Definitions in Folland, Real Analysis

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Key Definitions in *Real Analysis* by G. Folland written during the course of MAT457/MAT1000 in Fall 2023.

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Chapter 0: Preliminaries

Fact 1. Cauchy-Schwarz Inequality

$$|u \cdot v| \leq ||u||||v||$$

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right)$$

Definition 1. Given a collection of sets $\{E_n\}$

$$\limsup_{j\to\infty} E_j := \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} E_n = \{x : x \in E_n \text{ for infinitely many } n\}$$

and

$$\liminf_{j\to\infty} E_j := \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} E_n = \{x : x \in E_n \text{ for all but finitely many } n\}$$

Chapter 1: Measures

1.2 σ-algebras

Let X be a nonempty set.

Definition 2. An **Algebra** of sets on X is a nonempty collection $\mathcal{A} \subset \mathcal{P}(X)$ such that

- $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$
- If $E_1, ..., E_n \in \mathcal{A}$ Then $\cup_1^n E_j \in \mathcal{A}$

A **σ-Algebra** is an Algebra that is closed under countable unions. $<\mathcal{E}>$ denotes the $\sigma-$Algebra generated by the collection of sets $\mathcal{E}$

Lemma 1. Very Unassuming Let $\mathcal{E}$,$\mathcal{F} \subset \mathcal{P}(X)$

$$\mathcal{E} \subset <\mathcal{F}> \Rightarrow <\mathcal{E}> \subset <\mathcal{F}>$$

Definition 3. An **elementary family** $\mathcal{E} \subset \mathcal{P}(X)$ is a collection such that

- $\emptyset \in \mathcal{E}$
- If $E, F \in \mathcal{E}$ Then $E \cap F \in \mathcal{E}$
- If $E \in \mathcal{E}$ Then E^c is a finite disjoint union of members of $\mathcal{E}$

The collection of finite disjoint unions of members of an elementary family is an algebra

1.3 Measures

Let X be a set equipped with a $\sigma-$algebra $\mathcal{M}$.

Definition 4. A **measure** on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$
2. If $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$ then $\mu(\cup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$

A measure that only satisfies (2) for finite unions is referred to as a **finitely additive measure**.

Definition 5. On a measure space $(X, \mathcal{M}, \mu)$ μ is called

- **finite** if $\mu(X) < \infty$
- **σ-finite** if $X = \cup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$
- **semifinite** if for each $E \in \mathcal{M}$ with $\mu(E) < \infty$ $\exists F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$
- **complete** if its domain includes all subsets of $\mu-$null sets

1.4 Outer Measures

Definition 6. An **outer measure** on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

- $\mu^*(\emptyset) = 0$
- $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$
- $\mu^*(\cup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$

A set $A \subset X$ is μ^*-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subset X$$

Theorem 1. Caratheodory's Theorem: If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a $\sigma-$algebra and the restriction of μ^* to $\mathcal{M}$ is a complete measure

Definition 7. Given $\mathcal{A} \subset \mathcal{P}(X)$ an algebra, a function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called a **premeasure** if

- $\mu_0(\emptyset) = 0$
- If $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\cup_1^\infty \in \mathcal{A}$ then $\mu_0(\cup_1^\infty) = \sum_1^\infty \mu_0(A_j)$

Theorem 2. Hahn-Kolmogorov: Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the $\sigma-$algebra generated by $\mathcal{A}$. There exists a measure on μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 ($\mu = \mu^*|\mathcal{M}$). Further if ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu \leq \mu$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < \infty$. If μ_0 is $\sigma-$finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$

1.5 Borel Measures on the Real Line

Let $(\mathbb{R}, \mathcal{M}_\mu, \mu)$ denote a Lebesgue-Stieltjes measure space.

Lemma 2. For any $E \in \mathcal{M}_\mu$

$$\mu(E) = \inf\{\sum_1^\infty \mu((a_j, b_j) : E \subset \cup_1^\infty (a_j, b_j)\}$$

Theorem 3. If $E \in \mathcal{M}_\mu$, then

$$\begin{aligned} \mu(E) &= \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\} \\ &= \sup\{\mu(K) : E \supset K \text{ and } K \text{ is compact}\} \end{aligned}$$

Theorem 4. If $E \subset \mathbb{R}$ the following are equivalent:

- $E \in \mathcal{M}_\mu$
- $E = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$
- $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$

Proposition 1. If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$ then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \epsilon$

Let $(\mathbb{R}, \mathcal{L}, m)$ denote the Lebesgue measure space

Theorem 5. If $E \in \mathcal{L}$ then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ for all $s, r \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$ and $m(E) = |r|m(E)$

Chapter 2: Integration

Let $(X, \mathcal{M}, \mu)$ be a measure space.

