ECE 541 Project 3

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Question 1

For the system of $N+1 (N \geq 1)$ charged conducting bodies in a linear dielectric shown in the figure below, we have that

$$V_1 = a_{11}Q_1 + a_{12}Q_2 + ... + a_{1N}Q_N$$

$$V_2 = a_{21}Q_1 + a_{22}Q_2 + ... + a_{2N}Q_N$$

$$\vdots$$

$$V_N = a_{N1}Q_1 + a_{N2}Q_2 + ... + a_{NN}Q_N$$

where the constrains a_{ij} (i,j = 1,2,. . . ,N) are termed the coefficients of potential or potential coefficients. Derive this linear matrix relationship, i.e., explain why such a relationship holds. Also explain how the potential coefficients can be evaluated (by computations or measurements) for a given system.

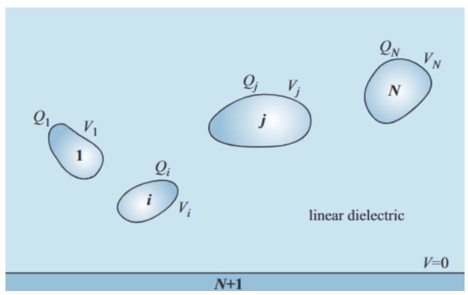

Figure 1: Multi-body conductor system

Solution

Voltage, when related to charge on a single body, is represented by the equation

$$V = \frac{E}{Q}$$

or

$$E = QV$$

Using this equation, we can derive an equation for multi-body systems. Considering a conductor has a charge Q, the first step is to find a way to find the total voltage on that body with respect to the other bodies in the system. Selecting a coefficient "a", we can directly relate the voltage (V) to the charge (Q), creating a sum of all voltages (2 to N) to find the total voltage on a particular body.

$$\sum_{j=2}^{N} a_{1j}Q_j$$

This only works for a singular body, labeled "1" in the above equation. Additionally, this does not account for the conductor's relationship to ground and the voltage induced through this relationship. Consequently, the relationship should now be altered to include this.

$$\sum_{j=1}^{N} a_{1j} Q_j$$

This is significantly better, but it does not provide us with the voltage of the total system, but rather only for a single body. We want something more along the lines of

$$V_1 = a_{11}Q_1 + a_{12}Q_2 + ... + a_{1N}Q_N$$

$$V_2 = a_{21}Q_1 + a_{22}Q_2 + ... + a_{2N}Q_N$$

$$\vdots$$

$$V_N = a_{N1}Q_1 + a_{N2}Q_2 + ... + a_{NN}Q_N$$

This relationship can be given as a matrix

$$[V_i] = [a_{ij}][Q_j]$$

resulting in

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & & & \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{bmatrix}$$

Where $V_i$is the voltage on the body being observed and $Q_j$is the body being compared. Furthermore, it is good to notice that the a matrix is $N \times N$ while the other matrices are $1 \times N$. This gives us the equation

$$a_{ij} = \frac{V_i}{Q_j}\Big|_{Q_k=0, k\neq j}$$

$k \neq j$ because this coefficient only refers to the potential between two distinct conductors. For this particular potential coefficient, all other charges on the remaining conductors are assumed to be zero. Additionally, i can be the same as j, again addressing the situation of a self induced voltage.

Question 2

A metallic sphere of radius a is positioned inside a very thin spherical metallic shell of radius b ($b > a$), concentrically with it. The medium is air everywhere. The sphere is connected via a very thin conductor to the reference point with zero potential, which is very far away, as indicated in the figure. The shell is charged by a charge Q. Find (a) the potential coefficients of the system, (b) the charge of the sphere, and (c) the potential of the shell.

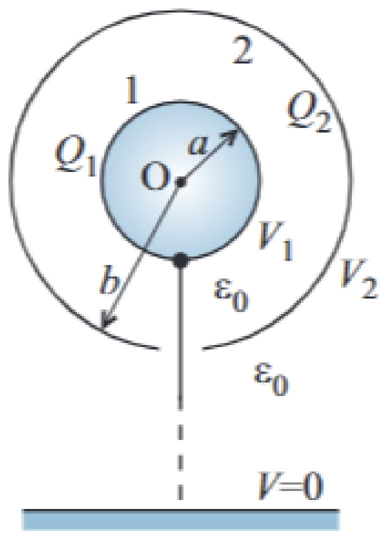

Figure 2: A metallic sphere positioned inside a very thin spherical metallic shell

Solution

Part A). Find the potential coefficients of the system

Starting with the equation for the electric field of a sphere

$$E = \frac{Q_1}{4\pi\epsilon r^2}$$

We can then find the voltage of the interior sphere by integrating over the distance from between the metallic sphere and the thing spherical metallic shell

$$V = \int_a^b E(r)dr = \frac{Q}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})$$

Applying this to the problem

$$V_1 = \int_a^b \frac{Q_1}{4\pi\epsilon r^2}dr = \frac{Q_1}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})$$

Using the theory of capacitors, we then know that $Q_1 = -Q_2$. Implementing this into the equation and flipping the bounds, we get

$$V_1 = -\int_a^b \frac{Q_2}{4\pi\epsilon r^2}dr = \int_b^a \frac{Q_2}{4\pi\epsilon r^2}dr = \frac{Q_2}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a})$$

Moreover, we can solve for V_2

$$V_2 = \int_a^b \frac{Q_2}{4\pi\epsilon r^2}dr = V_1 = \frac{Q_2}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})$$

Using this information, we can begin to calculate the potential coefficients of the system a_{11}, a_{12} and a_{22} using the equations for question 1.

$$a_{ij} = \frac{V_i}{Q_j}\Big|_{Q_k=0,k\neq j}$$

$$a_{11} = \frac{V_1}{Q_1} = \frac{Q_1}{4\pi\epsilon Q_1}(\frac{1}{a} - \frac{1}{b}) = \frac{1}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})$$

$$a_{12} = \frac{V_1}{Q_2} = \frac{Q_1}{4\pi\epsilon Q_2}(\frac{1}{a} - \frac{1}{b}) = -\frac{1}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a}) = \frac{1}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a})$$

$$a_{22} = \frac{V_2}{Q_2} = \frac{Q_2}{4\pi\epsilon Q_2}(\frac{1}{a} - \frac{1}{b}) = \frac{1}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})$$

$$a_{21} = \frac{V_2}{Q_1} = \frac{Q_2}{4\pi\epsilon Q_1}(\frac{1}{a} - \frac{1}{b}) = -\frac{1}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a}) = \frac{1}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a})$$

Note:

$$a_{12} = \frac{V_1}{Q_2} = \frac{Q_2}{4\pi\epsilon Q_2}(\frac{1}{b} - \frac{1}{a}) = \frac{1}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a})$$

(Same answer with two approaches)

Additionally, the system is geometric so $a_{11} = a22$ and $a_{12} = a_{21}$.

