Are All Convergences Topological?

Elijah French

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1 Introduction

An introductory look into topology and its connections to analysis seems to imply that all forms of convergence are topological. However, is this true? Can the form of "closeness" given by the concept of neighborhoods around a point be used to define all forms of convergence? It turns out that measure theory will yield an appropriate answer. In this report, metric spaces and their connections to topology will be explored and a form of convergence that is fundamental to probability theory will be defined.

2 Metric Spaces

Metric spaces generalize the idea of distance from Euclidean space to distinguish between points. This notion of distance is given by the spaces metric.

Definition 2.1. *Given a set X, a* ***metric***[5] *on X is a function $d : X \times X \to [0, \infty)$ s.t $\forall x, y, z \in X$*

- $d(x, y) = d(y, x)$
- $d(x, y) \geq 0$ *and equality holds* $\iff x = y$
- $d(x, z) \leq d(x, y) + d(y, z)$

The pair (X, d) is referred to as a metric space.

In an extension to the definition of balls in euclidean space,

Definition 2.2. *Given X and a metric on X, d, the* ***open ball***[5] *centered at x with radius ϵ is the set $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$*

Open balls around points are the fundamental neighborhoods with which one can ascertain various characteristics of sets and distinguish between elements. The set of all open balls in X can act as a generator basis for a topology on X. This topology, denoted $\mathcal{T}_d$, is referred to as the topology generated by the metric d. For the rest of the report, a metric space (X, d) is assumed to be endowed with $\mathcal{T}_d$. Given this topology, the following necessary and sufficient condition for openness is satisfied:

Proposition 2.1. *A set $O \subseteq X$ is in $\mathcal{T}_d$ iff $\forall x \in O$ $\exists \epsilon > 0$ s.t $B_\epsilon(x) \subseteq O$*

This is consistent with the classic definition of openness from analysis. With this fundamental property of open sets in metric spaces, we can explore more of their properties. It turns out that metric spaces satisfy certain characteristics that are often deemed desirable in topological spaces. For instance, one can separate points with disjoint open sets:

Theorem 2.1. *A metric space (X, d) is both Hausdorff and first countable.*[4]

Proof. Let $x, y \in X$ $x \neq y$. By Definition 2.1, $d(x, y) > 0$. Set $\epsilon = \frac{d(x,y)}{2}$. $B_\epsilon(x)$ and $B_\epsilon(y)$ are both neighborhoods of x and y respectively. If $z \in B_\epsilon(x) \cap B_\epsilon(y)$ the triangle inequality gives that

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, y) \\ &< \frac{d(x, y)}{2} + \frac{d(x, y)}{2} \\ &= d(x, y) \end{aligned}$$

Which implies that $d(x, y) < d(x, y)$ a contradiction and so $B_\epsilon(x) \cap B_\epsilon(y) = \emptyset$. Thus, X is Hausdorff by definition.

Let $x \in X$. It will be shown that $\{B_{\frac{1}{n}}(x) : n \in \mathbb{N}\}$ is a countable neighborhood basis of x. Given a neighborhood U of x, Prop 2.1 implies that $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq U$. As $\epsilon > 0$ the Archimedean Property gives that $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \epsilon$. Thus, $\forall n \geq N$ $B_{\frac{1}{n}}(x) \subseteq B_{\frac{1}{N}}(x) \subseteq B_\epsilon(x) \subseteq U$ and so X is first countable by definition.

□

With the added assumption of separability, metric spaces satisfy a fundamental property of Euclidean space:

Theorem 2.2. *A separable metric space (X, d) is second countable*

Proof. Assume (X, d) is separable. That is, $\exists Q \subseteq X$ s.t. Q is countable and $\overline{Q} = X$ or equivalently, $\forall x \in X$ every neighborhood of x has a point in Q. Claim: $\{B_k(q) : q \in Q, k \in \mathbb{Q}^+\}$ is a basis for (X, d).
It suffices, by the arbitrary union property of open sets, to show that every open ball in X is a union of a balls from this collection. Let $x \in X$ and $\epsilon > 0$. Let $y \in B_\epsilon(x)$. Setting $\delta = \frac{\epsilon - d(x,y)}{2}$ gives that $B_\delta(y) \subseteq B_\epsilon(x)$. Since Q is dense in X, $\exists q \in Q$ s.t. $q \in B_\delta(y)$. Then since $\mathbb{Q}^+$ is dense in $\mathbb{R}^+$, $\exists k \in \mathbb{Q}^+$ s.t. $d(q, y) \leq k \leq \delta$. Then, $y \in B_k(q) \subseteq B_{\epsilon - d(x,y)}(y) \subseteq B_\epsilon(x)$. As y was arbitrary, it follows that $B_\epsilon(x) \subseteq \cup_{\lambda \in \Lambda, \alpha \in \mathcal{A}} B_\alpha(\lambda)$ for some $\Lambda \subseteq Q$ and $\mathcal{A} \subseteq \mathbb{Q}^+$. Thus, since $\{B_k(q) : q \in Q, k \in \mathbb{Q}^+\}$ is countable and is a basis for $\mathcal{T}_d$, (X, d) is second countable.

