0^t F(\gamma(s)) - F(\gamma(s))ds \right| \\ &\leq \tau \sup_s |F(\gamma(s)) - F(\sigma(s))| \\ &\leq \tau \sup_s L|\gamma(s) - \sigma(s)| \\ &\leq \tau L d(\gamma, \sigma) \end{aligned}$$

Since $\tau L < 1$ it follows that Φ is a contraction and so a unique fixed point, solution to the ODE on $[-\tau, \tau]$ exists $\square$

4 Lebesgue Theory

Measuring simple sets is easy, for instance, the length of (a, b) is $b - a$. How do we measure more complicated sets?

4.1 Outer Measure

Definition 4.1. The **length** of the interval I is $|I| = b - a$. The **Lebesgue outer measure** of $A \subset \mathbb{R}$ is

$$m^*A = \inf\{\Sigma_k |I_k| : \{I_k\} \text{ is a covering of A by open intervals}\}$$

If for every covering of A, $\{I_k\}$, $\Sigma_k |I_k|$ diverges then $m^*A := \infty$. There are three immediate consequences of the above definition.

Theorem 4.2.
1. $m^*\emptyset$=0
2. If $A \subset B$ then $m^*A \leq m^*B$
3. If $A = \cup_{n=1}^{\infty} A_n$ then $m^*A \leq \sum\limits_{n=1}^{\infty} m^*A_n$

Proof. For $\epsilon > 0$ there exists for each n a covering $\{I_{k,n} : k \in \mathbb{N}\}$ such that

$$\sum_{n=1}^{\infty} |I_{k,n}| < m^*A_n + \frac{\epsilon}{2^n}$$

The collection $\{I_{k,n} : k, n \in \mathbb{N}\}$ covers A and

$$\sum_{k,n} |I_{k,n}| = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |I_{k,n}| \leq \sum_{k=1}^{\infty}(m^*A_n + \frac{\epsilon}{2^n}) = \sum_{k=1}^{\infty} m^*A_n + \epsilon$$

□

To assert 3., it suffices by definition of the infimum (think reverse of the epsilon definition of the supremum), to find a covering of A which can in fact be made smaller than the series presented plus an additional arbitrary epsilon. To do this we can use the definition of infimums on each one of the A_n and essentially squeeze the covering onto A.

Definition 4.3. If $Z \subset \mathbb{R}^n$ has outer measure zero, then it is a **zero set**

Proposition 4.4. Every subset of a zero set is a zero set. The countable union of zero sets is a zero set. Each plane $P_i(a)$ is a zero set in $\mathbb{R}^n$

This follows immediately from the above theorem. It seems that this theorem will be quite important.

Theorem 4.5. The linear outer measure of a closed interval is its length. The n-dimensional outer measure of a closed box is its volume.

Proof. Let [a,b] be a closed interval. Note that $(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$ is a cover for [a,b] and so $m^*([a, b]) \leq b - a + \epsilon$ It follows by the epsilon definition of the infimum that $m^*([a, b]) \leq b - a$.

To get the reverse inequality, we need to show that for every covering of [a,b] its total length is greater than or equal to $b - a$. Firstly, since **[a,b] is compact it suffices to only consider finite open coverings of [a,b]**. We then reason inductively.

The base case follows immediately as the endpoints of the open interval that contains [a,b] must be further apart than a and b themselves. Now, assume that the claim holds for an arbitrary n. Let $I_1, ..., I_{n+1}$ be an open covering of [a,b] with $I_i = (a_i, b_i)$. We claim that $\sum\limits_{i=1}^{n+1} |I_i| > b - a$. One of the intervals contains a, say (a_1, b_1)

If $b_1 \geq b$ then $\sum\limits_{i=1}^{n} |I_i| \geq |I_1| = b_1 - a_1 > b - a$

If $b_1 < b$ then

$$[a, b] = [a, b_1) \cup [b_1, b]$$

and $|I_1| > b_1 - a$. $[b_1, b]$ is a closed interval covered by n many open intervals, $I_2, ..., I_{n+1}$. It follows by the induction hypothesis that $\sum\limits_{i=2}^{n+1} |I_i| > b - b_1$. Thus,

$$\sum_{i=1}^{n+1} |I_i| = |I_1| + \sum_{i=2}^{n+1} |I_i| > (b_1 - a) + (b - b_1) = b - a$$

