Real Mathematical Analysis

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1 Chapter 6: Lebesgue Theory

Measuring simple sets is easy, for instance, the length of (a, b) is $b-a$. How do we measure more complicated sets?

1.1 Outer Measure

Definition 1.1. *The **length** of the interval I is $|I| = b - a$. The **Lebesgue outer measure** of $A \subset \mathbb{R}$ is*

$$m^*A = \inf\{\Sigma_k |I_k| : \{I_k\} \text{ is a covering of } A \text{ by open intervals}\}$$

If for every covering of A, $\{I_k\}$, $\Sigma_k |I_k|$ diverges then $m^*A := \infty$. There are three immediate consequences of the above definition.

Theorem 1.1. *1. $m^*\emptyset = 0$*

2. If $A \subset B$ then $m^*A \leq m^*B$

3. If $A = \cup_{n=1}^{\infty} A_n$ then $m^*A \leq \sum_{n=1}^{\infty} m^*A_n$

Proof. For $\epsilon > 0$ there exists for each n a covering $\{I_{k,n} : k \in \mathbb{N}\}$ such that

$$\sum_{n=1}^{\infty} |I_{k,n}| < m^*A_n + \frac{\epsilon}{2^n}$$

The collection $\{I_{k,n} : k, n \in \mathbb{N}\}$ covers A and

$$\sum_{k,n} |I_{k,n}| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |I_{k,n}| \leq \sum_{k=1}^{\infty} (m^*A_n + \frac{\epsilon}{2^n}) = \sum_{k=1}^{\infty} m^*A_n + \epsilon$$

□

To assert 3., it suffices by definition of the infimum (think reverse of the epsilon definition of the supremum), to find a covering of A which can in fact be made smaller than the series presented plus an additional arbitrary epsilon. To do this we can use the definition of infimums on each one of the A_n and essentially squeeze the covering onto A.

Definition 1.2. *If $Z \subset \mathbb{R}^n$ has outer measure zero, then it is a* **zero set**

Proposition 1.1. *Every subset of a zero set is a zero set. The countable union of zero sets is a zero set. Each plane $P_i(a)$ us a zero set in $\mathbb{R}^n$*

This follows immediately from the above theorem. It seems that this theorem will be quite important.

Theorem 1.2. *The linear outer measure of a closed interval is its length. The n-dimensional outer measure of a closed box is its volume.*

Proof. Let [a,b] be a closed interval. Note that $(a-\frac{\epsilon}{2}, b+\frac{\epsilon}{2})$ is a cover for [a,b] and so $m^*([a,b]) \leq b-a+\epsilon$ It follows by the epsilon definition of the infimum that $m^*([a,b]) \leq b-a$.

To get the reverse inequality, we need to show that for every covering of [a,b] its total length is greater than or equal to $b-a$. Firstly, since **[a,b] is compact it suffices to only consider finite open coverings of [a,b]**. We then reason inductively.
The base case follows immediately as the endpoints of the open interval that contains [a,b] must be further apart than a and b themselves. Now, assume that the claim holds for an arbitrary n. Let $I_1, ..., I_{n+1}$ be an open covering of [a,b] with $I_i = (a_i, b_i)$. We claim that $\sum_{i=1}^{n+1} |I_i| > b-a$.
One of the intervals contains a, say (a_1, b_1)
If $b_1 \geq b$ then $\sum_{i=1}^{n} |I_i| \geq |I_1| = b_1 - a_1 > b-a$
If $b_1 < b$ then

$$[a,b] = [a,b_1) \cup [b_1,b]$$

and $|I_1| > b_1 - a$. $[b_1, b]$ is a closed interval covered by n many open intervals, $I_2, ..., I_{n+1}$. It follows by the induction hypothesis that $\sum_{i=2}^{n+1} |I_i| > b - b_1$. Thus,

$$\sum_{i=1}^{n+1} |I_i| = |I_1| + \sum_{i=2}^{n+1} |I_i| > (b_1 - a) + (b - b_1) = b - a$$

□

An interesting proof. Try and remember all the tools you have in $\mathbb{R}$, in particular that of compactness and the results that follow from its formal definition.

Proof. Let $R = [a,b]x[c,d] \subset \mathbb{R}^2$. Using a similar covering as above, it is easy to see that $m^*R \leq (b-a)*(d-c)$. R has a Lebesgue number λ. Break R up into a finite set of rectangles $S_j \subset R$ where each S_j has a diameter less than λ. By their construction, $\sum |S_j| = (b-a)*(d-c)$. It follows that

$$\sum_j |S_j| \leq \sum_i \sum_{S_j \subset R_i} |S_j| \leq \sum |R_i|$$

which implies that $(b-a)*(d-c) \leq \sum |R_i|$. Thus, $(b-a)*(d-c) = m^*R$ □

1.2 Measurability

If A and B are disjoint intervals in $\mathbb{R}$

$$m^*(A \sqcup B) = m^*A + m^*B$$

It seems like this should follow to arbirary disjoint sets. However, this is not the case in general and we need an added condition.

Definition 1.3. *A set $E \subset \mathbb{R}$ is* ***(Lebesgue) measurable*** *if the division $E|E^c$ of $\mathbb{R}$ is such that for each $X \subset \mathbb{R}$ (test set) we have*

$$m^*X = m^*(X \cap E) + m^*(X \cap E^c)$$

Next, we denote $\mathcal{M} = \mathcal{M}(\mathbb{R}^n)$ as the collection of all Lebesgue measurable subsets of $\mathbb{R}^n$. If E is measurable, its **Lebesgue Measure** $mA = m^*A$.
We now, move out of $\mathbb{R}^n$ and into abstract sets.

Definition 1.4. *Let M be a set. An* ***Outer Measure*** *on M is a function $\omega : 2^M[0, \infty]$ such that*

1. $\omega(\emptyset) = 0$
2. $A \subseteq B \to \omega(A) \leq \omega(B)$