Efficient Matrix Multiplication

Team 12

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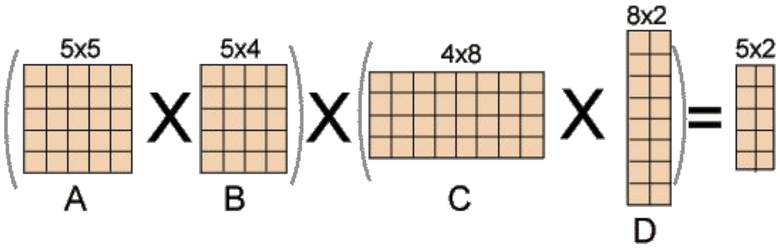
Our Project Goals

1. Parallelize matrix multiplication to speed up multiplication of large matrices
2. Create a simple and efficient library in Java for computing matrix chain multiplication

Background

Matrix Multiplication

- Res[i][j] = Row i of M1 · Col j of M2
- For a chain of n matrices multiplied together:
  - For each pair of adjacent matrices, inner dimensions must match!
  - Final matrix: (#rows M1 x #cols Mn)
- Associative



$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix}$$

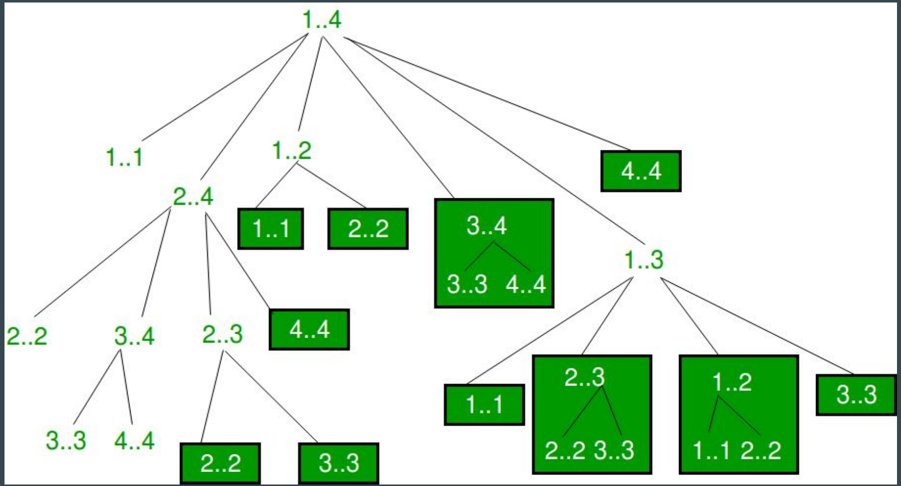
$$= \begin{bmatrix} 1x10 + 2x20 + 3x30 & 1x11 + 2x21 + 3x31 \\ 4x10 + 5x20 + 6x30 & 4x11 + 5x21 + 6x31 \end{bmatrix}$$

$$= \begin{bmatrix} 10+40+90 & 11+42+93 \\ 40+100+180 & 44+105+186 \end{bmatrix} = \begin{bmatrix} 140 & 146 \\ 320 & 335 \end{bmatrix}$$

Sequential Matrix Multiplication

```java
public Matrix multiply(Matrix other) {
    Matrix newMatrix = new Matrix(this.rows, other.cols);
    int innerDim = this.cols;

    for (int thisR = 0; thisR < this.rows; thisR++) {
        for (int otherC = 0; otherC < other.cols; otherC++) {
            for (int i = 0; i < innerDim; i++) {
                newMatrix.values[thisR][otherC] = newMatrix.values[thisR][otherC]
                        .add(this.values[thisR][i].multiply(other.values[i][otherC]));
            }
        }
    }

    return newMatrix;
}
```

Matrix Chain Multiplication Problem



- ➢ Optimization problem solved with **Dynamic Programming**
- ➢ Order in which the matrices are parenthesized affects the number of simple arithmetic operations needed to compute the product.
- ➢ Solution finds the optimal way to multiply a given sequence of matrices

Why optimize?

Matrix A: 10×30
Matrix B: 30×5
Matrix C: 5×60

$$(AB)C : (10 \times 30 \times 5) + (10 \times 5 \times 60) = 4500$$

$$A(BC) : (30 \times 5 \times 60) + (10 \times 30 \times 60) = 27000$$

Matrix multiplication is associative!

