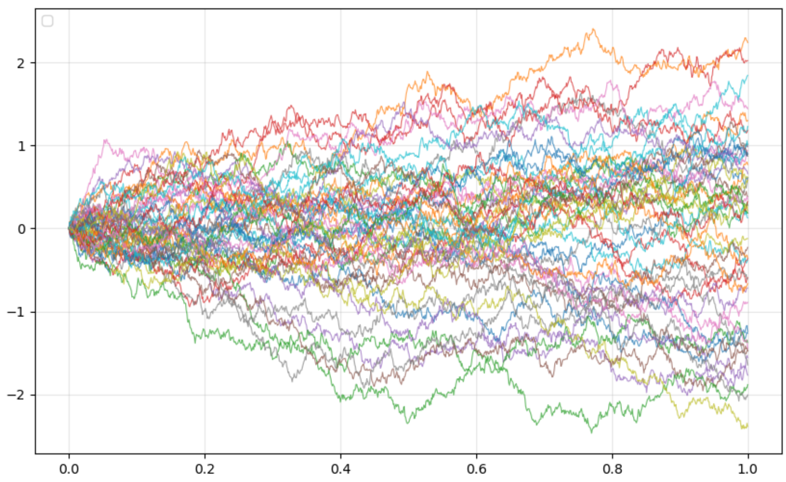
The Black–Scholes Model: A Probabilistic and Differential Approach

By Bernard Tao, Gabarret Périne and Oster Eliott

INSA Toulouse — GMM Department (4th year)

Supervisor: Aldéric Joulin



Academic year 2025-2026

Abstract

This report presents the Black–Scholes model from both a probabilistic and a partial differential equation (PDE) perspective. Starting from the definition of Brownian motion, we introduce the main tools of stochastic calculus, including martingales, quadratic variation, Itô's formula, and change of measure. These concepts are then used to derive the dynamics of geometric Brownian motion and to establish the Black–Scholes pricing framework under the no-arbitrage principle. We show how the Black–Scholes partial differential equation arises from a replication argument and how its solution coincides with the risk-neutral valuation formula for European options. Numerical simulations are provided to illustrate the theoretical results and to highlight the impact of the model assumptions. Finally, we discuss the main limitations of the Black–Scholes model, such as constant volatility, continuous price paths, and idealized trading conditions.

Contents

1 Introduction

The concept of Brownian motion originates from the work of *Robert Brown* (1773–1858), a Scottish botanist who, in 1827, reported the irregular motion of pollen particles suspended in water when observed under a microscope [**?**]. Although Brown did not propose a mathematical explanation, this phenomenon later became a fundamental object of study in probability theory and statistical physics.

The first attempt to model random fluctuations in a quantitative framework appeared at the beginning of the twentieth century with the work of *Louis Bachelier* (1870–1946). In his 1900 doctoral thesis *Théorie de la spéculation*, Bachelier introduced a continuous-time stochastic model for asset prices based on random movements with independent increments [1]. This pioneering work, largely overlooked at the time, is now recognized as the foundation of modern mathematical finance.

A rigorous probabilistic construction of Brownian motion was developed shortly thereafter, notably through the work of *Albert Einstein* in 1905 [**?**] and *Norbert Wiener* in 1923 [**?**], who provided a measure-theoretic definition of the process now known as the Wiener process. These advances enabled the development of stochastic calculus, most notably by *Kiyosi Itô* in the 1940s, who introduced Itô's formula and laid the foundations of stochastic differential equations [2].

The link between Brownian motion and financial markets was firmly established in the early 1970s with the seminal works of *Fischer Black*, *Myron Scholes*, and *Robert C. Merton* [3, 4]. Their model provides an explicit formula for pricing European options under assumptions of continuous trading, no arbitrage, and log-normal asset price dynamics. In recognition of this contribution, Scholes and Merton were awarded the Nobel Prize in Economic Sciences in 1997 (Fischer Black having passed away in 1995), marking the definitive entry of stochastic calculus into financial theory [10].

2 Preliminaries

The primary objective of this report is to introduce the probabilistic tools underlying the Black–Scholes framework—specifically Brownian motion, martingales, Itô calculus, and change of measure. We aim to demonstrate how these concepts lead to the Black–Scholes Partial Differential Equation (PDE) and the closed-form pricing formula for European options. Before diving into the stochastic theory, we establish the fundamental dynamics and definitions used throughout this paper.

Asset Dynamics and Stochastic Perturbation A common model for the evolution of a risky asset price $(X_t)_{t\geq 0}$ is a Stochastic Differential Equation (SDE) of the form:

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t, \tag{1}$$

where $\mu \in \mathbb{R}$ is the drift (representing the trend), $\sigma > 0$ is the volatility, and $(B_t)_{t\geq 0}$ is a standard Brownian motion.

The logic behind this modeling is straightforward: if we formally remove the random term ($\sigma = 0$), we recover a classical Ordinary Differential Equation (ODE):

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t \quad \Longrightarrow \quad X_t = X_0 \mathrm{e}^{\mu t}.$$

Thus, an SDE can be naturally interpreted as a deterministic ODE perturbed by a continuous random noise. In finance, this noise captures the inherent uncertainty of market price movements.

Option Payoff and Pricing Functional We consider a European call option with maturity $T > 0$ and strike price $K > 0$. Its payoff at time T is defined by the function:

$$f(u) \coloneqq (u - K)^+ = \max(u - K, 0).$$

To determine the fair value of this claim at any time $t \in [0, T]$ given the current price $X_t = x$, we define the discounted conditional valuation functional:

$$g(x, t) \coloneqq \mathrm{e}^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}\big[\, (X_T - K)^+ \,\big|\, X_t = x \big]\,. \tag{2}$$

This functional represents the expected future payoff, discounted back to the present time at the risk-free rate r.

The Risk-Neutral Framework A fundamental result in arbitrage pricing theory is that the drift μ of the physical measure is replaced by the risk-free rate r when pricing derivatives. Under this so-called **risk-neutral measure** $\mathbb{Q}$, the risky asset follows:

$$\begin{cases} \mathrm{d}X_t = rX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t, \\ X_0 = x. \end{cases} \tag{3}$$

Consequently, the time-0 price of a European call is given by the simplified expectation:

$$C(r,T,K,x,\sigma) = \mathrm{e}^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[(X_T-K)^+\right].$$

In the following sections, we provide the rigorous mathematical foundations necessary to solve these equations and prove the consistency between this probabilistic approach and the replication-based PDE approach.

3 Brownian Motion and Martingale Theory

In this section, we construct the probabilistic foundation of the model. We start by defining the Standard Brownian Motion, which models the random noise of the market, and then explore its associated martingales, which are crucial for the risk-neutral valuation framework.

3.1 Standard Brownian Motion

A real-valued stochastic process $(B_t)_{t\geq 0}$ is a **standard Brownian motion** if:

1. $t \mapsto B_t$ is almost surely continuous on $\mathbb{R}_+$;
2. $(B_t)_{t\geq 0}$ is a centered Gaussian process:

$$\forall t \geq 0, \qquad \mathbb{E}[B_t] = 0;$$

3. its covariance function satisfies, for all $s,t \geq 0$,

$$\mathrm{Cov}(B_s,B_t) = \min(s,t).$$

Remark 1. Taking $s=t$ gives

$$\mathrm{Var}(B_t) = \mathrm{Cov}(B_t,B_t) = t.$$

Since $\mathbb{E}[B_t]=0$, it follows that $\mathbb{E}[B_t^2]=t$. In particular, for $t=0$ we get $\mathbb{E}[B_0^2]=0$, hence $B_0=0$ almost surely.