Part B). Find the charge of the sphere

Using the equation from part a, we know that

$$a_{11} = \frac{V_1}{Q_1} = \frac{Q_1}{4\pi\epsilon Q_1}(\frac{1}{a} - \frac{1}{b})$$

We can solve for the charge of the sphere Q_1 by dividing a_{11} from V_1. In the picture provided, the sphere is connected to ground with $V = 0$. This would result in the charge on the sphere to be zero as well ($Q_1 = 0$).

$$Q_1 = \frac{V_1}{a_{11}} = \frac{0}{a_{11}} = 0$$

Part C). Find the potential of the shell

To solve for the potential (V) on the shell, we can use the equation for a_{11} which utilizes the newly known charge on the sphere to solve for V_2.

$$a_{21} = \frac{V_2}{Q_1}$$

$$V_2 = a_{21}Q_1 = a_{21} \times 0 = 0$$

Problem 3

We can also write

$$Q_1 = b_{11}V_1 + b_{12}V_2 + ... + b_{1N}V_N$$
$$Q_2 = b_{21}V_1 + b_{22}V_2 + ... + b_{2N}V_N$$
$$\vdots$$
$$Q_N = b_{N1}V_1 + b_{N2}V_2 + ... + b_{NN}V_N$$

or

$$[Q] = [b][V]$$

where the constants b_{ij} $(i, j = 1, 2, \ldots, N)$ are termed the coefficients of electrostatic induction, and the unit is F. Prove that the self coefficients of electrostatic induction (the coefficients with equal indices, $i = j$) are positive, while the mutual coefficients are all negative.

Solution

Building off of the previous problem, and considering that a and b are directly related, we can look at the derived equations for the potential coefficients.

$$a_{11} = \frac{1}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})$$

$$a_{12} = \frac{1}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a})$$

$$a_{22} = \frac{1}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})$$

$$a_{21} = \frac{1}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a})$$

In the context of the previous problem, b is always $> a$, resulting in $\frac{1}{b} < \frac{1}{a}$. Consequently, in all cases where $i \neq j$ the equation for a will be $\frac{1}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a})$ resulting in a negative value. Vise versa, in all cases where $i = j$, a will be equal to $\frac{1}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})$, which will result in a positive value.
With $b = a^{-1}$, whatever sign a has, b will also have. Therefore, for any situation where $i \neq j,\ b < 0$ and for any situation where $i = j,\ b > 0$.

Question 4

In order to avoid dealing with negative coefficients of electrostatic induction, we can rearrange the system of equations to get the system with so-called partial capacitances:

$$Q_1 = C_{11}V_1 + C_{12}(V_1 - V_2) + \cdots + C_{1N}(V_1 - V_N)$$

$$Q_2 = C_{21}V_1 + C_{22}(V_2) + \cdots + C_{1N}(V_2 - V_N)$$

$$\vdots$$

$$Q_N = C_{N1}(V_N - V_1) + C_{N2}(V_N - V_2) + \cdots + C_{NN}(V_N)$$

Note that at the right-hand side of the new equations are voltages between the bodies of the system. Find the relationships between coefficients of electrostatic induction and partial capacitances, bs in terms of Cs, and vice versa, Cs in terms of bs.

Solution

We can generalize the equations given in the problem to be

$$Q_i = C_{i1}(V_i - V_1) + C_{i2}(V_i - V_2) + \cdots + C_{ii}V_i + \cdots + C_{ij}(V_i - V_j) + \cdots + C_{iN}(V_i - V_N)$$

Knowing that b_{ij} is the inverse of a_{ij}, we can derive the equation

$$[b] = [a]^{-1}$$

$$b_{ij} = \frac{Q_i}{V_j}\Big|_{V_k=0, k\neq j}$$

Taking the term for Q_i and implementing this into the equation for b_{ij}, we can derive

$$b_{ij} = \frac{C_{i1}(V_i - V_1) + C_{i2}(V_i - V_2) + \cdots + C_{ii}V_i + \cdots + C_{ij}(V_i - V_j) + \cdots + C_{iN}(V_i - V_N)}{V_j}\Big|_{V_k=0, k\neq j}$$

To write this out in a way that is easier to understand

$$b_{ij} = \frac{C_{ij}(V_i - V_j)}{V_j}\Big|_{V_k=0, k\neq j}$$

for j = 1, 2, . . . , N This allows the derivation of

$$b_{ii} = \sum_{j=1}^{N} C_{ij}$$

because for any instance where $j = i$, the difference in voltage $V_i - V_j = V_i$; $V_j = V_i$, causing the dependency on voltage to disappear.
Using this same concept, we can switch b_{ij} and C's position to derive

$$C_{ii} = \sum_{j=1}^{N} b_{ij}$$

When $i \neq j$, from the previous problem, we know that b_{ij} will be negative. Therefore, for the general case

$$b_{ij} = -C_{ij}$$

Question 5

Let us observe the equations defining partial capacitances from another physical point of view. Note that each of these equations actually represents the generalized Gauss' law applied to a closed surface positioned about that particular conducting body. The charge of the body is on the left-hand side of the equation, while the sum on the right-hand side of the equation can be interpreted as the net outward flux of the electric flux density vector, **D**, through the surface. The individual terms in the sum are the parts of the flux showing how the flux is distributed towards other conducting bodies in the system. Each term represents a flux tube containing the lines of vector **D** that start at a part of the surface of one body and end at a part of the surface of another body. Using this reasoning, prove that

$$C_{ij} = C_{ji}.$$

Then prove that

$$b_{ij} = b_{ji}.$$

Finally, prove that

$$a_{ij} = a_{ji}$$

for all linear systems.

Solution

We can start the solution for $C_{ij} = C_{ji}$ we should start with the general formula for capacitance

$$C = \frac{Q}{V}$$

I am going to use the theory from question 3 to prove this.

$$E = \frac{Q}{4\pi\epsilon r^2}$$

$$V = \int_a^b E(r)dr = \frac{Q}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})$$

$$C_{ij} = \frac{Q_i}{\frac{Q_j}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})}$$

From problem 3, I am going to assume that $Q_1 = -Q_2$. Using this, the equation for C_{ij} evaluates to

$$C_{12} = \frac{Q_1}{\frac{Q_2}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})}$$

Using that $\frac{Q_1}{Q_2} = -1$, this simplifies to

$$C_{12} = 4\pi\varepsilon\frac{ab}{a-b}$$

We can use this to illustrate that $C_{ij} = C_{ji}$ by calculating

$$C_{21} = \frac{Q_2}{\frac{Q_1}{4\pi\epsilon}(\frac{1}{a} - \frac{1}{b})}$$

Using the same identity as for C_{12} , this simplifies to

$$C_{21} = 4\pi\varepsilon\frac{ab}{a-b}$$

Therefore, $C_{ij} = C_{ji}$ and this can be applied to all cases.

We can then use this same principle to solve that $b_{ij} = b_{ji}$. From question 4, we know that

$$b_{ij} = -C_{ij}$$

and since this is a linear operation, this instantly proves that

$$b_{ij} = b_{ji}$$

Finally, we need to calculate the $a_{ij} = a_{ji}$ which can be solved using the exact same method. Looking back at problem 3, we already know that

$$a_{12} = \frac{1}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a})$$

and

$$a_{21} = \frac{1}{4\pi\epsilon}(\frac{1}{b} - \frac{1}{a})$$

which are identical and prove that $a_{ij} = a_{ji}$.