2.1 Measurable Functions

Definition 8. A function $f : X \to Y$ where $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces is called **measurable** if

$$f^{-1}(E) \in \mathcal{M} \text{ for all } E \in \mathcal{N}$$

Proposition 2. If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measurable then so are $f + g$ and fg and if $\mathbb{C}$ is replaced by $\bar{\mathbb{R}}$ then $\max\{f, g\}$ and $\min\{f, g\}$ are measurable

This proposition implies that the positive and negative parts of f are measurable iff f is.

Proposition 3. If $\{f_j\}$ is a sequence of $\bar{\mathbb{R}}$-valued measurable functions on $(X, \mathcal{M})$ then the functions

- $g_1(x) = \sup_j f_j(x)$
- $g_2(x) = \inf_j f_j(x)$
- $g_3(x) = \limsup_{j\to\infty} f_j(x)$
- $g_4(x) = \liminf_{j\to\infty} f_j(x)$
- $g_5(x) = \lim_{j\to\infty} f_j(x)$ is measurable if the limit exists for every $x \in X$

Are all measurable

Definition 9. A **simple function** on a space X is a finite linear combination with complex coefficients of characteristic functions of sets in X. Equivalently a simple function is a measurable function whose range is a finite subset of $\mathbb{C}$ i.e.

$$f = \sum_{1}^{n} z_j \mathcal{X}_{E_j} \text{ where } E_j = f^{-1}(\{z_j\}) \text{ and } range(f) = \{z_1, ..., z_n\}$$

Theorem 6. If $f : X \to \mathbb{C}$ is measurable then there is a sequence $\{\phi_n\}$ of simple functions such that $0 \leq |\phi_1| \leq |\phi_2| \leq ... \leq |f|$ $\phi_n \to f$ point-wise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

Proposition 4. The following propositions are valid iff the measure μ is complete

- If f is measurable and $f = g$ $\mu-$a.e. then g is measurable
- If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ $\mu-$a.e. then f is measurable

Note the comparison to proposition 2 where convergence is pointwise rather than a.e.

Proposition 5. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \bar{\mathcal{M}}, \bar{\mu})$ be its completion. If f is a $\bar{\mathcal{M}}-$measurable function on X, there is an $\mathcal{M}-$measurable function g such that $f = g$ $\bar{\mu}-$almost everywhere

2.2 Integration of Non-negative Functions

Definition 10. If ϕ is a simple function in L^+ with representation $\phi = \sum_{i=1}^n a_i \chi_{E_i}$ then the **integral** of ϕ is defined as

$$\int \phi d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$$

Extending to all positive measurable functions $f \in L^+$,

$$\int f d\mu = \sup\{\int \phi d\mu : 0 \leq \phi \leq f, \phi simple\}$$

Theorem 7. Monotone Convergence Theorem: If $\{f_n\}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{n\to\infty} f_n = \sup_n f_n$ then

$$\int f = \lim_{n\to\infty} \int f_n$$

Proposition 6. If $f \in L^+$ then $\int f = 0$ if and only if $f = 0$ a.e.

Note that this can be used to show the MCT for a sequence of functions that increase only a.e. to f.

Lemma 3. Fatou's Lemma: If $\{f_n\}$ is any sequence in L^+, then

$$\int (\liminf f_n) \leq \liminf \int f_n$$

Proposition 7. If $f \in L^+$ and $\int f < \infty$ then $\{x : f(x) = \infty\}$ is a null set and $\{x : f(x) > 0\}$ is σ-finite

Fact 2. Markov Inequality: For $f \in Ł^+(\mu)$ and $\lambda \in (0, \infty)$

$$\mu(\{x : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_X f d\mu$$

It follows that if $\int f < \infty$ then f is a.e. bounded

2.3 Integration of Complex Functions

Definition 11. Given a real-valued function f the **integral** of f is defined as

$$\int f = \int f^+ - \int f^-$$

where the main consideration is of **integrable** (L^1) functions where $\int |f| = \int f^+ + f^- < \infty$

Proposition 8. If $f \in L^1$ then $|\int f| \leq \int |f|$

Proposition 9. If $f, g \in L^1$ then

- $\{x : f(x) \neq 0\}$ is $\sigma-$finite
- $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ iff $\int |f - g| = 0$ iff $f = g$ a.e.