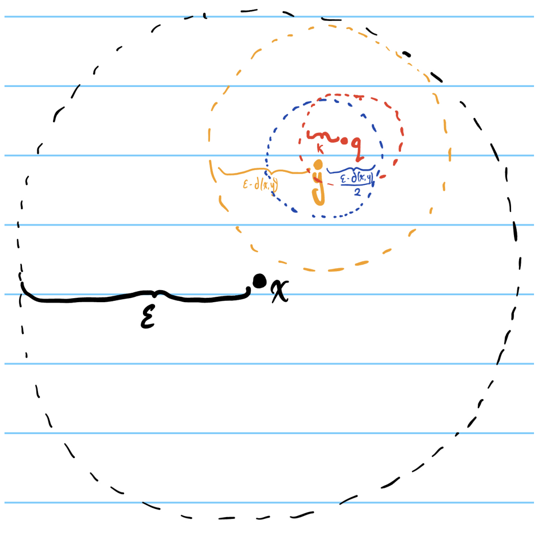

Figure 1: A representation of q and k given a $y \in B_\epsilon(x)$

□

In a way analogous to convergence in Euclidean space, the convergence of a sequence in a metric space can be defined in the following way:

Proposition 2.2. *In a metric space (X, d), a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x iff $\forall \epsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \; d(x_n, x) < \epsilon$*

Proof. ($\Rightarrow$) Let $\epsilon > 0$. $B_\epsilon(x)$ is a neighborhood of x and so since $x_n \to x \; \exists K \in \mathbb{N}$ s.t. $x_n \in B_\epsilon(x)$ $\forall n \geq K$. Then, setting $N = K$ implies that $d(x_n, x) < \epsilon \; \forall n \geq N$.
($\Leftarrow$) Let U be a neighborhood of x. Since U is open, $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. With this epsilon, the assumption gives a $N \in \mathbb{N}$ such that $\forall n \geq N \; d(x_n, x) < \epsilon$ which implies that $x_n \in B_\epsilon(x) \subseteq U \; \forall n \geq N$ and so $x_n \to x$ by definition.

□

This implies that in a metric space, convergence is uniquely determined by its metric. Now that convergence in metric spaces has been defined, one can explore more their properties. In particular, all forms of compactness are equivalent in a metric space, incredible!
Prior to presenting this result, the following proposition is necessary:

Proposition 2.3. *A sequentially compact metric space is separable and second countable.*[4]

Theorem 2.3. *For metric and second countable Hausdorff spaces, (a) limit point compactness, (b) sequential compactness, and (c) compactness are all equivalent properties.*[4]

Proof.
($(a) \Rightarrow (b)$): Assume X is limit point compact. Note that if X is a metric space or a second countable Hausdorff space, it is first countable and Hausdorff by Theorem 2.1. It suffices to assume that X is merely first countable and Hausdorff.
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Set $I = \{x_n : n \in \mathbb{N}\}$.
If I is finite, some point i in I must be repeatedly hit by the sequence and so a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ exists s.t. $x_{n_j} = i \; \forall j \in \mathbb{N}$ which converges to i by X's Hausdorffness.
If I is infinite it can be assumed WLOG that all the elements in the set are unique (take a subsequence of $(x_n)_{n\in\mathbb{N}}$ in which all the terms are different). By assumption, I has a limit point i. As X is second countable, i has a nested neighborhood basis $\{O_t : t \in \mathbb{N}\}$ s.t. $O_{t+1} \subseteq O_t$ $\forall t \in \mathbb{N}$. Since i is a limit point of I, $\forall t \in \mathbb{N} \; \exists x_j \in I$ s.t. $x_j \in O_t$. Now, define the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ recursively as follows: Set $x_{n_1} = x_j$ s.t. $x_j \in O_1$. and given x_{n_k}, $x_{n_k} \in O_m$ for some $m \in O_m$. Note that Hausdorffness implies that O_{m+1} has infinitely many points in I (or else a neighborhood would exist that contains no points in I by the finite intersection property of topologies) As O_{m+1} has infinitely many points from I, $\exists x_l \in O_{m+1}$ distinct from x_{n_k} s.t. $l > n_k$. Now set $x_{n_{k+1}} = x_l$.
Since each consecutive point in $(x_{n_k})_{k\in\mathbb{N}}$ is in a neighborhood further down the sequence of the neighborhood basis, $x_{n_k} \to i$.

