Real Mathematical Analysis

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1 Real Numbers

1.1 Cuts

Although we have used them since primary school, how are real numbers even defined? We were given a set of axioms about real numbers to solve problems involving them, but what if no such set satisfies these axioms? We would have been solving problems on the empty set! The following is a basic construction of the real numbers using Dedekind cuts.

Definition 1.1. A **cut** in $\mathbb{Q}$ is a pair of subsets A, B of $\mathbb{Q}$ such that

- $A \cup B = \mathbb{Q}$, $A, B \neq \emptyset$, and $A \cap B = \emptyset$
- If $a \in A$ and $b \in B$ Then $a < b$
- A contains no largest element

We denote the cut as $x = A|B$.

Definition 1.2. If $x = A|B$ and $y = C|D$ such that $A \subset C$ then x is **less than or equal** to y and we write $x \leq y$. If in addition, $A \neq C$ then x is **less than** y and we write $x < y$.

The most important property that distinguishes $\mathbb{Q}$ from $\mathbb{R}$ involves bounds.

Definition 1.3. $M \in \mathbb{R}$ is an **upper bound** for a set $S \subset \mathbb{R}$ if each $s \in S$ satisfies

$$s \leq M$$

We could also say that S is **bounded above** by M.

Definition 1.4. An upper bound for S that is less than all other upper bounds for S is a **least upper bound** for S. Denoted by $l.u.b.(S)$

Note that a least upper bound for S may or may not belong to S. This is why you should say "for S" rather than "of S"

Theorem 1.5. The set $\mathbb{R}$ is complete in that is satisfies the
Least Upper Bound Property: If S is a nonempty subset of $\mathbb{R}$ and is bounded above then there exists a least upper bound $S \in \mathbb{R}$

Proof. Let $\mathcal{C} \subseteq \mathbb{R}$ be bounded above. Define

$$C = \{a \in \mathbb{Q} : \text{ for some } A|B \in \mathcal{C} \text{ we have } a \in A\} \text{ and } D = \mathbb{Q} \setminus C$$

with $z = C|D$ which is clearly an upper bound for $\mathcal{C}$. Let $z' = C'|D'$ be any upper bound for $\mathcal{C}$. By the assumption that $A|B \leq C'|D'$ for all $A|B \in \mathcal{C}$, we see that the A for every member of $\mathcal{C}$ is contained in C'. Hence $C \subset C'$, so $z \leq z'$. That is, among all upper bounds for $\mathcal{C}$, z is least. $\square$

Arithmetic on cuts is defined to match out intuition from arithmetic on $\mathbb{R}$. The ordering defined by cuts allows for the following to be true:

Theorem 1.6. The set $\mathbb{R}$ of all cuts in $\mathbb{Q}$ is a complete ordered field that contains $\mathbb{Q}$ as an ordered subfield.

Proposition 1.7. Triangle Inequality: For all $x, y \in \mathbb{R}$ we have $|x + y| \leq |x| + |y|$

This definition of cuts in $\mathbb{Q}$ continues to be robust to further cutting. Defining further cuts on $\mathbb{R}$ does not result in a more "refined" set since the cuts would be at a real number by the $l.u.b.$ property.

1.2 Cauchy Sequences

$\mathbb{R}$ is also **complete** in the sense that every Cauchy sequence also converges.

Theorem 1.8. A Cauchy sequence $(a_n) \subseteq \mathbb{R}$ converges

Theorem 1.9. Every interval (a, b), no matter how small, contains both infinitely many rational and irrational numbers

Proof. (Sketch): Since $a < b$ are cuts, there exists $r < s \in \mathbb{Q}$ s.t. $r, s \in (a, b)$. Now, consider the transformation $T : [0, 1] \to (a, b)$ defined by $T(t) = r + (r - s)t$. Since $[0, 1]$ has infinitely many rationals and irrationals, (a, b) does as well. □

1.3 Euclidean Space

Given two sets A, B the **Cartesian Product** of A and B is the product $A \times B = \{(a, b) : a \in A \text{and} b \in B\}$

$\mathbb{R}^m$ is the m-th cartesian product of $\mathbb{R}$ on itself. Given $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, the dot product is defined to be,

$$\langle x, y \rangle = x_1 y_1 + ... + x_m y_m$$

Which is bilinear, symmetric, and positive definite. That is,

2 Topology

2.1 Metric Spaces

Metric spaces are a specific example of a topological space. These notes will focus solely on them, when discussing topology.

