Mean-field method for handling null constraints on positive-semidefinite operators

Garry Goldstein[1], Claudio Chamon[2], Claudio Castelnovo[1]
[1] *TCM Group, Cavendish Laboratory, University of Cambridge, J. J. Thomson Avenue, Cambridge CB3 0HE, United Kingdom and*
[2] *Department of Physics, Boston University, Boston, Massachusetts 02215, USA*

We present an approach for studying systems with hard constraints that certain positive-semidefinite operators must vanish. The difficulty with mean-field treatments of such cases is that imposing that the constraint is zero only in average is problematic for a quantity that is always non-negative. We reformulate the hard constraints by adding an auxiliary system such that some of the states to be projected out from the total system are at finite negative energy and the rest at finite positive energy. This auxiliary system comes with an extra coupling that we are free to vary and that parametrizes a whole family of mean-field theories. We argue that this variational-type parameter for the family of mean-field theories should be fixed by matching a given experimental observation, with the quality of the resulting mean-field approximation measured by how it fits other data. We test these ideas in the well-understood single-impurity Kondo problem, where we fix the parameter via the T_K obtained from the magnetic susceptibility value, and score the quality of the approximation by its predicted Wilson ratio.

I. INTRODUCTION

One of the key theoretical challenges in condensed matter physics is to understand systems with strong correlations, when interactions become larger than the kinetic energy dispersion bandwidth. Physical systems which fall in this important class include: the cuprate superconductors, where strong interactions between electrons in copper's 3d shells lead to an antiferromagnetic Mott insulator at half-filling and superconductivity upon hole doping; heavy-electron compounds where localized d and f orbitals of rare earth and transition metal atoms interact with itinerant electrons, leading to a renormalization of the electron mass by as much as a thousand times; and systems of cold atomic gases, where interactions may be tuned by a Feshbach resonance to realize strongly interacting multispecies systems [1–5, 9, 10, 16, 17]. In these and other examples of strongly correlated systems of interest in modern condensed matter physics, it is customary to encounter a broad range of energy scales that make any theoretical study a tall order. One can often make substantial progress by producing effective models where the largest energy scales are "projected out"; namely, the system is assumed to be in the lowest energy states of these large energy terms, thus introducing strict conditions (or constraints) in the effective models. These constraints lead to effective models such as the t-J model, the Heisenberg model, and the Kondo model, to name a few.

In this work we propose a method, which we call the soft constraint, to study systems where the constraint is that a positive-semidefinite operator annihilates the physical states. A mean-field treatment of the constraint, where it is satisfied in average, not identically, is that the average is already lower-bounded by zero, because of the positive-semidefiniteness of the operator. Therefore, the mean-field theory would be biased towards failure. Here we propose a method to circumvent this problem: we introduce an auxiliary degree of freedom with its own constraint, and construct a combined constraint that is satisfied only when both the original and new constraints are satisfied. The new quantity is no longer positive semidefinite, and hence the combined constraint can now be satisfied only in average. This method comes with an additional parameter than can be varied freely. We thus have not one, but instead a family of theories that, when treated exactly, reproduce the correct constraints. When treated within a mean-field approximation, the family of theories become different approximations to the same physical problem. We propose to fix the value of the parameter as follows: we choose one physical observable, and find the value of the "variational mean-field" parameter to best match the observable. The usefulness of the approximation with this fixed parameter can be measured by how it fits other observables. We choose as a benchmark the single-impurity Kondo problem, where we fix the parameter via the T_K obtained from the magnetic susceptibility value, and grade the quality of the approximation by the deviation of the predicted Wilson ratio from the exact Bethe Ansatz result.

The introduction of the auxiliary system allows us to impose the zero-average condition in the combined operator, but as we shall see, the combined operator is often not quadratic. To deal with quartic terms, we use the Kotliar-Ruckenstein approach. The combination of these two methods produces better results then the conventional Read-Newns meanfield for the Kondo model [1] for the Wilson ratio, and it reproduces the width of the Abrikosov-Suhl resonance to two loop level for the Kondo RG [1].

In addition to the Kondo problem, we show that the method gives the exact density of states for the infinite U Anderson model. In fact, we use this system as a simple way to first present, in section II, the basics of the soft-constraint method. Other systems that could be studied using these techniques include hard-core bosons, spinor condensates, t-J, Heisenberg, two-channel Kondo and two-channel Anderson models.

II. A FIRST EXAMPLE: THE INFINITE U ANDERSON MODEL

Let us present, as a first simple example of the method, a soft constraint formulation of the infinite U Anderson model for one spin 1/2 impurity. Define $c^\dagger_{k,\sigma}, c_{k,\sigma}$ and $f^\dagger_\sigma, f_\sigma$ as the creation and annihilation operators for the itinerant and impurity electrons, respectively. The impurity occupation number for spin state σ is $n_\sigma = f^\dagger_\sigma f_\sigma$. Within this notation, the Hamiltonian for the Anderson model is given by:

$$H_A = \sum_{k,\sigma} (\epsilon_k - \mu)\, c^\dagger_{k,\sigma} c_{k,\sigma} + \sum_{k,\sigma} \left(\bar{V}\ c^\dagger_{k,\sigma} f_\sigma + V\ f^\dagger_\sigma c_{k,\sigma} \right) + U\, n_\uparrow n_\downarrow\ , \tag{1}$$

where the limit $U \to \infty$ is understood.

We now consider an auxiliary system, unrelated to the impurity, which is used to satisfy the constraint. A variety of auxiliary systems are possible, slave bosons, slave fermions, slave spins and even slave compound systems; we shall focus on the case of the slave fermion h. Let

$$H = H_A + H_h \tag{2a}$$

with

$$H_h = \tilde{U}\, h^\dagger h\ , \tag{2b}$$

where the limit $\tilde{U} \to \infty$ is understood. In this limit, we are constraining the auxiliary fermion to lie in its ground state, so we are effectively adding a constant the initial Anderson model H_A. We now have two hard constrains in this system:

$$h^\dagger h = 0 \tag{3a}$$

$$n_\uparrow n_\downarrow = 0\ , \tag{3b}$$

which we would like to replace with an equivalent constraint

$$\Pi = n_\uparrow n_\downarrow - K\, h^\dagger h = 0\ , \tag{3c}$$

so long as we choose $K > 0$ and $K \neq 1$. The equivalence of combined constraints (3a) and (3b) to constraint (3c) is reasoned as follows: both $n_\uparrow n_\downarrow$ and $h^\dagger h$ take values in $\{0, 1\}$, and hence $n_\uparrow n_\downarrow - K\, h^\dagger h$ takes values in $\{-K, 0, 1-K, 1\}$; as long as $K \neq 0, 1$, all these four values are distinct, and hence enforcing the constraint (3c) implies that both (3a) and (3b) are simultaneously satisfied.

Now, while constraints (3a) and (3b) can be individually imposed via couplings $U, \tilde{U} \to \infty$, constraint (3c) cannot for our choice of $K > 0$. The reason is that the minimum attained value for Π is $-K < 0$. To impose constraint (3c), we use a Lagrange multiplier instead. The partition function for the combined Anderson model and auxiliary system, in the limit $U, \tilde{U} \to \infty$ is given by:

$$Z = Z_A(U \to \infty) \times Z_h(\tilde{U} \to \infty) = \int \mathcal{D}[c, c^\dagger] \int \mathcal{D}[f, f^\dagger] \int \mathcal{D}[\lambda]\, e^{-S_{Ah}}\ , \tag{4a}$$

where the imaginary-time action

$$S_{Ah} = \int_0^\beta d\tau\ L_{Ah}(\tau)\ , \tag{4b}$$

with Lagrangian

$$L_{Ah} = \sum_{k,\sigma} c^\dagger_{k\sigma} \left(\frac{d}{d\tau} + \epsilon_k \right) c_{k\sigma} + \sum_\sigma f^\dagger_\sigma \frac{d}{d\tau} f_\sigma - V \sum_{k,\sigma} \left(c^\dagger_{k,\sigma} f_\sigma + f^\dagger_\sigma c_{k,\sigma} \right) + h^\dagger \frac{d}{d\tau} h + \lambda(\tau) \left(n_\uparrow n_\downarrow - K\, h^\dagger h \right)\ . \tag{4c}$$

Thus far, there are no approximations, and K remains an arbitrary parameter.

We now take the stationary phase with respect to the parameter $\lambda(\tau)$:

$$\frac{\delta Z}{\delta \lambda(\tau)} = 0 \Rightarrow \left\langle n_\uparrow(\tau) n_\downarrow(\tau) - K h^\dagger(\tau) h(\tau) \right\rangle = 0\ . \tag{5}$$

Notice that, had we not introduced the slave fermion h, or equivalently had chosen $K = 0$, we would not have obtained a sensible or solvable meanfield equation, since $n_\uparrow n_\downarrow$ is a positive semidefinite operator and requiring that its average vanish is tantamount to enforcing the exact constraint and not an approximate one.

For our next step we take $\lambda(\tau) = \bar{\lambda}$ independent of time, and maximize the partition function with respect to K,

$$\frac{dZ}{dK} = 0 \Rightarrow \bar{\lambda} \left\langle h^\dagger h \right\rangle = 0\ , \tag{6}$$

from which we get that the optimum meanfield (with the largest partition function) is given by $\bar{\lambda} = 0$. This means that at meanfield the partition function for the combined Anderson model and slave fermion is given by the partition function with $U = \tilde{U} = 0$,

$$Z_{\mathrm{MF}} = Z_A(U = 0) \times Z_h(\tilde{U} = 0) \tag{7}$$

For the case of the single impurity with energy E_f it is known that the physics of the non-interacting model is controlled by the physics of the single impurity (with no free fermions) where the action is given by:

$$S_A = \sum_n \sum_\sigma \left(-i\omega_n + E_f - i\Delta\, \mathrm{sgn}\, \omega_n \right)\, f^\dagger_\sigma f_\sigma\ , \tag{8}$$

where $\Delta = \pi\, |V|^2 \rho$, where ρ is the density of states of the free fermions. The density of states for the impurity is given by

$$\rho_s(\omega) = \frac{1}{\pi} \frac{\Delta}{\Delta^2 + (\omega - E_f)^2}\ . \tag{9}$$

This approach does not capture the Abrikosov-Suhl resonance [1] but provides a good starting point when considering the density of states near the resonance at $-|E_f|$. We will therefore keep $\bar{\lambda}$ general as every $\bar{\lambda}$ produces a proper meanfield and study the Abrikosov-Suhl resonance [1] using this more general formulation. Furthermore for general K the Abriksov-Suhl resonance is restored and even satisfies the exact identity [9, 16]

$$\hat{\rho} = \frac{1}{\pi\Delta}\sin^2(\pi n_\uparrow) \ , \tag{10}$$

where $\hat{\rho}$ is the density of sates at zero energy.

III. THE KONDO MODEL

In order to study the physics of the Abrikosov-Suhl resonance in its simplest form, and to study the soft constraint for another model, we will now switch gears and study the Kondo model, which is related to the infinite U Anderson model [1]. The Kondo model is given by the following Hamiltonian:

$$H_{Kondo} = \sum_{k,\sigma} \epsilon_k \, c^\dagger_{k,\sigma} c_{k,\sigma} + J \sum_k \vec{S}_k \cdot \vec{S}_0 \tag{11}$$

Where $\vec{S}_0$ is a localized spin located at the origin. For a uniform density of states for the fermions $c_{k,\sigma}$, this model has an exact solution by the Bethe Ansatz leading to exact results for the magnetic susceptibility, density of states and heat capacity [8, 9]. We will strive to approximate these results using a simpler meanfield method.

A. Read-Newns approach

As a first step let us consider the Read-Newns formulation of the single impurity Kondo model, with Lagrangian given by:

$$\begin{aligned} L_{RN} = \quad & \sum_{k,\sigma} c^\dagger_{k\sigma}\left(\frac{d}{d\tau} + \epsilon_k\right) c_{k\sigma} \\ & + \sum_\sigma f^\dagger_\sigma \left(\frac{d}{d\tau} + \lambda_{RN}\right) f_\sigma - \lambda_{RN} \\ & + \frac{2}{J}\bar{V}V - \sum_{k,\sigma}\left[\bar{V}\, c^\dagger_{k,\sigma}\, f_\sigma + V\, f^\dagger_\sigma\, c_{k,\sigma}\right] \ , \end{aligned} \tag{12}$$

where λ_{RN} is a multiplier that enforces single occupancy $\sum_\sigma f^\dagger_\sigma f_\sigma = 1$ in the spin-1/2 fermion representation of the magnetic impurity.

At meanfield level, the Read-Newns formulation predicts a Kondo temperature

$$T_K \sim D\, e^{-\frac{1}{J\rho}} \ , \tag{13}$$

where D is the bandwidth. The exact Kondo temperature, however, is given by:

$$T_K \sim D\sqrt{J\rho}\ \ e^{-\frac{1}{J\rho}} \ , \tag{14}$$

where the factor of $\sqrt{J\rho}$, missing in the mean field result, can easily lead to a substantial correction to the value of T_K. This discrepancy has measurable consequences, since for instance the Kondo temperature is directly related to the susceptibility:

$$\chi \sim \frac{(g\mu_B)^2}{T_K} \ . \tag{15}$$

Furthermore, the meanfield Wilson ratio within the Read-Newns formulation is 1, indicating that this meanfield scheme also greatly underestimates the heat capacity.