Remark 2. Recall that a stochastic process $(X_t)_{t \geq 0}$ is said to be a **Gaussian process** if, for any finite set of times $0 \leq t_1 < t_2 < \cdots < t_n$, the random vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ follows a multivariate normal distribution. Such a process is entirely characterized by its mean function $m(t) = \mathbb{E}[X_t]$ and its covariance function $K(s,t) = \mathrm{Cov}(X_s, X_t)$.

To complement the definition, Figure 1 shows a simple numerical simulation of Brownian motion.

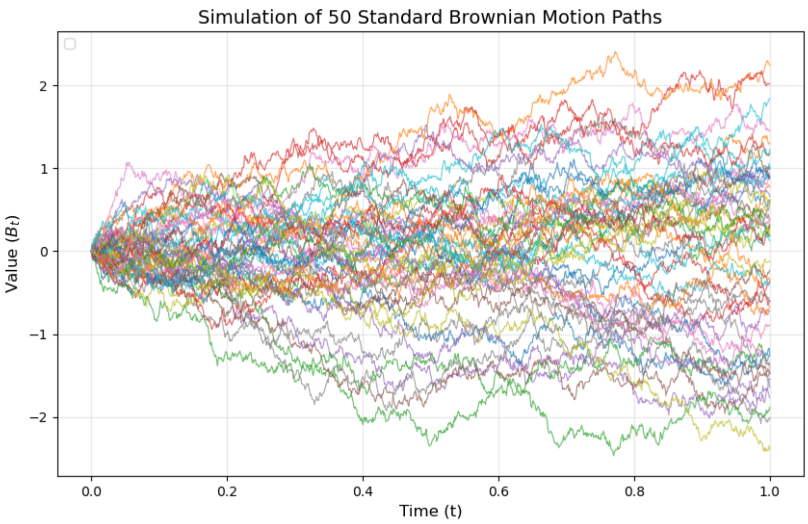


Figure 1: Simulated Brownian motion.

Link with a discrete Gaussian walk Before turning to continuous time, it is useful to keep in mind a discrete analogue: a sum of i.i.d. Gaussian increments already produces a centered Gaussian process with covariance $\min(n,m)$, mirroring the covariance structure of Brownian motion.

Let $(U_n)_{n \geq 1}$ be i.i.d. random variables with $U_n \sim \mathcal{N}(0,1)$, and define

$$B_n := \sum_{i=1}^{n} U_i, \qquad n \in \mathbb{N}, \qquad B_0 := 0.$$

Then $(B_n)_{n \in \mathbb{N}}$ is a centered Gaussian process and

$$\mathrm{Cov}(B_n, B_m) = \min(n,m), \qquad \forall n, m \in \mathbb{N}.$$

Thus, (B_n) provides a discrete-time analogue of Brownian motion (without continuity).

Equivalent characterization The following characterization highlights the two key features that make Brownian motion suitable for stochastic calculus and for modeling: Gaussian marginals and independent stationary increments.

Proposition 1 (Equivalent characterization)**.** *A process $(B_t)_{t\geq 0}$ is a standard Brownian motion if and only if:*

1. *$B_0 = 0$ almost surely and the paths are almost surely continuous;*
2. *for every $t > 0$, $B_t \sim \mathcal{N}(0, t)$;*
3. *it has independent and stationary increments: for $0 \leq s < t$,*

$$B_t - B_s \text{ is independent of } \mathcal{F}_s \coloneqq \sigma(B_u : 0 \leq u \leq s), \qquad B_t - B_s \sim \mathcal{N}(0, t-s).$$

Joint density of (B_s, B_t) Let $0 < s < t < T$. The joint density of (B_s, B_t) is, for $(x, y) \in \mathbb{R}^2$,

$$f_{(B_s,B_t)}(x, y) = \frac{1}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{1}{2}\left[\frac{x^2}{s} + \frac{(y-x)^2}{t-s}\right]\right). \tag{4}$$

Proof.

Step 1: (B_s, B_t) is a Gaussian vector. Fix $(a, b) \in \mathbb{R}^2$ and consider $aB_s + bB_t$. Write

$$aB_s + bB_t = (a + b)B_s + b(B_t - B_s).$$

By Brownian properties,

$$B_s \sim \mathcal{N}(0, s), \qquad B_t - B_s \sim \mathcal{N}(0, t-s),$$

and $B_t - B_s$ is independent of $\mathcal{F}_s$, hence independent of B_s. Therefore $(a + b)B_s$ and $b(B_t - B_s)$ are independent Gaussian random variables, so their sum is Gaussian. This proves (B_s, B_t) is Gaussian.

Step 2: Mean vector. Since Brownian motion is centered,

$$\mathbb{E}[B_s] = \mathbb{E}[B_t] = 0 \quad \Rightarrow \quad m \coloneqq \mathbb{E}\begin{pmatrix} B_s \\ B_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Step 3: Covariance matrix. Using $\mathrm{Cov}(B_u, B_v) = \min(u, v)$,

$$\mathrm{Var}(B_s) = s, \qquad \mathrm{Var}(B_t) = t, \qquad \mathrm{Cov}(B_s, B_t) = s \quad (s < t).$$

Hence

$$\Sigma \coloneqq \mathrm{Cov}\begin{pmatrix} B_s \\ B_t \end{pmatrix} = \begin{pmatrix} s & s \\ s & t \end{pmatrix}, \qquad \det(\Sigma) = st - s^2 = s(t-s) > 0,$$

and

$$\Sigma^{-1} = \frac{1}{s(t-s)} \begin{pmatrix} t & -s \\ -s & s \end{pmatrix}.$$

Step 4: Density formula. Since (B_s, B_t) is Gaussian with mean $m = (0,0)$ and covariance Σ, its density is

$$f_{(B_s,B_t)}(x,y) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^\top \Sigma^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right).$$

Compute the quadratic form:

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \Sigma^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{s(t-s)} \big(tx^2 - 2sxy + sy^2 \big).$$

Thus

$$f_{(B_s,B_t)}(x,y) = \frac{1}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{1}{2} \cdot \frac{tx^2 - 2sxy + sy^2}{s(t-s)} \right).$$

Finally,

$$\frac{tx^2 - 2sxy + sy^2}{s(t-s)} = \frac{x^2}{s} + \frac{(y-x)^2}{t-s},$$

which gives (4). □

3.2 Martingales Associated with Brownian Motion

Martingales formalize the idea of a "fair game" and play a central role in modern probability. In mathematical finance, they will later appear as the natural objects describing discounted asset prices under no-arbitrage (risk-neutral) valuation.

Definition (Continuous Martingale) : A real-valued stochastic process $(M_t)_{t\geq 0}$ adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a **martingale** if:

1. It is integrable: for all $t \geq 0$, $\mathbb{E}[|M_t|] < \infty$;

2. It satisfies the martingale property: for all $0 \leq s \leq t$,

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s \quad \text{a.s.}$$

We start with the most classical martingales derived from Brownian motion. Let $(B_t)_{t\geq 0}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with its natural filtration

$$\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t).$$

1. The process (B_t) is a martingale *Integrability and measurability.* B_t is $\mathcal{F}_t$-measurable by definition. Furthermore, B_t is integrable because, by the Cauchy-Schwarz (CS) inequality:

$$\mathbb{E}[|B_t|] \leq \sqrt{\mathbb{E}[B_t^2]} = \sqrt{t} < +\infty \tag{5}$$

The finitude is guaranteed since $B_t \sim \mathcal{N}(0, t)$, ensuring $B_t \in L^1(\Omega)$.