The first parenthesization is 6x more efficient than the other in terms of number of multiplication operations.

Optimization Sequential Algorithm

```java
// https://home.cse.ust.hk/~dekai/271/notes/L12/L12.pdf
protected int[][] getMinimumOrdering(int[] dims, int N) {
    int[][] dp = new int[N + 1][N + 1];
    int[][] s = new int[N + 1][N + 1];
    // l is the length of each matrix chain subproblem we tackle this iteration
    // Start smaller so that when we divide larger chains, answer is already done
    for (int l = 2; l <= N; l++) {
        for (int i = 1; i <= N - l + 1; i++) {
            int j = i + l - 1;
            dp[i][j] = Integer.MAX_VALUE;
            // For each possible partition point in this chain
            for (int k = i; k <= j - 1; k++) {
                // Total operations if we use this partition is the cost of multing left and
                // right + cost of combining at partition
                int q = dp[i][k] + dp[k + 1][j] + dims[i - 1] * dims[k] * dims[j];
                if (q < dp[i][j]) {
                    dp[i][j] = q;
                    s[i][j] = k;
                }
            }
        }
    }

    // Print minimum number of operations required to out file
    System.out.println(dp[1][N]);
    return s;
}
```

Parallelization

Parallel Objectives

...

Explore methods of parallelization of the DP Optimization Algorithm

Efficiently multiply matrix pairs with a speedup compared to the standard sequential algorithm.

Implementation: Matrix Chain Optimization

- Use two threads to calculate subproblems for chains of length 2 -> N
  - Wait on
  - When using `dp[i][k] + dp[k + 1][j]` to calculate the total operations for each k, wait for previous thread to calculate either value if it is still null
    - Ignore if l == 2, and just use zero instead
  - Poll for a new value of l and start again

Implementation: Matrix Chain Optimization

Proposed implementation:

- Initialize two threads
- Each thread polls the next lowest matrix length uncalculated
- Conduct standard DP algorithm using this value of l
- When using `dp[i][k] + dp[k + 1][j]` to calculate the total operations for each k, wait for previous thread to calculate either value if it is still null
    - Ignore if l == 2, and just use zero instead
- Poll for a new value of l and start again

Why This Is Infeasible

- Cost outweighs benefit unless chain length is thousands long
  - This is unrealistic
  - Resultant matrices would hold enormous values
- Much smaller problem than actual matrix computations

```java
@Override
public void run() {
    while (true) {
        // Grab the next available length on the queue
        int l;
        try {
            l = spQueue.poll();
        } catch (NullPointerException e) {
            // If all subproblem lengths have been calculated, stop
            return;
        }
        for (int i = 1; i <= N - l + 1; i++) {
            int j = i + l - 1;
            for (int k = i; k <= j - 1; k++) {
                while (dp[i][k] == null && l > 2) {
                }
                while (dp[k + 1][j] == null && l < 2) {
                }
                BigInteger v1 = dp[i][k] == null ? BigInteger.ZERO : dp[i][k];
                BigInteger v2 = dp[k + 1][j] == null ? BigInteger.ZERO : dp[k + 1][j];
                BigInteger q = v1.add(v2)
                        .add(BigInteger.valueOf(dims[i - 1] * dims[k] * dims[j]));
                if (k == i || q.compareTo(dp[i][j]) < 0) {
                    dp[i][j] = q;
                    s[i][j] = k;
                }
            }
        }
    }

}
```

Implementation: Matrix Multiplication

- 1st Task: How to split up matrix chain multiplication work across threads?
  - One approach: Split up each matrix pair multiplication on different threads
    - BUT Conflicts with efficient parenthesization of chain
      - Ex: A(B(CD))
  - **Final Approach:** Split up the row/column multiplication work in single matrix pair multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} j & k \\ l & m \\ n & o \end{bmatrix} = \begin{bmatrix} aj + bl + cn & ak + bm + co \\ dj + el + fn & dk + em + fo \\ gj + hl + in & gk + hm + io \end{bmatrix}$$