B. Soft constraint approach

Here we consider a different approach, which starts by re-writing the condition $\sum_\sigma f^\dagger_\sigma f_\sigma = 1$ as

$$(1 - n_\uparrow - n_\downarrow)^2 = n_\uparrow n_\downarrow + (1-n_\uparrow)(1-n_\downarrow) = 0 \,. \tag{16}$$

Whereas this approach is identical to the Read-Newns formulation in the exact Lagrangian, it must be handled differently at meanfield level. Indeed, the constraint is expressed now in terms of a positive semidefinite function that needs to be set to zero, and this cannot be achieved at meanfield level as it would necessarily give a diverging meanfield parameter when solved self-consistently. We introduce instead an auxiliary non-interacting fermionic Hilbert space that is constrained to be trivially empty: $h^\dagger h = 0$. For any non-integer positive parameter K, we can then combine the constraints in the original problem and in the auxiliary fermions by imposing:

$$n_\uparrow n_\downarrow + (1-n_\uparrow)(1-n_\downarrow) - K\, h^\dagger h = 0 \,, \tag{17}$$

withe $K > 0$, $K \neq 1$. Notice that this is exactly equivalent to enforcing $n_\uparrow n_\downarrow + (1-n_\uparrow)(1-n_\downarrow) = 0$ and $h^\dagger h = 0$ separately. However, we are now able to treat the combined constraint at meanfield level and look for finite parameters at saddle point.

The soft constraint Lagrangian can then be written as

$$\begin{aligned} L_{SC} = \quad & \sum_{k,\sigma} c^\dagger_{k\sigma}\left(\frac{d}{d\tau} + \epsilon_k\right) c_{k\sigma} \\ & + \sum_\sigma f^\dagger_\sigma \frac{d}{d\tau} f_\sigma + h^\dagger \frac{d}{d\tau} h \\ & + \lambda_{SC}\left[n_\uparrow n_\downarrow + (1-n_\uparrow)(1-n_\downarrow) - K\, h^\dagger h\right] \\ & + \frac{2}{J}\bar{V}V - \sum_{k,\sigma}\left[\bar{V}\, c^\dagger_{k,\sigma}\, f_\sigma + V\, f^\dagger_\sigma\, c_{k,\sigma}\right] \,. \end{aligned} \tag{18}$$

Notice that Eq.(18) is not a meanfield Lagrangian because of the terms $\propto n_\uparrow n_\downarrow$. It is possible to use the Kotliar Ruckenstein formulation of the Anderson model to convert our Kondo path integral into a meanfield [12, 17]. We recall that the Kotliar Ruckenstein slave formulation uses four slave bosons, e, d and $p_{\uparrow,\downarrow}$. These represent the empty, doubly occupied and spin up and spin down singly occupied states, which are then subject to the constraint that there is one physical state:

$$e^\dagger e + \sum_\sigma p_\sigma^\dagger p_\sigma + d^\dagger d = 1\,, \tag{19}$$

Furthermore to ensure that the state of the fermion is correlated with the state of the boson, one needs to impose two constraints, one per spin species:

$$f_\sigma^\dagger f_\sigma = p_\sigma^\dagger p_\sigma + d^\dagger d\,. \tag{20}$$

The electron operator is then given by:

$$f_\sigma \to z_\sigma\, f_\sigma, \qquad z_\sigma = e^\dagger p_\sigma + p_{-\sigma}^\dagger d\,. \tag{21}$$

It is also conventional (see Ref. [12]) to transform

$$z_\sigma \to \left(1 - d^\dagger d - p_\sigma^\dagger p_\sigma\right)^{-1/2}\, z_\sigma\, \left(1 - e^\dagger e - p_{-\sigma}^\dagger p_{-\sigma}\right)^{-1/2}. \tag{22}$$

The operators z_σ move between the four allowed boson states while f_σ moves between the fermion states, this identity insures that the states of the bosons and fermions remain correlated and that the constraint in Eq.(20) is satisfied. The Lagrangian for this formulation is given by:

$$\begin{aligned} L = \; & \sum_{k,\sigma} c_{k\sigma}^\dagger \left(\frac{d}{d\tau} + \epsilon_k - \mu\right) c_{k\sigma} + \sum_\sigma f_\sigma^\dagger \frac{d}{d\tau} f_\sigma + h^\dagger \frac{d}{d\tau} h \; + e^\dagger \frac{d}{d\tau} e + d^\dagger \frac{d}{d\tau} d + \sum_\sigma p_\sigma^\dagger \frac{d}{d\tau} p_\sigma \\ & + \sum_\sigma \lambda_\sigma \left(f_\sigma^\dagger f_\sigma - p_\sigma^\dagger p_\sigma - d^\dagger d\right) + \lambda_{KR} \left(e^\dagger e + \sum_\sigma p_\sigma^\dagger p_\sigma + d^\dagger d - 1\right) + \lambda_{SC} \left(e^\dagger e + d^\dagger d - K h^\dagger h\right) \\ & + 2\frac{V\bar{V}}{J} + \sum_{k,\sigma} \left[\bar{V}\, c_{k,\sigma}^\dagger\, z_\sigma\, f_\sigma + V\, f_\sigma^\dagger\, z_\sigma^\dagger\, c_{k,\sigma}\right]\,. \end{aligned} \tag{23}$$

The bosonic variables can be treated as complex numbers when searching for the saddle-point of the action. Within this approximation, we find that the zero temperature meanfield free energy is given by:

$$\begin{aligned} F = \; & F_o(iz^2\Delta + \lambda) + \frac{2\Delta}{\pi J \rho} \\ & -\lambda\left(2p^2 + 2d^2\right) + \lambda_{KR}\left(2p^2 + d^2 + e^2 - 1\right) + \lambda_{SC}\left(d^2 + e^2 - K\left\langle h^\dagger h\right\rangle\right)\,, \end{aligned} \tag{24}$$

where

$$F_o(\xi) = \frac{2}{\pi}\,\mathcal{I}m\left[\xi \ln\left(\frac{\xi}{\mathrm{e}D}\right)\right]\,, \tag{25}$$

with $\xi = iz^2\Delta + \lambda$, is the contribution from the integration of the itinerant and impurity electrons, and spin-rotation symmetry implies

$$p \equiv p_\uparrow = p_\downarrow \tag{26a}$$
$$\lambda \equiv \lambda_\uparrow = \lambda_\downarrow \tag{26b}$$
$$z \equiv z_\uparrow = z_\downarrow = (1 - d^2 - p^2)^{-1/2}\, p\,(e + d)\,(1 - e^2 - p^2)^{-1/2}\,. \tag{26c}$$

The self consistent meanfield conditions are

$$\frac{\partial F}{\partial \Delta} = 0 = \left(\frac{\partial F_0}{\partial \xi} - \frac{\partial F_0}{\partial \bar{\xi}}\right) i\, z^2 + \frac{2}{\pi J \rho} \tag{27a}$$

$$\frac{\partial F}{\partial p^2} = 0 = \left(\frac{\partial F_0}{\partial \xi} - \frac{\partial F_0}{\partial \bar{\xi}}\right) i\, \Delta\, \frac{\partial z^2}{\partial p^2} - 2\,\lambda + 2\,\lambda_{KR} \tag{27b}$$

$$\frac{\partial F}{\partial d^2} = 0 = \left(\frac{\partial F_0}{\partial \xi} - \frac{\partial F_0}{\partial \bar{\xi}}\right) i\, \Delta\, \frac{\partial z^2}{\partial d^2} - 2\,\lambda + \lambda_{KR} + \lambda_{SC} \tag{27c}$$

$$\frac{\partial F}{\partial e^2} = 0 = \left(\frac{\partial F_0}{\partial \xi} - \frac{\partial F_0}{\partial \bar{\xi}}\right) i\, \Delta\, \frac{\partial z^2}{\partial e^2} + \lambda_{KR} + \lambda_{SC} \tag{27d}$$

$$\frac{\partial F}{\partial \lambda} = 0 = \left(\frac{\partial F_0}{\partial \xi} + \frac{\partial F_0}{\partial \bar{\xi}}\right) - 2(d^2 + p^2) \tag{27e}$$

$$\frac{\partial F}{\partial \lambda_{KR}} = 0 = 2p^2 + d^2 + e^2 - 1 \tag{27f}$$

$$\frac{\partial F}{\partial \lambda_{SC}} = 0 = d^2 + e^2 - K\,\langle h^\dagger h\rangle\,. \tag{27g}$$

Particle-hole symmetry further imposes $e = d$, which, using Eqs. (27f,27g), allows us to parametrize

$$e^2 = d^2 = \frac{1}{2}\,\kappa \quad \text{and} \quad p^2 = \frac{1}{2}\,(1-\kappa)\,, \tag{28}$$

where $\kappa \equiv K\,\langle h^\dagger h\rangle$ is a parameter at our disposal. Using the relations

$$\frac{\partial \ln z^2}{\partial p^2} = p^{-2} + (1-d^2-p^2)^{-1} + (1-e^2-p^2)^{-1} = \frac{2}{1-\kappa} + 4 \tag{29a}$$

$$\frac{\partial \ln z^2}{\partial d^2} = [d\,(e+d)]^{-1} + (1-d^2-p^2)^{-1} = \kappa^{-1} + 2 \tag{29b}$$

$$\frac{\partial \ln z^2}{\partial e^2} = [e\,(e+d)]^{-1} + (1-e^2-p^2)^{-1} = \kappa^{-1} + 2 \tag{29c}$$

$$z^2 = (1-d^2-p^2)^{-1}\, p^2\, (e+d)^2\, (1-e^2-p^2)^{-1} = 4\,\kappa\,(1-\kappa)\,, \tag{29d}$$

where in the last equalities we used the particle-hole symmetry condition, and substituting in Eqs. (27c,27d), we conclude that $\lambda = 0_+$, which is consistent with Eq. (27e) if we observe that $d^2 + p^2 = 1$ and use that

$$\left(\frac{\partial F_0}{\partial \xi} + \frac{\partial F_0}{\partial \bar{\xi}}\right) = \frac{2}{\pi}\ \arctan\left(\frac{z^2\,\Delta}{\lambda}\right) \xrightarrow[\lambda\to 0_+]{} 1\,. \tag{30}$$

Notice that fixing a value for κ fixes e, d, p and z, and with values we can solve the remaining independent Eqs. (27a,27b,27c). Let us focus on (27a), which gives the solution for Δ. (The values of the λ_{KR} and λ_{SC} can be obtained from (27b,27c) once Δ is known.) Using that

$$\left(\frac{\partial F_0}{\partial \xi} - \frac{\partial F_0}{\partial \bar{\xi}}\right) = \frac{2}{i\pi}\ \ln\left(\frac{|\xi|}{\mathrm{e}D}\right)\,, \tag{31}$$

in Eq. (27a) gives us

$$\Delta = \mathrm{e}D\,\frac{1}{z^2}\ \exp\left(-\frac{1}{z^2 J\rho}\right)\,, \tag{32}$$

or

$$\Delta = \mathrm{e}D\,\frac{1}{4\,\kappa\,(1-\kappa)}\ \exp\left(-\frac{1}{4\,\kappa\,(1-\kappa)\,J\rho}\right)\,. \tag{33}$$

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$$\frac{K}{2} = e^2 = d^2 \tag{34}$$

$$= \frac{4\frac{\Delta}{\pi\rho J} + \lambda_{SC} - \sqrt{\left(4\frac{\Delta}{\pi\rho J} + \lambda_{SC}\right)^2 - 8\lambda_{SC}\frac{\Delta}{\pi\rho J}}}{4\lambda_{SC}}\,.$$

We note that for large $\lambda_{SC} \to \infty$, $K, e^2, d^2 \to 0$ and indeed we obtain a good projection onto the spin 1/2 subspace.

We also get that

$$p_\uparrow = p_\downarrow\,, \qquad p^2 = \frac{1}{2} - d^2\,, \tag{35}$$

and z given by Eq. (26c) can be further simplified to:

$$z = 4dp = -2\,(p-d)^2 + 2p^2 + 2d^2 = 1 - 2\,(p-d)^2\,. \tag{36}$$

Finally, we obtain, $\lambda_\uparrow, \lambda_\downarrow$, $\lambda_{KR} = \frac{1+4d^2}{2d^2}\,\frac{\Delta}{\pi\rho J}$ and

$$\Delta = z^{-2} D \exp\left(-\frac{z^{-2}}{\rho J}\right)\,. \tag{37}$$

We have now obtained a family of meanfield solutions as a function of the parameter K that we introduced in our approach. We can take advantage of this freedom for instance to best fit one of the known properties of the Kondo model, for instance $T_K = \sqrt{\rho J} D \exp\left(\frac{-1}{\rho J}\right)$, which we can do analytically perturbatively for small ρJ. We now use the identify $\Delta \sim T_k = \sqrt{\rho J} D \exp\left(\frac{-1}{\rho J}\right)$, $\rho J \ll 1$, then we get:

$$\exp\left(-\frac{z^{-2}}{\rho J}\right) \simeq \sqrt{\rho J} \exp\left(\frac{-1}{\rho J}\right), \tag{38}$$

and

$$z^{-2} \simeq 1 - \frac{1}{2}\rho J \ln(\rho J), \quad z \simeq 1 + \frac{1}{4}\rho J \ln(\rho J). \tag{39}$$

We then get:

$$d^2 \simeq \frac{1}{4} - \frac{1}{32}\frac{\lambda_{SC}}{\frac{\Delta}{\pi\rho J}}, \quad 2(p-d)^2 \simeq \frac{1}{16}\left(\frac{\lambda_{SC}}{\frac{\Delta}{\pi\rho J}}\right)^2. \tag{40}$$

This means that:

$$\left(\frac{\lambda_{SC}}{\frac{\Delta}{\pi\rho J}}\right)^2 = -4\rho J \ln(\rho J), \quad \lambda_{SC} = \frac{2\Delta}{\pi}\sqrt{\frac{4\ln(1/\rho J)}{\rho J}}. \tag{41}$$

Once we have tuned K to match the Kondo temperature, we can obtain the heat capacity

$$C_v = 2k_B^2 T \frac{\pi^2}{3}\frac{1}{z^2\Delta}(-\ln z + 1) \simeq 2k_B^2 T \frac{\pi^2}{3}\frac{1}{z^2\Delta}, \tag{42}$$

and the magnetic susceptibility

$$\begin{aligned}\chi &\simeq \frac{g^2\mu_B^2}{-\pi\lambda_{SC}z^{-2} + 2\pi p z^2\Delta + 3\lambda_{SC}} \\ &\simeq \frac{g^2\mu_B^2}{2\pi p z^2\Delta_0} \simeq \frac{g^2\mu_B^2}{\pi\Delta}.\end{aligned} \tag{43}$$

Correspondingly, the Wilson ratio is

$$w = \frac{2}{\pi} + \ldots \simeq 0.64, \tag{44}$$

which is also improved with respect to the conventional Read-Newns meanfield value of one.