($(b) \Rightarrow (c)$) : Assume X is sequentially compact. Note that a sequentially compact metric space is separable by Prop 2.3 and so is second countable by Theorem 2.2. It then suffices to assume X is second countable.
Let $\{U_\alpha : \alpha \in \mathcal{A}\}$ be an open cover of X. By X's second countability, $\{U_\alpha : \alpha \in \mathcal{A}\}$ has a countable subcover $\{U_{a_j} : j \in \mathbb{N}\}$. Assume $\{U_\alpha : \alpha \in \mathcal{A}\}$ does not have a finite subcover (implying that $\{U_{a_j} : j \in \mathbb{N}\}$ doesn't have one either). It must then be that $\forall n \in \mathbb{N}$, $\{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n}\}$ is not a cover of X. Define a sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n \in X \setminus \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n}\}$. The sequential compactness assumption then gives that $\exists (x_{n_j})_{j\in\mathbb{N}}$ with a limit, say x. Note that $x \in U_{\alpha_t}$ for some $t \in \mathbb{N}$. Since $x_{n_j} \to x \; \exists N \in \mathbb{N}$ s.t. $k \geq N \Rightarrow x_{n_k} \in U_{\alpha_t}$. Then, if $n_l > \alpha_t$ and $l \geq N$, $x_{n_l} \in U_{\alpha_t} \subseteq U_1 \cup ... \cup U_{\alpha_t} \cup ... \cup U_{n_l}$. So, $x_{n_l} \notin X \setminus \{U_{\alpha_1}, ..., U_{\alpha_{n_l}}\}$ a contradiction. Thus,

$\{U_\alpha : \alpha \in \mathcal{A}\}$ has a finite subcover and so X is compact by definition.

$\underline{((c) \Rightarrow (a))}$: Assume X is a compact metric or second countable hausdorff space. Assume to the contrary that X is not limit point compact. That is, $\exists I \subseteq X$ infinite such that $\forall x \in X$ there exists a neighborhood O_x of x that contains no other points of I. $\{O_x : x \in X\}$ is a cover of X. Thus, some finite subcover $\{O_{x_1}, ..., O_{x_n}\}$ exists by X's compactness. Then, $I \subseteq X \subseteq \cup_{i=1}^n O_{x_i}$. However, $I \cap O_{x_i} \subseteq x_i$ by construction. Thus, $I = I \cap (\cup_{i=1}^n O_{x_i}) = \cup_{i=1}^n I \cap O_{x_i} \subseteq \cup_{i=1}^n x_i = \{x_1, ...x_n\}$ which is a finite set implying I is finite, a contradiction. Thus, X is limit point compact.

□

Function spaces can be endowed with metrics. Such metrics attempt to measure the total deviation between two functions in their co-domains. A common metric on a function space is that of the sup norm error metric. The following is defined for continuous functions on [0,1], but can be generalized further.

Definition 2.3. *S: $C([0,1])^2 \to [0,\infty)$ is defined as*

$$S(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$$

Proposition 2.4. *S is a metric on $C([0,1])$*

Proof. In reference to definition 2.1, let $f, g, h \in C([0,1])$. Firstly,

$$\begin{aligned} S(f,g) &:= \sup\{|f(x) - g(x)| : x \in [0,1]\} \\ &= \sup\{|-f(x) + g(x)| : x \in [0,1]\} \\ &= \sup\{|g(x) - f(x)| : x \in [0,1]\} \\ &=: S(g,f) \end{aligned} \qquad \begin{aligned} S(f,g) &:= \sup\{|f(x) - g(x)| : x \in [0,1]\} \\ &\geq \sup\{0 : x \in [0,1]\} \\ &= 0 \end{aligned}$$

Secondly, if $f = g$

$$\begin{aligned} S(f,g) &:= \sup\{|f(x) - g(x)| : x \in [0,1]\} \\ &= \sup\{|f(x) - f(x)| : x \in [0,1]\} \\ &= \sup\{0\} \\ &= 0 \end{aligned}$$