Theorem 8. Dominated Convergence Theorem: Let $\{f_n\}$ be a sequence in L^+ such that

- $f_n \to f$ a.e.

- there exists a non-negative $g \in L^1$ such that $|f_n| \leq g$ *a.e.*

Then $f \in L^1$ and

$$\int f = \lim_{n\to\infty} \int f_n$$

Theorem 9. Given $\{f_j\} \subset L^1$ such that $\sum_1^\infty \int |f_j| < \infty$ then $\sum_1^\infty f_j$ converges a.e. to a function in L^1 and $\int \sum_1^\infty f_j = \sum_1^\infty \int f_j$

Theorem 10. Density of Simple (continuous) Functions in L^1 $((\mathbb{R}, \mathcal{M}, m))$ If $f \in L^1(\mu)$ and $\epsilon > 0$ then there is an integrable simple function ϕ such that $\int |f - \phi| d\mu < \epsilon$. Further, in $(\mathbb{R}, \mathcal{L}, m)$ there is a continuous function g that vanishes outside a bounded interval such that $\int |f - g| < \epsilon$

Theorem 11. Let f be a bounded real-valued function on $[a, b]$

- If f is Riemann integrable then f is Lebesgue measurable (and hence integrable on $[a, b]$ since it is bounded), and $\int_a^b f(x)dx = \int_{[a,b]} f dm$
- f is Riemann integrable iff $\{x \in [a, b] : f \text{ discontinuous at } x\}$ has Lebesgue measure 0

2.4 Modes of Convergence

Definition 12. A sequence $\{f_n\}$ of measurable complex-valued functions on $(X, \mathcal{M}, \mu)$ is **Cauchy in measure** if $\forall \epsilon > 0$

$$\mu(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0 \text{ as } m.n \to \infty$$

and **converges in measure** to f if $\forall \epsilon > 0$

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \text{ as } n \to \infty$$

Proposition 10. If $f_n \to f$ in L^1 then $f_n \to f$ in measure

Theorem 12. If $\{f_n\}$ is Cauchy in measure, then there is a measurable function f such that $f_n \to f$ in measure, and there is a subsequence $\{f_{n_j}\}$ that converges to f a.e. Moreover, if also $f_n \to g$ in measure then $g = f$ a.e. (**a.e. uniqueness of measure convergence**)

Proposition 11. If $f_n \to f$ in L^1 then there is a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \to f$

Theorem 13. Egoroff's Theorem Suppose that $\mu(X) < \infty$ and $f_1, f_2, ...$ and f are measurable complex-valued functions on X such that $f_n \to f$ a.e. Then $\forall \epsilon > 0$ there exists $E \subset X$ such that $\mu(X) < \epsilon$ and $f_n \to f$ uniformly on E^c (referred to as **almost uniform convergence**)

Chapter 3: Signed Measures and Differentiation

3.1 Signed Measures

Let $(X, \mathcal{M})$ be a measurable space.

Definition 13. A **signed measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$
- ν assumes at most one of the values $\pm\infty$

- If $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$ then $\nu(\cup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$ where the latter sum converges absolutely if $\nu(\cup_1^\infty E_j)$ is finite

Definition 14. If ν is a signed measure on $(X, \mathcal{M})$ a set $E \in \mathcal{M}$ is called **positive (negative, null)** if $\nu(F) \geq 0$ ($\nu(F) \leq 0, \nu(F) = 0$) for all $F \in \mathcal{M}$ such that $F \subset E$

Lemma 4. Any measurable subset of a positive set is positive and the union of any countable family of positive sets is positive

Theorem 14. The Hahn Decomposition Theorem: If ν is a signed measure on $(X, \mathcal{M})$ there exists a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \emptyset$. If P', N' are another such pair then $P \triangle P'$ and $N \triangle N'$ are null for ν.

