Fintech545 Week4 Project

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1 Problem 1

Assume P_{t-1} is known and $r_t \sim N(0, \sigma^2)$ we can calculate the expectations and standard deviation for P_{t+1} given the 3 types of returns:

- **Classic Brownian Motion:** $P_t = P_{t-1} + r_t$
  Calculate the Expectation:

$$
\begin{aligned}
E\left[P_t|P_{t-1}\right] &= E[P_{t-1} + r_t] \\
&= P_{t-1} + E(r_t) \\
&= P_{t-1}.
\end{aligned}
$$

  Calculate the Standard Deviation:

$$
\begin{aligned}
Var[P_t|P_{t-1}] &= Var[P_{t-1} + r_t] \\
&= Var(P_{t-1}) + Var(r_t) \\
&= E(r_t)^2 \\
&= \sigma^2, \\
Std(P_t|P_{t-1}) &= \sigma.
\end{aligned}
$$

- **Arithmetic Return System:** $P_t = P_{t-1}(1 + r_t)$
  Calculate the Expectation:

$$
\begin{aligned}
E\left[P_t|P_{t-1}\right] &= E[P_{t-1}(1 + r_t)] \\
&= P_{t-1} + P_{t-1}E(r_t) \\
&= P_{t-1}.
\end{aligned}
$$

  Calculate the Standard Deviation:

$$
\begin{aligned}
Var[P_t|P_{t-1}] &= E\left[P_{t-1}(1 + r_t) - E[P_{t-1}(1 + r_t)]^2\right]^2 \\
&= E\left[P_{t-1}r_t - E(P_{t-1}r_t)^2\right]^2 \\
&= P_{t-1}^2 E(r_t)^2 \\
&= P_{t-1}^2 \sigma^2, \\
Std(P_t|P_{t-1}) &= P_{t-1}\sigma.
\end{aligned}
$$

- **Geometric Brownian Motion:** $P_t = P_{t-1}e^{r_t}$
  Note that the moment generation function for normal distribution $N(0, \sigma)$ is :

$$
E[e^{st}] = e^{\frac{1}{2}\sigma^2 t^2}
$$

We can calculate the Expectation:

$$\begin{aligned}
E\left[P_t|P_{t-1}\right] &= E[P_{t-1}e^{r_t}] \\
&= P_{t-1}E[e^{r_t}] \\
&= P_{t-1}e^{\frac{1}{2}\sigma^2}.
\end{aligned}$$

Calculate the Standard Deviation:

$$\begin{aligned}
Var[P_t|P_{t-1}] &= E\left[P_{t-1}e^{r_t} - E[P_{t-1}e^{r_t}]^2\right]^2 \\
&= E[P_{t-1}e^{r_t}]^2 - E(P_{t-1}r_t)^2 \\
&= P_{t-1}^2\left(e^{\frac{1}{2}\sigma^2 * 2^2} - e^{\sigma^2}\right) \\
&= P_{t-1}^2\left(e^{2\sigma^2} - e^{\sigma^2}\right), \\
Std(P_t|P_{t-1}) &= P_{t-1}\sqrt{\left(e^{2\sigma^2} - e^{\sigma^2}\right)}.
\end{aligned}$$

Assume that $P_{t_1} = 30$, $\sigma = 0.5$, we can calculate the numerical expectation and standard deviation as:

Type of Return	Expectation	Standard Deviation
Classic	30.0	0.5
Discrete	30.0	15.0
Geometric	33.99445359200479	18.11701599632644

Simulate the return using `np.random.normal` 2000 times and calculate the mean and standard deviation:

Type of Return	Expectation	Standard Deviation
Classic	29.99489568675101	0.49704791031134987
Discrete	29.84687060253026	14.911437309340497
Geometric	33.79028077895063	18.188632338032303

The mean, standard deviation of simulation results are quite closed to the numerical calculations.