Conditional expectation. Let $0 \leq s \leq t$. Write

$$B_t = B_s + (B_t - B_s).$$

Taking conditional expectation with respect to $\mathcal{F}_s$,

$$\mathbb{E}(B_t \mid \mathcal{F}_s) = \mathbb{E}(B_s \mid \mathcal{F}_s) + \mathbb{E}(B_t - B_s \mid \mathcal{F}_s).$$

Since B_s is $\mathcal{F}_s$-measurable, $\mathbb{E}(B_s \mid \mathcal{F}_s) = B_s$. Also, $B_t - B_s$ is independent of $\mathcal{F}_s$ and has mean 0, hence $\mathbb{E}(B_t - B_s \mid \mathcal{F}_s) = 0$. Therefore,

$$\mathbb{E}(B_t \mid \mathcal{F}_s) = B_s,$$

so (B_t) is a martingale. □

2. The process $(B_t^2 - t)$ is a martingale *Integrability and measurability.* $B_t^2 - t$ is $\mathcal{F}_t$-measurable by construction. Moreover, it is integrable since by the triangle inequality:

$$\mathbb{E}[|B_t^2 - t|] \leq \mathbb{E}[B_t^2] + t = 2t < +\infty \tag{6}$$

where we used the fact that $\mathbb{E}[B_t^2] = t$. This confirms $B_t^2 - t \in L^1(\Omega)$. Furthermore, its centered property is verified by:

$$\mathbb{E}[B_t^2 - t] = \mathbb{E}[B_t^2] - t = t - t = 0 \tag{7}$$

Conditional expectation. For $0 \leq s \leq t$,

$$B_t = B_s + (B_t - B_s) \quad \Rightarrow \quad B_t^2 = B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2.$$

Taking $\mathbb{E}(\cdot \mid \mathcal{F}_s)$,

$$\mathbb{E}(B_t^2 \mid \mathcal{F}_s) = \mathbb{E}(B_s^2 \mid \mathcal{F}_s) + 2\mathbb{E}(B_s(B_t - B_s) \mid \mathcal{F}_s) + \mathbb{E}((B_t - B_s)^2 \mid \mathcal{F}_s).$$

Now $\mathbb{E}(B_s^2 \mid \mathcal{F}_s) = B_s^2$. Since B_s is $\mathcal{F}_s$-measurable and $B_t - B_s$ is independent of $\mathcal{F}_s$ with mean 0,

$$\mathbb{E}(B_s(B_t - B_s) \mid \mathcal{F}_s) = B_s\,\mathbb{E}(B_t - B_s) = 0.$$

Finally, $(B_t - B_s)$ is independent of $\mathcal{F}_s$. By the stationarity of increments, $(B_t - B_s)$ and B_{t-s} follow the same law ($B_t - B_s \sim \mathcal{N}(0, t-s)$), which leads to:

$$\mathbb{E}((B_t - B_s)^2 \mid \mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2) = \mathrm{Var}(B_t - B_s) = t - s \tag{8}$$

Hence

$$\mathbb{E}(B_t^2 \mid \mathcal{F}_s) = B_s^2 + (t-s), \qquad \mathbb{E}(B_t^2 - t \mid \mathcal{F}_s) = B_s^2 - s,$$

so $(B_t^2 - t)$ is a martingale. $\square$

3. Exponential martingale Fix $\theta \in \mathbb{R}$ and define

$$M_t^\theta := \exp\left(\theta B_t - \frac{1}{2}\theta^2 t\right), \qquad t \geq 0.$$

Let $0 \leq s \leq t$. Then

$$M_t^\theta = \exp\left(\theta B_s - \frac{1}{2}\theta^2 s\right) \exp\left(\theta(B_t - B_s) - \frac{1}{2}\theta^2(t-s)\right) = M_s^\theta \, X_{t,s},$$

where

$$X_{t,s} := \exp\left(\theta(B_t - B_s) - \frac{1}{2}\theta^2(t-s)\right).$$

Since $B_t - B_s$ is independent of $\mathcal{F}_s$,

$$\mathbb{E}(M_t^\theta \mid \mathcal{F}_s) = M_s^\theta \, \mathbb{E}(X_{t,s}).$$

Let $X = B_t - B_s \sim \mathcal{N}(0, t-s)$. Then

$$\mathbb{E}(X_{t,s}) = \mathrm{e}^{-\frac{1}{2}\theta^2(t-s)} \mathbb{E}(\mathrm{e}^{\theta X}) = \mathrm{e}^{-\frac{1}{2}\theta^2(t-s)} \mathrm{e}^{\frac{1}{2}\theta^2(t-s)} = 1,$$

Indeed, let $X = B_t - B_s \sim \mathcal{N}(0, t-s)$. Recalling the Moment Generating Function (MGF) of a Gaussian random variable, we have $\mathbb{E}[e^{\theta X}] = \exp\left(\frac{1}{2}\theta^2(t-s)\right)$. It follows that:

$$\mathbb{E}(X_{t,s} \mid \mathcal{F}_s) = e^{-\frac{1}{2}\theta^2(t-s)} \mathbb{E}(e^{\theta X}) = e^{-\frac{1}{2}\theta^2(t-s)} e^{\frac{1}{2}\theta^2(t-s)} = 1 \tag{9}$$

so $\mathbb{E}(M_t^\theta \mid \mathcal{F}_s) = M_s^\theta$, and (M_t^θ) is a martingale. $\square$

4 Stochastic Calculus and Change of Measure

The next step is to understand why classical calculus is not directly applicable to Brownian paths. Since Brownian motion is almost surely nowhere differentiable and has infinite variation, one needs a new integration theory and a modified chain rule: this is precisely the role of Itô calculus. Finally, to move from a physical description of price dynamics to a pricing framework, we introduce Girsanov's theorem, which describes how drifts can be modified by an equivalent change of measure.

4.1 Integration Theory and Quadratic Variation

Fundamental theorem of calculus and chain rule Let $f, g : \mathbb{R} \to \mathbb{R}$ be C^1 functions. Then the chain rule stands as

$$(f \circ g)'(t) = f'(g(t))g'(t),$$

and integrating from 0 to t,

$$f(g(t)) = f(g(0)) + \int_0^t f'(g(s))g'(s) \, \mathrm{d}s.$$

If g is not differentiable, this classical argument breaks down.

Bounded variation and Stieltjes integrals Let $f \in C^1(\mathbb{R})$ and let $g : \mathbb{R} \to \mathbb{R}$ be of finite (bounded) variation on $[0, t]$, i.e.,

$$\sup_{\pi_t} \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| < \infty, \tag{10}$$

where the supremum is over all partitions $\pi_t = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[0, t]$. Then the Stieltjes integral exists and:

$$f(g(t)) = f(g(0)) + \int_0^t f'(g(s)) \, \mathrm{d}g(s), \tag{11}$$

where $\mathrm{d}g$ denotes the Radon measure associated with g.

Why Brownian motion does not fit this framework Brownian motion has almost surely infinite variation on every interval: for all $t > 0$,

$$\mathbb{P}\left(\sup_{\pi_t} \sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}| = +\infty\right) = 1, \tag{12}$$

where the supremum is taken over all partitions $\pi_t = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of $[0, t]$. Consequently, an integral such as $\int_0^t H_s \, \mathrm{d}B_s$ cannot be defined as a classical Stieltjes integral in general. The Itô integral is instead defined as an L^2 limit of Riemann sums.

Quadratic variation of Brownian motion A key feature distinguishing Brownian motion from smooth paths is its non-trivial quadratic variation. This phenomenon is exactly what generates the additional second-order term in Itô's formula.