Question 6

Shown in the figure below is a detail of a triaxial cable, which consists of three coaxial cylindrical conductors, with radii $a = 1$mm, $b = 2$mm, $c = 2.5$mm, $d = 5$mm, and $e = 5.5$mm. The dielectric inside the cable is homogeneous, of relative permittivity $\epsilon_r = 4$. Designating the inner conductor of the cable by 1 and the middle conductor by 2, and taking the outer conductor as a reference conductor, find (a) the potential coefficients, (b) the coefficients of electrostatic induction, and (c) the partial capacitances per unit length of the cable. (d) If the potentials of the inner conductor and the middle conductor with respect to the outer conductor are $V_1 = 4V$ and $V_2 = 3V$, respectively, find the charges per unit length of the inner and middle conductor, Q'_1 and Q'_2

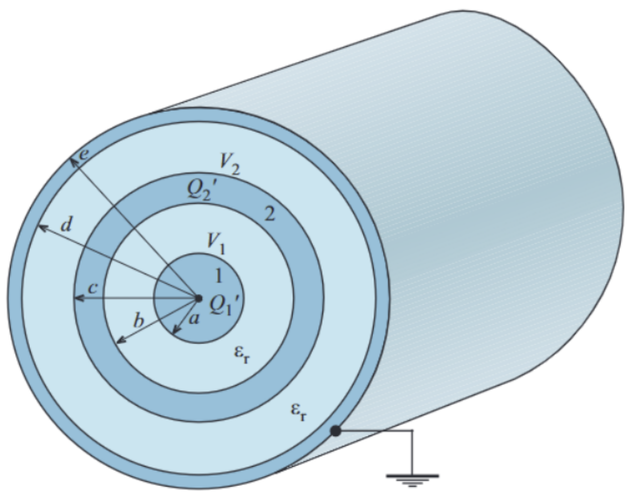

Figure 3: Triaxial cable

Solution

Part A). Find the potential coefficients

Starting with the equation for electric field

$$E(r) = \frac{Q'}{2\pi\epsilon r}$$

Integrating this with respect to the dielectrics we are able to obtain the voltage at a point

$$V = \int_a^b E(r)dr = \int_a^b \frac{Q'}{2\pi\epsilon r}dr = \frac{Q'}{2\pi\epsilon}\int_a^b \frac{1}{r}dr$$

This evaluates to

$$V = \frac{Q'}{2\pi\epsilon} ln(\frac{b}{a})$$

However, this is for a single dielectric coax cable. We can rearrange this equation to be applicable to our triaxial cable.

$$V_1 = \frac{Q_1'}{2\pi\epsilon}[\int_a^b \frac{1}{r}dr + \int_c^d \frac{1}{r}dr] = \frac{Q_1'}{2\pi\epsilon}[ln(\frac{b}{a}) + ln(\frac{d}{c})]$$

The electric field propagates radially from the center. From the center conductor, the electric field passes through two dielectrics which is why we integrate over two integrals to find V_1. From the central conductor, the electric field only passes through one dielectric. This changes the integral for V_2 to only include one integral.

$$V_2 = \frac{Q_2'}{2\pi\epsilon} \int_c^d \frac{1}{r}dr = \frac{Q_2'}{2\pi\epsilon} ln(\frac{d}{c})$$

Using the equations for the potential coefficients and assuming that $Q_1' = Q_2'$

$$a_{11} = \frac{V_1}{Q_1'} = \frac{ln(\frac{b}{a}) + ln(\frac{d}{c})}{2\pi\epsilon} = 6.2298 \times 10^9$$

$$a_{21} = \frac{V_2}{Q_1'} = \frac{ln(\frac{d}{c})}{2\pi\epsilon} = a_{12} = 3.1149 \times 10^9$$

$$a_{22} = \frac{V_2}{Q_2'} = \frac{ln(\frac{d}{c})}{2\pi\epsilon} = 3.1149 \times 10^9$$

Part B). Find the coefficients of electrostatic induction

From the previous problems, we know that

$$[a] = \begin{bmatrix} 6.2298 \times 10^9 & 3.1149 \times 10^9 \\ 3.1149 \times 10^9 & 3.1149 \times 10^9 \end{bmatrix}$$

$$[a] = [b]^{-1}$$

$$[b] = [a]^{-1} = \begin{bmatrix} 3.2104 \times 10^{-10} & -3.2104 \times 10^{-10} \\ -3.2104 \times 10^{-10} & 6.4207 \times 10^{-10} \end{bmatrix}$$

Part C). Find the partial capacitances per unit length of the cable

From previous problems, we know that

$$C_{ij} = -b_{ij}$$

and when $i = j$

$$C_{ii} = \sum_{j=1}^{N} b_{ij}$$

this results in

$$[c] = \begin{bmatrix} 0 & 3.2104 \times 10^{-10} \\ 3.2104 \times 10^{-10} & 3.2104 \times 10^{-10} \end{bmatrix}$$

Part D). If the potentials of the inner conductor and the middle conductor with respect to the outer conductor are $V_1 = 4V$ and $V_2 = 3V$, respectively, find the charges per unit length of the inner and middle conductor, Q'_1 and Q'_2.

From earlier, we know that

$$Q'_1 = b_{11}V_1 + b_{12}V_2 = 3.2104 \times 10^{-10}C$$

$$Q'_2 = b_{21}V_1 + b_{22}V_2 = 6.4206 \times 10^{-10}C$$

Question 7

Energy of a two-body system ($N = 1$) can be expressed in terms of the charges and potentials of individual bodies as

$$W_e = \frac{1}{2}QV = \frac{1}{2}Q(V_1 - V_2) = \frac{1}{2}(Q_1V_1 + Q_2V_2)$$

Generally, for a linear multibody system with $N + 1$ charged conducting bodies, where the potential of the $(N + 1)$th body (reference body) is zero,

$$W_e = \frac{1}{2}\sum_{i=1}^{N} Q_iV_i \quad \text{(multibody system)}$$

By employing the coefficients of electrostatic induction, find the energy per unit length of the triaxial cable defined in the previous problem, if the potentials of the inner conductor and the middle conductor with respect to the outer conductor are $V_1 = 3\text{V}$ and $V_2 = 1\text{V}$, respectively.