□

An interesting proof. Try and remember all the tools you have in $\mathbb{R}$, in particular that of compactness and the results that follow from its formal definition. The same can be said for higher dimensions:

Proof. Let $R = [a,b] \times [c,d] \subset \mathbb{R}^2$. Using a similar covering as above, it is easy to see that $m^*R \leq (b-a)*(d-c)$. R has a Lebesgue number λ. Break R up into a finite set of rectangles $S_j \subset R$ where each S_j has a diameter less than λ. By their construction, $\sum |S_j| = (b-a)*(d-c)$. It follows that

$$\sum_j |S_j| \leq \sum_i \sum_{S_j \subset R_i} |S_j| \leq \sum |R_i|$$

which implies that $(b-a)*(d-c) \leq \sum |R_i|$. Thus, $(b-a)*(d-c) = m^*R$ □

4.2 Measurability

If A and B are disjoint intervals in $\mathbb{R}$

$$m^*(A \sqcup B) = m^*A + m^*B$$

It seems like this should follow to arbitrary disjoint sets. However, this is not the case in general and we need an added condition.

Definition 4.6. A set $E \subset \mathbb{R}$ is **(Lebesgue) measurable** if the division $E|E^c$ of $\mathbb{R}$ is such that for each $X \subset \mathbb{R}$ (test set) we have

$$m^*X = m^*(X \cap E) + m^*(X \cap E^c)$$

Next, we denote $\mathcal{M} = \mathcal{M}(\mathbb{R}^n)$ as the collection of all Lebesgue measurable subsets of $\mathbb{R}^n$. If E is measurable, its **Lebesgue Measure** $mA = m^*A$.

We now move out of $\mathbb{R}^n$ and into more abstract sets. For the rest of this section we will consider and arbitrary measure as the results have nothing to do with euclidean space.

Definition 4.7. Let M be any set. An **abstract outer measure** on M is a function $\omega : 2^M \to [0,\infty]$ that satisfies the three axioms of outer measure.

1. $\omega(\emptyset) = 0$
2. ω is monotone
3. ω is countably subadditive

Further, a set $E \subset M$ is **measurable** with respect to ω if $E|E^c$ is so clean that for each test set $X \subset M$ we have

$$\omega(X) = \omega(X \cap E) + \omega(X \cap E^c)$$

For any set M there are two trivial outer measures. The counting measure (assigns cardinality for measure) and the zero measure (empty set zero measure, else infinity). All sets are measurable with respect to these outer measures.

Theorem 4.8. The collection $\mathcal{M}$ of measurable sets with respect to any outer measure on any set M is a σ-algebra and the outer measure restricted to this σ-algebra is countably additive. All zero sets are measurable and have no effect on measurability. In particular Lebesgue measure has these properties.

Note that a **σ-algebra** is a collection of sets that includes the empty set, is closed under complement, and is closed under countable union. **Countable additivity** of ω means that if $E_1, E_2, \ldots$ are measurable with respect to ω then

$$E = \sqcup_i E_i \Rightarrow \omega(E) = \sum_i \omega(E_i)$$

From countable additivity we deduce a very useful fact about measures. It applies to any outer measure ω, in particular to the Lebesgue outer measure.

Theorem 4.9. If $\{E_k\}$ and $\{F_k\}$ are sequences of measurable sets then the following holds:

upward measure continuity	$E_k \uparrow E \Rightarrow \omega(E_k) \uparrow \omega(E)$
downward measure continuity	$F_k \downarrow F$ and $\omega(F_1) < \infty \Rightarrow \omega(F_k) \downarrow \omega(F)$

4.3 Meseomorphism

We first define a general space

Definition 4.10. A **measure space** is a triple $(M, \mathcal{M}, \mu)$ where M is a set, $\mathcal{M}$ is a σ-algebra of subsets of M, and μ is a measure on M. That is, $\mu : \mathcal{M} \to [0, \infty]$ has the three properties

- $\mu(\emptyset) = 0$
- μ is monotone: $A \subset B$ implies $\mu(A) \leq \mu(B)$
- μ is countably additive on $\mathcal{M}$: $E = \sqcup E_i$ implies $\mu(E) = \sum \mu(E_i)$

By the definition of outer measure given in the previous section, any outer measure restricted to the σ-algebra of measurable sets is a measure. Similar to how isomorphisms preserve algebraic structure and homeomorphisms preserve topological structure, meseomorphisms preserve measure structure.