Fix $t > 0$ and consider the uniform partition $t_k = \frac{kt}{n}$. Let $\Delta B_k = B_{t_k} - B_{t_{k-1}}$. Then

$$\Delta B_k \sim \mathcal{N}\left(0, \frac{t}{n}\right), \qquad \mathbb{E}[(\Delta B_k)^2] = \frac{t}{n}.$$

Thus

$$\mathbb{E}\left[\sum_{k=1}^{n} (\Delta B_k)^2\right] = t.$$

Moreover, using independence of increments and $\mathrm{Var}\left((\Delta B_k)^2\right) = 2\left(\frac{t}{n}\right)^2$ for a centered Gaussian, we have:

$$\mathbb{E}\left[\left|\sum_{k=1}^{n} (\Delta B_k)^2 - t\right|^2\right] = \mathrm{Var}\left(\sum_{k=1}^{n} (\Delta B_k)^2\right) \tag{13}$$

By the independence of increments and the fact that $(\Delta B_k)^2$ follows a scaled χ^2 distribution with $\mathrm{Var}((\Delta B_k)^2) = 2(\frac{t}{n})^2$, we obtain:

$$\sum_{k=1}^{n} 2\left(\frac{t}{n}\right)^2 = \frac{2t^2}{n} \xrightarrow[n\to\infty]{} 0 \tag{14}$$

Hence

$$\sum_{k=1}^{n} (\Delta B_k)^2 \longrightarrow t \quad \text{in } L^2 \text{ and in probability.}$$

We define the *quadratic variation* $(\langle B, B\rangle_t)_{t\geq 0}$ by $\langle B, B\rangle_0 = 0$ and, for all $t > 0$,

$$\langle B, B\rangle_t \overset{\mathbb{P}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} (B_{t_k} - B_{t_{k-1}})^2. \tag{15}$$

The previous computation shows that:

$$\langle B, B\rangle_t = t, \quad \forall t \geq 0. \tag{16}$$

4.2 Itô's Formula and Applications

Itô's formula (Brownian case) Let $f \in C^2(\mathbb{R})$. Then for all $t \geq 0$:

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}\langle B, B\rangle_s. \tag{17}$$

Since $\langle B, B\rangle_t = t$, this becomes:

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s. \tag{18}$$

Remark 3. The additional second-order term is exactly the correction coming from quadratic variation; it has no analogue in classical calculus.

We now extend Itô's formula to functions depending both on space and time. Let $f \in \mathcal{C}^{2,1}(\mathbb{R}\times\mathbb{R}_+)$ and let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Applying Itô's formula to the process $X_t = f(B_t, t)$, we obtain:

$$\begin{aligned} f(B_t, t) = f(B_0, 0) &+ \int_0^t \partial_x f(B_s, s)\,\mathrm{d}B_s + \int_0^t \partial_t f(B_s, s)\,\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \partial^2_{xx} f(B_s, s)\,\mathrm{d}\langle B, B\rangle_s, \end{aligned} \tag{19}$$

where $\partial_t f$, $\partial_x f$, and $\partial^2_{xx} f$ stand for the partial derivatives with respect to the time and space parameters, respectively. Since the quadratic variation of Brownian motion satisfies $\mathrm{d}\langle B, B\rangle_s = \mathrm{d}s$, this simplifies to:

$$f(B_t, t) = f(B_0, 0) + \int_0^t \partial_x f(B_s, s)\,\mathrm{d}B_s + \int_0^t \left(\partial_t f(B_s, s) + \frac{1}{2}\partial^2_{xx} f(B_s, s)\right)\mathrm{d}s. \tag{20}$$

Remark 4. In particular, we can show that a process adapted to the Brownian filtration is a martingale if and only if it can be written as a stochastic integral with respect to a Brownian motion (by the Martingale Representation Theorem). Hence, $(X_t)_{t\geq 0}$ is a martingale if and only if the drift term vanishes, meaning the function f satisfies the following Partial Differential Equation (PDE):

$$\partial_t f + \frac{1}{2}\partial^2_{xx} f = 0. \tag{21}$$

Application: Retrieving Martingales via Itô The martingales introduced earlier can also be obtained in a systematic way using Itô's formula.

1. The martingale $B_t^2 - t$. Apply Itô's formula to the function $f(x) = x^2$ (so $f'(x) = 2x$ and $f''(x) = 2$). Using (18),

$$B_t^2 = B_0^2 + \int_0^t 2B_s\ \mathrm{d}B_s + \frac{1}{2}\int_0^t 2\ \mathrm{d}s = \int_0^t 2B_s\ \mathrm{d}B_s + t,$$

since $B_0 = 0$. Therefore $B_t^2 - t = \int_0^t 2B_s\ \mathrm{d}B_s$. The right-hand side is an Itô integral, hence a (continuous) martingale. □

2. The exponential martingale M_t^θ. Fix $\theta \in \mathbb{R}$ and define $M_t^\theta := \exp\big(\theta B_t - \frac{1}{2}\theta^2 t\big)$. Set

$Y_t \coloneqq \theta B_t - \frac{1}{2}\theta^2 t$, so that $M_t^\theta = \mathrm{e}^{Y_t}$. To compute the quadratic variation of Y, we recall that only the stochastic part (the term in $\mathrm{d}B_t$) contributes to the bracket. Given:

$$\mathrm{d}Y_t = \theta \,\mathrm{d}B_t - \frac{1}{2}\theta^2 \,\mathrm{d}t \tag{22}$$

the quadratic variation is obtained by the rule $(\mathrm{d}B_t)^2 = \mathrm{d}t$:

$$\mathrm{d}\langle Y, Y\rangle_t = \theta^2 \,\mathrm{d}\langle B, B\rangle_t = \theta^2 \,\mathrm{d}t \tag{23}$$

Apply Itô's formula to $f(y) = \mathrm{e}^y$ (so $f' = f'' = \mathrm{e}^y$):

$$\mathrm{d}M_t^\theta = \mathrm{e}^{Y_t} \ \mathrm{d}Y_t + \frac{1}{2}\mathrm{e}^{Y_t} \ \mathrm{d}\langle Y, Y\rangle_t = M_t^\theta \left(\theta \ \mathrm{d}B_t - \frac{1}{2}\theta^2 \ \mathrm{d}t\right) + \frac{1}{2}M_t^\theta \left(\theta^2 \ \mathrm{d}t\right) = \theta M_t^\theta \ \mathrm{d}B_t.$$

Thus $M_t^\theta = 1 + \int_0^t \theta M_s^\theta \ \mathrm{d}B_s$. The Itô integral is a martingale, hence $\mathbb{E}[M_t^\theta] = 1$ for all $t \geq 0$. □

Remark 5. Alternatively, one could directly apply Itô's formula in dimension two to the function $f(x,t) = \exp\bigl(\theta x - \frac{1}{2}\theta^2 t\bigr)$, evaluated along (B_t, t). This yields the same result in a single step.

4.3 Girsanov's Theorem and Change of Measure

To move from a physical (historical) description of price dynamics to a pricing framework, we need a way to change probability measures while keeping a Brownian structure. Girsanov's theorem provides exactly this.

Stochastic exponential Let $(X_t)_{t\geq 0}$ be a continuous martingale and define

$$Z_t \coloneqq \exp\left(X_t - \frac{1}{2}\langle X, X\rangle_t\right).$$

Then (Z_t) is a positive martingale and satisfies (formally)

$$\mathrm{d}Z_t = Z_t \ \mathrm{d}X_t.$$

Change of measure To rigorously define the new probability measure $\mathbb{Q}$, we first fix a finite time horizon $T > 0$. We define $\mathbb{Q}$ on $\mathcal{F}_T$ by the Radon-Nikodym derivative:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Z_T.$$

Let $t \in [0,T]$ and let L_t be any bounded $\mathcal{F}_t$-measurable random variable. By definition of the change of measure, we have:

$$\mathbb{E}_{\mathbb{Q}}[L_t] = \mathbb{E}_{\mathbb{P}}[L_t Z_T].$$

Using the tower property of conditional expectation, we can write:

$$\mathbb{E}_{\mathbb{P}}[L_t Z_T] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[L_t Z_T \mid \mathcal{F}_t]] .$$

Since L_t is $\mathcal{F}_t$-measurable, it acts as a constant with respect to the conditional expectation given $\mathcal{F}_t$ and can be factored out:

$$\mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[L_t Z_T \mid \mathcal{F}_t]] = \mathbb{E}_{\mathbb{P}}[L_t \, \mathbb{E}_{\mathbb{P}}[Z_T \mid \mathcal{F}_t]] .$$

Finally, since $(Z_t)_{t\in[0,T]}$ is a martingale under $\mathbb{P}$, we have $\mathbb{E}_{\mathbb{P}}[Z_T \mid \mathcal{F}_t] = Z_t$. This yields the fundamental consistency relation:

$$\mathbb{E}_{\mathbb{Q}}[L_t] = \mathbb{E}_{\mathbb{P}}[L_t Z_t].$$

Since the horizon T was chosen arbitrarily, this formula holds for any $T > 0$. This justifies limiting the Radon-Nikodym derivative to $\mathcal{F}_t$ as Z_t for processes observed up to time t.