Definition 2.1. A **Metric Space** is a set M, the elements of which are referred to as points of M, together with a metric d s.t. $\forall x, y, z \in M$

- **Positive Definiteness:** $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$
- **Symmetry:** $d(x, y) = d(y, x)$
- **Triangle Inequality:** $d(x, z) \leq d(x, y) + d(y, z)$

Remember that for a sequence of points in a metric space, $(p_n)_{n \in \mathbb{N}}$ is different than $\{p_n : n \in \mathbb{N}\}$. The former having an ordering for the points and the latter, a set where they are all jumbled. Formally, a sequence in M is a function $f : \mathbb{N} \to M$. The nth term in the sequence is $f(n) = p_n$.

The sequence (p_n) **converges to the limit** p in M if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; n \geq N \Rightarrow d(p_n, p) < \epsilon$$

2.2 Continuity

Linear Algebra focuses on "Linear" objects. The main focus of the topic being that of linear transformations. In Analysis, the main focus is on "smooth" objects, in particular that of continuous functions.

Definition 2.2. A function $f : M \to N$ is **continuous** if it **preserves sequential convergence:** (p_n) converges in M to $p \Rightarrow (f(p_n))$ converges in N to $f(p)$

In Linear Algebra, spaces that look and behave the same are called isomorphic, that is, a linear bijection exists between them. In metric spaces, a bijection $f : M \to N$ is called a **Homeomorphism** if f and f^{-1} are continuous. M and N are then called **Homeomorphic** which is an equivalence relation and is denoted as $M \cong N$.

Note that the above definition of

Theorem 2.3. $f : M \to N$ is continuous if and only if it satisfies the **(ϵ, δ)-condition**:

$$\forall \epsilon > 0 \text{ and } p \in M \; \exists \delta > 0 \text{ such that } x \in M \text{ and } d_M(x, p) < \delta \Rightarrow d_N(f(x), f(p)) < \epsilon$$

Proof. Forwards via contradiction. Backwards direct. □

2.3 The Topology of a Metric Space

We now move onto two fundamental concepts in Topology. Let M be a metric space and let $S \subseteq M$. A point $p \in M$ is a **limit** of S if there exists a sequence (p_n) in S that converges to it.

Definition 2.4. S is a **closed set** if it contains all its limits

Definition 2.5. S is an **open set** if for each $p \in S$ there exists an $r > 0$ such that

$$d(p, q) < r \Rightarrow q \in S$$

Theorem 2.6. The complement of an open set is a closed set and the complement of a closed set is an open set

The **Topology** of M is the collection $\mathcal{T}$ of all open subsets of M.

Theorem 2.7. $\mathcal{T}$ is closed under arbitrary union, finite intersection, and it contains $\emptyset$, M.

Two sets, uniquely defined for a set S and point p are

$$limS = \{p \in M : p \text{ is a limit of } S\}$$

$$M_r p = \{q \in M : d(p, q) < r\}$$

The former being the **limit set** of S and the latter being the **r-neighborhood** of p

Theorem 2.8. $limS$ is a closed set and $M_r p$ is an open set

Proof. Suppose that $p_n \to p$ and each $p_n \in limS$. Then, $\exists (p_{n,k})_{k\in\mathbb{N}}$ in S that converges to p_n as $k \to \infty$. Then, consider the sequence $q_n = p_{n,k_n}$ such that

$$d(p_n, q_n) < \frac{1}{n}$$

Then as $n \to \infty$ we have

$$d(p, q_n) \leq d(p, p_n) + d(p_n, q_n) \to 0$$

□

Corollary 2.9. The interval (a, b) is open in $\mathbb{R}$ and so are $(-\infty, b)$, (a, ∞), and $(-\infty, \infty)$.

Proof. (a, b) is the r-neighborhood of its midpoint $m = \frac{a+b}{2}$ where $r = \frac{b-a}{2}$. Therefore (a, b) is open in $\mathbb{R}$ □

Note that $limS$ is the "smallest" closed set that contains S in the sense that it is a subset of all closed sets that contain S. Also, since limits of limits are limits, $\lim(\lim S) = \lim S$ (lim operator is idempotent). $limS$ will be referred to as the **closure** of S and will be denoted as $\bar{S}$.