Definition 4.11. Given $(M, \mathcal{M}, \mu)$ and $(M', \mathcal{M}', \mu')$ then a mapping $T : M \to M'$ is a

- **mesemorphism** if T sends each $E \in \mathcal{M}$ to $T(E) \in \mathcal{M}'$
- **meseomorphism** if T is a bijection and both T and T^{-1} are mesemorphisms
- **mesisometry** if T is a meseomorphism and $\mu'(T(E)) = \mu(E)$ for each $E \in \mathcal{M}$

4.4 Affine Motions

An affine motion of $\mathbb{R}^n$ is an invertible linear transformation T followed by a translation. Translation does not affect Lebesgue measure.

Theorem 4.12. An affine motion $T : \mathbb{R}^n \to \mathbb{R}^n$ is a meseomorphism. It multiplies measure by $|\det(T)|$

In order to prove this, we need some preliminary results. Firstly,

Proposition 4.13. Every open set in n-space is a countable disjoint union of open cubes plus a zero set

Proof. Cover the set with unit dyadic open squares. Accept all the squares that are fully contained inside the set. For all the rejected squares, bisect them into four equal sub squares. Continue this process. The "lines" are a measure zero set. □

This then implies that

Proposition 4.14. Every open set is a countable disjoint union of balls plus a zero set.

Now for the proof of Theorem 4.19

Proof. T is Lipschitz with Lipschitz constant $||T||$. Its inverse has a Lipschitz constant $||T^{-1}||$ □

4.5 Regularity

We now turn to the question, which subsets of Euclidean space are measurable?

Theorem 4.15. Open and closed sets in $\mathbb{R}^n$ are Lebesgue measurable

Proposition 4.16. The half spaces $[a, \infty) \times \mathbb{R}^{n-1}$ and $(a, \infty) \times \mathbb{R}^{n-1}$ are measurable in $\mathbb{R}^n$. So are all open boxes.

Proof. In $n = 2$. $a \times \mathbb{R}$ is a zero set. Now define

$$X^- = \{(x, y) \in X : x < a\} \quad \text{and} \quad X^+ = \{(x, y) \in X : x > a\}$$

Then $X = X^- \sqcup X^+$. Given an $\epsilon > 0$ there is a countable covering $\mathcal{R}$ by rectangles R with $\sum_{\mathcal{R}} |R| \leq m^*(X) + \epsilon$. Let $R^{\pm} = \{(x, y) \in \mathcal{R} : R \in \mathcal{R} \text{ and } \pm (x - a) > 0\}$ Then

$$\begin{aligned} m^*(X) &\leq m^*(X \cap H) + m^*(X \cap H^c) \\ &\leq \sum_{\mathcal{R}^+} |R^+| + \sum_{\mathcal{R}^-} |R^-| \\ &= \sum_{\mathcal{R}} |R| \\ &\leq m^*(X) + \epsilon \end{aligned}$$

It follows immediately that (a, ∞) is measurable and that $(a, b) \times \mathbb{R}$ is measurable as it is the intersection of two measurable sets. Interchanging order gives that $\mathbb{R} \times (c, d)$ is also measurable and so is the intersection $R = (a, b) \times (c, d)$ $\square$

The general result for open and closed sets being measurable follows from the fact that every open set is a countable union of open boxes and so since σ-algebras are closed with respect to countable unions and complements, every open and closed set is measurable.

Definition 4.17. A countable intersection of open sets is called a G_δ-set
A countable union of closed sets is called a F_σ-set

It is clear that both of these sets are measurable themselves.