Theorem 1 (Girsanov)**.** *Let (B_t) be a Brownian motion under $\mathbb{P}$, and let $(u_t)_{t\geq 0}$ be an adapted process. Define*

$$\widetilde{B}_t \coloneqq B_t - \int_0^t u_s \, \mathrm{d}s.$$

Under the measure $\mathbb{Q}$, defined by the Radon-Nikodym derivative restricted to $\mathcal{F}_t$:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t u_s \, \mathrm{d}B_s - \frac{1}{2}\int_0^t u_s^2 \, \mathrm{d}s\right), \tag{24}$$

the process $(\widetilde{B}_t)_{t\geq 0}$ is a standard Brownian motion by Girsanov's theorem.

5 Derivation of the Black–Scholes PDE by Replication

In this section, we derive the Black–Scholes partial differential equation (PDE) using a replication argument. The key idea is to construct a self-financing portfolio that replicates the payoff of a European claim. By absence of arbitrage, the replicating portfolio must earn the risk-free rate once the stochastic risk has been hedged away.

5.1 Dynamics and Itô's Expansion

Assume that under the physical measure $\mathbb{P}$ the risky asset satisfies

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t, \qquad X_0 = x > 0, \tag{25}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants and (B_t) is a Brownian motion.

Let $T > 0$ and let $g : \mathbb{R}_+ \to \mathbb{R}$ be a payoff function (for instance $g(x) = (x - K)^+$). We denote by $V(t, x)$ the time-t value of the claim when $X_t = x$:

$$V(t, x) = \text{price at time } t \text{ of the claim with payoff } g(X_T).$$

We assume that $V \in C^{1,2}([0, T) \times \mathbb{R}_+)$ (once differentiable in t and twice in x), so that Itô's lemma applies to the process $V(t, X_t)$.

Here and throughout the paper, subscripts denote partial derivatives. For a sufficiently smooth function $V(t, x)$, we write:

$$V_t = \frac{\partial V}{\partial t}, \qquad V_x = \frac{\partial V}{\partial x}, \qquad V_{xx} = \frac{\partial^2 V}{\partial x^2}. \tag{26}$$

In particular, in finance, the quantity V_x (the sensitivity of the option price to the underlying price) corresponds to the *Delta* hedge ratio, and V_{xx} (the sensitivity of the Delta to the underlying price) to the *Gamma*.

By Itô's formula defined earlier, we can consider the detailed equation

$$\mathrm{d}V(t, X_t) = V_t(t, X_t) \,\mathrm{d}t + V_x(t, X_t) \,\mathrm{d}X_t + \frac{1}{2} V_{xx}(t, X_t) \,\mathrm{d}\langle X, X\rangle_t.$$

From (25) we have $\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t$ and

$$\mathrm{d}\langle X, X\rangle_t = (\sigma X_t)^2 \,\mathrm{d}t = \sigma^2 X_t^2 \,\mathrm{d}t.$$

Therefore

$$\mathrm{d}V(t, X_t) = \Big(V_t + \mu X_t V_x + \tfrac{1}{2}\sigma^2 X_t^2 V_{xx}\Big)(t, X_t) \,\mathrm{d}t + \sigma X_t V_x(t, X_t) \,\mathrm{d}B_t. \tag{27}$$

5.2 Building the Replicating Portfolio

We now want to build a self-financing replication portfolio. Consider a portfolio holding Δ_t units of the risky asset X_t. Let Π_t be the portfolio value. If the strategy is self-financing, then

$$\mathrm{d}\Pi_t = \Delta_t \,\mathrm{d}X_t + r(\Pi_t - \Delta_t X_t) \,\mathrm{d}t. \tag{28}$$

We aim to replicate the claim, i.e.

$$\Pi_t = V(t, X_t) \quad \text{for all } t \in [0, T].$$

We choose the hedging ratio

$$\Delta_t := V_x(t, X_t). \tag{29}$$

Using (28) and (25), we can rephrase it as

$$\mathrm{d}\Pi_t = \Delta_t(\mu X_t \ \mathrm{d}t + \sigma X_t \ \mathrm{d}B_t) + r(\Pi_t - \Delta_t X_t) \ \mathrm{d}t.$$

Substituting $\Pi_t = V(t, X_t)$ and $\Delta_t = V_x(t, X_t)$ gives

$$\mathrm{d}\Pi_t = \Big(\mu X_t V_x + r(V - X_t V_x)\Big)(t, X_t) \ \mathrm{d}t + \sigma X_t V_x(t, X_t) \ \mathrm{d}B_t. \tag{30}$$

5.3 The Black–Scholes PDE

Comparing (30) with (27), we see that the terms in $\mathrm{d}B_t$ coincide by construction. Hence the difference process $\Pi_t - V(t, X_t)$ has no stochastic part, which is a great leap forward. Since $\Pi_t = V(t, X_t)$, their drifts must also match. Therefore, comparing the $\mathrm{d}t$ coefficients in (27) and (30) yields

$$\Big(V_t + \mu X_t V_x + \tfrac{1}{2}\sigma^2 X_t^2 V_{xx}\Big) = \Big(\mu X_t V_x + r(V - X_t V_x)\Big).$$

The $\mu X_t V_x$ terms cancel, and we obtain

$$V_t + \tfrac{1}{2}\sigma^2 X_t^2 V_{xx} = rV - rX_t V_x.$$

Since this holds for all values of $X_t = x > 0$, we get the Black–Scholes PDE:

$$\boxed{V_t(t,x) + \frac{1}{2}\sigma^2 x^2 V_{xx}(t,x) + rxV_x(t,x) - rV(t,x) = 0, \qquad (t,x) \in [0,T) \times \mathbb{R}_+.} \tag{31}$$

The terminal condition comes from the payoff at maturity:

$$V(T, x) = g(x). \tag{32}$$

Remark 6. A notable consequence of the replication argument is that the drift μ does not appear in the pricing PDE. Only the risk-free rate r matters, reflecting the fact that in an arbitrage-free complete market, prices are determined under risk-neutral valuation.

6 Risk-Neutral Valuation and Explicit Formula

We now turn to the probabilistic approach. We apply the tools of stochastic calculus to the Black–Scholes model. First, Itô's formula yields the explicit solution of the geometric Brownian motion. Then, combining no-arbitrage arguments with martingale ideas leads to closed-form option pricing formulas via the risk-neutral measure.

6.1 Solution of the geometric Brownian motion

Consider

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t, \qquad X_0 = x_0 > 0.$$

Apply Itô's formula to $\log X_t$:

$$\mathrm{d}(\log X_t) = \frac{1}{X_t}\,\mathrm{d}X_t - \frac{1}{2}\frac{1}{X_t^2}\,\mathrm{d}\langle X, X\rangle_t.$$

Here $\mathrm{d}\langle X, X\rangle_t = \sigma^2 X_t^2 \,\mathrm{d}t$, hence

$$\mathrm{d}(\log X_t) = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\,\mathrm{d}B_t.$$

Integrating from 0 to t,

$$\log X_t = \log x_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t,$$

so

$$X_t = x_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right). \tag{33}$$

6.2 Link between mathematical tools and finance concepts

A market is arbitrage-free if it does not allow any arbitrage opportunity. An arbitrage opportunity is defined as a self-financing strategy that requires no initial investment, has no risk of loss, and has a positive probability of making a profit. Essentially, it is a strategy that never loses. For example, if a stock is traded at different prices on two different markets, one could buy the asset at the lower price and instantly sell it at the higher price; this constitutes a classic arbitrage strategy.