Note that in Topology, a property of a metric space or of a mapping connecting them that can be solely described with respect to open sets is called a **topological property**. Let $f : M \to N$ be given. The **preimage** of a set $V \subset N$ is defined to be

$$f^{-1}(V) = \{p \in M : f(p) \in V\}$$

Theorem 2.10. The following are equivalent for continuity of $f : M \to N$

1. The (ϵ, δ)-condition
2. The sequential convergence preservation condition
3. The **closed set condition**: The preimage of each closed set in N is closed in M
4. The **open set condition**: The preimage of each open set in N is open in M

Note that the set $\{x \in \mathbb{Q} : -\pi < x < \pi\}$ is open and closed in $\mathbb{Q}$, but is neither open nor closed in $\mathbb{R}$.

Proposition 2.11. Every metric subspace N of M inherits its topology from M in the sense that each subset $V \subset N$ which is open in N is actually the intersection $V = N \cap U$ for some $U \subset M$ that is open in M, and vice versa

The same can be said for the closed sets of N. With this, it is easy to see why $S = \{x \in \mathbb{Q} : -\pi < x < \pi\}$ is both open and closed. $S = \mathbb{Q} \cap K$ and $S = \mathbb{Q} \cap U$ where $U = (-\pi, \pi)$ and $K = [-\pi, \pi]$.

Corollary 2.12. Assume N is a metric subspace of M and is also a closed subset of M. A set $L \subset N$ is closed in N if and only if it is closed in M. Similarly if N is a metric subspace of M and is also an open subset of M then $U \subset N$ is open in N if and only if it is open in M

2.4 Product Metrics

We now define a metric on the cartesian product of two metric spaces $M = X \times Y$. There are three natural choices with $p = (x, y)$ and $p' = (x', y')$

$$d_E(p, p') = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}$$
$$d_{max}(p, p') = \max d_X(x, x'), d_Y(y, y')$$
$$d_{sum}(p, p') = d_X(x, x') + d_Y(y, y')$$

Proposition 2.13. $d_{max} \leq d_E \leq d_{sum} \leq 2d_{max}$

Proof. Trivial □

This result about the above metrics imply that in a topological sense they are equivalent. That is,

Corollary 2.14. Let $p_n = (p_{1n}, p_{2n})$ be a sequence in $M = M_1 \times M_2$:

a) (p_n) converges with respect to d_{max}
b) (p_n) converges with respect to d_E
c) (p_n) converges with respect to d_{sum}
d) (p_{1n}) and p_{2n} converge in M_1 and M_2 respectively

Corollary 2.15. If $f : M \to N$ and $g : X \to Y$ are continuous then so is their cartesian product $f \times g$ which sends $(p, x) \in M \times X$ to $(f(p), g(x)) \in N \times Y$.

Proof. Follows immediately from 2.14 d) □

Note that proposition 2.13 can be generalized to an m-cartesian product of space implying that $d_{max} \leq d_E \leq d_{sum} \leq md_{max}$. In particular this implies that a sequence in $\mathbb{R}^m$ converges if and only if each of its components does.

Further, the metric $d : M \times M \to \mathbb{R}$ is continuous.

Theorem 2.16. d is continuous

Proof. Checking (ϵ, δ)-continuity with respect to the metric d_{sum} on $M \times M$. Given an $\epsilon > 0$ take $\delta = \epsilon$. If $d_{sum}((p, q), (p', q')) < \delta$ the triangle inequality gives that

$$d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < d(p', q') + \epsilon$$
$$d(p', q') \leq d(p', p) + d(p, q) + d(q, q') < d(p, q) + \epsilon$$

which gives that

$$d(p, q) - \epsilon < d(p', q') < d(p, q) + \epsilon$$

Thus $|d(p', q') - d(p, q)| < \epsilon$ and we get continuity with respect to the metric d_{sum}. By theorem 17 it does not matter which metric we use on $M \times M$. □

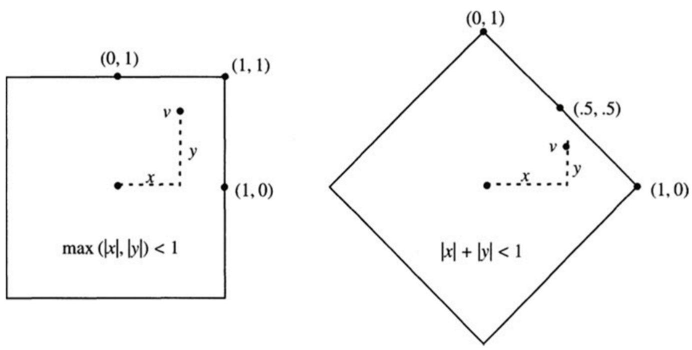

Figure 1: The unit disk in the max metric and sum (taxi-cab metric)