Solution

Using the information already acquired in problem 6

$$[b] = \begin{bmatrix} 3.2104 \times 10^{-10} & -3.2104 \times 10^{-10} \\ -3.2104 \times 10^{-10} & 6.4207 \times 10^{-10} \end{bmatrix}$$

Using this matrix [b], we can then find the charge of the system. Again using equations for problem 6

$$Q'_1 = b_{11}V_1 + b_{12}V_2$$

$$Q_2' = b_{21}V_1 + b_{22}V_2$$

Applying the given values for V_1 and V_2 we can derive a value for Q_1 and Q_2

$$Q_1' = 6.4207 \times 10^{-10}$$

$$Q_2' = -3.2104 \times 10^{-10}$$

From here we can apply the equation given in the problem

$$W_e = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i$$

$$W_e = \frac{1}{2} \times \left(Q_1'V_1 + Q_2'V_2 \right) = 8.0259 \times 10^{-10}$$

Problem 8

Shown in the figure below is a system consisting of three concentric spherical conductors (the inner conductor is a solid sphere, while the remaining two are spherical shells). The radius of the inner conductor is $a = 1$mm. The inner radius of the middle conductor is $b = 2$mm, and outer $c = 2.5$mm. The inner radius of the outer conductor is $d = 5$mm. The space between the conductors is air-filled. The outer conductor is grounded, and the potentials of the inner and middle conductors with respect to the ground are $V_1 = 15$V and $V_2 = 10$V, respectively. Label the inner conductor as 1 and the middle conductor as 2, and take the outer conductor as a reference conductor. With this notation, find (a) the potential coefficients, (b) the coefficients of electrostatic induction, and (c) the partial capacitances of the system. (d) Then find the charges Q_1 and Q_2 for given potentials $V_1 = 15$V and $V_2 = 10$V, using the coefficients of electrostatic induction found in (b). (e) Finally, find the energy of the system of three conductors (for $V_1 = 15$V and $V_2 = 10$V).

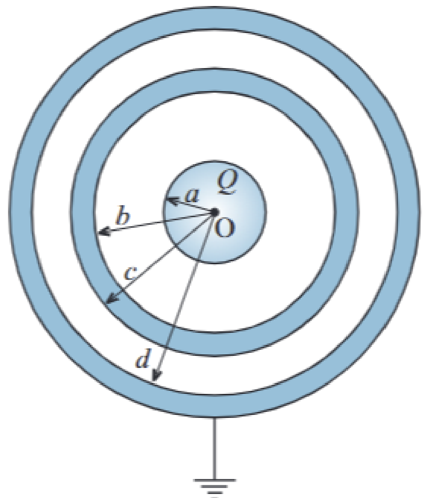

Figure 4: Enter Caption

Solution

Part A). Find the potential coefficients

To begin solving for the potential coefficients, we need to first find the correct equation for the electric field of a spherical capacitor

$$E(r) = \frac{Q}{4\pi\epsilon r^2} \quad (a < r < b)$$

Using this we will want to find the voltage of the capacitor

$$V = \int_a^b E(r)dr = \int_a^b \frac{Q}{4\pi\epsilon r^2}dr = \frac{Q}{4\pi\epsilon}\Big(\frac{1}{a} - \frac{1}{b}\Big)$$

Using the same theory as for in question 6, the expression for the voltages of the inner and middle conductor can be found. Note: even though the voltages are given, the expression must be derived to find the potential coefficients.

$$V_1 = \frac{Q_1}{4\pi\epsilon}\left[\int_a^b \frac{1}{r^2}dr + \int_c^d \frac{1}{r^2}dr\right] = \frac{Q_1}{4\pi\epsilon}\left[\Big(\frac{1}{a} - \frac{1}{b}\Big) + \Big(\frac{1}{c} - \frac{1}{d}\Big)\right]$$

$$V_2 = \frac{Q_2}{4\pi\epsilon}\int_c^d \frac{1}{r^2}dr = \frac{Q_2}{4\pi\epsilon}\Big(\frac{1}{c} - \frac{1}{d}\Big)$$

Again, for this derivation we can assume that $Q_1 = Q_2$. Using this, we can derive formulas for the potential coefficients.

$$a_{ij} = \frac{V_i}{Q_j}\bigg|_{Q_k=0, k\neq j}$$

$$a_{11} = \frac{V_1}{Q_1} = \frac{\frac{Q_1}{4\pi\epsilon}[(\frac{1}{a}-\frac{1}{b})+(\frac{1}{c}-\frac{1}{d})]}{Q_1} = \frac{1}{4\pi\epsilon}\Big[(\frac{1}{a}-\frac{1}{b})+(\frac{1}{c}-\frac{1}{d})\Big]$$

$$a_{22} = \frac{V_2}{Q_2} = \frac{\frac{Q_2}{4\pi\epsilon}[\frac{1}{c}-\frac{1}{d}]}{Q_2} = \frac{1}{4\pi\epsilon}\Big[\frac{1}{c}-\frac{1}{d}\Big]$$

$$a_{21} = \frac{V_2}{Q_1} = \frac{\frac{Q_2}{4\pi\epsilon}[\frac{1}{c}-\frac{1}{d}]}{Q_1} = \frac{1}{4\pi\epsilon}\Big[\frac{1}{c}-\frac{1}{d}\Big] = a_{12}$$

$$[a] = \begin{bmatrix} 6.2914\times 10^{12} & 1.7975\times 10^{12} \\ 1.7975\times 10^{12} & 1.7975\times 10^{12} \end{bmatrix}$$

Part B). Find the coefficients of electrostatic induction

Using the matrix we found for in part a, we can easily find the b matrix [b]

$$[a] = [b]^{-1}$$

$$[b] = [a]^{-1} = \begin{bmatrix} 2.2253\times 10^{-13} & -2.2253\times 10^{-13} \\ -2.2253\times 10^{-13} & 7.7884\times 10^{-13} \end{bmatrix}$$

Part C). Find the partial capacitances of the system

From previous problems, we know that

$$C_{ij} = -b_{ij}$$

and when $i = j$

$$C_{ii} = \sum_{j=1}^{N} b_{ij}$$

this results in

$$[c] = \begin{bmatrix} 0 & 2.2253\times 10^{-13} \\ 2.2253\times 10^{-13} & 5.5631\times 10^{-13} \end{bmatrix}$$

Part D). Find the charges Q_1 and Q_2 for given potentials $V_1 = 15$V and $V_2 = 10$V, using the coefficients of electrostatic induction found in (b).

$$Q_1 = b_{11}V_1 + b_{12}V_2 = 1.1126\times 10^{-12}C$$

$$Q_2 = b_{21}V_1 + b_{22}V_2 = 4.4505\times 10^{-12}C$$

Part E). Find the energy of the system of the three conductors (for $V_1 = 15$V and $V_2 = 10$V).

$$W_e = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i$$

$$W_e = \frac{1}{2}\times\Bigg(Q_1V_1 + Q_2V_2\Bigg) = 3.0597\times 10^{-11}$$

Question 9

Consider four parallel large metallic electrodes situated in a vacuum, as shown in the figure below. The thickness of each electrode, as well as the distance between each two adjacent electrodes, is $d = 2$ cm. The surface area of sides of electrodes facing each other is $S = 1$ m^2. Fringing effects can be neglected. Taking the fourth electrode as a reference conductor, find (a) the potential coefficients, (b) the coefficients of electrostatic induction, and (c) the partial capacitances of this system. (d) If the first and the fourth electrode are grounded, the charge of the second electrode is Q = 10 µC, and the potential of the third electrode with respect to the ground is V = 10 kV, find the electric field intensity between the electrodes. (e) Find the electric energy stored in the system of four electrodes using the coefficients of electrostatic induction.