IV. ALTERNATIVE FORMULATION TO THE SOFT CONSTRAINT APPROACH

The main weakness of the above derivation is that it contained an arbitrary constant K that had to be tuned by hand. To show that there is arbitrariness to a lot of meanfields in particular the Read-Newns meanfield and to rederive the results of the soft constraint in a simpler fashion for the case of the Kondo model we will rederive the results of the previous section using a modified Read-Newns meanfield. We will introduce irrelevant terms $U n_\uparrow n_\downarrow$ and $W(1-n_\uparrow)(1-n_\downarrow)$ into the Read-Newns formulations that vanish trivially in the exact partition function but change the Lagrangian thereby changing the meanfield (the Read-Newns meanfield is recovered at the arbitrary choice of $U = W = 0$). The Lagrangian for this formulation is given by:

$$\begin{aligned} &e^\dagger \frac{d}{d\tau} e + d^\dagger \frac{d}{d\tau} d + \sum_\sigma p_\sigma^\dagger \frac{d}{d\tau} p_\sigma + \sum_\sigma f_\sigma^\dagger \frac{d}{d\tau} f_\sigma + \\ &\lambda_{RN}\left(\sum_\sigma f_\sigma^\dagger f_\sigma - 1\right) + \sum_{k,\sigma} c_{k,\sigma}^\dagger \frac{d}{d\tau} c_{k,\sigma} + 2\frac{V\bar{V}}{J} + \\ &U d^\dagger d + V \sum_{k,\sigma} c_{k,\sigma}^\dagger z_\sigma f_\sigma + h.c. + \sum (\epsilon_k - \mu) c_{k,\sigma}^\dagger c_{k,\sigma} + \\ &+W e^\dagger e + \lambda_{KR}\left(e^\dagger e + \sum_\sigma p_\sigma^\dagger p_\sigma + d^\dagger d - 1\right) + \\ &+\sum_\sigma \lambda_\sigma \left(f_\sigma^\dagger f_\sigma - p_\sigma^\dagger p_\sigma - d^\dagger d\right) \end{aligned} \tag{45}$$

As we noted above we have added for generality two free parameters $U d^\dagger d$ and $W e^\dagger e$ which do nothing as they are projected out by the Read Newns constraint. We find that the zero temperature meanfield partition function is given by [1]:

$$\begin{aligned}\frac{1}{\beta}\ln(Z) = &\sum_\sigma \frac{1}{\pi}\mathrm{Im}\left[\xi_\sigma \ln\left(\frac{\xi_\sigma}{\mathrm{e}D}\right)\right] - \lambda_{RN} - \frac{2\Delta}{\pi J\rho} \\ &-\lambda_{KR}\left(\sum_\sigma p_\sigma^2 + 2d^2 - 1\right) - U d^2 - W e^2 \\ &-\sum_\sigma \lambda_\sigma\left(p_\sigma^2 + d^2\right)\end{aligned} \tag{46}$$

Here $\xi_\sigma = \lambda_{RN} + iz^2\Delta + \lambda_\sigma$. Furthermore:

$$\begin{aligned} z_\sigma &= \left(1 - d^2 - p_\sigma^2\right)^{-1/2} 2d\left(p_\sigma + p_{-\sigma}\right)\left(1 - d^2 - p_{-\sigma}^2\right)^{-1/2} \\ &\equiv z. \end{aligned} \tag{47}$$

We find that $K = e^2 = d^2 =$

$$\frac{4\Delta\frac{N}{\pi\rho J} + U - \sqrt{\left(4\frac{N\Delta}{\pi\rho J} + U\right)^2 - 8U\left(\frac{N\Delta}{\pi\rho J}\right)}}{4U}. \tag{48}$$

Notice that for large U, $e^2, d^2 \to 0$ showing that we obtain good projection onto the spin 1/2 subspace. From this we get that

$$p^2 = \frac{1}{2} - d^2. \tag{49}$$

z is now given by Eq.(47) which can be further simplified to:

$$z = 4dp = -2(p-d)^2 + 2p^2 + 2d^2 = 1 - 2(p-d)^2 \tag{50}$$

and for the rest of the values we have $\lambda_{RN} = -\lambda = \frac{U}{2}$, $\lambda_{KR} = \frac{1+4d^2}{2d^2}\Delta\frac{N}{\pi\rho J} + W$ and

$$\Delta = z^{-2} D \exp\left(-\frac{z^{-2}}{\rho J}\right) \tag{51}$$

We now use the identify $\Delta \sim T_k = \sqrt{\rho J} D \exp\left(\frac{-1}{\rho J}\right)$, $\rho J \ll 1$, then we get:

$$\exp\left(-\frac{z^{-2}}{\rho J}\right) \simeq \sqrt{\rho J} \exp\left(\frac{-1}{\rho J}\right) \tag{52}$$

And

$$z^{-2} \simeq 1 - \frac{1}{2}\rho J \ln(\rho J), \quad z \simeq 1 + \frac{1}{4}\rho J \ln(\rho J) \tag{53}$$

We then get:

$$d^2 \simeq \frac{1}{4} - \frac{1}{32}\frac{U}{\frac{2\Delta}{\pi\rho J}}, \quad 2(p-d)^2 \simeq \frac{1}{16}\left(\frac{U}{\frac{2\Delta}{\pi\rho J}}\right)^2 \tag{54}$$

This means that:

$$\left(\frac{U}{\frac{2\Delta}{\pi\rho J}}\right)^2 = -4\rho J \cdot \ln(\rho J), \; U = \frac{2\Delta}{\pi}\sqrt{\frac{4\ln(1/\rho J)}{\rho J}} \tag{55}$$

Furthermore the heat capacity is given by:

$$C_v = 2k_B^2 T \frac{\pi^2}{3}\frac{1}{z^2\Delta}(-\ln z + 1) \simeq 2k_B^2 T \frac{\pi^2}{3}\frac{1}{z^2\Delta} \tag{56}$$

And the magnetic susceptibility is given by:

$$\begin{aligned} \chi &\simeq \frac{g^2\mu_B^2}{-\pi U z^{-2} + 2pz^2\Delta\frac{2\pi}{2} + 3U} \\ &\simeq \frac{g^2\mu_B^2}{2\pi p z^2 \Delta_0} \simeq \frac{g^2\mu_B^2}{\pi\Delta} \end{aligned} \tag{57}$$

The Wilson ratio is:

$$w = \frac{2}{\pi} + ... \simeq 0.64 \tag{58}$$

All the results are highly similar to the soft constraint formulation.

V. OUTLOOK

In this work we have presented a new method to study strongly correlated systems, the soft constraint. While the soft-constraint approach to the Kondo model was merely a proof of principle to show its potential to resolve a highly controlled and well studied model the method is applicable to a wide variety of systems including hard core bosons, spinor condensates, t-J model, Heisenberg model, two channel Kondo and two channel Anderson models as well as Kondo model and infinite U Anderson model presented in the main text. The authors are currently in the process of applying this method to these models with promising results [16]. One of the main drawbacks of the soft constraint method is its arbitrariness, there is an arbitrary parameter denoted by K in the main text with each K yielding a perfectly good meanfield. While there are some first principles ways to choose an optimal K [16] our approach was to use this arbitrariness to our advantage and select the K that reproduces the Bethe ansatz as closely as possible. We also noted that there is an arbitrariness to almost every meanfield that includes constraints. As an example we pointed out the arbitrariness of the Read-Newns meanfield for the Kondo model where it is possible to add terms to the meanfield without changing the action, once again this arbitrariness can be exploited in our favor to select the meanfield which best matches the Bethe ansatz. In the future we will apply these approaches to many open problems [16].

Appendix A: Considerations on the soft constraint approach

As an example we could impose the two constraints $n_\uparrow n_\downarrow = 0$ and $(1-n_\uparrow)(1-n_\downarrow) = 0$ with two separate fermionic spaces and two relevant coupling constants. To do this we will consider a Lagrangian for the system given by:

$$\begin{aligned}
&e^\dagger \frac{d}{d\tau} e + \sum_\sigma p_\sigma^\dagger \frac{d}{d\tau} p_\sigma + \sum_\sigma f_\sigma^\dagger \frac{d}{d\tau} f_\sigma + \sum_{k,\sigma} c_{k,\sigma}^\dagger \frac{d}{d\tau} c_{k,\sigma} + \\
&+h_2^\dagger \frac{d}{d\tau} h_2 + 2\frac{V\bar{V}}{J} + \lambda_{SC,1}\left(n_\uparrow n_\downarrow - K_1 h_1^\dagger h_1\right) + \\
&+\lambda_{SC,2}\left(\left(1-n_\uparrow\right)\left(1-n_\downarrow\right) - K_2 h_2^\dagger h_2\right) + \\
&d^\dagger \frac{d}{d\tau} d + W e^\dagger e + U d^\dagger d + V \sum_{k,\sigma} c_{k,\sigma}^\dagger z_\sigma f_\sigma + h.c. + \\
&+\lambda_{KR}\left(e^\dagger e + \sum_\sigma p_\sigma^\dagger p_\sigma + d^\dagger d - 1\right) + \sum\left(\epsilon_k - \mu\right) c_k^\dagger c_k + \\
&h_1^\dagger \frac{d}{d\tau} h_1 + \sum_\sigma \lambda_\sigma\left(f_\sigma^\dagger f_\sigma - p_\sigma^\dagger p_\sigma - d^\dagger d\right)
\end{aligned} \tag{A1}$$

Where we have done both the soft constraint and added the trivially vanishing terms $U n_\uparrow n_\downarrow$ and $W\left(1-n_\uparrow\right)\left(1-n_\downarrow\right)$. The meanfield partition function at zero temperature zero magnetic field is given by:

$$\begin{aligned}
\frac{1}{\beta}\ln\left(Z\right) = &\frac{N}{\pi} Im\left[\xi \ln\left(\frac{\xi}{eD}\right)\right] - \frac{N\Delta}{\pi J\rho} - U d^2 - W e^2 - \\
&-\lambda_{SC,1}\left(d^2 - K_1 h_1^\dagger h_1\right) - \lambda_{SC,2}\left(e^2 - K_2 h_2^\dagger h_2\right) - \\
&-\lambda_{KR}\left(\sum_\sigma p_\sigma^2 + d^2 + e^2 - 1\right) - \sum_\sigma \lambda\left(p_\sigma^2 + d^2\right)
\end{aligned} \tag{A2}$$

Compare with Eq. (17.107) in Ref [1]. Here $\xi_\sigma = iz^2\Delta + \lambda$ where $\lambda_\uparrow = \lambda_\downarrow \equiv \lambda$. Furthermore $p_\uparrow = p_\downarrow \equiv p$ and $z_\uparrow = z_\downarrow \equiv z$. At meanfield level $e = K_2$, $d = K_1$ $p^2 = \frac{1}{2}\left(1 - K_1 - K_2\right)$. Furthermore if we define $\tilde{\lambda}_{SC,1} = \lambda_{SC,1} + U$ and $\tilde{\lambda}_{SC,2} = \lambda_{SC,2} + W$ and the terms U and W drop out of the calculation and we get

$$z = \frac{\sqrt{2}\left(\sqrt{K_1} + \sqrt{K_2}\right)\left(1 - K_1 - K_2\right)^{1/2}}{\left(1 - K_1 + K_2\right)^{1/2}\left(1 + K_1 - K_2\right)^{1/2}} \tag{A3}$$

This means that

$$\Delta = z^{-2}\sin\left(\frac{\pi}{N}\left(1 + K_1 - K_2\right)\right) D \exp\left(\frac{-z^2}{\rho J}\right)$$

To restore particle hole symmetry we need to take $K_1 = K_2$ and the results reduce to those in Sec. III B. It is not too hard to see that the heat capacity and magnetic susceptibility stay the same as in Sec. III B in the symmetric case.