2.5 Completeness

There is a natural extension to cauchy sequences in $\mathbb{R}$ to metric spaces. The sequence (p_n) in M satisfies the **Cauchy Criterion** provided that for each $\epsilon > 0$ there is an integer N such that for all $k, n \geq N$ we have $d(p_k, p_n) < \epsilon$, and (p_n) is said to be a Cauchy sequence. It follows immediately by use of the triangle inequality that convergence $\to$ Cauchy.

Note that in $\mathbb{R}$, cauchiness implies convergence, however, in a general metric space, this need not be true. Consider $\mathbb{Q}$ with the inherited metric from $\mathbb{R}$ and the sequence $(p_n) = (3, 3.1, 3.14, \dots)$. This sequence is cauchy, but does not converge to a point in $\mathbb{Q}$.

Definition 2.17. A metric space M is **complete** if each Cauchy sequence in M converges to a limit in M

Theorem 2.18. $\mathbb{R}^n$ is complete

Proof. This follows immediately as a result of Corrollary 2.14 and the fact that $\mathbb{R}$ is complete □

Note that completeness is not a topological property as $(-1, 1)$ is homeomorphic to $\mathbb{R}$, but $(-1, 1)$ is not complete.

2.6 Compactness

Compactness is one of the most important concepts in real analysis, it reduces the infinite to the finite.

Definition 2.19. A subset A of a metric space M is (sequentially) **compact** if every sequence (a_n) in A has a subsequence (a_{n_k}) that converges to a limit in A

Note that it immediately follows that all finite sets and the empty set are compact.

Theorem 2.20. Every compact set is closed and bounded

Proof. Every convergent subsequence most converge to the same point as the convergent mother sequence and convergent sequences are bounded (a contradiction would occur if a compact set was unbounded as a unbounded sequence would exist). □

Theorem 2.21. The closed interval $[a, b] \subset \mathbb{R}$ is compact

Proof. Let (x_n) be a sequence in $[a, b]$ and set

$$C = \{x \in [a, b] : x_n < x \text{ only finitely often}\}$$

$a \in C$ and b is an upper bound so the L.U.B propery implies $\exists c \in [a, b]$ supremum for C. We claim a subsequence (a_{n_k}) converges to c. Suppose not, that no subsequence does. Then, for some $r > 0$, x_n lies in $(c-r, c+r)$ only finitely many times. But then since there are finitely many points smaller than c, there are finitely many points smaller than $c + r$ implying that c is not the least upper bound of C, thus some subsequence of (x_n) converges to c and therefore $[a, b]$ is compact. □

Theorem 2.22. Every closed subset of a compact set is compact

Theorem 2.23. Heine-Borel Theorem: Every closed and bounded subset of $\mathbb{R}^n$ is compact

Proof. Let $A \subset \mathbb{R}^m$ be closed and bounded. Boundedness implies that A is contained in some box, which is compact. Since A is closed, A is compact by the above. □

Note that this is not true for general metric spaces M. Consider $\mathbb{N}$ equipped with the discrete metric. Then $\mathbb{N}$ is complete, closed, and bounded, but the sequence $(1, 2, 3, 4, \dots)$ does not have any convergent subsequence. Further, consider the metric space $C([0, 1], \mathbb{R})$ whose metric is $d(f, g) = \max\{|f(x) - g(x)|\}$. The space is complete but its closed unit ball is not compact. For example, consider the sequence of functions $f_n = x^n$ has no subsequence that converges with respect to the metric d. In fact, every closed ball is noncompact...

Theorem 2.24. The intersection of a nested sequence of compact nonempty sets is compact and nonempty

The **diameter** of a nonempty set $S \subset M$ is the supremum of the distance $d(x, y)$ between points of S

Theorem 2.25. If in addition to being nested, nonempty, and compact, the sets A_n have diameter that tends to 0 as $n \to \infty$ then $A = \cap A_n$ is a singleton

2.6.1 Continuity and Compactness

Theorem 2.26. If $f : M \to N$ is continuous and A is a compact subset of M then $f(A)$ is a compact subset of N. That is, the continuous image of compact is compact.