Lastly,

$$\begin{aligned} S(f,g) &:= \sup\{|f(x) - g(x)| : x \in [0,1]\} \\ &= \sup\{|f(x) - h(x) + h(x) - g(x)| : x \in [0,1]\} \\ &\leq \sup\{|f(x) - h(x)| + |h(x) - g(x)| : x \in [0,1]\} \\ &= \sup(\{|f(x) - h(x)| : x \in [0,1]\} + \{|h(x) - g(x)| : x \in [0,1]\}) \\ &= \sup\{|f(x) - h(x)| : x \in [0,1]\} + \sup\{|h(x) - g(x)| : x \in [0,1]\} \\ &=: S(f,h) + S(h,g) \end{aligned}$$

Where the second to last step is from properties of the supremum. Thus, S is a metric by definition. □

This metric and the convergences that it induces in $C([0,1])$ has an interesting relationship with that of uniform convergence. It is expressed in the following theorem:

Theorem 2.4. *A sequence of functions $(f_n)_{n\in\mathbb{N}} \subseteq C([0,1])$ converges uniformally to f on $[0,1]$ if and only if $(f_n)_{n\in\mathbb{N}} \subseteq C([0,1])$ converges to f with respect to S.*[7]

As an example, consider the sequence $(x^n)_{n\in\mathbb{N}}$. Given an x in [0,1) $\lim_{n\to\infty} x^n = 0$. As a result of this, it may appear initially that x^n converges to 0 with respect to this metric. However, at x=1, $\lim_{n\to\infty} 1^n = 1$. As the sup norm takes the supremum of the difference between functions on a set, $x = 1$ results in the sequence of functions to not converge to 0 in S.

Proposition 2.5. *The sequence $(x^n)_{n\in\mathbb{N}} \subseteq C([0,1])$ does not converge to 0 uniformally.*

Proof. Firstly, set $\epsilon = \frac{1}{2}$ and let $N \in \mathbb{N}$. Set $n = N$

$$\begin{aligned} S(0, x^n) &:= \sup\{|0 - x^N| : x \in [0,1]\} \\ &= |0 - 1^N| && \text{as } x^N < 1^N \ \forall x \in [0,1) \\ &= 1 \\ &\geq \frac{1}{2} \end{aligned}$$

Thus, $(x^n) \not\to 0$ in $(C([0,1]), S)$ by the negation of proposition 2.2. It then follows from theorem 2.3 that $(x^n) \not\to 0$ uniformally.

□

3 Measure Theory

Measure theory attempts to generalize the ideas of length, volume, and area to arbitrary spaces.[3] In order to match the intuitive idea of what a measure is, some restrictions on the sorts of objects that can be measured need to be made.

Definition 3.1. *Given a set X, a σ-**Algebra**[2] $\mathcal{X} \subseteq X$ is a collection of sets of X such that*

- $\emptyset, X \in \mathcal{X}$
- $A \in \mathcal{X} \Rightarrow A^c \in \mathcal{X}$
- $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{X} \Rightarrow \cup_{i=1}^{\infty} A_n \in \mathcal{X}$

The collection (X, $\mathcal{X}$) is referred to as a measurable space.

The power set $P(X)$ is clearly a σ-Algebra, however, it turns out that this collection is often too large and smaller collections are preferred. An example of this is the collection $B(\mathbb{R})$ in $\mathbb{R}$. $B(\mathbb{R})$ is defined to be the smallest $\sigma-$Algebra of $\mathbb{R}$ that contains all of its open subsets. A measure can now be defined which acts on a set's σ-Algebra.

Definition 3.2. *Given a measurable space (X, $\mathcal{X}$) a **measure**[2] $\mu : \mathcal{X} \to [0,\infty)$ is a function where*

- $\mu(\emptyset) = 0$
- $\forall (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{X}$ *where* $A_i \cap A_j = \emptyset$ $\forall i, j \in \mathbb{N}$ *with* $i \neq j$ $\mu(\cup_{i=1}^{\infty} A_n) = \sum_{i=1}^{\infty} \mu(A_n)$

The collection (X, $\mathcal{X}$, μ) is referred to as a measure space.

It is often important to compare two measurable spaces. One way to do this is to define a function between them that connects their σ-Algebras.