Appendix B: Large N formulation

We will consider the Lagrangian:

$$\begin{aligned}
L_{SC} = &\sum_{k,\alpha} c_{k\alpha}^\dagger\left(\frac{d}{d\tau} + \epsilon_k\right) c_{k\alpha} + \sum_\sigma f_\alpha^\dagger \frac{d}{d\tau} f_\alpha \\
&- \sum_\sigma\left\{\bar{V}\sum_k\left(c_{k,\alpha}^\dagger f_\alpha\right) + V\left(f_\alpha^\dagger c_{k,\alpha}\right)\right\} + N\frac{V\bar{V}}{J} \\
&+ \lambda_{SC}\left[\left(\sum_\alpha f_\alpha^\dagger f_\alpha - Q\right)^2 - K\sum h^\dagger h\right] \\
&+ h^\dagger \frac{d}{d\tau} h\,.
\end{aligned} \tag{B1}$$

This is a soft constraint formulation of the Coqblin-Schrieffer Hamiltonian. Because of the interaction term it is not in meanfield formulation yet. However we can use the multiparticle Kotliar Ruckenstein formulation to put it in meanfield form [12, 17]. We would like to introduce slave bosons for the model of the model. They are based on:

$$\psi_1, \psi_2, \ldots.\psi_N \equiv \psi_\alpha \tag{B2}$$

The slave bosons would be all symmetrized combinations of the form $\psi_{\alpha,1}^{(m)}\cdots_{\alpha,m}$, $m = 0, 1, 2, 3..., N$. The electron operators are given by:

$$f_\alpha \to z_\alpha f_\alpha \tag{B3}$$

Where

$$z_\alpha = \sum_{m=1}^{4}\sum_{\alpha,1<\ldots<\alpha,m} \psi_{\alpha,1\ldots\alpha,m-1}^{(m-1)\dagger}\psi_{\alpha,\alpha1\ldots\alpha,m-1}^{(m)}, \qquad \alpha_i \neq \alpha$$

One needs to introduce the constraints:

$$\begin{aligned}
1 &= \sum_{m=0}^{N} \psi_{\alpha,1..\alpha,m}^{(m)\dagger}\psi_{\alpha,1..\alpha,m}^{(m)} \\
f_\alpha^\dagger f_\alpha &= \sum_{m=1}^{n} \psi_{\alpha,\alpha,1..\alpha,m}^{(m)\dagger}\psi_{\alpha,\alpha,1..\alpha,m}^{(m)}
\end{aligned} \tag{B4}$$

It is also conventional to transform:

$$z_\alpha = \sum_{m=1}^{4}\sum_{\alpha,1<\ldots<\alpha,m} \psi_{\alpha,1\ldots\alpha,m-1}^{(m-1)\dagger} L_\alpha R_\alpha \psi_{\alpha,1\ldots\alpha,m}^{(m)}, \qquad \alpha_i \neq \alpha \tag{B5}$$

with:

$$\begin{aligned}
R_\alpha &= \left[1 - \sum_{m=0}^{N-1} \psi_{\alpha,1..\alpha,m}^{(m)\dagger}\psi_{\alpha,1..\alpha,m}^{(m)}\right]^{-1/2} \qquad \alpha_i \neq \alpha \\
L_\alpha &= \left[1 - \sum_{m=1}^{N} \psi_{\alpha,\alpha,1..\alpha,m}^{(m)\dagger}\psi_{\alpha,\alpha,1..\alpha,m}^{(m)}\right]^{-1/2}
\end{aligned} \tag{B6}$$

The Kotliar Ruckenstein Lagrangian is given by:

$$\sum \psi^{(m)\dagger}_{\alpha,1}{}_{\cdots\alpha,m}\frac{d}{d\tau}\psi^{(m)}_{\alpha,1}{}_{\cdots\alpha,m}+\sum_{\sigma}c^{\dagger}_{k,\alpha}\frac{d}{d\tau}c_{k,\alpha}+$$
$$\sum_{\sigma}f^{\dagger}_{\alpha}\frac{d}{d\tau}f_{\alpha}+\sum_{\sigma}\lambda_{\alpha}\left(f^{\dagger}_{\alpha}f_{\alpha}-\sum_{m=1}^{n}\psi^{(m)\dagger}_{\alpha,\alpha,1..\alpha,m}\psi^{(m)}_{\alpha,\alpha,1..\alpha,m}\right)+$$
$$V\sum_{k,\sigma}c^{\dagger}_{k,\alpha}z_{\alpha}f_{\alpha}+h.c.+\sum\left(\epsilon_k-\mu\right)c^{\dagger}_{k,\alpha}c_{k,\alpha}+$$
$$+N\frac{V\bar{V}}{J}+\lambda_{KR}\left(\sum_{m=0}^{N}\psi^{(m)\dagger}_{\alpha,1..\alpha,m}\psi^{(m)}_{\alpha,1..\alpha,m}-1\right)+$$
$$+\lambda_{SC}\left(\sum_{m=0}^{N}\left(m-Q\right)^2\psi^{(m)\dagger}_{\alpha,1..\alpha,m}\psi^{(m)}_{\alpha,1..\alpha,m}-Kh^{\dagger}h\right) \quad \text{(B7)}$$

Appendix C: Derivations of various formulas in the main text

1. Solutions at zero temperature

As a warm up we will consider the Lagrangian:

$$e^{\dagger}\frac{d}{d\tau}e+d^{\dagger}\frac{d}{d\tau}d+\sum_{\sigma}p^{\dagger}_{\sigma}\frac{d}{d\tau}p_{\sigma}+\sum_{\sigma}f^{\dagger}_{\sigma}\frac{d}{d\tau}f_{\sigma}+\sum_{k,\sigma}c^{\dagger}_{k,\sigma}\frac{d}{d\tau}c_{k,\sigma}+2\frac{V\bar{V}}{J}+$$
$$\lambda_{RN}\left(\sum_{\sigma}f^{\dagger}_{\sigma}f_{\sigma}-Q\right)+Ud^{\dagger}d+V\sum_{k,\sigma}c^{\dagger}_{k,\sigma}z_{\sigma}f_{\sigma}+h.c.+\sum\left(\epsilon_k-\mu\right)c^{\dagger}_{k}c_{k}+$$
$$+\lambda_{KR}\left(e^{\dagger}e+\sum_{\sigma}p^{\dagger}_{\sigma}p_{\sigma}+d^{\dagger}d-1\right)+\sum_{\sigma}\lambda_{\sigma}\left(f^{\dagger}_{\sigma}f_{\sigma}-p^{\dagger}_{\sigma}p_{\sigma}-d^{\dagger}d\right)+g\mu_B B\left(n_{\uparrow}-n_{\downarrow}\right) \quad \text{(C1)}$$

We will solve for the meanfield. We will focus on the case where $\beta\to\infty$ and $B=0$. This implies that $\langle p_{\uparrow}\rangle=\langle p_{\downarrow}\rangle\equiv p$. Furthermore $z_{\uparrow}=z_{\downarrow}\equiv z$. At meanfield level:

$$z=\frac{\left(e+d\right)p}{\left(1-d^2-p^2\right)^{1/2}\left(1-e^2-p^2\right)^{1/2}} \quad \text{(C2)}$$

With these simplifications the meanfield Lagrangian becomes:

$$\sum_{\sigma}f^{\dagger}_{\sigma}\frac{d}{d\tau}f_{\sigma}+\sum_{k,\sigma}c^{\dagger}_{k,\sigma}\frac{d}{d\tau}c_{k,\sigma}+\lambda_{RN}\left(\sum_{\sigma}f^{\dagger}_{\sigma}f_{\sigma}-Q\right)+2\frac{V\bar{V}}{J}$$
$$+Ud^2+V\sum_{k,\sigma}c^{\dagger}_{k,\sigma}zf_{\sigma}+h.c.+\sum\left(\epsilon_k-\mu\right)c^{\dagger}_{k}c_{k}+$$
$$+\lambda_{KR}\left(e^2+2p^2+d^2-1\right)+\sum_{\sigma}\lambda_{\sigma}\left(f^{\dagger}_{\sigma}f_{\sigma}-p^2-d^2\right) \quad \text{(C3)}$$

We then can obtain the Gibbs free energy at zero temperature following [1]. It is given by:

$$\frac{1}{\beta}\ln\left(Z\right)=\sum_{\sigma}\frac{N}{2\pi}Im\left[\xi_{\sigma}\ln\left(\frac{\xi_{\sigma}}{\mathrm{e}D}\right)\right]-\lambda_{RN}Q-\frac{N\Delta}{\pi J\rho}-Ud^2-\lambda_{GR}\left(e^2+2p^2+d^2-1\right)-\sum_{\sigma}\lambda_{\sigma}\left(p^2+d^2\right) \quad \text{(C4)}$$

Compare with Eq. (17.107) in [1]. Here $\xi_{\sigma}=\lambda_{RN}+iz^2\Delta+\lambda_{\sigma}$. Furthermore at $B=0$, $\lambda_{\uparrow}=\lambda_{\downarrow}=\lambda$, this then further simplifies to:

$$\frac{1}{\beta}\ln\left(Z\right)=\frac{N}{\pi}Im\left[\xi\ln\left(\frac{\xi}{\mathrm{e}D}\right)\right]-\lambda_{RN}Q-\frac{N\Delta}{\pi J\rho}-Ud^2-\lambda_{GR}\left(e^2+2p^2+d^2-1\right)-2\lambda\left(p^2+d^2\right) \quad \text{(C5)}$$

The meanfield equations are given by:

$$\begin{aligned}
\frac{N}{\pi}\cot^{-1}\left(\frac{\lambda_{RN}+\lambda}{z^2\Delta}\right) &= Q \\
\frac{N}{\pi}\cot^{-1}\left(\frac{\lambda_{RN}+\lambda}{z^2\Delta}\right) &= 2\left(p^2+d^2\right) \\
z^2\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^2\Delta^2}}{D}\right) &= -\frac{N}{\pi\rho J} \\
e^2+2p^2+d^2 &= 1 \\
2z\frac{\partial z}{\partial e}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^2\Delta^2}}{D}\right) &= -2e\lambda_{KR} \\
2z\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^2\Delta^2}}{D}\right) &= -2d\lambda_{KR}-4d\lambda-2Ud \\
2z\frac{\partial z}{\partial p}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^2\Delta^2}}{D}\right) &= -4p\lambda_{KR}-4p\lambda
\end{aligned} \tag{C6}$$

These further simplify to:

$$\begin{aligned}
Q &= 2\left(p^2+d^2\right) \\
e^2+2p^2+d^2 &= 1 \\
2\frac{\partial z}{\partial e}\cdot\Delta\frac{N}{z\pi\rho J} &= 2e\lambda_{KR} \\
2\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{z\pi\rho J} &= 2d\lambda_{KR}+4d\lambda+2Ud \\
2\frac{\partial z}{\partial p}\cdot\Delta\frac{N}{z\pi\rho J} &= 4p\lambda_{KR}+4p\lambda
\end{aligned} \tag{C7}$$

Comparing the first two equations we get that $e=d$ and $p^2=\frac{1}{2}\left(1-2d^2\right)$, we have used that $Q=1$. Now $\frac{1}{z}\frac{\partial z}{\partial e}=\frac{\partial}{\partial e}\ln z$ and similarly for p and d. Now

$$\ln z=\ln\left(e+d\right)+\ln\left(p\right)-\frac{1}{2}\ln\left(1-d^2-p^2\right)-\frac{1}{2}\ln\left(1-e^2-p^2\right) \tag{C8}$$

With this:

$$\begin{aligned}
\frac{\partial\ln z}{\partial e} &= \frac{1}{e+d}+\frac{e}{\left(1-e^2-p^2\right)} \\
\frac{\partial\ln z}{\partial d} &= \frac{1}{e+d}+\frac{d}{\left(1-d^2-p^2\right)} \\
\frac{\partial\ln z}{\partial p} &= \frac{1}{p}+\frac{p}{\left(1-d^2-p^2\right)}+\frac{p}{\left(1-e^2-p^2\right)}
\end{aligned} \tag{C9}$$

Now $e=d$ and $e^2+p^2=e^2+d^2=\frac{1}{2}$, so:

$$\begin{aligned}
\frac{\partial\ln z}{\partial e} &= \frac{1}{2d}+2d \\
\frac{\partial\ln z}{\partial d} &= \frac{1}{2d}+2d \\
\frac{\partial\ln z}{\partial p} &= \frac{1}{p}+4p
\end{aligned} \tag{C10}$$

Now for simplicity we will consider the case where $U = 0$. With this the equations simplify to:

$$
\begin{aligned}
2 \cdot \left(\frac{1}{2d} + 2d\right) \cdot \Delta \frac{N}{\pi\rho J} &= 2d\lambda_{KR} \\
2\left(\frac{1}{2d} + 2d\right) \cdot \Delta \frac{N}{\pi\rho J} &= 2d\lambda_{KR} + 4d\lambda \\
2\left(\frac{1}{p} + 4p\right) \cdot \Delta \frac{N}{z\pi\rho J} &= 4p\lambda_{KR} + 4p\lambda
\end{aligned}
\tag{C11}
$$

From this we see that $\lambda = 0$, and we have that:

$$
\begin{aligned}
\left(\frac{1}{d} + 4d\right) \Delta \frac{N}{\pi\rho J} &= 2d\lambda_{KR} \\
p^2 + d^2 &= \frac{1}{2} \\
\left(\frac{1}{p} + 4p\right) \cdot \Delta \frac{N}{\pi\rho J} &= 2p\lambda_{KR}
\end{aligned}
\tag{C12}
$$

One solution by inspection is $e = p = d = \frac{1}{2}$ and $\lambda_{KR} = 4\Delta\frac{N}{\pi\rho J}$, $\lambda = 0$. To check that this is the only solution we write:

$$
\begin{aligned}
\frac{d^2}{p^2} &= \frac{4d^2+1}{4p^2+1} \\
p^2 + d^2 &= \frac{1}{2}
\end{aligned}
\tag{C13}
$$

The we get

$$
\frac{d^2}{1/2 - d^2} = \frac{4d^2+1}{3-4d^2}
\tag{C14}
$$

and $d = \frac{1}{2}$. To get the rest of the values we note that $z = 1$ and we get two equations:

$$
\begin{aligned}
\frac{N}{\pi} \cot^{-1}\left(\frac{\lambda_{RN}}{\Delta}\right) &= Q \\
\frac{N}{\pi} \ln\left(\frac{\sqrt{\lambda_{RN}^2 + \Delta^2}}{D}\right) &= -\frac{N}{\pi\rho J}
\end{aligned}
\tag{C15}
$$

For $Q/N = \frac{1}{2}$, $\lambda_{RN} = 0$ and

$$
\Delta = D \exp\left(-\frac{1}{\rho J}\right)
\tag{C16}
$$

And we get back to the regular Read-Newns meanfield. We get new insight into the Read newns results in that $e^2 = d^2 = p^2 = \frac{1}{4}$, or all four states of the Kondo spin are equally occupied at zero temperature, which means that for $N = 2$ the meanfield projector does a rather poor job.