Mathematically, a market is *arbitrage-free* if there exists an equivalent probability measure $\mathbb{Q}$ such that the discounted risky asset is a martingale under $\mathbb{Q}$. To prove this, let us assume that the discounted risky asset is a $\mathbb{Q}$-martingale and that an arbitrage opportunity exists. We consider a self-financing trading strategy $(\theta_t)_{t\in[0,T]}$ whose portfolio value (V_t) satisfies:

$$V_0 = 0, \tag{34}$$

$$V_T \geq 0 \quad \mathbb{P}\text{-a.s.}, \tag{35}$$

$$\mathbb{P}(V_T > 0) > 0. \tag{36}$$

Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ (denoted $\mathbb{Q} \sim \mathbb{P}$), which means:

$$\forall A \in \mathcal{F}, \quad \mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0. \tag{37}$$

This implies that $\mathbb{P}$ and $\mathbb{Q}$ share the same null sets and, by extension, the same support. In particular:

$$\mathbb{P}(V_T > 0) > 0 \quad \Longrightarrow \quad \mathbb{Q}(V_T > 0) > 0. \tag{38}$$

Define the discounted portfolio value by $\tilde{V}_t := e^{-rt}V_t$. Since the discount factor is strictly positive, we have:

$$\tilde{V}_T \geq 0 \quad \mathbb{Q}\text{-a.s.}, \qquad \mathbb{Q}(\tilde{V}_T > 0) > 0. \tag{39}$$

Therefore, by the properties of the expectation for non-negative random variables:

$$\mathbb{E}^{\mathbb{Q}}[\tilde{V}_T] > 0. \tag{40}$$

However, since the discounted risky asset price is a martingale under $\mathbb{Q}$, the discounted value of any self-financing portfolio starting at zero must also be a $\mathbb{Q}$-martingale. Consequently:

$$\mathbb{E}^{\mathbb{Q}}[\tilde{V}_T] = \tilde{V}_0 = V_0 = 0. \tag{41}$$

This yields a contradiction $(0 > 0)$. Hence, no arbitrage opportunity can exist under these assumptions.
In the Black–Scholes model, the measure $\mathbb{Q}$ exists and is unique. Different equivalent martingale measures correspond to different possible prices for the same payoff. Here, the uniqueness of $\mathbb{Q}$ implies that for every contingent claim, there exists a unique arbitrage-free price.

For a contingent claim H with maturity T, its time-t price is given by the risk-neutral expectation:

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}H \mid \mathcal{F}_t\right]. \tag{42}$$

6.3 Explicit Derivation of the Black–Scholes Formula

There are two complementary viewpoints to obtain the Black–Scholes price: the PDE approach (replication) and the probabilistic approach (risk-neutral expectation). We now derive explicitly the closed-form formula for a European call by computing the expectation under $\mathbb{Q}$; this computation will later match the solution of the PDE.

Recall that under the risk-neutral measure, the price of the underlying asset $(X_t)_{t\geq 0}$

follows a Geometric Brownian Motion (GBM) described by the following Stochastic Differential Equation (SDE):

$$\mathrm{d}X_t = rX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t, \tag{43}$$

where r is the risk-free rate, σ is the constant volatility, and (B_t) is a standard Brownian motion under $\mathbb{Q}$.

Let X_T denote the price of the underlying asset at maturity T. The payoff of a European call option with strike K is given by:

$$(X_T - K)^+ = (X_T - K)\,\mathbf{1}_{\{X_T \geq K\}}. \tag{44}$$

Therefore,

$$\mathbb{E}^{\mathbb{Q}}\big[(X_T - K)^+\big] = \mathbb{E}^{\mathbb{Q}}\big[X_T \mathbf{1}_{\{X_T \geq K\}}\big] - K\,\mathbb{Q}(X_T \geq K).$$

Under the risk-neutral measure $\mathbb{Q}$, the asset price follows

$$\mathrm{d}X_t = rX_t\ \mathrm{d}t + \sigma X_t\ \mathrm{d}B_t, \qquad X_0 = x_0.$$

which has an explicit solution

$$X_T = x_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T\right).$$

If we consider

$$Z := \frac{B_T}{\sqrt{T}} \sim \mathcal{N}(0,1).$$

Then the solution can be written as

$$X_T = x_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,Z\right).$$

From the definition of the European call,

$$X_T \geq K \quad \Longleftrightarrow \quad Z \geq \frac{\ln(K/x_0) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.$$

Hence,

$$\mathbb{Q}(X_T \geq K) = \int_{\frac{\ln(K/x_0) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}^{+\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}\ \mathrm{d}x.$$

and on the other part,

$$\mathbb{E}^{\mathbb{Q}}\big[X_T\mathbf{1}_{\{X_T\geq K\}}\big] = x_0\mathrm{e}^{(r-\frac{\sigma^2}{2})T}\mathbb{E}\left[\mathrm{e}^{\sigma\sqrt{T}Z}\mathbf{1}_{\left\{Z\geq\frac{\ln(K/x_0)-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right\}}\right].$$

Using the density of Z, we obtain

$$x_0\mathrm{e}^{(r-\frac{\sigma^2}{2})T}\int_{d_1}^{+\infty}\mathrm{e}^{\sigma\sqrt{T}x}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x,$$

where

$$d_1 := \frac{\ln(K/x_0)-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

We rewrite the exponent as

$$\sigma\sqrt{T}x-\frac{x^2}{2} = -\frac{1}{2}(x-\sigma\sqrt{T})^2+\frac{1}{2}\sigma^2T.$$

Thus,

$$\mathbb{E}^{\mathbb{Q}}\big[X_T\mathbf{1}_{\{X_T\geq K\}}\big] = x_0\mathrm{e}^{rT}\int_{d_1}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-\sigma\sqrt{T})^2}{2}}\,\mathrm{d}x.$$

Performing the change of variable $y = x-\sigma\sqrt{T}$,

$$\mathbb{E}^{\mathbb{Q}}\big[X_T\mathbf{1}_{\{X_T\geq K\}}\big] = x_0\mathrm{e}^{rT}\int_{d_1-\sigma\sqrt{T}}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y.$$

As found earlier, let's define

$$d_1 = \frac{\ln(x_0/K)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1-\sigma\sqrt{T}.$$

which gives us a better expression of the esperance under $\mathbb{Q}$ of a European call

$$\mathbb{E}^{\mathbb{Q}}\big[(X_T-K)^+\big] = x_0\mathrm{e}^{rT}\Phi(d_1)-K\Phi(d_2).$$

Discounting back to time 0, we obtain the Black–Scholes price

$$\boxed{C_0 = x_0\Phi(d_1)-K\mathrm{e}^{-rT}\Phi(d_2).}$$

This completes the explicit probabilistic derivation of the Black–Scholes formula.

6.4 Probabilistic Representation and Time-Homogeneity

To compute the price $V(t,x)$, we rely on the risk-neutral representation. Here, $(\tilde{B}_t)_{t\geq 0}$ denotes a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. It is defined via Girsanov's theorem by the transformation $\tilde{B}_t = B_t + \int_0^t \theta \, \mathrm{d}s$, where $\theta = \frac{\mu - r}{\sigma}$ is the market price of risk, effectively shifting the drift of the underlying asset to the risk-free rate r.