Theorem 4.18. Lebesgue measure is **regular** in the sense that each measurable set E can be sandwiched between an F_σ-set and a G_δ-set, $F \subset E \subset G$, such that $G \setminus F$ is a zero set. Conversely if there is F, G such that $F \subset E \subset G$ then E is measurable.

This results in the following corollary

Corollary 4.19. A bounded subset $E \subset \mathbb{R}^n$ is measurable if and only if it has a **regularity sandwich** $F \subset E \subset G$ such that F is a F_σ-set, G is a G_δ-set, and $m(F) = m(G)$

This property can be used to show an important class of functions are meseomorphisms.

Corollary 4.20. A lipeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ is a meseomorphism

Proof. Lipeomorphisms send regularity sandwiches to regularity sandwiches $\square$

4.6 Inner Measures, Hulls, and Kernels

Consider any bounded $A \subset \mathbb{R}^n$, measurable or not. m^*A is the infimum of the measure of open sets that contain A. The infimum is acheived by a G_δ-set that contains A. We call it a **hull** of A and denote it as H_A. It is unique up to a zero set. Dually, the inner measure of A is the supremum of the measure of closed sets it contains. The supremum is achieved by an F_σ-set contained in A. We call it a **kernel** of A and denote it as K_A.

Definition 4.21. We denote the **inner measure** of A as $m_*(A) = m(K_A)$ clearly $m_*(A) \leq m^*(A)$ and m_* measures A from inside.

The hull and kernel are the measure theoretic analogs of closure and interior in topology.

Definition 4.22. The **measure theoretic boundary** of A is $\partial_m(A) = H_A \mathbb{K}_A$

Theorem 4.23. If $A \subset B \subset \mathbb{R}^n$ and B is a box then A is measurable if and only if it divides B cleanly.

4.7 Products and Slices

The regularity of Lebesgue measure has a number of uses. The following are some:

Theorem 4.24. If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^k$ are measurable thn $A \times B$ is measurable and $m(A \times B) = m(A) \cdot m(B)$

Where $0 \cdot \infty = 0 = \infty \cdot 0$
To prove this, we need some lemmas. In particular,

Proposition 4.25. If U and V are open then $U \times V$ is measurable and $m(U \times V) = m(E)m(V)$.

Proof. A previous lemma can be applied to find that $U = \sqcup_i I_i \cup Z_U$ and $V = \sqcup_j J_j \cup Z_V$ where Z are zero sets and I and J are intervals. Then

$$U \times V = \sqcup_{i,j} I_i \times J_j \cup Z$$

$Z = (Z_U \times V) \cup (U \times Z_V)$ Then,

$$\left(\sum_i m(I_i)\right)\left(\sum_j m(J_j)\right) = \sum_{i,j} m(I_i)m(J_j) = \sum_{i,j} M(I_i \times J_j)$$

we conclude that $m(U \times V) = m(U)m(V)$ $\square$

4.8 Lebesgue Integrals

We can now the fully realize the idea from first year Analysis that the integral of a function is the area under its graph.

Definition 4.26. The **undergraph** of f is

$$\mathcal{U}(f) = \{(x,y) \in \mathbb{R} \times [0,\infty) : 0 \leq y < f(x)\}$$

The function f is called **measurable** if $\mathcal{U}(f)$ is measurable with respect to planar Lebesgue measure. If it is then

Definition 4.27. The **Lebesgue integral** of f is the measure of the undergraph

$$\int f = m(\mathcal{U}(f))$$

Since a measurable set can have infinite measure, the definition allows for $\int f = \infty$. We now define a common requirement for functions in Real Analysis

Definition 4.28. The function $f : \mathbb{R} \to [0,\infty)$ is **Lebesgue integrable** if (it is measurable and) its integral is finite. The set of all integrable functions is denoted by $\mathcal{L}^1$

Most convergence proofs are easy from a geometric perspective when considering undergraphs. We firstly define a new form of convergence for sequences of functions. (f_n) converges to f **almost everywhere (a.e.)** if $\lim_{n\to\infty} f_n(x) = f(x)$ for **almost every** x. That is, for all x not belonging to some zero set.