Definition 3.3. *A function $f : X \to Y$ where (X, $\mathcal{X}$) and (Y, $\mathcal{Y}$) are measurable spaces is said to be $\mathcal{X}$-$\mathcal{Y}$ measurable if $f^{-1}(A) \in \mathcal{X}$ $\forall A \in \mathcal{Y}$* [2]

In the next section, two types of convergences for measurable functions are defined.

4 Convergence Almost Everywhere and In Measure

Let $(X, \mathcal{X}, \mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}}$ a sequence of real valued, $\mathcal{X}$-$B(\mathbb{R})$ measurable functions.

Definition 4.1. *$(f_n)_{n\in\mathbb{N}}$ is said to converge to f* ***almost everywhere***[2] *if*

$$\mu(\{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\}^c) = 0$$

Almost everywhere convergence is fundamental to probability theory. For instance, the strong law of large numbers states that the sample average of a sequence of independent, identically distributed random variables converges almost everywhere to their mean.[2]

Definition 4.2. *$(f_n)_{n\in\mathbb{N}}$ is said to converge to f* ***in measure***[2] *if $\forall\epsilon > 0$,*

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) = 0$$

Initially, the above two forms of convergences may appear identical. In fact, this is not the case, but the two are related. The following two propositions explore this relationship.

Proposition 4.1. *If $(f_n)_{n\in\mathbb{N}}$ converges to f in measure, then there exists a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ that converges to f almost everywhere.*[2]

Proof. Set $n_1 = 1$. Recursively choose n_j in the following manner. Given an n_j, set $\epsilon = \frac{1}{j+1}$ and choose n_{j+1} using the assumption of convergence in measure where

$$\mu(\{x : |f_{n_{j+1}}(x) - f(x)| > \epsilon\}) \leq \frac{1}{2^{j+1}}$$

Now, with the sequence chosen, define $A_j = \{x : |f_{n_j}(x) - f(x)| > \frac{1}{j}\}$. Note that $A = \{x : |f_{n_j}(x) - f(x)| = 0\}$ is such that $(\cup_{k\geq n} A_j)_{n\in\mathbb{N}}$ decreases to A ($A = \cap_{n\in\mathbb{N}} \cup_{j\geq n} A_j$). It then follows from elementary results of measures and convergence that

$$\mu(A) = \lim_{n\to\infty} \mu(\cup_{j\geq n} A_j) \leq \lim_{n\to\infty} \sum_{j=n}^{\infty} \mu(A_j) \leq \lim_{n\to\infty} \frac{1}{2^{n+1}}$$

Now, for $x \notin A$ $\exists n \in \mathbb{N}$ s.t. $|f_{n_j}(x) - f(x)| \leq \frac{1}{j}$ (negating $A = \cap_{n\in\mathbb{N}} \cup_{j\geq n} A_j$) and so as n is fixed and j can be made arbitrarily large $\lim_{j\to\infty} f_{n_j}(x) = f(x)$. It then follows that $\{x : \lim_{j\to\infty} f_{n_j}(x) = f(x)\}^c \subseteq A$ which implies that

$$\mu(\{x : \lim_{j\to\infty} f_{n_j}(x) = f(x)\}^c) \leq \mu(A) = 0$$

Since $\mu(B) \geq 0$ $\forall B \in \mathcal{X}$ it follows that $(f_{n_j})_{j\in\mathbb{N}}$ converges to f almost everywhere by definition. □

With an added assumption on the measure space, convergence almost everywhere implies convergence in measure.

Proposition 4.2. *If $(f_n)_{n\in\mathbb{N}}$ converges to f almost everywhere and $\mu(X) = 1$ then $(f_n)_{n\in\mathbb{N}}$ converges to f in measure.*[2]

With reference to metric spaces and the convergences that they imply, convergence almost everywhere is in fact induced by a metric. The following proposition uses a Lebesgue integral which can be thought of as a generalized Riemman integral.

Proposition 4.3. *$(f_n)_{n\in\mathbb{N}}$ converges to f in measure if and only if $(f_n)_{n\in\mathbb{N}}$ converges to f with respect to the metric defined by $d(f, g) = \int_X min\{|f - g|, 0\}\mu(dx)$* [1]

Proposition 2.2 then gives that convergence in measure is topological (by the topology induced by d).

5 Convergence Not Related to Any Topology

With this background information, the question “Are All Convergences Topological?” can be answered. Throughout this section the Lebesgue measure, λ, on $(\mathbb{R}, B(\mathbb{R}))$ is used. This measure satisfies $\lambda([a,b]) = b - a \ \forall a, b \in \mathbb{R}$ with $a < b$.