Lets return to the general U case and obtain a better projection operator. In this case by comparing with Eq. (C11) we get that

$$
\begin{aligned}
2 \cdot \left(\frac{1}{2d} + 2d\right) \Delta \frac{N}{\pi\rho J} &= 2d\lambda_{GR} \\
2\left(\frac{1}{2d} + 2d\right) \cdot \Delta \frac{N}{\pi\rho J} &= 2d\lambda_{GR} + 4d\lambda + 2Ud \\
2\left(\frac{1}{p} + 4p\right) \cdot \Delta \frac{N}{\pi\rho J} &= 4p\lambda_{GR} + 4p\lambda
\end{aligned}
\tag{C17}
$$

By comparing the first two equations we get that $\lambda = -\frac{U}{2}$. With this the equations simplify to:

$$\left(1+4d^2\right)\Delta\frac{N}{\pi\rho J} = 2d^2\lambda_{KR}$$
$$p^2 + d^2 = \frac{1}{2}$$
$$\left(1+4p^2\right)\cdot\Delta\frac{N}{z\pi\rho J} = 2p^2\lambda_{KR} - Up^2 \tag{C18}$$

Using the second equation we get that:

$$\left(1+4d^2\right)\Delta\frac{N}{\pi\rho J} = 2d^2\lambda_{GR}$$
$$\left(3-4d^2\right)\Delta\frac{N}{\pi\rho J} = \left(1-2d^2\right)\left(\lambda_{GR} - U/2\right) \tag{C19}$$

Now we write $x = d^2$ and we get:

$$\left(1+4x\right)\Delta\frac{N}{\pi\rho J} = 2x\lambda_{GR}$$
$$\left(3-4x\right)\Delta\frac{N}{\pi\rho J} = \left(1-2x\right)\left(\lambda_{GR} - U/2\right) \tag{C20}$$

From this we get that $\lambda_{GR} = \frac{(1+4x)}{2x}\Delta\frac{N}{\pi\rho J}$ and

$$2x\left(3-4x\right)\Delta\frac{N}{\pi\rho J} = \left(1-2x\right)\left(\left(1+4x\right)\Delta\frac{N}{\pi\rho J} - xU\right) \tag{C21}$$

or

$$-2Ux^2 + \left(4\Delta\frac{N}{\pi\rho J} + U\right)x - \Delta\frac{N}{\pi\rho J} = 0 \tag{C22}$$

$$e^2 = d^2 = \frac{4\Delta\frac{N}{\pi\rho J} + U - \sqrt{\left(4\frac{N\Delta}{\pi\rho J} + U\right)^2 - 8U\left(\frac{N\Delta}{\pi\rho J}\right)}}{4U} \tag{C23}$$

We note that for large U, $e^2, d^2 \to 0$ showing that we obtain good projection onto the spin 1/2 subspace. From this we get that

$$p^2 = \frac{1}{2} - d^2 \tag{C24}$$

z is now given by Eq. (C2) which can be further simplified to:

$$z = 4dp = -2\left(p-d\right)^2 + 2p^2 + 2d^2 = 1 - 2\left(p-d\right)^2 \tag{C25}$$

and to get the rest of the values we have

$$\frac{N}{\pi}\cot^{-1}\left(z^{-2}\frac{\lambda_{RN}+\lambda}{\Delta}\right) = Q$$
$$z^2\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2 + z^2\Delta^2}}{D}\right) = -\frac{N}{\pi\rho J} \tag{C26}$$

For $Q/N = \frac{1}{2}$, $\lambda_{RN} = -\lambda = \frac{U}{2}$ and

$$\Delta = z^{-2}D\exp\left(-\frac{z^{-2}}{\rho J}\right) \tag{C27}$$

[I AM NOT SURE HOW TO SOLVE FOR THE LAST SET F CONSISTENCY RELATIONS...WE SHOULD TALK, E.G. z AND Δ ARE CONNECTED....]

One option is to identify $\Delta \sim T_k = \sqrt{\rho J} D \exp\left(\frac{-1}{\rho J}\right)$, $\rho J \ll 1$, then we get:

$$\exp\left(-\frac{z^{-2}}{\rho J}\right) \simeq \sqrt{\rho J} \exp\left(\frac{-1}{\rho J}\right) \tag{C28}$$

And

$$z^{-2} \simeq 1 - \frac{1}{2}\rho J \ln(\rho J), \qquad z \cong 1 + \frac{1}{4}\rho J \ln(\rho J) \tag{C29}$$

We then get:

$$d^2 \simeq \frac{1}{4} - \frac{1}{32}\frac{U}{\frac{N\Delta}{\pi\rho J}}, \qquad d \cong \frac{1}{2} - \frac{1}{32}\frac{U}{\frac{N\Delta}{\pi\rho J}}, \qquad 2(p-d)^2 \cong \frac{1}{16}\left(\frac{U}{\frac{N\Delta}{\pi\rho J}}\right)^2 \tag{C30}$$

This means that:

$$\left(\frac{U}{\frac{N\Delta}{\pi\rho J}}\right)^2 = -4\rho J \cdot \ln(\rho J), \qquad U = \frac{N\Delta}{\pi}\sqrt{\frac{4\ln(1/\rho J)}{\rho J}} \tag{C31}$$

2. Heat Capacity Kotliar Rukenstein

We would like to calculate the heat capacity of the system near zero temperature. It is known that the partition function at non-zero temperature for the system is given by [1]:

$$\frac{1}{\beta}\ln Z = \frac{N}{\pi}\int d\epsilon f(\epsilon)\, Im\left[\ln(\xi-\epsilon)\right] - \lambda_{RN} Q - \frac{N\Delta}{\pi J\rho} - Ud^2 - \lambda_{KR}\left(e^2+2p^2+d^2-1\right) - 2\lambda\left(p^2+d^2\right) \tag{C32}$$

Here $\xi_\sigma = \lambda_{RN} + iz^2\Delta + \lambda$. and $f(\epsilon)$ is the fermi Dirac distribution. Now using Eq. (16) in Appendix E in [13] we have that:

$$f(\epsilon) = \theta(-\epsilon) - k_B^2 T^2 \frac{\pi^2}{6}\delta'(\epsilon) + ... \tag{C33}$$

Therefore

$$\begin{aligned}\frac{1}{\beta}\ln Z = {}& \frac{N}{\pi} Im\left[\xi \ln\left(\frac{\xi}{\mathrm{e}D}\right)\right] - \lambda_{RN} Q - \frac{N\Delta}{\pi J\rho} - Ud^2 - \lambda_{KR}\left(e^2+2p^2+d^2-1\right) - 2\lambda\left(p^2+d^2\right)\\ & - k_B^2 T^2 \frac{N\pi}{6}\frac{z^2\Delta}{\left(z^4\Delta^2 + (\lambda_{RN}+\lambda)^2\right)}\end{aligned} \tag{C34}$$

The meanfield equations are given by:

$$
\begin{aligned}
\frac{N}{\pi}\cot^{-1}\left(\frac{\lambda_{RN}+\lambda}{z^2\Delta}\right)+k_B^2T^2\frac{N\pi}{6}\frac{2z^2\Delta\left(\lambda_{RN}+\lambda\right)}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)}&=Q\\
\frac{N}{\pi}\cot^{-1}\left(\frac{\lambda_{RN}+\lambda}{z^2\Delta}\right)+k_B^2T^2\frac{N\pi}{6}\frac{2z^2\Delta\left(\lambda_{RN}+\lambda\right)}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)}&=2\left(p^2+d^2\right)\\
z^2\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right)-k_B^2T^2\frac{N\pi}{6}\frac{z^2\Delta}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)}-&\\
+k_B^2T^2\frac{N\pi}{6}\frac{2z^6\Delta^2}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)^2}&=-\frac{N}{\pi\rho J}\\
e^2+2p^2+d^2&=1\\
2z\frac{\partial z}{\partial e}\Delta\cdot\left\{\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right)-k_B^2T^2\frac{N\pi}{6}\frac{z^2\Delta}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)}-\right.&\\
\left.+k_B^2T^2\frac{N\pi}{6}\frac{2z^6\Delta^2}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)^2}\right\}&=-2e\lambda_{KR}\\
2z\frac{\partial z}{\partial d}\Delta\cdot\left\{\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^2\Delta^2}}{D}\right)-k_B^2T^2\frac{N\pi}{6}\frac{z^2\Delta}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)}-\right.&\\
\left.+k_B^2T^2\frac{N\pi}{6}\frac{2z^6\Delta^2}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)^2}\right\}&=-2d\lambda_{KR}-4d\lambda-2Ud\\
2z\frac{\partial z}{\partial p}\Delta\cdot\left\{\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^2\Delta^2}}{D}\right)-k_B^2T^2\frac{N\pi}{6}\frac{z^2\Delta}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)}-\right.&\\
\left.+k_B^2T^2\frac{N\pi}{6}\frac{2z^6\Delta^2}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)^2}\right\}&=-4p\lambda_{KR}-4p\lambda
\end{aligned}
\tag{C35}
$$

We first obtain that $e^2=d^2$ and $p^2=\frac{1}{2}-e^2$. We then simplify the equations to:

$$
\begin{aligned}
Q&=2\left(p^2+d^2\right)\\
e^2+2p^2+d^2&=1\\
2\frac{\partial z}{\partial e}\cdot\Delta\frac{N}{z\pi\rho J}&=2e\lambda_{KR}\\
2\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{z\pi\rho J}&=2d\lambda_{KR}+4d\lambda+2Ud\\
2\frac{\partial z}{\partial p}\cdot\Delta\frac{N}{z\pi\rho J}&=4p\lambda_{KR}+4p\lambda
\end{aligned}
\tag{C36}
$$

From this Eqs. (C17) to (C25) follow. In particular $\lambda = -\frac{U}{2}$. We now obtain:

$$\frac{N}{\pi}\cot^{-1}\left(z^{-2}\frac{\lambda_{RN}+\lambda}{\Delta}\right) + k_B^2T^2\frac{N\pi}{6}\frac{2z^2\Delta\left(\lambda_{RN}+\lambda\right)}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)} = Q$$

$$z^2\frac{N}{\pi}\ln\left(\frac{\sqrt{\lambda_{RN}^2+z^4\Delta^2}}{D}\right) - k_B^2T^2\frac{N\pi}{6}\frac{z^2\Delta}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)} \tag{C37}$$

$$+k_B^2T^2\frac{N\pi}{6}\frac{2z^6\Delta^2}{\left(z^4\Delta^2+\left(\lambda_{RN}+\lambda\right)^2\right)^2} = -\frac{N}{\pi\rho J} \tag{C38}$$

Furthermore whenever we see terms $\sim k_B^2T^2$ we may as well replace them with the zero temperature solutions. This means that:

$$\frac{N}{\pi}\cot^{-1}\left(z^{-2}\frac{\lambda_{RN}+\lambda}{\Delta}\right) = Q$$

$$z^2\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right) + k_B^2T^2\frac{\pi^2}{6}\frac{1}{z_0^2\Delta_0^2} = -\frac{1}{\rho J}$$

From this we obtain that $\lambda_{RN} = -\lambda = \frac{U}{2}$. and

$$\Delta = z^{-2}D\exp\left(-\frac{z^{-2}}{\rho J} - k_B^2T^2\frac{\pi^2}{6}\frac{1}{z_0^4\Delta_0^2}\right) \cong \Delta_0 - k_B^2T^2\frac{\pi^2}{6}\frac{1}{z_0^4\Delta_0} \tag{C39}$$

This means that:

$$-\frac{1}{\beta}\ln Zccx = \frac{N}{\pi}Im\left[\xi\ln\left(\frac{\xi}{\mathrm{e}D}\right)\right] - k_B^2T^2\frac{N\pi}{6z_0^2\Delta_0} + \frac{Nk_B^2T^2\frac{\pi^2}{6z_0^4\Delta_0}}{\pi J\rho} + ... \tag{C40}$$