Thread 1 Thread 4

Thread 2 Thread 5

Thread 3 Thread 6

Implementation: Matrix Multiplication

- 2nd Task: How to parallelize multiplication between two matrices?
  - Use/reuse thread pool with fixed number of threads for each matrix pair multiplication
  - Steps:
    - Create empty result matrix
    - Assign tasks to thread pool to populate result matrix
    - Each task responsible for assigning value to cell (i, j) in result matrix
    - Wait until all tasks have finished
    - Return result matrix that was populated by different threads

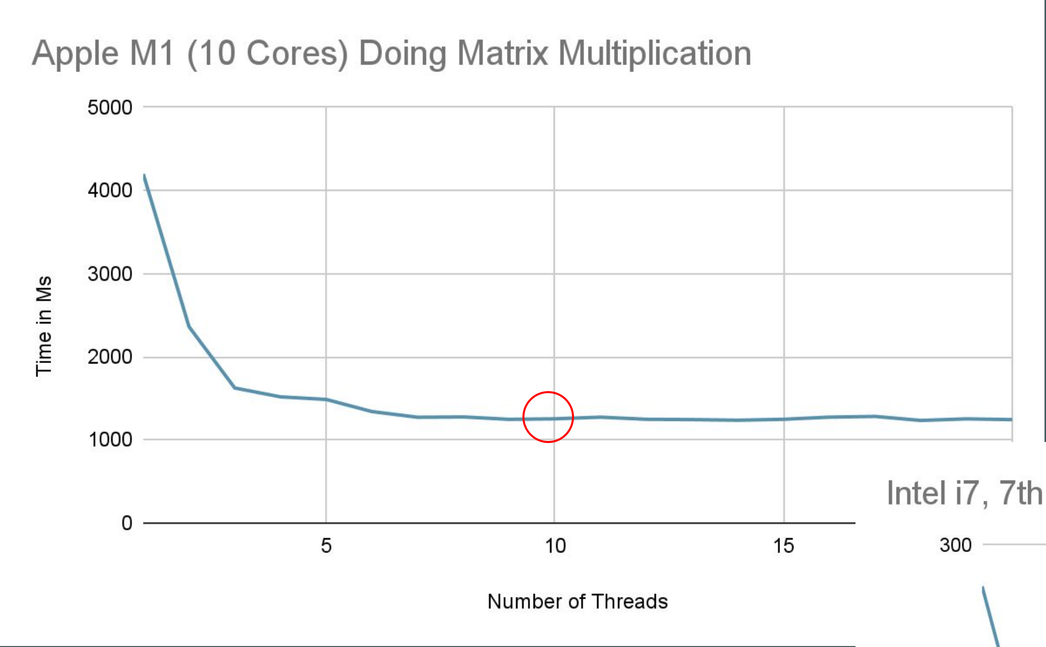
```
class MultThread implements Runnable {
    Matrix a;
    Matrix b;
    Matrix result;
```

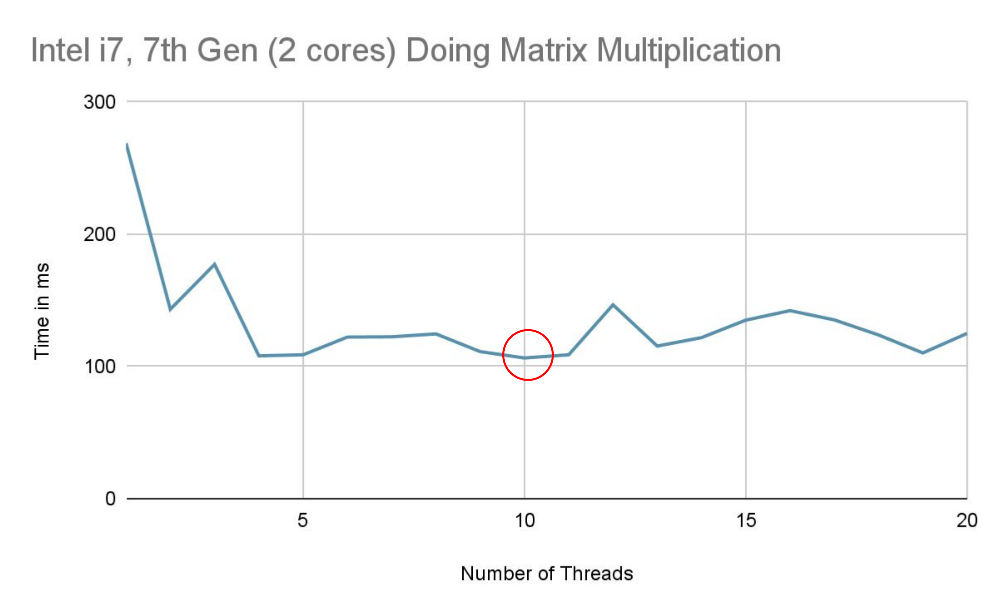
Implementation: Testing & Matrix Generation