Definition 15. We say that two signed measures μ and ν on $(X, \mathcal{M})$ are **mutually singular** or that ν is **mutually singular with respect** to μ or vice verse if $\exists E, F \in \mathcal{M}$ such that $E \cap F = \emptyset$, $E \cup F = X$, E is null for μ, and F is null for ν

Theorem 15. The Jordan Decomposition Theorem: If ν is a signed measure there exists unique positive measures ν^+ and ν^- such that

$$\nu = \nu^+ - \nu^- \text{ and } \nu^+ \perp \nu^-$$

The **total variation** of ν is defined as

$$|\nu| = \nu^+ + \nu^-$$

ν is called **finite** if $|\nu|$ is finite

Note that ν is of the form $\nu(E) = \int_E f d\mu$ where $\mu = |\nu|$ and $f = \mathcal{X}_P - \mathcal{X}_N$ $X = P \cup N$
Integration with respect to a signed measure is defined simply as

$$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$$

and

$$\int f d\mu = \int f d\nu^+ - \int f d\nu^-$$

3.2 The Lebesgue-Radon-Nikodym Theorem

Definition 16. If ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$ we say that ν is **absolutely continuous** with respect to μ and write

$$\nu \ll \mu$$

if $\nu(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$

Theorem 16. Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$ then $\nu \ll \mu$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta \Rightarrow |\nu(E)| < \epsilon$

Definition 17. A function f is called $\mu-$**extended** if at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite. This is denoted by

$$L^1_{ext}(\mu)$$

With such a function, you can define the signed measure ν by $\nu(E) = \int f d\mu$ which is clearly absolutely continuous with respect to μ and is finite iff $f \in L^1(\mu)$

The relationship $\nu(E) = \int_E f d\mu$ is denoted by $d\nu = f d\mu$

Theorem 17. The Lebesgue-Radon-Nikodym Theorem: Let ν be a $\sigma-$finite signed measure and μ a $\sigma-$finite positive measure on $(X, \mathcal{M})$. Then there exist unique $\sigma-$finite signed measures λ, ρ on $(X, \mathcal{M})$ such that

$$\lambda \perp \mu, \rho \ll \mu, \text{ and } \nu = \lambda + \rho$$

Moreover, there is an extended $\mu-$integrable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$ and any two such functions are equal $\mu-$a.e.

The above decomposition is called the **Lebesgue decomposition** of ν with respect to μ. f is called the **Radon-Nikodym-derivative** of ν with respect to μ. It is denoted by $d\nu/d\mu$

Chapter 5: Elements of Functional Analysis

5.1 Normed Vector Spaces

Let K denote either $\mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{X}$ be a vector space over K.

Definition 18. A **seminorm** on $\mathcal{X}$ is a function $x \to \|x\|$ from $\mathcal{X}$ to $[0, \infty)$ such that

- $\|x\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$ (**triangle inequality**)
- $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in \mathcal{X}$ and $\lambda \in K$

If $\|x\|= 0$ if and only if $x = 0$ then a seminorm is a called a **norm**

A vector space $\mathcal{X}$ with a norm is referred to as a **normed vector space**

Definition 19. Norm Topology: If $\mathcal{X}$ then the function $\rho(x, y) = \|x - y\|$ is a metric on $\mathcal{X}$ which induces a topology referred to as the **norm topology**
Banach Space: A normed vector space that is complete with respect to the norm topology

Theorem 18. A normed vector space $\mathcal{X}$ is complete if and only if every absolutely convergent series in $\mathcal{X}$ converges

Definition 20. A linear map $T : \mathcal{X} \to \mathcal{Y}$ is called **bounded** if $\exists C \geq 0$ such that

$$\|Tx\| \leq C\|x\| \text{ for all } x \in \mathcal{X}$$

The space of all bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $L(\mathcal{X}, \mathcal{Y})$ which is a vector space itself. This leads to the following definition of an **operator norm** for L:

$$\|T\| = \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x\}$$

Proposition 12. If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a linear map, the following are equivalent:

- T is continuous
- T is continuous at 0
- T is bounded

Proposition 13. If $\mathcal{Y}$ is complete, so is $L(\mathcal{X}, \mathcal{Y})$

If $\mathcal{X}$ is complete, $L(\mathcal{X}, \mathcal{X})$ is in fact a **Banach algebra** (a Banach space that is also an algebra).

Definition 21. Given $T \in L(\mathcal{X}, \mathcal{Y})$

- T is said to be **invertible** or an **isomorphism** if T is bijective and T^{-1} is bounded
- T is called an **isometry** if $\|Tx\| = \|x\|$

5.2 Linear Functionals

Let $\mathcal{X}$ be a vector space over K, where $K = \mathbb{C}$ or $\mathbb{R}$.