The rest of the terms don't depend on the temperature. Now $Im\left(\frac{d}{d\Delta}\left(\xi\ln\left(\frac{\xi}{\mathrm{e}D}\right)\right)\right) = Im\left(i\ln\left(\frac{\xi}{\mathrm{e}D}\right)+i\right)$. Now $Re\left(\ln\left(\frac{\xi}{\mathrm{e}D}\right)\right)+1 = \ln\left(\frac{\Delta_0}{\mathrm{e}D}\right)+1 = \ln\left(\frac{\Delta_0}{D}\right) = -\frac{z_0^{-2}}{\rho J} - \ln z$. This means that:

$$\frac{1}{\beta}\ln Z = Const - k_B^2T^2\frac{\pi^2}{6}\frac{z_0^2}{z_0^4\Delta_0}\left(\frac{1}{\rho J}+\ln z_0\right) + k_B^2T^2\frac{N\pi}{6z_0^2\Delta_0} + \frac{Nk_B^2T^2\frac{\pi}{6z_0^4\Delta_0}}{J\rho}$$

$$= Const + Nk_B^2T^2\frac{\pi^2}{6}\frac{1}{z_0^2\Delta_0}\left(-\frac{1}{z_0^2\rho J} - \ln z_0 + \frac{1}{z_0^2\rho J} + 1\right) \tag{C41}$$

From this we obtain that

$$C_v = Nk_B^2T\frac{\pi^2}{3}\frac{1}{z_0^2\Delta_0}\left(-\ln z_0+1\right) \cong Nk_B^2T\frac{\pi^2}{3}\frac{1}{z_0^2\Delta_0} \tag{C42}$$

3. Magnetic susceptibility Kotliar Rukenstein

We would like to calculate the magnetic susceptibility at zero temperature. Recall that the Lagrangian is given by:

$$e^\dagger\frac{d}{d\tau}e + d^\dagger\frac{d}{d\tau}d + \sum_\sigma p_\sigma^\dagger\frac{d}{d\tau}p_\sigma + \sum_\sigma f_\sigma^\dagger\frac{d}{d\tau}f_\sigma + \sum_{k,\sigma}c_{k,\sigma}^\dagger\frac{d}{d\tau}c_{k,\sigma} + 2\frac{V\bar{V}}{J} +$$

$$\lambda_{RN}\left(\sum_\sigma f_\sigma^\dagger f_\sigma - Q\right) + Ud^\dagger d + V\sum_{k,\sigma}c_{k,\sigma}^\dagger z_\sigma f_\sigma + h.c. + \sum\left(\epsilon_k-\mu\right)c_k^\dagger c_k +$$

$$+\lambda_{KR}\left(e^\dagger e + \sum_\sigma p_\sigma^\dagger p_\sigma + d^\dagger d - 1\right) + \sum_\sigma\lambda_\sigma\left(f_\sigma^\dagger f_\sigma - p_\sigma^\dagger p_\sigma - d^\dagger d\right) + g\mu_B B\left(n_\uparrow - n_\downarrow\right) \tag{C43}$$

Now it is easy to see that $g\mu_B B\left(n_\uparrow - n_\downarrow\right) = g\mu_B B\left(\sum_\sigma \sigma p_\sigma^\dagger p_\sigma\right)$. We would like to focus on a subset of the meanfield equations:

$$\begin{aligned}\sum_\sigma \left\langle f_\sigma^\dagger f_\sigma\right\rangle &= Q = 1\\ \left\langle e^\dagger e\right\rangle + \sum_\sigma \left\langle p_\sigma^\dagger p_\sigma\right\rangle + \left\langle d^\dagger d\right\rangle &= 1\\ \left\langle p_\sigma^\dagger p_\sigma\right\rangle + \left\langle d^\dagger d\right\rangle &= \left\langle f_\sigma^\dagger f_\sigma\right\rangle\end{aligned} \tag{C44}$$

Summing the two spin dependent equations in Eq. (C44) and substituting the first equation we get that:

$$\left\langle e^\dagger e\right\rangle + \sum_\sigma \left\langle p_\sigma^\dagger p_\sigma\right\rangle + \left\langle d^\dagger d\right\rangle = \sum_\sigma \left\langle p_\sigma^\dagger p_\sigma\right\rangle + 2\left\langle d^\dagger d\right\rangle \tag{C45}$$

Or equivalently $\left\langle e^\dagger e\right\rangle = \left\langle d^\dagger d\right\rangle$ this holds for any magnetic fields and temperatures. The meanfield Lagrangian is given by:

$$\begin{aligned}&\sum_\sigma f_\sigma^\dagger \frac{d}{d\tau} f_\sigma + \sum_{k,\sigma} c_{k,\sigma}^\dagger \frac{d}{d\tau} c_{k,\sigma} + \lambda_{RN}\left(\sum_\sigma f_\sigma^\dagger f_\sigma - Q\right) + 2\frac{V\bar{V}}{J}\\ &+Ud^2 + V\sum_{k,\sigma} c_{k,\sigma}^\dagger z_\sigma f_\sigma + h.c. + \sum\left(\epsilon_k - \mu\right) c_{k,\sigma}^\dagger c_{k,\sigma} +\\ &+\lambda_{KR}\left(2p^2 + 2d^2 - 1\right) + \sum_\sigma \lambda_\sigma\left(f_\sigma^\dagger f_\sigma - p^2 - d^2\right) + g\mu_B B\left(\sum_\sigma \sigma p_\sigma^2\right)\end{aligned}$$

Furthermore:

$$z_\sigma = \left(1 - d^2 - p_\sigma^2\right)^{-1/2} 2d\left(p_\sigma + p_{-\sigma}\right)\left(1 - d^2 - p_{-\sigma}^2\right)^{-1/2} \equiv z \tag{C46}$$

Is now spin independent. We then can obtain the Gibbs free energy at zero temperature following [1]. It is given by:

$$\frac{1}{\beta}\ln(Z) = \sum_\sigma \frac{N}{2\pi} Im\left[\xi_\sigma \ln\left(\frac{\xi_\sigma}{\mathrm{e}D}\right)\right] - \lambda_{RN}Q - \frac{N\Delta}{\pi J\rho} - Ud^2 \tag{C47}$$

$$-\lambda_{KR}\left(\sum_\sigma p_\sigma^2 + 2d^2 - 1\right) - \sum_\sigma \lambda_\sigma\left(p_\sigma^2 + d^2\right) - g\mu_B B\left(\sum_\sigma \sigma p_\sigma^2\right) \tag{C48}$$

Compare with Eq. (17.107) in [1]. Here $\xi_\sigma = \lambda_{RN} + iz^2\Delta + \lambda_\sigma$. The meanfield equations are given by:

$$\begin{aligned}\frac{N}{2\pi}\sum_\sigma \cot^{-1}\left(\frac{\lambda_{RN} + \lambda_\sigma}{z^2\Delta}\right) &= Q\\ \frac{N}{2\pi}\cot^{-1}\left(\frac{\lambda_{RN} + \lambda_\sigma}{z^2\Delta}\right) &= \left(p_\sigma^2 + d^2\right)\\ z^2\frac{N}{2\pi}\sum_\sigma \ln\left(\frac{\sqrt{\left(\lambda_{RN} + \lambda_\sigma\right)^2 + z^4\Delta^2}}{D}\right) &= -\frac{N}{\pi\rho J}\\ \sum_\sigma p_\sigma^2 + 2d^2 &= 1\\ 2z\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{2\pi}\sum_\sigma \ln\left(\frac{\sqrt{\left(\lambda_{RN} + \lambda_\sigma\right)^2 + z^4\Delta^2}}{D}\right) &= -4d\lambda_{GR} - 2d\sum_\sigma \lambda_\sigma - 2Ud\\ 2z\frac{\partial z}{\partial p_\sigma}\cdot\Delta\frac{N}{2\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN} + \lambda_\sigma\right)^2 + z^4\Delta^2}}{D}\right) &= -2p_\sigma\lambda_{KR} - 2p_\sigma\lambda_\sigma - g\mu_B B\sigma p_\sigma\end{aligned} \tag{C49}$$

next we note that to leading order in B it is safe to replace $g\mu_B B p_\sigma \cong g\mu_B B p_0$ where p_0 is the zero field solution given in Eq. (C24). Now if we sum the two equations written in the last line of Eq. (C49) and the two equations written on line 2 in Eq. (C49) we obtain the equations:

$$\begin{aligned}
\frac{N}{2\pi}\sum_\sigma \cot^{-1}\left(\frac{\lambda_{RN}+\lambda_\sigma}{z^2\Delta}\right) &= Q \\
\sum_\sigma \frac{N}{2\pi}\cot^{-1}\left(\frac{\lambda_{RN}+\lambda_\sigma}{z^2\Delta}\right) &= \left(\sum_\sigma p_\sigma^2 + 2d^2\right) \\
z^2\frac{N}{2\pi}\sum_\sigma \ln\left(\frac{\sqrt{(\lambda_{RN}+\lambda_\sigma)^2+z^4\Delta^2}}{D}\right) &= -\frac{N}{\pi\rho J} \\
\sum_\sigma p_\sigma^2 + 2d^2 &= 1 \\
2z\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{2\pi}\sum_\sigma \ln\left(\frac{\sqrt{(\lambda_{RN}+\lambda_\sigma)^2+z^4\Delta^2}}{D}\right) &= -4d\lambda_{GR} - 2d\sum_\sigma \lambda_\sigma - 2Ud \\
2z\sum_\sigma \frac{\partial z}{\partial p_\sigma}\cdot\Delta\frac{N}{2\pi}\ln\left(\frac{\sqrt{(\lambda_{RN}+\lambda_\sigma)^2+z^4\Delta^2}}{D}\right) &= -2\sum_\sigma p_\sigma\lambda_{KR} - 2\sum_\sigma p_\sigma\lambda_\sigma
\end{aligned} \tag{C50}$$

Now compare with Eq. (C6). Now we notice that $\frac{\partial z}{\partial p_\sigma} = \frac{\partial z}{\partial p} + O(B)$ and $\lambda_\sigma = \lambda + O(B)$ which means that to linear order in the magnetic field we may as well replace the last line with:

$$\begin{aligned}
\frac{N}{2\pi}\sum_\sigma \cot^{-1}\left(\frac{\lambda_{RN}+\lambda_\sigma}{z^2\Delta}\right) &= Q \\
\sum_\sigma \frac{N}{2\pi}\cot^{-1}\left(\frac{\lambda_{RN}+\lambda_\sigma}{z^2\Delta}\right) &= \left(\sum_\sigma p_\sigma^2 + 2d^2\right) \\
z^2\frac{N}{2\pi}\sum_\sigma \ln\left(\frac{\sqrt{(\lambda_{RN}+\lambda_\sigma)^2+z^4\Delta^2}}{D}\right) &= -\frac{N}{\pi\rho J} \\
\sum_\sigma p_\sigma^2 + 2d^2 &= 1 \\
2z\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{2\pi}\sum_\sigma \ln\left(\frac{\sqrt{(\lambda_{RN}+\lambda_\sigma)^2+z^4\Delta^2}}{D}\right) &= -4d\lambda_{KR} - 2d\sum_\sigma \lambda_\sigma - 2Ud \\
z\sum_\sigma \frac{\partial z}{\partial p_\sigma}\cdot\sum_{\sigma'}\Delta\frac{N}{2\pi}\ln\left(\frac{\sqrt{(\lambda_{RN}+\lambda_{\sigma'})^2+z^4\Delta^2}}{D}\right) &= -2\sum_\sigma p_\sigma\lambda_{KR} - \sum_{\sigma'} p_{\sigma'}\sum_\sigma \lambda_\sigma
\end{aligned} \tag{C51}$$

Now we substitute the third line in Eq. (C51) into the fifth and sixth lines to obtain:

$$\begin{aligned}
\sum_\sigma p_\sigma^2 + 2d^2 &= 1 \\
\frac{2}{z}\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{\pi\rho J} &= -4d\lambda_{GR} - 2d\sum_\sigma \lambda_\sigma - 2Ud \\
\frac{1}{z}\sum_\sigma \frac{\partial z}{\partial p_\sigma}\cdot\Delta\frac{N}{\pi\rho J} &= -2\sum_\sigma p_\sigma\lambda_{KR} - \sum_{\sigma'} p_{\sigma'}\sum_\sigma \lambda_\sigma
\end{aligned} \tag{C52}$$

Now

$$\ln z = \ln(d) + \ln\left(\sum_\sigma p_\sigma\right) - \sum_\sigma \frac{1}{2}\ln\left(1-d^2-p_\sigma^2\right) \tag{C53}$$

Therefore

$$
\begin{aligned}
\frac{\partial \ln z}{\partial d} &= \frac{1}{d} + \sum_{\sigma} \frac{d}{1-d^2-p_\sigma^2} \cong \frac{1}{d} + d\frac{2}{1-d_0^2-p_0^2} \\
\frac{\partial \ln z}{\partial p_\sigma} &= \frac{p_\sigma}{1-d^2-p_\sigma^2} + \frac{1}{\sum_\sigma p_\sigma}
\end{aligned}
\tag{C54}
$$