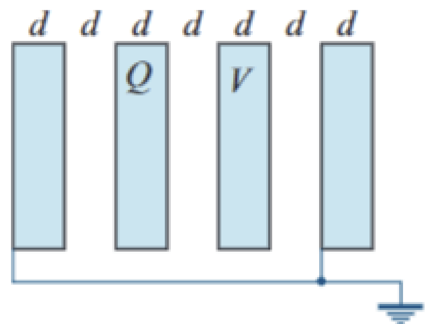

Figure 5: Four parallel large metallic electrodes situated in a vacuum

Part A). Find the potential coefficients

For a parallel plate capacitor

$$E = \frac{Q}{\epsilon S}$$

$$V = Ed = \frac{Qd}{\epsilon S}$$

We can use this to solve for the potential coefficients

$$a_{ij} = \frac{V_i}{Q_j}\bigg|_{Q_k=0, k\neq j}$$

Assuming that all charges are equal $Q_1 = Q_2 = Q_3 = Q_4$, we can solve for the potential coefficients.

$$a_{11} = \frac{V_1}{Q_1} = \frac{3d}{\epsilon s}$$

$$a_{12} = \frac{V_1}{Q_2} = \frac{2d}{\epsilon s} = a_{21}$$

$$a_{13} = \frac{V_1}{Q_3} = \frac{d}{\epsilon s} = a_{31}$$

$$a_{22} = \frac{V_2}{Q_2} = \frac{2d}{\epsilon s}$$

$$a_{23} = \frac{V_2}{Q_3} = \frac{d}{\epsilon s}$$

$$a_{33} = \frac{V_3}{Q_3} = \frac{d}{\epsilon s}$$

$$[a] = \frac{d}{\epsilon_0 S}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6.7766 \times 10^9 & 2.2589 \times 10^9 & 4.5177 \times 10^9 \\ 2.2589 \times 10^9 & 4.5177 \times 10^9 & 2.2589 \times 10^9 \\ 4.5177 \times 10^9 & 2.2589 \times 10^9 & 2.2589 \times 10^9 \end{bmatrix}$$

Part B). Find the coefficients of electrostatic induction

$$[a] = [b]^{-1}$$

$$[b] = [a]^{-1} = \begin{bmatrix} 4.4270 \times 10^{-10} & -4.4270 \times 10^{-10} & 0 \\ -4.4270 \times 10^{-10} & 8.88540 \times 10^{-10} & -4.4270 \times 10^{-10} \\ 0 & -4.4270 \times 10^{-10} & 8.88540 \times 10^{-10} \end{bmatrix}$$

Part C). Find the partial capacitances of this system

From previous problems, we know that

$$C_{ij} = -b_{ij}$$

and when $i = j$

$$C_{ii} = \sum_{j=1}^{N} b_{ij}$$

this results in

$$[c] = \begin{bmatrix} 0 & 4.4270 \times 10^{-10} & 0 \\ 4.4270 \times 10^{-10} & 5.1699 \times 10^{-23} & 4.4270 \times 10^{-10} \\ 0 & 4.4270 \times 10^{-10} & 4.4270 \times 10^{-10} \end{bmatrix}$$

Part D). If the first and the fourth electrodes are grounded, the charge of the second electrode is $Q = 10\ \mu$C, and the potentials of the third electrode with respect to the ground is $V = 10$kV, find the electric field intensity between the electrodes.

From earlier in the problem we identified the electric field of a parallel plate capacitor to be

$$E = \frac{Q}{\epsilon S}$$

We can use this to solve for the electric fields. The total charge of the system is Q so there will be no other Q values. Additionally, the electric field on the third electrode should be calculated from the equation

$$E = \frac{V}{d}$$

Part E). Find the electric energy stored in the system of four electrodes using the coefficients of electrostatic induction.

Question 10

Shown in the figure below is a multiconductor transmission line composed of N straight wire conductors parallel to a reference conducting plane (ground plane). The medium above the plane is a homogeneous dielectric of permittivity ϵ. Derive the approximate expressions for the coefficients of potential of the line assuming that the wires are thin, i.e., that all wire radii are small compared to the distances between the conductors and their heights with respect to the ground plane,

$$a_{jj} = \frac{1}{2\pi\epsilon} ln \frac{2h_j}{a_j}, \quad a_{ij} = \frac{1}{2\pi\epsilon} ln \frac{D_{ij}}{d_i j} \ (i \neq j),$$

where a_j is the radius of the jth conductor and h_j is its height with respect to the ground plane, d_{ij} is the distance between the axes of the ith and jth conductors, and D_{ij} is the distance between the axes of the ith conductor and the image of the jth conductor. These expressions can, in turn, be used in many practical applications.

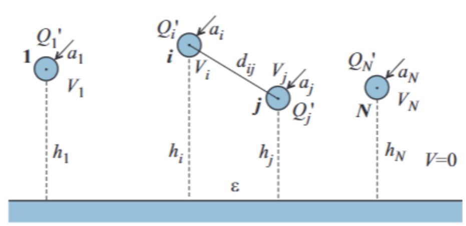

Figure 6: multiconductor transmission line

Solution

To start this problem, we will want to use this equation for voltage

$$V = \frac{Q'}{2\pi\epsilon} ln \frac{r_2}{r_1}$$

Using this we can solve for the self potential coefficient

$$a_{jj} = \frac{V_j}{Q'_j} = \frac{1}{2\pi\epsilon} ln \frac{2h_j}{a_j}$$

where we use image theorem to acquire the $2h_j$ However, this expression changes when it is comparing two different capacitors. We can again use the image theorem again to solve for this. Finding the distance between the reference's image and the other capacitor can be found using the equation:

$$D_{ij} = \sqrt{d_{ij}^2 + 4h_j^2}$$

Apply this equation to our potential coefficient equation

$$a_{ij} = \frac{V_i}{Q'_j} = \frac{1}{2\pi\epsilon} ln \frac{D_{ij}}{d_{ij}}$$

Problem 11

Specially for $N = 2$ (two-wire line above a conducting plane) and using the notation given in figure (a) below, find the potential coefficients of the system, the coefficients of electrostatic induction, and the partial capacitances per unit length of the system. Then, based on the equivalent circuit scheme shown in figure (b), find the total capacitance between the line conductors in the presence of the conducting plane, C', per unit length of the line. What does C' amount to for the symmetrical two-wire line ($a = b$) with the conductors at the same height above the plane ($H = h$)? What is the expression for C' in the case when $h \gg d$.