Definition 22. A linear map from $\mathcal{X}$ to K is called a **linear functional** on $\mathcal{X}$

Definition 23. If $\mathcal{X}$ is a normed vector space, the space $L(\mathcal{X}, K)$ of bounded linear functionals on $\mathcal{X}$ is called the **dual space** of $\mathcal{X}$ and is denoted by X^* Note that $\mathcal{X}^*$ is a Banach space with the operator norm.

Definition 24. If $\mathcal{X}$ is real vector space, a **sublinear functional** on $\mathcal{X}$ is a map $p : \mathcal{X} \to \mathbb{R}$ such that

$$p(x + y) \leq p(x) + p(y) \text{ and } p(\lambda x) = \lambda p(x) \text{ for all } x, y \in \mathcal{X} \text{ and } \lambda \geq 0$$

Note that every seminorm is a sublinear functional.

Theorem 19. The Hahn-Banach Theorem Let $\mathcal{X}$ be a real vector space, p a sublinear functional on $\mathcal{X}$, $\mathcal{M}$ a subspace of $\mathcal{X}$, and f a linear functional on $\mathcal{M}$ such that $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional F on $\mathcal{X}$ such that $F(x) \leq p(x)$ for all $x \in \mathcal{M}$ and $F|\mathcal{M} = f$

The main applications of Hahn-Banach are:

Theorem 20. Let $\mathcal{X}$ be a normed vector space

- If $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $x \in \mathcal{X} \setminus \mathcal{M}$ there exists $f \in \mathcal{X}^*$ such that $f(x) \neq 0$ and $f|\mathcal{M} = 0$. In fact, if $\delta = \inf_{y \in \mathcal{M}} |x - y|$, f can be taken to satisfy $|f| = 1$ and $f(x) = \delta$
- If $x \neq 0 \in \mathcal{X}$, there exists $f \in \mathcal{X}^*$ such that $|f| = 1$ and $f(x) = |x|$
- The bounded linear functionals on $\mathcal{X}$ separate points
- If $x \in \mathcal{X}$ define $\hat{x} : \mathcal{X}^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then the map $x \to \hat{x}$ is a linear isometry from $\mathcal{X}$ into $\mathcal{X}^{**}$ (the dual of $\mathcal{X}^*$)

Definition 25. Let $\mathcal{X}$ be a normed vector space. With $\hat{\mathcal{X}} = \{\hat{x} : x \in \mathcal{X}\}$. The closure of $\hat{\mathcal{X}}$ $\bar{\hat{\mathcal{X}}}$ is called the **completion** of $\mathcal{X}$

Definition 26. If $\hat{\mathcal{X}} = \mathcal{X}^{**}$ $\mathcal{X}$ is called **reflexive**. One can identify $\hat{x}$ with x and thus regard $\mathcal{X}^{**}$ as a superspace of $\mathcal{X}$ reflexivity then means that $\mathcal{X}^{**} = \mathcal{X}$

The Baire Category Theorem and Its Consequences

This section presents the BCT and uses it to conclude some fundamental results for linear maps between Banach spaces.

Theorem 21. The Baire Category Theorem: Let $\mathcal{X}$ be a complete metric space.

- If $\{U_n\}_1^\infty$ is a sequence of open dense sets in $\mathcal{X}$ then $\cap_1^\infty U_n$ is dense in $\mathcal{X}$
- $\mathcal{X}$ is not a countable union of nowhere dense sets

Note that this is a purely topological proof and so the conclusion holds for spaces meerly homeomorphic to complete metric spaces.

Definition 27. A set $E \subset \mathcal{X}$ is **meager** if it is a countable union of nowhere dense sets. The complement of a meager set is called **residual**

Theorem 22. The Open Mapping Theorem: Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. If $T \in L(\mathcal{X}, \mathcal{Y})$ is surjective, then T is open. A consequence of this is that if T is also injective then T is an isomorphism

Theorem 23. The Closed Graph Theorem If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a closed linear map then T is bounded

Theorem 24. The Uniform Boundedness Principle Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and that $\mathcal{A}$ is a subset of $L(\mathcal{X}, \mathcal{Y})$ then

- If $\sup_{T\in\mathcal{A}}|Tx| < \infty$ for all x in some nonmeager subset of $\mathcal{X}$ then $\sup_{T\in\mathcal{A}}|T| < \infty$
- If $\mathcal{X}$ is a Banach space and $\sup_{T\in\mathcal{A}}|Tx|$ for all $x \in \mathcal{X}$ then $\sup_{T\in\mathcal{A}}|T| < \infty$