Let $\tau = T - t$ be the time to maturity. We aim to show that the pricing function depends only on τ and the current price x:

$$V(t,x) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[(x\mathcal{X}_\tau - K)^+\right] \tag{45}$$

where $\mathcal{X}_\tau$ is a random variable independent of $\mathcal{F}_t$.

Proof Under the measure $\mathbb{Q}$, the solution to the SDE for the underlying asset at maturity T can be written in terms of X_t:

$$X_T = X_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(\tilde{B}_T - \tilde{B}_t)\right). \tag{46}$$

The term $(\tilde{B}_T - \tilde{B}_t)$ represents a future Brownian increment. By the fundamental properties of Brownian motion:

1. **Stationarity**: The increment $(\tilde{B}_T - \tilde{B}_t)$ has the same distribution as $\tilde{B}_{T-t}$.
2. **Independence**: This increment is independent of the sigma-algebra $\mathcal{F}_t$.

Let us define the random variable $\mathcal{X}_\tau$ representing the growth factor over the remaining time $\tau = T - t$:

$$\mathcal{X}_\tau \coloneqq \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma Z_\tau\right), \quad \text{where } Z_\tau \sim \mathcal{N}(0,\tau).$$

The value of the option at time t is defined as the conditional expectation of the payoff with respect to the filtration $\mathcal{F}_t$:

$$V_t = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[(X_T - K)^+ \mid \mathcal{F}_t\right].$$

First, we observe that (X_t) is a **Markov process**. Indeed, the future value X_T depends only on the current value X_t and the future Brownian increment, which is independent of the past $\mathcal{F}_t$. Consequently, conditioning on the entire history $\mathcal{F}_t$ is equivalent to conditioning on the current state X_t:

$$\mathbb{E}^{\mathbb{Q}}\left[(X_T - K)^+ \mid \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[(X_T - K)^+ \mid X_t\right].$$

To compute this explicitely, we use the functional representation. Since X_T can be written as $X_T = \varphi(X_t, Y)$ where Y is independent of $\mathcal{F}_t$, we treat X_t as a fixed constant x:

$$V(t,x) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[(x\mathcal{X}_\tau - K)^+\right]. \tag{47}$$

This confirms the time-homogeneity of the model: the option price depends strictly on the time remaining until maturity τ. □

Connection with the Black-Scholes PDE This probabilistic representation provides the fundamental link with the analytical approach. Specifically, the function $V(t,x)$ defined by the risk-neutral expectation above corresponds precisely to the unique solution of the **Black-Scholes PDE (31)** derived in Section 5 via the replication argument, subject to the terminal condition $V(T,x) = (x-K)^+$. This duality confirms that the no-arbitrage price obtained by dynamic hedging coincides perfectly with the expected discounted payoff under the risk-neutral measure.

7 Limits of the Black–Scholes model

The Black–Scholes model assumes that the asset price follows a geometric Brownian motion

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t, \tag{48}$$

where μ and σ are constant parameters and (B_t) is a Brownian motion.

While this model is mathematically tractable and leads to closed-form pricing formulas [3, 4, 8, 7], its assumptions are only approximations of real market behavior. In this section, we summarize the main limitations of the Black–Scholes framework and briefly indicate standard extensions used in practice [7, 8].

7.1 Constant volatility

A central assumption is that volatility is constant over time:

$$\sigma_t = \sigma.$$

This assumption is mainly made for analytical convenience [3, 8]. However, empirical evidence shows that volatility varies through time and across market regimes, and that implied volatilities depend on strike and maturity (the smile/skew effect) [7].

To illustrate this point, Figure 2 compares a Black–Scholes trajectory with constant volatility ($\sigma = 0.20$) to a trajectory generated using a time-dependent volatility function

$\sigma(t)$. In the latter case, the price dynamics exhibit periods of relative calm and periods of higher diffusion, reflecting the non-stationarity of market risk.

Figure 2: Simulated asset price paths: Constant volatility (left) vs. time-dependent volatility $\sigma(t) = 0.2 + 0.1\sin(2\pi t)$ (right).

Observed option prices imply an *implied volatility* depending on strike K and maturity T:

$$\sigma = \sigma(K, T), \tag{49}$$

which is known as the *volatility smile* (or *skew*) [7]. This implies that the market assigns different risk levels to different moneyness levels for the same maturity. As a consequence, a single constant σ cannot reproduce the complexity of the observed option surface.

7.2 Gaussian log-returns

From (48), the log-return satisfies

$$\log\left(\frac{S_T}{S_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma B_T,$$

hence

$$\log\left(\frac{S_T}{S_0}\right) \sim \mathcal{N}\left(\left(\mu - \frac{1}{2}\sigma^2\right)T,\ \sigma^2 T\right).$$

From (25), the log-return satisfies:

$$\ln\left(\frac{X_T}{X_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma B_T \sim \mathcal{N}\left(\left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right). \tag{50}$$

This implies a symmetric distribution with thin tails (exponential decay). In practice, empirical returns display **heavy tails** and asymmetry, meaning extreme events occur far more frequently than predicted by the Gaussian assumption [6].

To account for this *excess kurtosis*, one common approach is to model returns using a **Student's t-distribution**. Unlike the Gaussian law, its tails decay according to a power law, which significantly increases the probability of "black swan" events. For instance, while a 5-sigma event is virtually impossible in a normal model, it becomes statistically plausible under a Student distribution with low degrees of freedom, better matching the reality of market crashes..

7.3 Continuous paths and absence of jumps

Because Brownian motion has continuous trajectories, the Black–Scholes model satisfies

$$\mathbb{P}\big(S_t \text{ is continuous in } t\big) = 1.$$

In real markets, prices can jump due to earnings announcements, macroeconomic news, or crises. Such discontinuities cannot be captured by a pure diffusion model [**?**].

Jumps also break the ideal perfect-hedging argument: they cannot be anticipated nor hedged instantaneously, and the market is no longer complete in general [**?**, 8].

7.4 Continuous trading and perfect replication

The derivation of the Black–Scholes price relies on continuous-time trading and on the existence of a perfectly replicating portfolio [3, 4, 8].

In practice, trading is discrete and transaction costs exist, so exact replication is impossible and hedging strategies involve residual risk [7].

7.5 Interest Rate Dynamics

The standard Black–Scholes model assumes a constant risk-free interest rate:

$$r_t = r. \tag{51}$$

In reality, interest rates vary over time and exhibit stochastic behavior, which significantly impacts the pricing and hedging of derivatives, particularly for long-dated instruments where the sensitivity to rates (Rho) is higher [7, 8].

To illustrate this effect, Figure 3 compares an asset price path under a constant rate ($r = 5\%$) with a path where the rate follows a cyclical function $r(t) = 0.05 + 0.04\sin(2\pi t/T)$. Even when driven by the exact same Brownian motion ($\tilde{B}_t$), the trajectories diverge.

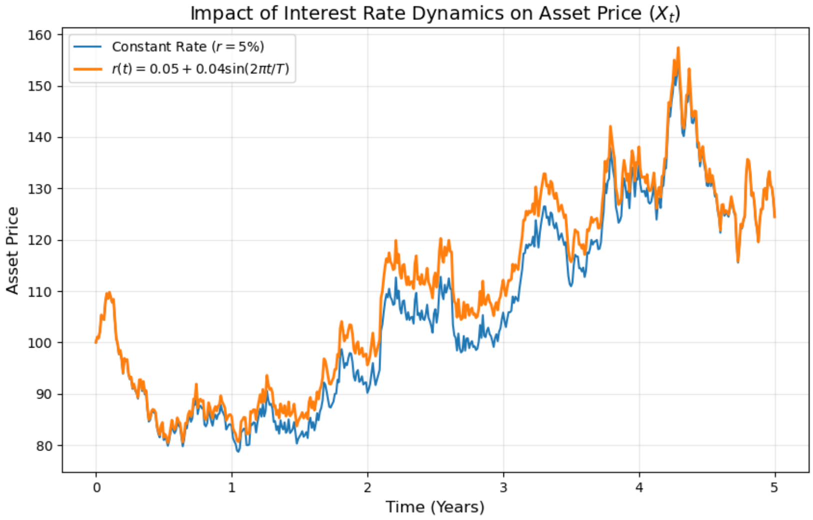


Figure 3: Impact of interest rate modeling: Constant rate vs. time-dependent drift $r(t)$.