- Created program to randomly generate matrices
- Finding Optimal Thread Count
  - Test program on same large test case (10-20 matrices, 1000 x 1000 max dimension)
  - Repeatedly run program on this test case but change number of threads used by thread pool each time
  - Pick the smallest thread count that achieves the best runtime and use for the rest of testing

```java
public class MatrixGenerator {
    static final int MAX_COLS = 100;
    static final int MAX_ROWS = 100;
    static final int MAX_VALUE = 100;
    static final int MIN_MATRICES = 50;
    static final int MAX_MATRICES = 50;
    static final String outPath = "in/";

    public static void main(String[] args) throws FileNotFoundException {
        int fileNum = 1;
        File f;
        do {
            f = new File(outPath + "test" + fileNum + ".txt");
            fileNum++;
        } while (f.exists() && !f.isDirectory());

        PrintStream stream = new PrintStream(f);
        System.setOut(stream);

        int lastC = -1;

        int numMatrices = (int) (Math.random() * (MAX_MATRICES - MIN_MATRICES + 1)) + MIN_MATRICES;
        System.out.println(numMatrices);
        for (int i = 0; i < numMatrices; i++) {
            int tr = (lastC != -1 ? lastC : (int) (Math.random() * MAX_ROWS + 1));
            int tc = (int) (Math.random() * MAX_COLS + 1);

            System.out.println(tr + " " + tc);
            for (int r = 0; r < tr; r++) {
                for (int c = 0; c < tc; c++) {
                    int val = (int) (Math.random() * (MAX_VALUE + 1));
                    System.out.print(val + (c < tc - 1 ? " " : ""));
                }
                System.out.println();
            }

            lastC = tc;
        }
    }
}
```

Thread Results





Note: Different Test Cases used, M1 using a much larger test case

Implementation: Testing & Matrix Generation

- 3 approaches to test
  - Brute force
  - Efficient ordering on single thread
  - Efficient ordering on multiple thread
- Testing Validity

```java
long bruteForceRuntimes = 0;
long syncSmartOrderingRuntimes = 0;
long multiThreadedSmartOrderingRuntimes = 0;

// Run multiple trials for each approach on same test case
for (int i = 0; i < NUM_PERFORMANCE_TRIALS; i++) {
    // Multiply chain out using brute force approach and record runtime
    long startTimeBruteForce = System.currentTimeMillis();
    Matrix resBruteForce = chain.multiplyOutBruteForce();
    long endTimeBruteForce = System.currentTimeMillis();
    bruteForceRuntimes += endTimeBruteForce - startTimeBruteForce;

    // Multiply chain out using efficient ordering on single thread approach and record
    runtime
    long startTimeSyncSmartOrdering = System.currentTimeMillis();
    Matrix resSyncSmartOrdering = chain.multiplyOut(useMultipleThreads:false);
    long endTimeSyncSmartOrdering = System.currentTimeMillis();
    syncSmartOrderingRuntimes += endTimeSyncSmartOrdering - startTimeSyncSmartOrdering;

    // Multiply chain out using efficient ordering and multiple threads approach and
    record runtime
    long startTimeMultiThreadedSmartOrdering = System.currentTimeMillis();
    Matrix resMultiThreadedSmartOrdering = chain.multiplyOut(useMultipleThreads:true);
    long endTimeMultiThreadedSmartOrdering = System.currentTimeMillis();
    multiThreadedSmartOrderingRuntimes += endTimeMultiThreadedSmartOrdering -
    startTimeMultiThreadedSmartOrdering;

    if (!matricesAreEqual(resBruteForce, resSyncSmartOrdering,
    resMultiThreadedSmartOrdering)) {
        System.out.println("Incorrect results from matrix multiplication.");
        return;
    }
```