Now we may as well replace to order B, $\sum_\sigma \frac{\partial \ln z}{\partial p_\sigma} \cong \frac{1}{2}\sum_\sigma p_\sigma \sum \frac{1}{1-d^2-p_\sigma^2} + \frac{2}{\sum p_\sigma}$ furthermore we may as well replace to leading order in B, $\sum_\sigma p_\sigma^2 = \frac{1}{2}\sum_\sigma p_\sigma \sum_{\sigma'} p_{\sigma'}$, with these simplifications we get that:

$$
\begin{aligned}
\frac{1}{2}\sum_\sigma p_\sigma \sum_{\sigma'} p_{\sigma'} + 2d^2 &= 1 \\
\left(2 + \frac{4d^2}{1-d_0^2-p_0^2}\right)\cdot \Delta\frac{N}{\pi\rho J} &= -4d^2\lambda_{KR} - 2d^2\sum_\sigma \lambda_\sigma - 2Ud^2 \\
\left(\frac{1}{2}\sum_\sigma p_\sigma \sum \frac{1}{1-d^2-p_\sigma^2} + \frac{2\sum_\sigma p_\sigma}{\sum_\sigma p_\sigma \sum_{\sigma'} p_{\sigma'}}\right)\Delta\frac{N}{\pi\rho J} &= -2\sum_\sigma p_\sigma \lambda_{KR} - \sum_{\sigma'} p_{\sigma'} \sum_\sigma \lambda_\sigma
\end{aligned}
\tag{C55}
$$

Next we substitute $\sum_\sigma p_\sigma \sum_{\sigma'} p_{\sigma'} = 2-4d^2$ into the third line in Eq. (C55). We can cancel the common factors of $\sum_\sigma p_\sigma$ in the last line of Eq. (C55) and get $\sum_\sigma \lambda_\sigma = -2\lambda_{KR} - \left(\frac{1}{2}\sum \frac{1}{1-d^2-p_\sigma^2} + \frac{2}{1-2d^2}\right)\Delta\frac{N}{\pi\rho J}$, substituting this into the second line in Eq. (C55) we get that:

$$
\begin{aligned}
\sum_\sigma p_\sigma^2 + 2d^2 &= 1 \\
\left(2 + \frac{4d^2}{1-d_0^2-p_0^2}\right)\cdot \Delta\frac{N}{\pi\rho J} &= -4d^2\lambda_{KR} - 2d^2\left(-2\lambda_{KR} - \left(\frac{1}{2}\sum_\sigma \frac{1}{1-d^2-p_\sigma^2} + \frac{1}{1-2d^2}\right)\Delta\frac{N}{\pi\rho J}\right) - 2Ud^2
\end{aligned}
\tag{C56}
$$

however we may replace to leading order in B $\sum_\sigma \frac{1}{1-d^2-p_\sigma^2} = \frac{2}{1-d_0^2-p_0^2}$ (where we have used the first line in Eq. (C56), so we obtain

$$
\left(2 + \frac{4d^2}{1-d_0^2-p_0^2}\right)\cdot \Delta\frac{N}{\pi\rho J} = -4d^2\lambda_{KR} - 2d^2\left(-2\lambda_{KR} - \left(\frac{1}{2}\frac{2}{1-d_0^2-p_0^2} + \frac{1}{1-2d^2}\right)\Delta\frac{N}{\pi\rho J}\right) - 2Ud^2 \tag{C57}
$$

now we notice that the terms proportional to λ_{KR} cancel and $1-d_0^2-p_0^2 = \frac{1}{2}$ leading to:

$$
\left(1+4d^2\right)\cdot \Delta\frac{N}{\pi\rho J} = 2d^2\left(2 + \frac{1}{1-2d^2}\right)\Delta\frac{N}{\pi\rho J} - Ud^2 \tag{C58}
$$

Setting $d^2 = x$ we get:

$$
(1+4x)(1-2x)\Delta\frac{N}{\pi\rho J} = 2x(3-4x)\Delta\frac{N}{\pi\rho J} - 2Ux(1-2x) \tag{C59}
$$

this simplifies to:

$$
-4Ux^2 + \left(4\Delta\frac{N}{\pi\rho J} + 2U\right)x - \Delta\frac{N}{\pi\rho J} = 0 \tag{C60}
$$

$$
e^2 = d^2 = \frac{2\Delta\frac{N}{\pi\rho J} - U - \sqrt{\left(2\frac{N\Delta}{\pi\rho J} - U\right)^2 - 4U\left(\frac{N\Delta}{\pi\rho J}\right)}}{4U} \tag{C61}
$$

We note that for large U, $e^2, d^2 \to 0$ showing that we obtain good projection onto the spin 1/2 subspace. From this we get that

$$
\sum_\sigma p_\sigma^2 = -\frac{N\Delta}{\pi\rho JU} + \frac{\sqrt{\left(2\frac{N\Delta}{\pi\rho J} - U\right)^2 - 4U\left(\frac{N\Delta}{\pi\rho J}\right)}}{2U} \tag{C62}
$$

z is now given by Eq. (C46) which can be further simplified to again to leading order in B:

$$z = 2d \sum_\sigma p_\sigma \tag{C63}$$

Note when taylor expanding this with respect to These equations are identical to Eqs. (C21) to (C25). From this we get that: $\lambda_{RN} = \frac{U}{2}$, $\lambda_\sigma = -\frac{U}{2} + \sigma\lambda_1$, $e = d = e_0 = d_0$, $p_\sigma = p_0 + \sigma p_1$ and $\Delta = \Delta_0$. From Eq. (C49) we see that:

$$\begin{aligned} -\frac{N}{2\pi}\frac{\lambda_1}{z^2\Delta_0} &= p_1 \\ 2p_1\left(1-4p_0^2\right)\frac{z^{-2}}{\rho J} &= 2p_1U - 2p_0\lambda_1 - 2p_1\lambda_0 - g\mu_B B\sigma p_0 \end{aligned} \tag{C64}$$

This simplifies to:

$$2p_1\left(1-4p_0^2\right)\Delta_0\frac{z^{-2}}{\rho J} = 2p_1U + 2p_0p_1z_0^2\Delta_0\frac{2\pi}{N} + p_1U - g\mu_B B\sigma p_0 \tag{C65}$$

Now $z_0 = 4p_0d_0$ so this simplifies to

$$\chi = g\mu_B\frac{p_1}{B} = \frac{g^2\mu_B^2}{-2\left(1-4p_0^2\right)\Delta_0\frac{z^{-2}}{\rho J} + 2p_0z_0^2\Delta_0\frac{2\pi}{N} + 3U} \tag{C66}$$

Where $N = 2$. Now using

$$p_0^2 \simeq \frac{1}{4} + \frac{1}{4}\frac{U}{\frac{N\Delta}{\pi\rho J}} \tag{C67}$$

We get that

$$\chi \cong \frac{g^2\mu_B^2}{-\frac{2}{N}\pi U z^{-2} + 2p_0z_0^2\Delta_0\frac{2\pi}{N} + 3U} \cong \frac{g^2\mu_B^2}{2p_0z_0^2\Delta_0\frac{2\pi}{N}} \simeq\cong \frac{g^2\mu_B^2}{\Delta_0\frac{2\pi}{N}} \tag{C68}$$

The Wilson ratio is:

$$w = \frac{\pi^2\chi}{\frac{1}{2}\frac{3}{2}C_v} = \frac{2}{\pi} \simeq 0.64 \tag{C69}$$

4. More on the zero temperature solution (a slight generalization)

We now notice that we can also add the term $W\left(1-n_\uparrow\right)\left(1-n_\downarrow\right) = We^\dagger e$ to the action given in Eq. (C1) without changing the partition function as $e^\dagger e = 0$ after projection. The meanfield equations for that action are given by:

$$\begin{aligned} \frac{N}{\pi}\cot^{-1}\left(\frac{\lambda_{RN}+\lambda}{z^2\Delta}\right) &= Q \\ \frac{N}{\pi}\cot^{-1}\left(\frac{\lambda_{RN}+\lambda}{z^2\Delta}\right) &= 2\left(p^2+d^2\right) \\ z^2\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right) &= -\frac{N}{\pi\rho J} \\ e^2+2p^2+d^2 &= 1 \\ 2z\frac{\partial z}{\partial e}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right) &= -2e\lambda_{KR} - 2eW \\ 2z\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right) &= -2d\lambda_{KR} - 4d\lambda - 2Ud \\ 2z\frac{\partial z}{\partial p}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right) &= -4p\lambda_{KR} - 4p\lambda \end{aligned} \tag{C70}$$

Following the derivation given in Eqs (C7) to (C17) we obtain that:

$$\begin{aligned} 2\cdot\left(\frac{1}{2d}+2d\right)\Delta\frac{N}{\pi\rho J} &= 2d\lambda_{KR}+2Wd \\ 2\left(\frac{1}{2d}+2d\right)\cdot\Delta\frac{N}{\pi\rho J} &= 2d\lambda_{KR}+4d\lambda+2Ud \\ 2\left(\frac{1}{p}+4p\right)\cdot\Delta\frac{N}{\pi\rho J} &= 4p\lambda_{KR}+4p\lambda \end{aligned} \tag{C71}$$

Once again $e=d$ and $e^2+p^2=e^2+d^2=\frac{1}{2}$, Now we obtain that:

$$\lambda=\frac{1}{2}\left(W-U\right) \tag{C72}$$

From this we obtain that

$$\begin{aligned} \left(1+4d^2\right)\Delta\frac{N}{\pi\rho J} &= 2d^2\left(\lambda_{KR}+W\right) \\ \left(1+4p^2\right)\cdot\Delta\frac{N}{\pi\rho J} &= p^2\left(2\lambda_{KR}+W-U\right) \end{aligned}$$

We notice that if we call

$$\tilde{\lambda}_{GR}=\lambda_{GR}+W \tag{C73}$$

we obtain the equations:

$$\begin{aligned} \left(1+4d^2\right)\Delta\frac{N}{\pi\rho J} &= 2d^2\tilde{\lambda}_{GR} \\ \left(3-4d^2\right)\Delta\frac{N}{\pi\rho J} &= \left(1-2d^2\right)\left(\tilde{\lambda}_{GR}-U\right) \end{aligned} \tag{C74}$$

which is identical to Eq. (C19) and the rest of Eqs. (C20) to (C31) follow.

5. Soft constraint Kotliar Ruckenstein Kondo model

We can freely add in the soft constraint to the above formulation of the Kondo model. It does nothing to the partition function but merely changes the Lagrangian and hence the meanfield. The Lagrangian for the system is given by:

$$\begin{aligned} h^\dagger\frac{d}{d\tau}h+e^\dagger\frac{d}{d\tau}e+d^\dagger\frac{d}{d\tau}d+\sum_\sigma p_\sigma^\dagger\frac{d}{d\tau}p_\sigma+\sum_\sigma f_\sigma^\dagger\frac{d}{d\tau}f_\sigma+\sum_{k,\sigma}c_{k,\sigma}^\dagger\frac{d}{d\tau}c_{k,\sigma}+2\frac{V\bar{V}}{J}+ \\ \lambda_{RN}\left(\sum_\sigma f_\sigma^\dagger f_\sigma-Q\right)+Ud^\dagger d+We^\dagger e+V\sum_{k,\sigma}c_{k,\sigma}^\dagger z_\sigma f_\sigma+h.c.+\sum\left(\epsilon_k-\mu\right)c_k^\dagger c_k+ \\ +\lambda_{KR}\left(e^\dagger e+\sum_\sigma p_\sigma^\dagger p_\sigma+d^\dagger d-1\right)+\sum_\sigma\lambda_\sigma\left(f_\sigma^\dagger f_\sigma-p_\sigma^\dagger p_\sigma-d^\dagger d\right)+\lambda_{SC}\left(d^\dagger d-Kh^\dagger h\right) \end{aligned} \tag{C75}$$

The meanfield equations for that action are given by:

$$\begin{aligned}
\frac{N}{\pi}\cot^{-1}\left(\frac{\lambda_{RN}+\lambda}{z^2\Delta}\right) &= Q \\
\frac{N}{\pi}\cot^{-1}\left(\frac{\lambda_{RN}+\lambda}{z^2\Delta}\right) &= 2\left(p^2+d^2\right) \\
z^2\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right) &= -\frac{N}{\pi\rho J} \\
e^2+2p^2+d^2 &= 1 \\
2z\frac{\partial z}{\partial e}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right) &= -2e\lambda_{GR}-2eW \\
2z\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right) &= -2d\lambda_{GR}-4d\lambda-2Ud-2\lambda_{SC}d \\
2z\frac{\partial z}{\partial p}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\left(\lambda_{RN}+\lambda\right)^2+z^4\Delta^2}}{D}\right) &= -4p\lambda_{GR}-4p\lambda \\
d^2 &= K
\end{aligned} \tag{C76}$$

Lets introduce $\tilde{\lambda}_{SC}=\lambda_{SC}+U$. Then we get:

$$\begin{aligned}
\frac{2}{z}\frac{\partial z}{\partial e}\cdot\Delta\frac{N}{\pi\rho J} &= 2e\lambda_{GR}+2eW \\
\frac{2}{z}\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{\pi\rho J} &= 2d\lambda_{GR}+4d\lambda+2\tilde{\lambda}_{SC}d \\
\frac{2}{z}\frac{\partial z}{\partial p}\cdot\Delta\frac{N}{\pi\rho J} &= 4p\lambda_{GR}+4p\lambda \\
d^2 &= K \\
p^2 &= \frac{Q}{2}-K \\
e^2 &= K
\end{aligned} \tag{C77}$$

Similarly to Section C 4 we obtain that:

$$\begin{aligned}
2\cdot\left(\frac{1}{2d}+2d\right)\Delta\frac{N}{\pi\rho J} &= 2d\lambda_{GR}+2Wd \\
2\left(\frac{1}{2d}+2d\right)\cdot\Delta\frac{N}{\pi\rho J} &= 2d\lambda_{GR}+4d\lambda+2\tilde{\lambda}_{SC}d \\
2\left(\frac{1}{p}+4p\right)\cdot\Delta\frac{N}{\pi\rho J} &= 4p\lambda_{GR}+4p\lambda
\end{aligned} \tag{C78}$$

Similarly we obtain that:

$$\lambda=\frac{1}{2}\left(\tilde{\lambda}_{SC}-U\right) \tag{C79}$$

From this we obtain that

$$\begin{aligned}
\left(1+4d^2\right)\Delta\frac{N}{\pi\rho J} &= 2d^2\left(\lambda_{GR}+W\right) \\
\left(1+4p^2\right)\cdot\Delta\frac{N}{\pi\rho J} &= p^2\left(2\lambda_{GR}+W-U\right)
\end{aligned}$$

We notice that if we call

$$\tilde{\lambda}_{KR} = \lambda_{KR} + W \tag{C80}$$

we obtain the equations:

$$\begin{aligned}\left(1+4d^2\right)\Delta\frac{N}{\pi\rho J} &= 2d^2\tilde{\lambda}_{KR}\\ \left(3-4d^2\right)\Delta\frac{N}{\pi\rho J} &= \left(1-2d^2\right)\left(\tilde{\lambda}_{KR}-\tilde{\lambda}_{SC}\right)\end{aligned} \tag{C81}$$

which is identical to Eq. (C19) and the rest of Eqs. (C20) to (C31) follow. The only extra relation is that:

$$d^2 = K \tag{C82}$$

Which can be obtained for the right K for every λ_{SC}, as such we gain nothing by our soft constraint formulation at zero field and zero temperature, however we reproduce all old results. It is not too hard to check that the results in Sections C 2 and C 3 are reproduced as well.