Analysis of the Impact The divergence observed in Figure 3 stems from the accumulation of the risk-neutral drift. Under a time-varying rate, the discount factor and the forward price depend on the integral $\int_0^T r(s)\mathrm{d}s$. In a market environment with fluctuating rates, a constant r fails to capture the term structure of interest rates, leading to systematic pricing errors. For a future quant, this highlights the necessity of more sophisticated models, such as the **Hull–White** or **Heath–Jarrow–Morton** (HJM) frameworks, which treat r_t as a stochastic process.

7.6 A calibrated pricing framework

Finally, the volatility parameter σ is not predicted by the model: it is calibrated from market prices.

Therefore, Black–Scholes should be viewed primarily as a no-arbitrage pricing framework, and as a starting point for more refined models (local volatility, stochastic volatility, jumps, ...) [7, 8, 9].

8 Conclusion

The Black–Scholes model occupies a central position in mathematical finance, not because of the realism of its assumptions, but because of the coherent and tractable framework it provides. By combining probabilistic tools, no-arbitrage arguments, and partial differential equations, the model establishes a clear link between asset dynamics, hedging strategies, and derivative pricing.

Throughout this report, we showed how Brownian motion and Itô calculus naturally lead to the Black–Scholes PDE and to the explicit pricing formula for European options. This dual probabilistic and analytical viewpoint highlights the internal consistency of the model and explains its enduring role as a reference in both theory and practice.

At the same time, the analysis of the model's limitations emphasizes that Black–Scholes should not be interpreted as a faithful description of real market dynamics. Rather, it serves as a foundational benchmark from which more realistic models—incorporating stochastic volatility, jumps, or market frictions—can be developed. In this sense, the Black–Scholes model remains an essential starting point for modern quantitative finance.

A Numerical Simulation Scripts

This appendix provides the Python source code used to generate the figures presented in this report. The simulations rely on the `numpy` package for stochastic increments and `matplotlib` for visualization.

A.1 Standard Brownian Motion Simulation

This script generates the trajectories of B_t used on the cover page and in Section 3.

```
import numpy as np
import matplotlib.pyplot as plt

# Simulation parameters
T = 1.0          # Time horizon
N = 1000         # Number of time steps
dt = T / N       # Time step size
num_paths = 50 # Number of paths to simulate
t = np.linspace(0, T, N + 1)

# Generate Brownian increments: dW ~ N(0, sqrt(dt))
np.random.seed(42)
dW = np.random.normal(0, np.sqrt(dt), size=(num_paths, N))

# Compute the Brownian paths W_t (starting at 0)
W = np.zeros((num_paths, N + 1))
W[:, 1:] = np.cumsum(dW, axis=1)

# Plotting
plt.figure(figsize=(10, 6))
for i in range(num_paths):
    plt.plot(t, W[i, :], linewidth=0.8, alpha=0.7)

plt.title('Simulation of Standard Brownian Motion Paths')
plt.xlabel('Time (t)')
plt.ylabel('Value ($B_t$)')
plt.grid(True, alpha=0.3)
plt.savefig('brownian_motion.png')
plt.show()
```

A.2 Time-Dependent Volatility Comparison

This script compares the GBM dynamics under constant σ versus time-varying $\sigma(t)$.

```
import numpy as np
import matplotlib.pyplot as plt

```

```python
# Parameters for the SDE
S0, r, T, N = 100, 0.05, 1.0, 252
dt = T / N

# Constant Volatility
sigma_const = 0.2

def simulate_gbm_const_vol(S0, r, sigma, dt, N, num_paths):
    S = np.zeros((N + 1, num_paths))
    S[0] = S0
    for t in range(1, N + 1):
        dW = np.random.normal(0, np.sqrt(dt), num_paths)
        S[t] = S[t-1] * np.exp((r - 0.5 * sigma**2) * dt + sigma * dW)
    return S

# Time-Dependent Volatility function
def sigma_time_dependent(t, T):
    return 0.2 + 0.1 * np.sin(2 * np.pi * t / T)

def simulate_gbm_time_dependent_vol(S0, r, sigma_func, dt, N, num_paths
    , T_total):
    S = np.zeros((N + 1, num_paths))
    S[0] = S0
    time_points = np.linspace(0, T_total, N + 1)
    for t_idx in range(1, N + 1):
        curr_sigma = sigma_func(time_points[t_idx-1], T_total)
        dW = np.random.normal(0, np.sqrt(dt), num_paths)
        S[t_idx] = S[t_idx-1] * np.exp((r - 0.5 * curr_sigma**2) * dt +
     curr_sigma * dW)
    return S

# Simulation & Plotting
paths_const = simulate_gbm_const_vol(S0, r, sigma_const, dt, N, 5)
paths_time_dep = simulate_gbm_time_dependent_vol(S0, r,
    sigma_time_dependent, dt, N, 5, T)

# Visualization logic... (refer to source for plt.subplot calls)
```

A.3 Interest Rate Dynamics

This script demonstrates the impact of a cyclical risk-free rate $r(t)$ on the asset drift.

```python
import numpy as np
import matplotlib.pyplot as plt

# Simulation parameters
S0, sigma, T, N = 100, 0.2, 5.0, 500
```

```python
dt = T / N
t = np.linspace(0, T, N+1)

def r_func(t):
    return 0.05 + 0.04 * np.sin(2 * np.pi * t / T)

np.random.seed(42)
dW = np.random.normal(0, np.sqrt(dt), N)
W = np.insert(np.cumsum(dW), 0, 0)

# Constant rate case
S_const = S0 * np.exp((0.05 - 0.5 * sigma**2) * t + sigma * W)

# Variable rate case (numerical integration)
r_integral = np.cumsum(r_func(t)) * dt
S_variable = S0 * np.exp(r_integral - 0.5 * sigma**2 * t + sigma * W)

# Plotting results
plt.plot(t, S_const, label='Constant Rate ($r=5\%$)')
plt.plot(t, S_variable, label='$r(t) = 0.05 + 0.04 \sin(2\pi t / T)$')
plt.savefig('interest_rate_impact.png')
```

References

[1] L. Bachelier, *Théorie de la spéculation*, Annales scientifiques de l'École normale supérieure, Série 3, Tome 17, pp. 21–86, 1900.

[2] K. Itô, *Stochastic integral*, Proceedings of the Imperial Academy, vol. 20, pp. 519–524, 1944.

[3] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, vol. 81, no. 3, pp. 637–654, 1973.

[4] R. C. Merton, *Theory of rational option pricing*, The Bell Journal of Economics and Management Science, vol. 4, no. 1, pp. 141–183, 1973.

[5] R. C. Merton, *Option pricing when underlying stock returns are discontinuous*, Journal of Financial Economics, vol. 3, no. 1-2, pp. 125–144, 1976.

[6] R. Cont, *Empirical properties of asset returns: stylized facts and statistical issues*, Quantitative Finance, vol. 1, no. 2, pp. 223–236, 2001.

[7] J. C. Hull, *Options, Futures, and Other Derivatives*, 11th Edition, Pearson Education, 2021.

[8] S. E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer Finance, 2004.

[9] M. S. Joshi, *The Concepts and Practice of Mathematical Finance*, 2nd Edition, Cambridge University Press, 2008.

[10] The Royal Swedish Academy of Sciences, *The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1997*, Press Release, 1997.