Consider the set of $B(\mathbb{R})$-$B(\mathbb{R})$ measurable functions f_m^n defined for $1 \leq m \leq n$ by

$$f_m^n(x) = \begin{cases} 1 & \frac{(m-1)}{n} \leq x \leq \frac{m}{n} \\ 0 & \text{otherwise} \end{cases}$$

Now, define $n : \mathbb{N} \to \mathbb{N}$ s.t. $\frac{(n(k)+1)(n(k))}{2} \leq k < \frac{(n(k)+2)(n(k)+1)}{2}$. Note that $\lim_{k\to\infty} n(k) = \infty$. Next, define $m : \mathbb{N} \to \mathbb{N}$ by $m(k) = k - \frac{(n(k)+1)n(k)}{2}$ and the sequence $(g_k)_{k\in\mathbb{N}}$ by

$$g_k = f_{m(k)}^{n(k)}$$

which is well defined as $1 \leq m(k) \leq n(k)$.

This sequence can be visualized as a box which discretely and repeatedly travels along the interval $[0,1]$ whose width gets thinner with each iteration. For g_k, the width of this interval is $\frac{1}{n(k)}$. Note that this is the entire length of where g_k differs from 0.

Proposition 5.1. *$(g_k)_{k\in\mathbb{N}}$ converges in measure but not almost everywhere to 0.*

Proof. Firstly, convergence in measure will be shown. Let $\epsilon > 0$.

$$\begin{aligned} \lim_{k\to\infty} \lambda(\{x \in \mathbb{R} : |g_k(x) - 0| > \epsilon\}) &= \lim_{k\to\infty} \lambda(\{x \in \mathbb{R} : |g_k(x)| > \epsilon\}) \\ &\leq \lim_{k\to\infty} \lambda\Big(\Big[\frac{(m(k)-1)}{n(k)}, \frac{m(k)}{n(k)}\Big]\Big) \\ &= \lim_{k\to\infty} \frac{1}{n(k)} \\ &= 0 \end{aligned}$$

However, $(g_k)_{k\in\mathbb{N}}$ does not converge almost everywhere to 0. Given an $x \in [0,1]$ and $n \in \mathbb{N}$ there is always an $m \leq n$ such that $\frac{(m-1)}{n} \leq x \leq \frac{m}{n}$. Then, as $n : \mathbb{N} \to \mathbb{N}$ is surjective, increase k until $n(k) = n$. Then, increase k again until $m(k) = m$. It follows that $x \in [\frac{(m(k)-1)}{n(k)}, \frac{m(k)}{n(k)}]$ and so $g_k(x) = 1 \neq 0$ This means that $\forall x \in [0,1]$, $\lim_{k\to\infty} g_k(x) \neq 0$. Thus,

$$\begin{aligned} \lambda(\{x \in X : \lim_{k\to\infty} g_k(x) = 0\}^c) &= \lambda([0,1]) \\ &= 1 \\ &\neq 0 \end{aligned}$$

and so $(g_k)_{k\in\mathbb{N}}$ does not converge to 0 almost everywhere by definition.

□

Now is time for the main result.

Proposition 5.2. *Convergence almost everywhere is not topological.*[6]

Proof. Assume to the contrary, that is, that convergence almost everywhere is a topological convergence. Thus, as $(g_k)_{k\in\mathbb{N}}$ does not converge almost everywhere to 0 there exists O, a neighborhood in this topology, such that $\forall N \in \mathbb{N}\ \exists k \geq N$ s.t $g_k \notin O$. Define the subsequence

$(g_{k_j})_{j\in\mathbb{N}}$ to be always outside of O. Since $(g_k)_{k\in\mathbb{N}}$ converges in measure to 0, so does $(g_{k_j})_{j\in\mathbb{N}}$. Then, by Proposition 4.1 there exists a subsequence of $(g_{k_j})_{j\in\mathbb{N}}$, $(g_{k_{j_t}})_{t\in\mathbb{N}}$ that converges to 0 in measure. Thus, $\exists z \in \mathbb{N}$ s.t. $g_{k_{j_z}} \in O$. However, this contradicts the assumption that $g_{k_j} \notin O$ $\forall j \in \mathbb{N}$. Thus, convergence almost everywhere is not a topological convergence. Note that this implies that convergence almost everywhere is not metrizable by Proposition 2.2. $\square$

Thus, there are in fact important convergences that are not topological!

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