Proof. Let $(b_n) \subseteq N$. For each b_n pick a single $a_n \in M$ such that $f(a_n) = b_n$. Using the compactness of A and continuity of f, the result follows. □

Corollary 2.27. A continuous real-valued function deifned on a compact set is bounded; it assumes maximum and minimum values.

Note that the above theorems imply that compactness is a topological property. Thus compactness can be used to prove spaces are not homeomorphic to each other.

Theorem 2.28. If M is compact then a continuous bijection $f : M \to N$ is a homeomorphism - its inverse bijection $f^{-1} : N \to M$ is automatically continuous

Proof. Suppose that $q_n \to q$ in N. $p_n = f^{-1}(q_n)$ and $p = f^{-1}(q)$ are well defined. Assume (p_n) does not converge to p. Then there is a subsequence (p_{n_k}) such that for all k we have $d(p_{n_k}, p) > \delta > 0$. Compactness implies that there exists a sub-subsequence (p_{k_l}) that converges to a point $p^* \in M$ as $l \to \infty$. Thus $p \neq p^*$. Since f is continuous we have

$$f(p_{n_{k_l}}) \to f(p^*)$$

. Then, $f(p^*) = f(p)$ and so by the bijectivity of f, $p_n \to p$ and therefore f^{-1} is continuous. □

Note that if M is not compact, this fact does not hold. Consider the continuous bijection from $[0, 2\pi)$ to S^1 mentioned previously.

One says that $h : M \to N$ **embeds** M into N if h is a homeomorphism from M onto $h(M)$. Topologically, M and $h(M)$ are then equivalent, a property of M that holds for every embedded copy of M is an **absolute** or **intrinsic** property of M.

Proposition 2.29. *A compact is absolutely closed and absolutely bounded.*

2.6.2 Uniform Continuity and Compactness

We can extend the definition of uniform continuity to metric spaces with an analogous definition.

Definition 2.30. A function $f : M \to N$ is **uniformly continuous** if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$p, q \in M \text{ and } d_M(p, q) < \delta \Rightarrow d_N(f(p), f(q)) < \epsilon$$

The following is one of the most important results in analysis:

Theorem 2.31. Every continuous function defined on a compact set is uniformly continuous.

Proof. Suppose not that $\exists f : M \to N$ continuous M compact which is not uniformly continuous. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$ $\exists p, q \in M$ with $d(p, q) < \delta$ but $d(f(p), f(q)) \geq \epsilon$. By taking $\delta = \frac{1}{n}$ and letting (p_n) and (q_n) being the sequence of points for such a delta. Compactness implies that there exists a subsequence p_{n_k} which converges to some $p \in M$. By their definition, (q_n) also converges to p. Continuity thus implies that $f(p_{n_k}) \to f(p)$ and $f(q_{n_k}) \to f(p)$. But then, if k is large enough,

$$d(f(p_{n_k}), f(q_{n_k})) \leq d(f(p_{n_k}), f(p)) + d(f(p), f(q_{n_k})) < \epsilon$$

contrary to how p_n and q_n were defined. Thus, f is uniformly continuous on M. $\square$

2.7 Compactness

3 Lebesgue Theory

Measuring simple sets is easy, for instance, the length of (a, b) is $b - a$. How do we measure more complicated sets?

3.1 Outer Measure

Definition 3.1. The **length** of the interval I is $|I| = b - a$. The **Lebesgue outer measure** of $A \subset \mathbb{R}$ is

$$m^*A = \inf\{\Sigma_k |I_k| : \{I_k\} \text{ is a covering of A by open intervals}\}$$

If for every covering of A, $\{I_k\}$, $\Sigma_k |I_k|$ diverges then $m^*A := \infty$. There are three immediate consequences of the above definition.

Theorem 3.2.
1. $m^*\emptyset$=0
2. If $A \subset B$ then $m^*A \leq m^*B$
3. If $A = \cup_{n=1}^{\infty} A_n$ then $m^*A \leq \sum_{n=1}^{\infty} m^*A_n$

Proof. For $\epsilon > 0$ there exists for each n a covering $\{I_{k,n} : k \in \mathbb{N}\}$ such that

$$\sum_{n=1}^{\infty} |I_{k,n}| < m^*A_n + \frac{\epsilon}{2^n}$$

The collection $\{I_{k,n} : k, n \in \mathbb{N}\}$ covers A and

$$\sum_{k,n} |I_{k,n}| = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |I_{k,n}| \leq \sum_{k=1}^{\infty}(m^*A_n + \frac{\epsilon}{2^n}) = \sum_{k=1}^{\infty} m^*A_n + \epsilon$$

□

To assert 3., it suffices by definition of the infimum (think reverse of the epsilon definition of the supremum), to find a covering of A which can in fact be made smaller than the series presented plus an additional arbitrary epsilon. To do this we can use the definition of infimums on each one of the A_n and essentially squeeze the covering onto A.