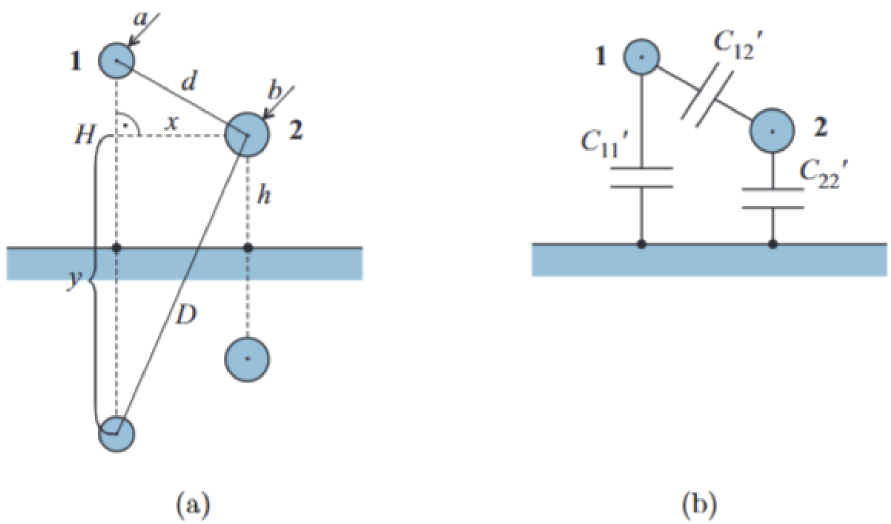

Figure 7: two-wire line above a conducting plane

Solution

Part A). Find the potential coefficients of the system, the coefficients of electrostatic induction, and the partial capacitances per unit length of the system.

This follows a similar thought process as the previous problem when solving for the potential coefficients. First we will want to identify the relationship for voltage between two wires.

$$V = \frac{Q'}{2\pi\epsilon} ln\frac{r_2}{r_1}$$

Using the equation for potential coefficients

$$a_{ij} = \frac{V_i}{Q_j}$$

This allows us to find the potential coefficients. The only difference between the coefficients is the distance between the reference's image.

$$a_{11} = \frac{\frac{Q_1'}{2\pi\epsilon} ln\frac{2H}{a}}{Q_1'} = \frac{1}{2\pi\epsilon} ln\frac{2H}{a}$$

where a is the radius of the Conductor 1 and $2H$ is the distance between it and its image. The self coefficient for body 2 is very similar.

$$a_{22} = \frac{1}{2\pi\epsilon} ln\frac{2h}{b}$$

where b is the radius of conductor 2 and $2h$ is the distance between it and its image. The only difference between these is the fact that they are not on an even plane, meaning that their heights and radii are not the same.
When solving for the potential coefficients with respect to one another, we can use the equation from the previous problem.

$$a_{ij} = \frac{V_i}{Q'_j} = \frac{1}{2\pi\epsilon} ln \frac{D_{ij}}{d_{ij}}$$

where d_{ij} is the distance between the two conductors and Dij is the distance between the reference's image and the compared conductor.

$$D_{ij} = \sqrt{d_{ij}^2 + 4h_j^2}$$

This means

$$a_{12} = \frac{V_1}{Q'_2} = \frac{1}{2\pi\epsilon} ln \frac{D_{12}}{d_{12}}$$

Using the law of potential coefficients, we know that $a_{12} = a_{21}$, resulting in

$$a_{21} = \frac{1}{2\pi\epsilon} ln \frac{D_{12}}{d_{12}}$$

From here, we can create an a matrix

$$[a] = \begin{bmatrix} \frac{1}{2\pi\epsilon} ln \frac{2H}{a} & \frac{1}{2\pi\epsilon} ln \frac{D_{12}}{d_{12}} \\ \frac{1}{2\pi\epsilon} ln \frac{D_{12}}{d_{12}} & \frac{1}{2\pi\epsilon} ln \frac{2h}{b} \end{bmatrix}$$

Find the coefficients of electrostatic induction, we can take the inverse equation for a_{ij} which looks like

$$b_{ij} = \frac{Q'_i}{V_j}$$

This gives us

$$b_{11} = \frac{Q'_1}{\frac{Q'_1}{2\pi\epsilon} ln \frac{2H}{a}} = \frac{2\pi\epsilon}{ln \frac{2H}{a}}$$

$$b_{22} = \frac{Q'_2}{\frac{Q'_2}{2\pi\epsilon} ln \frac{2h}{b}} = \frac{2\pi\epsilon}{ln \frac{2h}{b}}$$

$$b_{12} = b_{21} = \frac{Q'_1}{\frac{Q'_1}{2\pi\epsilon} ln \frac{D}{d}} = \frac{2\pi\epsilon}{ln \frac{D}{d}}$$

$$[b] = \begin{bmatrix} \frac{2\pi\epsilon}{ln \frac{2H}{a}} & \frac{2\pi\epsilon}{ln \frac{D}{d}} \\ \frac{2\pi\epsilon}{ln \frac{D}{d}} & \frac{2\pi\epsilon}{ln \frac{2h}{b}} \end{bmatrix}$$

Find the partial capacitances of the system From previous problems, we know that

$$C_{ij} = -b_{ij}$$

and when $i = j$

$$C_{ii} = \sum_{j=1}^{N} b_{ij}$$

this results in

$$[c] = \begin{bmatrix} \frac{2\pi\epsilon}{ln\frac{D}{d}} + \frac{2\pi\epsilon}{ln\frac{2H}{a}} & -\frac{2\pi\epsilon}{ln\frac{D}{d}} \\ -\frac{2\pi\epsilon}{ln\frac{D}{d}} & \frac{2\pi\epsilon}{ln\frac{2h}{b}} + -\frac{2\pi\epsilon}{ln\frac{D}{d}} \end{bmatrix}$$

Part B). Find the total capacitance between the line conductors in the presence of the conducting plane, C', per unit length of the line. What does C' amount to for the symmetrical two-wire line ($a = b$) with the conductors at the same height above the plane ($H = h$)? What is the expression for C' in the case when $h \gg d$.

Total capacitance is given by

$$C' = C'_{12} + C'_{11}||C'_{22} = C'_{12} + \frac{C'_{11}C'_{22}}{C'_{11} + C'_{22}}$$

$$C' = \frac{(\frac{2\pi\epsilon}{ln\frac{D}{d}} + \frac{2\pi\epsilon}{ln\frac{2H}{a}})(\frac{2\pi\epsilon}{ln\frac{2h}{b}} + -\frac{2\pi\epsilon}{ln\frac{D}{d}})}{\frac{2\pi\epsilon}{ln\frac{D}{d}} + \frac{2\pi\epsilon}{ln\frac{2H}{a}} + \frac{2\pi\epsilon}{ln\frac{2h}{b}} + -\frac{2\pi\epsilon}{ln\frac{D}{d}}} - \frac{2\pi\epsilon}{ln\frac{D}{d}}$$

If $a = b$ and $h = H$ then the expression simplifies because $a_{22} = a_{11}$ making the expression for the total capacitance

$$C' = \frac{1}{2}(\frac{2\pi\epsilon}{ln\frac{D}{d}} + \frac{2\pi\epsilon}{ln\frac{2h}{a}}) - \frac{2\pi\epsilon}{ln\frac{D}{d}}$$