Brute Force

1 Thread

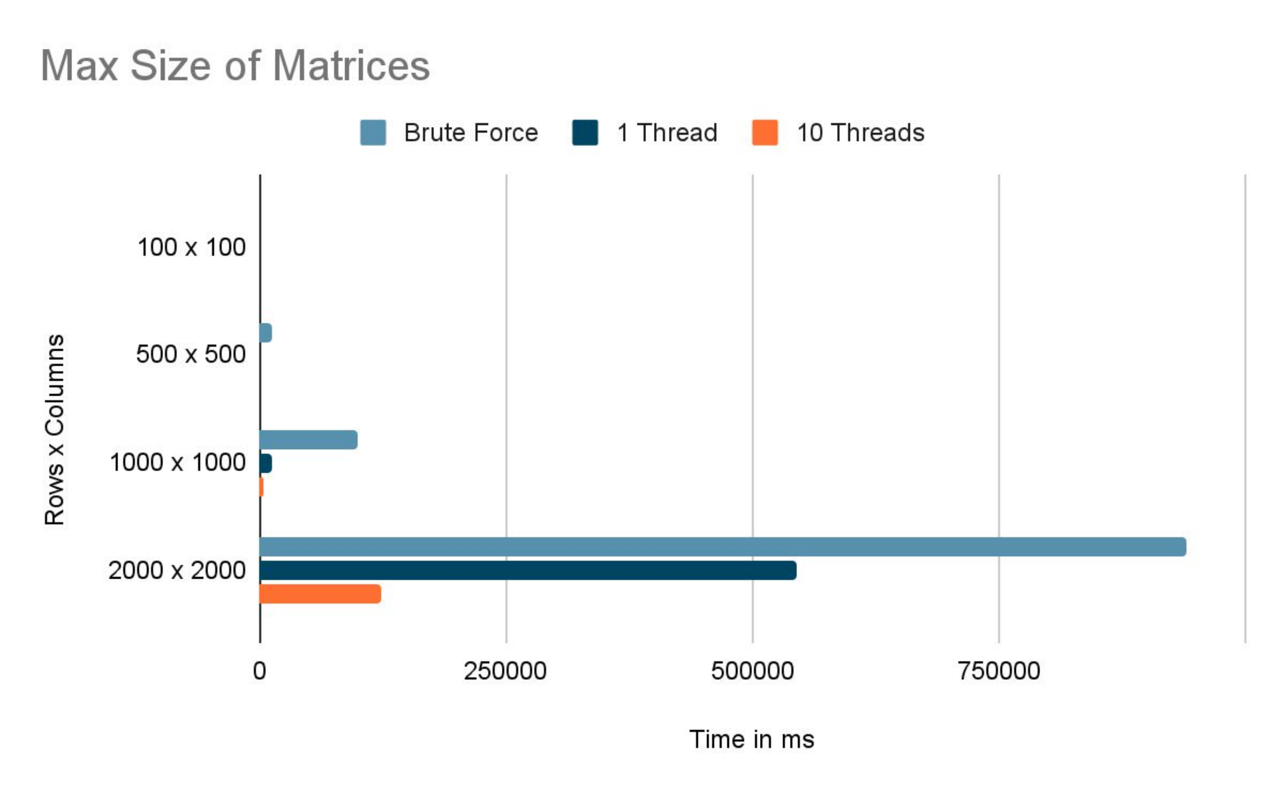
10 Threads

Check Results

Implementation: Testing & Matrix Generation

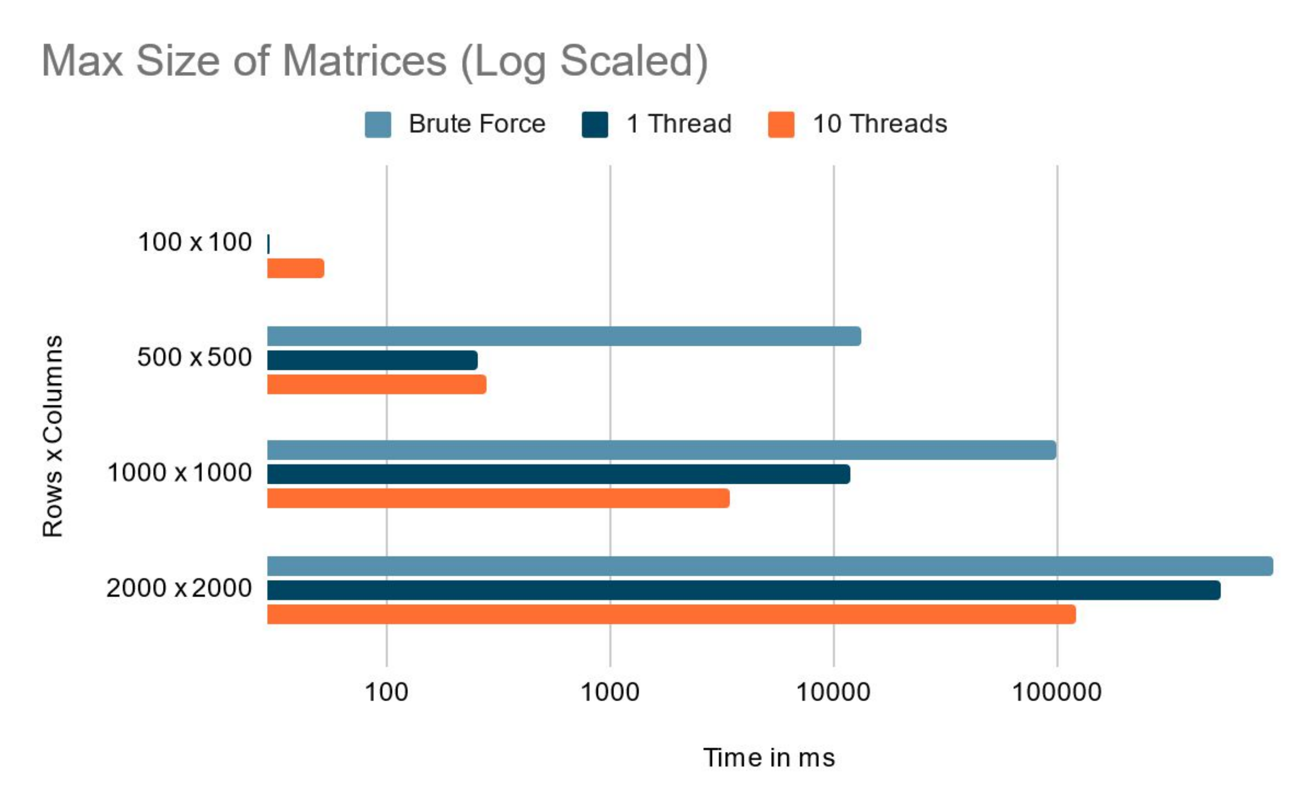
- Testing Performance
  - 2 independent variables: matrix chain size and individual matrix dimensions
  - Matrix Dimension Performance
    - Fix each matrix chain to size 8
    - Run trials on test cases with differing maximum matrix dimensions: (500 x 500), (1000 x 1000), (2000 x 2000)
  - Matrix Chain Size Performance
    - Fix max dimension of each matrix to (100 x 100)
    - Run trials on test cases with differing matrix chain lengths: 50, 100, 1000