Definition 3.3. If $Z \subset \mathbb{R}^n$ has outer measure zero, then it is a **zero set**

Proposition 3.4. Every subset of a zero set is a zero set. The countable union of zero sets is a zero set. Each plane $P_i(a)$ us a zero set in $\mathbb{R}^n$

This follows immediately from the above theorem. It seems that this theorem will be quite important.

Theorem 3.5. The linear outer measure of a closed interval is its length. The n-dimensional outer measure of a closed box is its volume.

Proof. Let [a,b] be a closed interval. Note that $(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$ is a cover for [a,b] and so $m^*([a, b]) \leq b - a + \epsilon$ It follows by the epsilon definition of the infimum that $m^*([a, b]) \leq b - a$.

To get the reverse inequality, we need to show that for every covering of [a,b] its total length is greater than or equal to $b - a$. Firstly, since **[a,b] is compact it suffices to only consider finite open coverings of [a,b]**. We then reason inductively.
The base case follows immediately as the endpoints of the open interval that contains [a,b] must be further apart than a and b themselves. Now, assume that the claim holds for an arbitrary n. Let $I_1, ..., I_{n+1}$ be an open covering of [a,b] with $I_i = (a_i, b_i)$. We claim that $\sum_{i=1}^{n+1} |I_i| > b - a$. One of the intervals contains a, say (a_1, b_1)
If $b_1 \geq b$ then $\sum_{i=1}^{n} |I_i| \geq |I_1| = b_1 - a_1 > b - a$
If $b_1 < b$ then

$$[a, b] = [a, b_1) \cup [b_1, b]$$

and $|I_1| > b_1 - a$. $[b_1, b]$ is a closed interval covered by n many open intervals, $I_2, ..., I_{n+1}$. It follows by the induction hypothesis that $\sum_{i=2}^{n+1} |I_i| > b - b_1$. Thus,

$$\sum_{i=1}^{n+1} |I_i| = |I_1| + \sum_{i=2}^{n+1} |I_i| > (b_1 - a) + (b - b_1) = b - a$$

□

An interesting proof. Try and remember all the tools you have in $\mathbb{R}$, in particular that of compactness and the results that follow from its formal definition.

Proof. Let $R = [a, b] \times [c, d] \subset \mathbb{R}^2$. Using a similar covering as above, it is easy to see that $m^*R \leq (b-a)*(d-c)$. R has a Lebesgue number λ. Break R up into a finite set of rectangles $S_j \subset R$ where each S_j has a diameter less than λ. By their construction, $\sum |S_j| = (b-a)*(d-c)$. It follows that

$$\sum_j |S_j| \leq \sum_i \sum_{S_j \subset R_i} |S_j| \leq \sum |R_i|$$

which implies that $(b-a)*(d-c) \leq \sum |R_i|$. Thus, $(b-a)*(d-c) = m^*R$ □

3.2 Measurability

If A and B are disjoint intervals in $\mathbb{R}$

$$m^*(A \sqcup B) = m^*A + m^*B$$

It seems like this should follow to arbirary disjoint sets. However, this is not the case in general and we need an added condition.

Definition 3.6. A set $E \subset \mathbb{R}$ is **(Lebesgue) measurable** if the division $E|E^c$ of $\mathbb{R}$ is such that for each $X \subset \mathbb{R}$ (test set) we have

$$m^*X = m^*(X \cap E) + m^*(X \cap E^c)$$

Next, we denote $\mathcal{M} = \mathcal{M}(\mathbb{R}^n)$ as the collection of all Lebesgue measurable subsets of $\mathbb{R}^n$. If E is measurable, its **Lebesgue Measure** $mA = m^*A$.

We now move out of $\mathbb{R}^n$ and into more abstract sets.

Definition 3.7. Let M be a set. An **Outer Measure** on M is a function $\omega : 2^M \to [0, \infty]$ such that

1. $\omega(\emptyset) = 0$
2. $A \subseteq B \Rightarrow \omega(A) \leq \omega(B)$

References

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