Finally, if $h \gg d$, then the previous expression becomes

$$C' = \frac{1}{2}(\frac{2\pi\epsilon}{ln\frac{D}{d}} + \frac{2\pi\epsilon}{ln\frac{2h}{a}}) - \frac{2\pi\epsilon}{ln\frac{D}{d}} \quad \text{I honestly don't know}$$

Question 12

A symmetrical two-wire line is positioned in air over a ground plane, as shown in the figure below. The wire radii are $a = 1$ cm, the height of the axes of both wires with respect to the plane is $h = 4$ m, and the distance between the wire axes is $d = 1$ m. One of the wires is grounded. (a) Find the capacitance per unit length of this system. (b) Find the breakdown voltage of the system. (c) Calculate the maximum energy that can be contained in this system per unit of its length

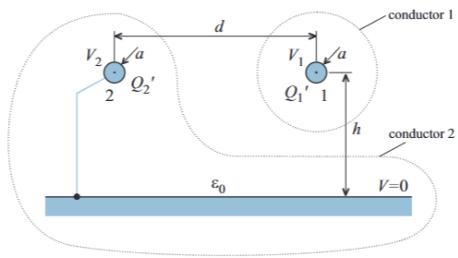

Figure 8: A symmetrical two-wire line positioned in air aver a ground plane

Solution

Part A).Find the capacitance per unit length of this system.

Using the same process we used in the previous problem, we can find the voltage

$$V = \frac{Q'}{2\pi\epsilon} ln \frac{r_2}{r_1}$$

where r_2 is is the distance from the compared conductor the reference conductor's image and r_1 is either the radius of the conductor (when compared to ground) or the distance between conductors.

$$a_{11} = a_{22} = \frac{Q_1'}{V_1} = \frac{1}{2\pi\epsilon} ln \frac{2h}{a}$$

$$a_{12} = a_{21} = \frac{Q_1'}{V_2} = \frac{1}{2\pi\epsilon} ln \frac{D}{d}$$

This will be important for later.

For now, we need to solve to get Q_2' in terms of Q_1'. We can do this by seeing that the second conductor is attached to ground resulting in $V_2 = 0$. The equation for V_1 and V_2 are

$$V_1 = a_{11}Q_1' + a_{12}Q_2'$$

$$V_2 = a_{21}Q_1' + a_{22}Q_2' = 0$$

Rearranging the equation for V_2 we can get Q_2' in terms of Q_1'.

$$Q_2' = -\frac{a_{21}}{a_{22}} Q_1'$$

Plugging this into the equation for V_1 we are able to derive

$$V_1 = (a_{11} - \frac{a_{12}a_{21}}{a_2 2}) Q_1'$$

The equation for capacitance is given as

$$C' = \frac{Q'}{V} = \frac{Q'_1}{V_1} = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{a_{22}}{2a_{11} - 2a_{12}} = \frac{1.2016 \times 10^{11}}{1.3031 \times 10^{22}} = 9.2213 \times 10^{-12} \text{F/m}$$

Part B).Find the breakdown voltage of the system.

The critical electric field in air has a given value of $3\,\frac{\text{MV}}{m}$. Using the equation

$$E_{cr0} = \frac{Q'_1}{2\pi\epsilon_0 a}$$

we can solve for the critical charge $(Q'_1)_{cr}$ which comes out to be 1.6689×10^{-6}C. We can then solve for the critical voltage by plugging this value and the value for our capacitance into the following equation, resulting in a value for breakdown voltage.

$$\frac{(Q'_1)_{cr}}{C'} = (V_1)_{cr} = 180.99 \text{ kV}$$

Part C).Calculate the maximum energy that can be contained in this system per unit of its length.

We can find the maximum energy of the system through the equation

$$w'_e = \frac{1}{2}C'V_1^2$$

where V_1 is the breakdown voltage because we are finding the maximum energy. This gives us that

$$w'_e = 0.1510 \text{ J/m}$$

Question 13

Shown in the figure below is the cross section of a system composed of a two-wire line whose conductors are galvanically connected to each other and a conducting plane. The line is in a vertical plane, the wire radii are $a = 1$ cm, the heights of the wire axes with respect to the plane are $H = 7$ m and $h = 3$ m, and the medium is air. Find the capacitance per unit length of the system.

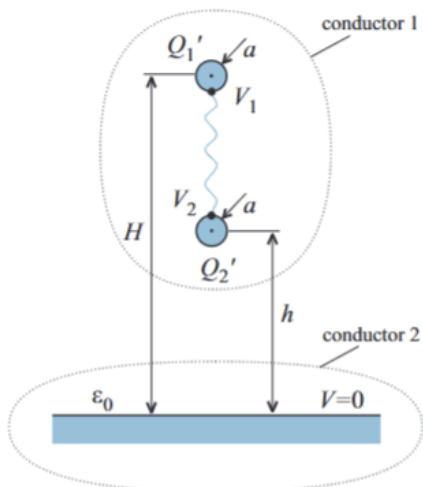

Figure 9: two-wire line whose conductors are galvanically connected to each other and a conducting plane

Solution

Using a similar theory to the one used in the previous problem we can reuse the voltage equation found in part a.

$$V = \frac{Q'}{2\pi\epsilon} ln \frac{r_2}{r_1}$$

where r_2 is is the distance from the compared conductor the reference conductor's image and r_1 is either the radius of the conductor (when compared to ground) or the distance between conductors. Hypothetically, we should be able to use image theorem here and do so in a nearly identical sense to the previous problem. We will start this solution by assuming $Q_1 = Q_2$ to derive the potential coefficients.

$$a_{ij} = \frac{V_1}{Q_1}$$

$$a_{11} = \frac{1}{2\pi\epsilon} ln \frac{2H}{a} = 1.3022 \times 10^{11}$$

$$a_{12} = a_{21} = \frac{1}{2\pi\epsilon} ln \frac{H+h}{H-h} = 1.6471 \times 10^{10}$$

$$a_{22} = \frac{1}{2\pi\epsilon} ln \frac{2h}{a} = 1.1499 \times 10^{11}$$

The constraints of this problem are that $V_1 = V_2$. Therefore, to solve for the capacitance, we will want the equation

$$C' = \frac{Q_1' + Q_2'}{V_1}$$

This leaves us with the issue of finding the values for Q_1' and Q_2' This can be done using the constraints given and setting the equations for V_1 and V_2 equal to each other. Those equations would be

$$V_1 = a_{11}Q_1' + a_{12}Q_2'$$

$$V_2 = a_{21}Q_1' + a_{22}Q_2'$$

$$a_{21}Q_1' + a_{22}Q_2' = a_{11}Q_1' + a_{12}Q_2'$$

$$(a_{11} - a_{21})Q_1' = (a_{22} - a_{12})Q_2'$$

$$1.1375 \times 10^{11} \times Q_1' = 9.8517 \times 10^{10} \times Q_2'$$

$$Q_1' = 1.1546Q_2'$$

$$Q_2' = 0.8661Q_1'$$

We can substitute these in for the V_1 equation.