Matrix Dimension Performance

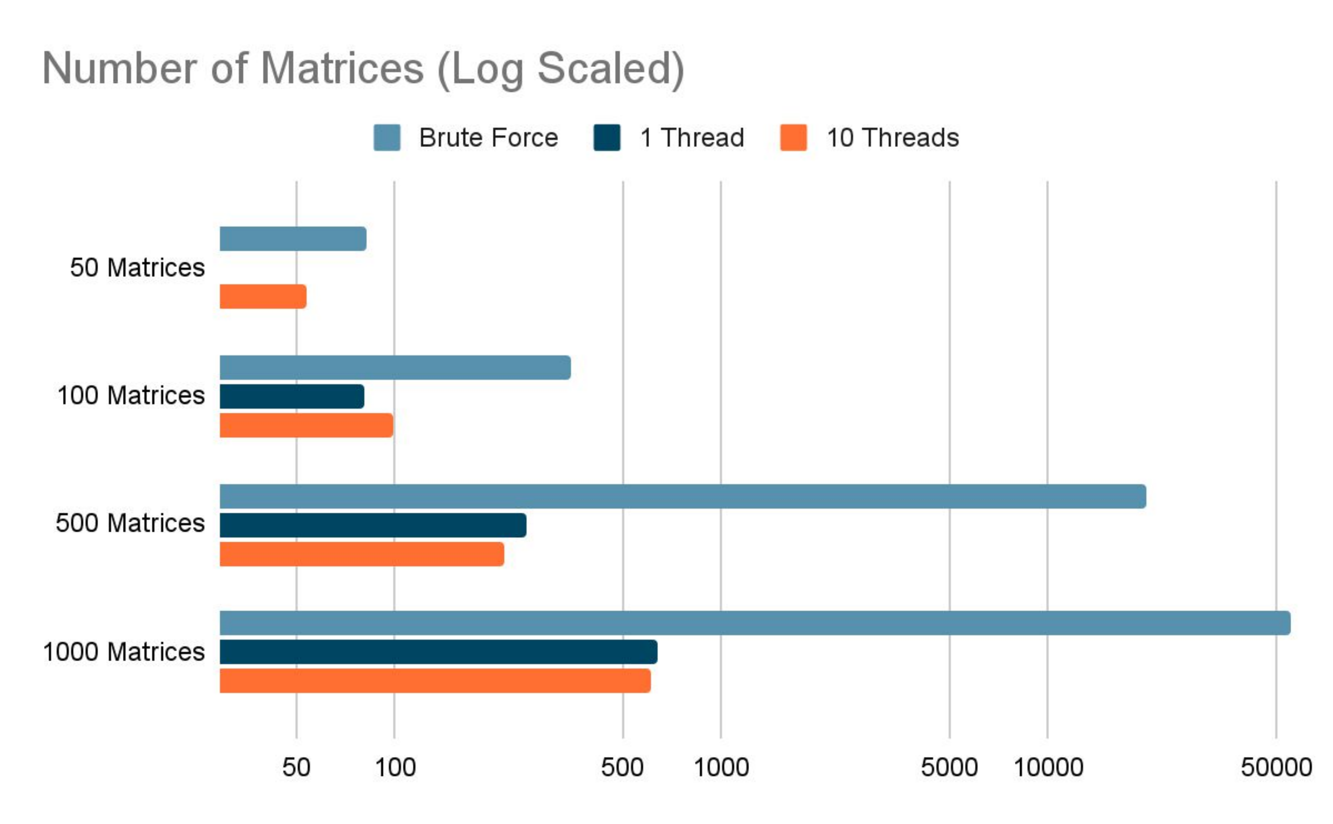


But wait! Brute Force hides all our data...

Matrix Dimension Performance (Log Scaled)



Matrix Chain Size Performance (Log Scaled)



Challenges

- Multiplication overflow
  - Converted matrices to use BigIntegers
- Identifying possible targets for parallelization
- Running large test cases on brute force approach

Conclusions

1. Stick to traditional DP optimization algorithm
2. Able to successfully parallelize matrix multiplication

Related Works for Further Study

- Hu, TC; Shing, MT (1981). Computation of Matrix Chain Products, Part I, Part II (PDF) (Technical report). Stanford University, Department of Computer Science. Part II, page 3. STAN-CS-TR-81-875.
- Wang, Xiaodong; Zhu, Daxin; Tian, Jun (April 2013). "Efficient computation of matrix chain". 2013 8th International Conference on Computer Science Education: 703–707. doi:10.1109/ICCSE.2013.6553999.
- Chin, Francis Y. (July 1978). "An O(n) algorithm for determining a near-optimal computation order of matrix chain products". Communications of the ACM. 21 (7): 544–549. doi:10.1145/359545.359556.
- Hu, T.C; Shing, M.T (June 1981). "An O(n) algorithm to find a near-optimum partition of a convex polygon". Journal of Algorithms. 2 (2): 122–138. doi:10.1016/0196-6774(81)90014-6