Theorem 4.29. Monotone Convergence Theorem Assume that (f_n) is a sequence of measurable functions $f_n : \mathbb{R} \to [0,\infty)$ and $f_n \uparrow f$ a.e. as $n \to \infty$. Then

$$\int f_n \uparrow \int f$$

The above follows from upward measure continuity. Note that it does not require the functions to be in $\mathcal{L}^1$.

Definition 4.30. The **completed undergraph** of $f : \mathbb{R} \to [0,\infty)$ is the set

$$\hat{\mathcal{U}}(f) = \{(x,y) \in \mathbb{R} \times [0,\infty) : 0 \leq y \leq f(x)\}$$

Corollary 4.31. $\hat{\mathcal{U}}(f)$ is measurable if and only if $\mathcal{U}(f)$ is measurable, and if f is measurable then their measures are equal

Proof. Follows immediately from Theorem 4.19 when the matrix representations of the functions $T_{\pm n}(x,y) = (x, (1 \pm \frac{1}{n})y)$ are considered □

The same concept can be used to define the downwards convergence of integrals. However, this would require the functions to be integrable.

Definition 4.32. If $f_n : X \to [0,\infty)$ is a sequence of functions then the **lower and upper envelope sequences** are

$$\underline{f}_n = \inf\{f_k(x) : k \geq n\} \text{ and } \bar{f}_n = \sup\{f_k(x) : k \geq n\}$$

Corollary 4.33. $\hat{\mathcal{U}}(\bar{f}_n) = \cup_{k\geq n}\mathcal{U}(f_k)$ and $\hat{\mathcal{U}}(\underline{f}_n) = \cap_{k\geq n}\mathcal{U}(f_k)$

Proof.

$$\begin{aligned}(x,y) \in \mathcal{U}(\bar{f}_n) &\Leftrightarrow y < \sup\{f_k(x) : k \geq n\} \\ &\Leftrightarrow \exists l \geq n \text{ such that } y < f_l(x) \\ &\Leftrightarrow \exists l \geq n \text{ such that } (x,y) \in \mathcal{U}(f_l) \\ &\Leftrightarrow (x,y) \in \cup_{k\geq n}\mathcal{U}(f_k)\end{aligned}$$

□

The next theorem is fundamental in Measure Theory and has many applications in Probability.

Theorem 4.34. Dominated Convergence Theorem If $f_n : \mathbb{R} \to [0, \infty)$ is a sequence of measurable functions such that $f_n \to f$ a.e. and if there exists a function $g : \mathbb{R} \to [0, \infty)$ whose integral is finite and which is an upper bound for all functions f_n then f is integrable and $\int f_n \to \int f$ as $n \to \infty$

Proof. Proposition 30 implies the envelope functions are measurable. Due to the dominator g they are integrable. The Monotone Convergence Theorem and Corollary 29 imply that their integrals converge to $\int f$. Since $\mathcal{U}(\underline{f}_n) \subset \mathcal{U}(f_n) \subset \hat{\mathcal{U}}(\bar{f}_n)$ the integral of f_n also converges to $\int f$ $\square$

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Figure 5: Dominated Convergence

Corollary 4.35. The pointwise limit of measurable functions is measurable

We firstly define two new forms of limits of sequences of functions.

Definition 4.36. Given a sequence $(f_n)_{n\in\mathbb{N}}$

$$\limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sup_{k\geq n} f_k(x) \text{ and}$$

$$\liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} \inf_{k\geq n} f_k(x)$$

Theorem 4.37. Fatou's Lemma If $f_n : \mathbb{R} \to [0, \infty)$ is a sequence of measurable functions then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

Proof. The assertion is really more about lim infs than integrals. The lim inf of the sequence (f_n) is $f = \lim_{n\to\infty} \underline{f}_n$. Since $\underline{f}_n \uparrow f$, the MCT implies that $\int \underline{f}_n \uparrow \int f$, and since $\underline{f}_n \leq f_n$ we have $\int f \leq \liminf \int f_n$ $\square$