2 Problem 2

Part of the arithmetic returns calculations are shown below:

```
discrete_return.head(10)
```

	Date	SPY	AAPL	MSFT	AMZN	TSLA	GOOGL	GOOG	META	NVDA	BRK-B
0	2/15/2022 0:00	0.016127	0.023152	0.018542	0.008658	0.053291	0.007987	0.008319	0.015158	0.091812	0.006
1	2/16/2022 0:00	0.001121	-0.001389	-0.001167	0.010159	0.001041	0.008268	0.007784	-0.020181	0.000604	-0.001
2	2/17/2022 0:00	-0.021361	-0.021269	-0.029282	-0.021809	-0.050943	-0.037746	-0.037669	-0.040778	-0.075591	-0.006
3	2/18/2022 0:00	-0.006475	-0.009356	-0.009631	-0.013262	-0.022103	-0.016116	-0.013914	-0.007462	-0.035296	0.003
4	2/22/2022 0:00	-0.010732	-0.017812	-0.000729	-0.015753	-0.041366	-0.004521	-0.008163	-0.019790	-0.010659	-0.002
5	2/23/2022 0:00	-0.017739	-0.025864	-0.025893	-0.035756	-0.069979	-0.017144	-0.014045	-0.017963	-0.042882	-0.013
6	2/24/2022 0:00	0.015049	0.016680	0.051093	0.045095	0.048073	0.039996	0.039883	0.046107	0.060794	-0.005
7	2/25/2022 0:00	0.022064	0.012966	0.009233	0.016058	0.011364	0.013328	0.013914	0.013873	0.017223	0.035
8	2/28/2022 0:00	-0.002558	0.001638	0.004978	-0.001466	0.074777	0.004444	0.002762	0.002613	0.009438	0.006

Figure 1: Caption

Calculate the VaR in the $\alpha\% = 5\%$ scenario using different methods, the results are shown in the following table:

Normal Distribution	Normal with EW variance	MLE fitted T	Fitted AR(1)	Historic Simulation
6.5602 %	9.1385 %	5.7580%	3.5558%	4.7523%

From the results, we can see that using AR(1) we obtain the smallest VaR. The reason might be that the AR(1) "smoothed" the trend by autocorrelation. Using AR(1) we get a trend that has a part that can be fixed based on the previous data (βy_{t-1}) and only live the rest noise as a random variable. The methd using Normal distribution with Exponentially Weighted variance gives the largest VaR. The reason might be that the fluctuation in the recent data affects the variance. The results for Normal Distribution, MLE fitted T distribution and Historical Simulation are relatively closed.

3 Problem 3

Using Monte Carlo Simulation calculate the VaR with Multivariate Normal Distribution and Multivariate T Distribution (Use MLE fit the degree of freedom) respectively, the VaR of different portfolios are calculated as:

		Multivariate Normal Distribution (EW Covariance)	Multivariate T Distribution (EW Covariance)	Multivariate Normal Distribution	Multivariate T Distribution
Portfolio A	VaR($)	5476.7251	9681.9645	7862.6238	8568.246
	VaR(%)	1.826%	3.2279%	2.6213%	2.8566%
Portfolio B	VaR($)	4385.8638	6892.2111	6637.4965	7573.7378
	VaR(%)	1.4898%	2.3412%	2.2547%	2.5727%
Portfolio C	VaR($)	3723.3961	7153.5154	5752.6525	6724.801
	VaR(%)	1.3788%	2.6490%	2.1303%	2.4903%
Total Holdings	VaR($)	13491.6581	224320.8905	19311.4454	19719.8069
	VaR(%)	1.5609%	25.9517%	2.2341%	2.2814%

From the table, we can see that in general, using Multivariate T distribution for simulation will get a relatively high VaR. This is a reasonable result since the T distribution has a heavier tail. However, while using Pearson Covariance for calculation, the results of Normal and T distribution are quite close, using Exponentially Weighted Covariance for simulation can get results that have much larger discrepancies. Especially for the VaR of the Total Holdings, the VaR calculated by T is much higher than the result of Normal Distribution. The reason might be that there might be more fluctuations in recent results that affect the outcomes.