Mathematics for raytracing

What do we need

Raytracing consits, in its most basic form, in simulating the path of rays of lights in a 3D environnement in order to deduce their color from the objects encountered.
Will will thus need to compute the intersection points of the rays with objects. In order to achieve that, it will be needed to know about the mathematical tools that can be used to model those objects, as well as the rays themselves.
In this workshop we will cover *vectors*, *equations of 3D surfaces* and the mathematical tools used to solve these equations.

Basic vocabulary:

- **vector** : Simply put, a vector is a point in a space. The space can be the 3D space, 2D space, or any vector space of your choice. (*what is a vector space? coming in another workshop :D*) A vector is defined by its coordinates in a basis of the space. For exemple, the 3D space (also called $\mathbb{R}^3$) has this basis (called *canonical basis*):

  $$\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$$

  A vector with the following coordinates:

  $$\vec{u} = \begin{pmatrix}3\\1\\4\end{pmatrix}$$

  could be written as

  $$\vec{u} = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix}$$

  in the canonical basis of $\mathbb{R}^3$. A basis of a space is any combination of vector such that all elements of the space can be written as a linear combination of the elements of the base, but no element of the base can be written as a linear combination of the other elements of the base.
  Alternatively, a vector can be seen as an arrow, the indication of a direction, an arrow pointing from any point in space towards somewhere else. It has a direction (its axis)

and a way (from left to right or from right to left for exemple). The length of a vector is called its norm and is written $||\vec{a}||$.

Exercise : What is the decomposition of the vector

$$\vec{u} = \begin{pmatrix} 9 \\ 5 \\ 3 \end{pmatrix}$$

in the non canonical basis

$$B = \{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} \}$$

Exercise : Try to find a non canonical basis of $\mathbb{R}^4$ to complete the following basis:

$$\{ \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} ? \\ \vdots \\ \vdots \\ ? \end{pmatrix}, \begin{pmatrix} ? \\ \vdots \\ \vdots \\ ? \end{pmatrix} \}$$

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- **set** : A set of points is an ensemble of elements choosen by following a certain rule. For exemple:

$$E = \{2k : k \in \mathbb{N}\}$$

  is the set of all positive even numbers. A set can also be defined by explicitely listing its elements.

Exercise : write an expression of the set of points describing a horizontal plane in 3D. Do the same for a plane tilted by 45 degrees around the z axis.

- **intersection** : Two sets are said to intersect if there exists at least one object that is an element of both sets. If A and B intersect, we call $A \cap B$ the intersection of A and B, and it is also a set.

*Exercise : find the elements that intersect the curve $y = x + 5$ and the cicrle centered on the origin (0, 0) and radius 1 (in 2D)

- **surface** : A surface in a 3D space is just a set of points, defined by a piecewise continuous function. This means that we won't really count individual points in space as a surface, but we want to be able to go from one to another without leaving the surface. Such objects can typically be represented by equations.

- **equation** : an equation is an expression containing one or more unknown (or variable) as well as an (in)equality statement.
  Exercise : solve the following equations (find x)

$$2x = 4 + 5x$$

$$x = \frac{2+x}{5}$$

$$\sqrt{x+2} = 42$$

- **parametrisation** : A parametrisation is a way of defining a set using one or more variables as parameters. For exemple, a parametrisation for a short line could be:

$$\{3t + 2, t \in [0, 1]\}$$

  Here, t will take every value between 0 and 1 in order to draw the whole line, and each point on the line can be uniquely represented by a value of t. This also let us go from a higher dimension space to a lower dimension space. (here, we go from 2D to 1D because the points on the line have two coordinates x and y, but the parameter t can be used as a unique coordinate.)

Exercise : find the parametrisation the the line passing by the points $a = (2, 3)$ and $b = (5, 10)$

- **distance** The distance between two points a and b is written as follows:

$$d = |a - b|$$

  As you can see this is the same notation as the absolute value between two numbers. This is actually the same thing. In 1D, there are two numbers that a_1 and a_2 that are at the same distance from any other number b. For exemple if $b = 0$, the equation

$$4 = |a - 0|$$

  has two solutions, 4 and -4. This relationship is true even for higher dimensions.

Defining our rays:

A ray of light has an origin and a direction. As such, it is possible to write a parametrisation for a ray.

Exercise : try to find a parametrisation for a ray of light in 3D. Then try to write a function that can describe a point in space that is part of the ray using the parametrisation you found.

Defining our objects:

*Exercise: Try to find an equation describing a sphere in 3 dimensions. You can do it two ways:

- *write an equation using the coordinates x, y and z of a point in space, the radius r of the sphere, and the coordinates x_0, y_0 and z_0 of the center of the sphere*
- *write an equation using only vectors to describe your sphere*

More mathematical tools:

- The dot product, or scalar product is an operation that takes to vectors and returns a scalar (a regular number, if you will). It has interesting characteristics:

  $$\vec{a} \cdot \vec{b} = ||a|| \times ||b|| \times cos\theta$$

  where θ is the angle between the two vectors. Note that if $\vec{a}$ and $\vec{b}$ are orthogonals (right angle), the dot product will be zero. The dot product can also be used to find the projection of a vector on another vector!
  *Exercise : let be

  $$\vec{u} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

  Find $\vec{v}$ so that $\vec{u} \cdot \vec{v} = 0$.
- The cross product between $\vec{a}$ and $\vec{b}$ produces a vector orthogonal to both $\vec{a}$ and $\vec{b}$.

  $$\vec{a} \times \vec{b} = ||a|| \times ||b|| \times sin\theta\vec{n}$$

  where $\vec{n}$ is a unit vector orthogonal to both $\vec{a}$ and $\vec{b}$. There are other ways to compute it, I encourage you to look into it! Note that if $\vec{a}$ and $\vec{b}$ are parallels, their cross product will be zero.
  Exercise: Implement a function that takes two vectors and returns the dot product between them.
  Exercise : let be

  $$\vec{u} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

Find $\vec{v}$ so that $\vec{u} \cdot \vec{v} = 0$.
Exercise: Implement a function that takes two vectors and returns the cross product between them. *

Solving the equation for the sphere

Exercice : Now that you found an equation, lets try to solve it. Using your equation of the sphere and your parametrisation of the ray, try to find the parameter(s) $t \in \mathbb{R}$ so that the point described by it are at the intersection of your ray and a sphere of radius R and of center $\vec{c}$.
*Hint: You will need to solve a second degree polynomial. If you don't remember how, here is a few tips:

- try to write your whole equation in the form $at^2 + bt + c = 0$ where a, b and c are vectors and t is your unknown.
- compute the **discriminant** of the equation: $\delta = b^2 - 4ac$.
- if $\delta < 0$ the equation has no solution. Else, the solutions are given by:

$$s_{1,2} = \frac{-b \pm \sqrt{\delta}}{2a}$$

- **Caution!** since a, b and c are vectors, You might find that confusing. Remember that member to member multiplication does not exist for vectors and that the *dot product* is used instead.
  Exercise : write a function that takes a ray of light and a sphere in input, and returns the parameter t of the ray such that the point of the ray described by t is part of the sphere.