Theorem 4.38. Let $f, g : \mathbb{R} \to [0, \infty)$ be measurable functions

- If $f \leq g$ Then $\int f \leq \int g$
- If $\mathbb{R} = \sqcup_{k\geq 1} X_k$ and each X_k is measurable then
$$\int f = \sum_{k\geq 1} \int_{X_k} f$$
- If $X \subset \mathbb{R}$ is measurable then $m(X) = \int \chi_X$
- If $m(X) = 0$ then $\int_X f = 0$
- If $f(x) = g(x)$ almost everywhere then $\int f = \int g$
- If $c \geq 0$ then $\int cf = c \int f$
- The integral of f is zero if and only if $f(x) = 0$ for almost every x
- $\int f + g = \int f + \int g$

All but the last result follows from theorems and propositions in measure. However, the last property is slightly more difficult to prove.

Definition 4.39. If $f : \mathbb{R} \to \mathbb{R}$ then the **f-translation** $T_f : \mathbb{R}^2 \to \mathbb{R}^2$ sends the point (x, y) to $(x, y + f(x))$

T_f is a bijection whose inverse is T_{-f}

Theorem 4.40. If $f : \mathbb{R} \to [0, \infty)$ is integrable then T_f preserves planar Lebesgue measure. That is, it is a mesisometry.

This theorem then allows us to show the linearity of the integral for positive functions.

Corollary 4.41. If $f, g : \mathbb{R} \to [0, \infty)$ are integrable then

$$\int f + g = \int f + \int g$$

Proof. Since $\mathcal{U}(f + g) = \mathcal{U}(f) \sqcup T_f(\mathcal{U}(g))$ and T_f is a mesisometry we see that $f + g$ is measurable and $m(\mathcal{U}(f + g)) = m(\mathcal{U}(f)) + m(\mathcal{U}(g))$. That is, the integral of the sum is the sum of the integrals. $\square$

This alongside the MCT gives a useful result when dealing with series.

Corollary 4.42. If $f_k : \mathbb{R} \to [0, \infty)$ is a sequence of integrable functions then

$$\sum_{k=1}^{\infty} \int f_k = \int \sum_{k=1}^{\infty} f_k$$

Proof. With $F_n(x) = \sum_{k=1}^{n} f_k(x)$ and $F(x) = \sum_{k=1}^{\infty} f_k(x)$ then $F_n(x) \uparrow F(x)$ and so MCT and the above corollary can be applied. $\square$

The above results have assumed that the functions of interest have been non-negative. To generalize the integral to arbitrary functions, we first define the positive and negative portions of a function.

Definition 4.43. For $f : \mathbb{R} \to \mathbb{R}$ define

$$f_+(x) = \begin{cases} f(x) \text{ if } f(x) \geq 0 \\ 0 \text{ if } f(x) < 0 \end{cases} \quad \text{and } f_-(x) \begin{cases} -f(x) \text{ if } f(x) < 0 \\ 0 \text{ if } f(x) \geq 0 \end{cases}$$

Then $f = f_+ - f_-$ and if $f_\pm$ is integrable then the integral of f is defined to be

$$\int f = \int f_+ - \int f_-$$

Proposition 4.44. The set of measurable functions $f : \mathbb{R} \to \mathbb{R}$ is a vector space, the set of integrable functions is a subspace, and the integral is a linear map from the latter onto $\mathbb{R}$

4.9 Italian Measure Theory

Figure 6: Italian Measure Theory

5 Fourier Analysis

Fourier series concerns functions defined on the circle or equivalently, periodic functions. The following extends analogous Analysis to functions defined on all of $\mathbb{R}$. However, for this we require the functions to "vanish" at infinity. Instead of associating each function with a sequence of coefficients as in Fourier series, the Fourier transform associates a function defined on $\mathbb{R}$ with a function defined on $\mathbb{R}$. The Fourier coefficients a_n of a function defined on $[0, 1]$ are

$$a_n = \int_0^1 f(x)e^{-2\pi inx}dx$$

then

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}$$

The Fourier transform is defined by extending this definition continuously.