$$V_1 = a_{11} * 1.1546Q_2' + a_{12}Q_2'$$

$$V_1 = a_{11}Q_1' + a_{12}0.8661Q_1'$$

This gives two equation

$$\frac{V_1}{Q_1} = a_{11} + a_{12}0.8661$$

$$\frac{V_1}{Q_2} = a_{11} * 1.1546 + a_{12}$$

The equation for the capacitance per unit length of the system is

$$C' = \frac{Q_1' + Q_2'}{V_1}$$

so we are going to want to take the inverse of these equations

$$\frac{Q_1'}{V_1} = \frac{1}{a_{11} + a_{12}0.8661}$$

$$\frac{Q_2'}{V_1} = \frac{1}{a_{11} * 1.1546 + a_{12}}$$

This means that

$$C' = \frac{Q_1' + Q_2'}{V_1} = \frac{1}{a_{11} + a_{12}0.8661} + \frac{1}{a_{11} * 1.1546 + a_{12}} = 1.2916 \times 10^{-11} \text{ F/m}$$

Question 14

A bus in a computer, in the form of a four-conductor microstrip transmission line, can be approximated by four horizontal wire conductors in a homogeneous dielectric over a conducting plane, as shown in the figure below. The wire radii are $a = 0.1$ mm, the height of conductor axes with respect to the plane is $h = 1$ mm, the distance between axes of adjacent conductors is d = 0.5 mm, and the relative permittivity of the dielectric is $\epsilon_r = 4$. Conductors 1 and 4 are uncharged. The potentials of conductors 2 and 3 with respect to the plane are $V_2 = 2$ V and $V_3 = -2$ V, respectively. Find the potentials of conductors 1 and 4 with respect to the plane.

You will see that the potentials of conductors 1 and 4 are quite large in magnitude compared to the potentials of charged conductors in the bus. This is an illustration of so-called capacitive "cross talk" in a multiconductor transmission line.

What is the electric energy per unit length of this multiconductor transmission line?

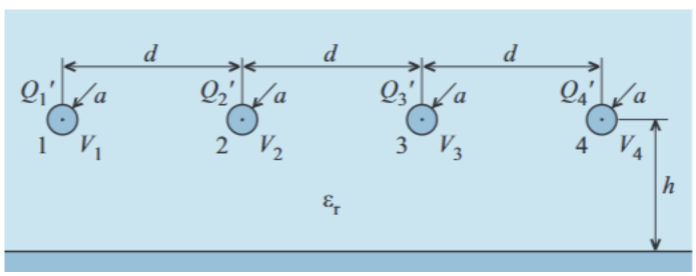

Figure 10: Multiconductor Transmission Line

Solution

We want to start with the equations for the voltages

$$V_1 = a_{11}Q_1' + a_{12}Q_2' + a_{13}Q_3' + a_{14}Q_4'$$

$$V_2 = a_{21}Q_1' + a_{22}Q_2' + a_{23}Q_3' + a_{24}Q_4'$$

$$V_3 = a_{31}Q_1' + a_{32}Q_2' + a_{33}Q_3' + a_{34}Q_4'$$

$$V_4 = a_{41}Q_1' + a_{42}Q_2' + a_{43}Q_3' + a_{44}Q_4'$$

However, the charges on Q_1' and Q_4' are zero, so we can delete these terms from those equations

$$V_1 = a_{12}Q_2' + a_{13}Q_3'$$

$$V_2 = a_{22}Q_2' + a_{23}Q_3'$$

$$V_3 = a_{32}Q'_2 + a_{33}Q'_3$$

$$V_4 = a_{42}Q'_2 + a_{43}Q'_3$$

Because the voltages are opposite meaning

$$V_2 = -V_3$$

it is safe to say that $Q'_2 = -Q'_3$. This results in a new voltage equation of

$$V_2 = a_{22}Q'_2 - a_{23}Q'_2 = (a_{22} - a_{23})Q'_2$$

this means that we only need to solve for a_{22} and a_{23} at this time

$$a_{22} = \frac{V_2}{Q'_2} = \frac{1}{2\pi\epsilon} ln\frac{2h}{a} = 1.3462 \times 10^{10}$$

$$a_{23} = \frac{V_2}{Q'_3} = -\frac{V_2}{Q'_2} = -\frac{1}{2\pi\epsilon} ln\frac{\sqrt{(2h)^2 + d^2}}{d} = -6.3660 \times 10^9$$

This gives us

$$Q'_2 = \frac{V_2}{a_{22} - a_{23}} = \frac{V_2}{1.3462 \times 10^{10} + 6.3660 \times 10^9} = 1.0087 \times 10^{-10}$$

With this value of Q'_2 I am able to obtain

$$V_1 = 0.2774 \text{ V}$$

$$V_4 = -0.2774 \text{ V}$$

Solving for the electric energy per unit length

$$w'_e = \frac{1}{2}(Q'_1V_1 + Q'_2V_2 + Q'_3V_3 + Q'_4V_4) = 0$$

Question 15

A symmetrical two-wire line is positioned in air over a PEC (perfect electric conductor) ground plane, as shown in the figure below. The wire radii are $a = 1$ cm, the height of the axes of both wires with respect to the plane is $h = 4$ m, and the distance between the wire axes is $d = 1$ m. One of the wires is grounded. Both wires are made from copper with conductivity $\sigma_c = 58$ MS/m. The operating frequency is $f = 1$ MHz. Find the capacitance per unit length C', the external inductance per unit length L'_{ext}, the internal inductance per unit length L'_{int}, and the high-frequency resistance per unit length R', of this transmission line (conductor 1 is wire 1, conductor 2 is wire 2 plus the ground plane). Calculate the characteristic impedance Z_0 and attenuation coefficient α of the transmission line.

Solution

From problem 12 we know that

$$C' = \frac{Q'}{V} = \frac{Q'_1}{V_1} = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{a_{22}}{2a_{11} - 2a_{12}} = \frac{1.2016 \times 10^{11}}{1.3031 \times 10^{22}} = 9.2213 \times 10^{-12}\text{F/m}$$

To solve for the high frequency resistance per unit length we need to first solve for the surface resistance.

$$R_S = \sqrt{\frac{\pi \mu_o f}{\sigma_c}} = 2.6090 \times 10^{-4}$$

Using this value we can solve for the resistance PUL

$$R' = \frac{R_S}{2\pi a}\left[1 + \left[-\frac{a_{21}}{a_{22}}\right]^2\right] = 0.0056\ \Omega/\text{m}$$

To find characteristic impedance, we can use the equation

$$Z_0 = \frac{1}{cC'} = 361.4816\ \Omega$$

To solve for the attenuation coefficient we want to use the equation

$$\alpha = \frac{R'}{2Z_0} + \frac{G'}{2Y_0}$$

where Y_0 is Z_0^{-1} and

$$G' = \frac{\sigma_d}{\epsilon}C' = 6.0406 \times 10^{7}\text{S/m}$$

This makes

$$\alpha = 1.0918 \times 10^{10}$$

Finally, I need to find the inductance

$$L' = \frac{\epsilon \mu_0}{C'} = 1.2066 \times 10^{-6}$$