Definition 5.1. The **Fourier transform** of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\eta}dx$$

then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{2\pi ix\zeta}d\zeta$$

5.1 Integration of Functions

We first define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{N\to\infty} \int_{-N}^{N} f(x)dx$$

However, this may not exist. We can impose restrictions on the size of f at infinity. In particular,

Definition 5.2. A function f on $\mathbb{R}$ is said to be of **moderate decrease** ($f \in \mathcal{M}(\mathbb{R})$) if f is continuous and there exists a constant $A > 0$ such that

$$|f(x)| \le \frac{A}{1 + x^2} \text{ for all } x \in \mathbb{R}$$

$\mathcal{M}$ is a vector space. The exponent of 2 can be replaced by $1 + \epsilon$ for any $\epsilon > 0$.

Proposition 5.3. The integral of a function of moderate decrease satisfies the following properties:

- Linearity
- Translation invariance
$$\int_{-\infty}^{\infty} f(x - h)dx = \int_{-\infty}^{\infty} f(x)dx$$
- Scaling under dilations for $\delta > 0$
$$\delta \int_{-\infty}^{\infty} f(\delta x)dx = \int_{-\infty}^{\infty} f(x)dx$$
- Continuity
$$\int_{-\infty}^{\infty} |f(x - h) - f(x)|dx \to 0 \text{ as } h \to 0$$

Proof. Take $|h| \le 1$. For $\epsilon > 0$ take N so large that

$$\int_{|x|\ge N} |f(x)|dx \le \frac{\epsilon}{4} \text{ and } \int_{|x|\ge N} |f(x - h)|dx \le \frac{\epsilon}{4}$$

Since f is uniformly continuous in the interval $[-N - 1, N + 1]$ $\sup_{|x|\le N} |f(x - h) - f(x)| \to 0$ as h tends to 0. So, take $h > 0$ to ensure that this supremum is smaller than $\frac{\epsilon}{4N}$. Altogether,

$$\begin{aligned}\int_{\infty}^{\infty} |f(x - h) - f(x)|dx &\le \int_{-N}^{N} |f(x - h) - f(x)|dx + \int_{|x|\ge N} |f(x - h)|dx + \int_{|x|\ge N} |f(x)|dx \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4}\end{aligned}$$

□

5.2 The Schwartz Space

The **Schwartz Space** on $\mathbb{R}$ consists of the set of all indefinitely differentiable functions f so that f and all its derivatives $f', f'', ..., f^{(l)}, ..$ are rapidly decreasing. That is,

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty \text{ for every } k, l \geq 0$$

This space is denoted by $\mathcal{S} = \mathcal{S}(\mathbb{R})$. $\mathcal{S}$ is a vector space. Moreover, $f \in \mathcal{S} \Rightarrow \frac{df}{dx} \in \mathcal{S}(\mathbb{R})$ and $xf(x) \in \mathcal{S}(\mathbb{R})$. The **Gaussian** defined by e^{-x^2} is in $\mathcal{S}$

5.3 The Fourier Transform on the Schwartz Space

With this rapid decay we can now formally define the Fourier transform of a function.

Definition 5.4. The **Fourier transform** of a function $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\zeta}dx$$

The notation $f(x) \to \hat{f}(\zeta)$ will be used to denote the transform.

The transform of a function has various properties. In particular,

Proposition 5.5. If $f \in \mathcal{S}(\mathbb{R})$ then

1. $f(x+h) \to \hat{f}(\zeta)e^{2\pi i h\zeta}$ whenever $h \in \mathbb{R}$
2. $f(x)e^{-2\pi i x h} \to \hat{f}(\zeta + h)$ whenever $h \in \mathbb{R}$
3. $f(\delta x) \to \frac{1}{\delta}\hat{f}(\frac{1}{\delta}\zeta)$ whenever $\delta > 0$
4. $f'(x) \to 2\pi i\zeta \hat{f}(\zeta)$
5. $-2\pi i x f(x) \to \frac{d}{d\zeta}\hat{f}(\zeta)$

Proof. Proof of 5.: We must show that $\hat{f}$ is differentiable and we need to find its derivative. Let $\epsilon > 0$ and consider

$$\frac{\hat{f}(\zeta+h) - \hat{f}(\zeta)}{h} - (-2\hat{\pi i x f})(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\zeta}\left[\frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x\right]dx$$

Since $f(x)$ and $xf(x)$ are of rapid decraese, there exists an integer N such that $\int_{|x|\geq N} |f(x)|dx \leq \epsilon$ and $\int_{|x|\geq N} |x||f(x)|dx \leq \epsilon$. Moreover, for $|x| \leq N$, there exists h_0 so that $|h| < h_0$ implies that

$$\left|\frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x\right| \leq \frac{\epsilon}{N}$$

Hence for $|h| < h_0$ we have

$$\begin{aligned}\left|\frac{\hat{f}(\zeta+h) - \hat{f}(\zeta)}{h} - (-2\hat{\pi i x f})(\zeta)\right| &\leq \int_{-N}^{N} \left|f(x)e^{-2\pi i x\zeta}\left[\frac{e^{-2\pi i x h} - 1}{h} + 2\pi i x\right]\right| \\ &\leq C'\epsilon\end{aligned}$$

□

Fourier transform preserves the nice properties of functions. In particular,

Theorem 5.6. If $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$

Further, some functions are unaffected by the transform. Consider the Gaussian kernel:

Theorem 5.7. If $f(x) = e^{-\pi x^2}$ then $\hat{f}(\zeta) = f(\zeta)$

This results in the following corollary:

Corollary 5.8. If $\delta > 0$ and $K_\delta(x) = \delta^{-\frac{1}{2}}e^{\frac{-\pi x^2}{\delta}}$ Then $\hat{K}_\delta(\zeta) = e^{-\pi\delta\zeta^2}$

As the variance of f converges to 0, the transform of f flattens out and so in either case, only one of f or $\hat{f}$ can peak. The functions $K_\delta(x) = \delta^{-\frac{1}{2}}e^{-\pi x^2/\delta}$ are good kernels. That is,

1. $\int_{-\infty}^{\infty} K_\delta(x)dx = 1$
2. $\int_{-\infty}^{\infty} |K_\delta(x)|dx \leq M$
3. For every $\eta > 0$, we have $\int_{|x|>\eta} |K_\delta(x)|dx \to 0$ as $\delta \to 0$

A direct corollary of the above results is that

Corollary 5.9. If $f \in \mathcal{S}(\mathbb{R})$ then

$$(f * K_\delta)(x) \to f(x)$$

uniformly in x as $\delta \to 0$

5.3.1 The Fourier Inversion

The next identity is often called the multiplication formula.

Proposition 5.10. If $f, g \in \mathcal{S}(\mathbb{R})$ then

$$\int_{-\infty}^{\infty} f(g)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)dy$$

To prove this, we first need to define the way in which to recover a function from its representation in Fourier space. In particular,

Theorem 5.11. Fourier Inversion If $f \in \mathcal{S}(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{2\pi i x\zeta}d\zeta$$

We now have a way to connect functions to their representations in Fourier space. In particular:

Corollary 5.12. The Fourier transform is a bijective mapping on the Schwartz space.

5.3.2 The Plancherel Formula

To prove the above require proposition, we need a few more results about convolutions. In particular,

Proposition 5.13. If $f, g \in \mathcal{S}(\mathbb{R})$ then

1. $f * g \in \mathcal{S}(\mathbb{R})$
2. $f * g = g * f$
3. $(f \hat{*} g)(\zeta) = \hat{f}(\zeta)\hat{g}(\zeta)$

We have restricted ourselves to "nice" functions. Equipping the Schwartz space with the Hermitian inner product

$$(f, g) = \int_{-\infty}^{\infty} f(g)\bar{g}(x)dx$$

whose associated norm is

$$||f|| = \sqrt{(f, f)}$$

We know that Fourier transforms remain in Schwartz spaces. The size of functions also doesn't change. That is,

Theorem 5.14. Plancherel If $f \in \mathcal{S}(\mathbb{R})$ then $||\hat{f}|| = ||f||$

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