6. Double soft constraint Kotliar Ruckenstein

We would like to do away with the Read Newns constraint and study the system using purely soft constraints. To do this we will add in the constraint $(1-n_\uparrow)(1-n_\downarrow) - K_2 h_2^\dagger h_2 = 0$ which enforces that the impurity site is never empty. The Lagrangian for the system is given by:

$$\begin{aligned} e^\dagger\frac{d}{d\tau}e + d^\dagger\frac{d}{d\tau}d + \sum_\sigma p_\sigma^\dagger\frac{d}{d\tau}p_\sigma + \sum_\sigma f_\sigma^\dagger\frac{d}{d\tau}f_\sigma + \sum_{k,\sigma} c_{k,\sigma}^\dagger\frac{d}{d\tau}c_{k,\sigma} + h_1^\dagger\frac{d}{d\tau}h_1 + h_2^\dagger\frac{d}{d\tau}h_2 + 2\frac{V\bar{V}}{J} + \\ +\lambda_{SC,1}\left(n_\uparrow n_\downarrow - K_1 h_1^\dagger h_1\right) + \lambda_{SC,2}\left((1-n_\uparrow)(1-n_\downarrow) - K_1 h_1^\dagger h_1\right) + \\ +We^\dagger e + Ud^\dagger d + V\sum_{k,\sigma} c_{k,\sigma}^\dagger z_\sigma f_\sigma + h.c. + \sum(\epsilon_k-\mu)c_k^\dagger c_k + \\ +\lambda_{KR}\left(e^\dagger e + \sum_\sigma p_\sigma^\dagger p_\sigma + d^\dagger d - 1\right) + \sum_\sigma \lambda_\sigma\left(f_\sigma^\dagger f_\sigma - p_\sigma^\dagger p_\sigma - d^\dagger d\right) + g\mu_B B(n_\uparrow - n_\downarrow)\end{aligned} \tag{C83}$$

The meanfield partition function at zero temperature zero magnetic field is given by:

$$\begin{aligned}\frac{1}{\beta}\ln(Z) = \frac{N}{\pi}Im\left[\xi\ln\left(\frac{\xi}{\mathrm{e}D}\right)\right] - \frac{N\Delta}{\pi J\rho} - Ud^2 - We^2 - \lambda_{SC,1}\left(d^2 - K_1 h_1^\dagger h_1\right)\\ -\lambda_{KR}\left(\sum_\sigma p_\sigma^2 + d^2 + e^2 - 1\right) - \sum_\sigma \lambda\left(p_\sigma^2 + d^2\right) - \lambda_{SC,2}\left(e^2 - K_2 h_2^\dagger h_2\right)\end{aligned} \tag{C84}$$

Compare with Eq. (17.107) in [1]. Here $\xi_\sigma = iz^2\Delta + \lambda$ where $\lambda_\uparrow = \lambda_\downarrow \equiv \lambda$. Furthermore $p_\uparrow = p_\downarrow \equiv p$ and $z_\uparrow = z_\downarrow \equiv z$. At meanfield level:

$$z = \frac{(e+d)\,p}{\left(1-d^2-p^2\right)^{1/2}\left(1-e^2-p^2\right)^{1/2}} \tag{C85}$$

The meanfield equations are given by:

$$
\begin{aligned}
e^2 &= K_2 \\
\frac{N}{\pi}\cot^{-1}\left(\frac{\lambda}{z\Delta}\right) &= 2\left(p^2+d^2\right) \\
z^2\frac{N}{\pi}\ln\left(\frac{\sqrt{\lambda^2+z^4\Delta^2}}{D}\right) &= -\frac{N}{\pi\rho J} && \text{(C86)} \\
e^2+2p^2+d^2 &= 1 && \text{(C87)} \\
2z\frac{\partial z}{\partial e}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\lambda^2+z^4\Delta^2}}{D}\right) &= -2e\lambda_{KR}-2eW-2e\lambda_{Soft,2} && \text{(C88)} \\
2z\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\lambda^2+z^4\Delta^2}}{D}\right) &= -2d\lambda_{KR}-4d\lambda-2Ud-2\lambda_{Soft,1}d && \text{(C89)} \\
2z\frac{\partial z}{\partial p}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\lambda^2+z^4\Delta^2}}{D}\right) &= -4p\lambda_{KR}-4p\lambda && \text{(C90)} \\
d^2 &= K_1 && \text{(C91)}
\end{aligned}
$$

We note that $p^2=\frac{1}{2}\left(1-K_1-K_2\right)$. Furthermore lets define $\tilde{\lambda}_{Soft,1}=\lambda_{Soft,1}+U$ and $\tilde{\lambda}_{Soft,2}=\lambda_{Soft,2}+W$ and the terms U and W drop out of the calculation. Therefore:

$$
z=\frac{\sqrt{2}\left(\sqrt{K_1}+\sqrt{K_2}\right)\left(1-K_1-K_2\right)^{1/2}}{\left(1-K_1+K_2\right)^{1/2}\left(1+K_1-K_2\right)^{1/2}} \tag{C92}
$$

The Eq. (C86) now simplifies to:

$$
\begin{aligned}
\frac{\lambda}{z^2\Delta} &= \cot\left(\frac{\pi}{N}\left(1+K_1-K_2\right)\right) \\
\sqrt{\lambda^2+z^4\Delta^2} &= D\exp\left(\frac{-z^2}{\rho J}\right) && \text{(C93)}
\end{aligned}
$$

This means that

$$
\Delta=z^{-2}\sin\left(\frac{\pi}{N}\left(1+K_1-K_2\right)\right)D\exp\left(\frac{-z^2}{\rho J}\right)
$$

To restore particle hole symmetry we need to take $K_1=K_2$ and the results reduce to those in Section C 1. It is not too hard to see that the heat capacity and magnetic susceptibility stay the same in the symmetric case.

7. Compound soft constraint Kotliar Ruckenstein formulation

We would like to do away with the Read Newns constraint and study the system using purely soft constraints. To do this we will add in the constraint $\left(1-n_\uparrow\right)\left(1-n_\downarrow\right)+n_\uparrow n_\downarrow-Kh^\dagger h=0$ which enforces that the impurity site is never empty and never doubly occupied. The Lagrangian for the system is given by:

$$
\begin{aligned}
& e^\dagger\frac{d}{d\tau}e+d^\dagger\frac{d}{d\tau}d+\sum_\sigma p^\dagger_\sigma\frac{d}{d\tau}p_\sigma+\sum_\sigma f^\dagger_\sigma\frac{d}{d\tau}f_\sigma+\sum_{k,\sigma}c^\dagger_{k,\sigma}\frac{d}{d\tau}c_{k,\sigma}+h^\dagger\frac{d}{d\tau}h+2\frac{V\bar{V}}{J}+ \\
& +\lambda_{SC}\left(n_\uparrow n_\downarrow+\left(1-n_\uparrow\right)\left(1-n_\downarrow\right)-K_1h^\dagger_1h_1\right)+ \\
& +We^\dagger e+Ud^\dagger d+V\sum_{k,\sigma}c^\dagger_{k,\sigma}z_\sigma f_\sigma+h.c.+\sum\left(\epsilon_k-\mu\right)c^\dagger_kc_k+ && \text{(C94)}
\end{aligned}
$$

$$
+\lambda_{KR}\left(e^\dagger e+\sum_\sigma p^\dagger_\sigma p_\sigma+d^\dagger d-1\right)+\sum_\sigma\lambda_\sigma\left(f^\dagger_\sigma f_\sigma-p^\dagger_\sigma p_\sigma-d^\dagger d\right)+g\mu_BB\left(n_\uparrow-n_\downarrow\right) \tag{C95}
$$

The meanfield partition function at zero temperature zero magnetic field is given by:

$$\frac{1}{\beta}\ln(Z) = \frac{N}{\pi}Im\left[\xi\ln\left(\frac{\xi}{\mathrm{e}D}\right)\right] - \frac{N\Delta}{\pi J\rho} - Ud^2 - We^2 - \lambda_{SC}\left(d^2+e^2-K_1h_1^\dagger h_1\right) \tag{C96}$$

$$-\lambda_{KR}\left(\sum_\sigma p_\sigma^2 + d^2 + e^2 - 1\right) - \sum_\sigma \lambda\left(p_\sigma^2+d^2\right) \tag{C97}$$

Compare with Eq. (17.107) in [1]. Here $\xi_\sigma = iz^2\Delta + \lambda$ where $\lambda_\uparrow = \lambda_\downarrow \equiv \lambda$. Furthermore $p_\uparrow = p_\downarrow \equiv p$ and $z_\uparrow = z_\downarrow \equiv z$. At meanfield level:

$$z = \frac{(e+d)\,p}{(1-d^2-p^2)^{1/2}(1-e^2-p^2)^{1/2}} \tag{C98}$$

The meanfield equations are given by:

$$\begin{aligned}
e^2+d^2 &= K\\
\frac{N}{\pi}\cot^{-1}\left(\frac{\lambda}{z\Delta}\right) &= 2\left(p^2+d^2\right)\\
z^2\frac{N}{\pi}\ln\left(\frac{\sqrt{\lambda^2+z^4\Delta^2}}{D}\right) &= -\frac{N}{\pi\rho J}\\
e^2+2p^2+d^2 &= 1\\
2z\frac{\partial z}{\partial e}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\lambda^2+z^4\Delta^2}}{D}\right) &= -2e\lambda_{KR} - 2eW - 2e\lambda_{SC}\\
2z\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\lambda^2+z^4\Delta^2}}{D}\right) &= -2d\lambda_{KR} - 4d\lambda - 2Ud - 2d\lambda_{SC}\\
2z\frac{\partial z}{\partial p}\cdot\Delta\frac{N}{\pi}\ln\left(\frac{\sqrt{\lambda^2+z^4\Delta^2}}{D}\right) &= -4p\lambda_{KR} - 4p\lambda
\end{aligned} \tag{C99}$$

We get $p^2 = \frac{1}{2}(1-K)$ and $e^2 = K - d^2$, furthermore substituting the third equation into the last three we get that:

$$\begin{aligned}
\frac{2}{z}\frac{\partial z}{\partial e}\cdot\Delta\frac{N}{\pi\rho J} &= 2e\lambda_{GR} + 2eW + 2e\lambda_{SC}\\
\frac{2}{z}\frac{\partial z}{\partial d}\cdot\Delta\frac{N}{\pi\rho J} &= 2d\lambda_{KR} + 4d\lambda + 2\lambda_{SC}d + 2Ud\\
\frac{2}{z}\frac{\partial z}{\partial p}\cdot\Delta\frac{N}{\pi\rho J} &= 4p\lambda_{GR} + 4p\lambda\\
d^2 &= K - e^2\\
p^2 &= \frac{1}{2}(1-K)
\end{aligned} \tag{C100}$$

Furthermore:

$$\ln z = \ln(e+d) + \ln(p) - \frac{1}{2}\ln\left(1-d^2-p^2\right) - \frac{1}{2}\ln\left(1-e^2-p^2\right) \tag{C101}$$

With this:

$$\begin{aligned}
\frac{\partial\ln z}{\partial e} &= \frac{1}{e+d} + \frac{e}{(1-e^2-p^2)}\\
\frac{\partial\ln z}{\partial d} &= \frac{1}{e+d} + \frac{d}{(1-d^2-p^2)}\\
\frac{\partial\ln z}{\partial p} &= \frac{1}{p} + \frac{p}{(1-d^2-p^2)} + \frac{p}{(1-e^2-p^2)}
\end{aligned} \tag{C102}$$

This simplifies to:

$$\begin{aligned}
\frac{\partial \ln z}{\partial e} &= \frac{1}{e+d} + \frac{e}{\left(\frac{1}{2}\left(1+K\right)-e^{2}\right)} \\
\frac{\partial \ln z}{\partial d} &= \frac{1}{e+d} + \frac{d}{\left(\frac{1}{2}\left(1-K\right)+e^{2}\right)} \\
\frac{\partial \ln z}{\partial p} &= \frac{1}{p} + \frac{p}{\left(\frac{1}{2}\left(1+K\right)-e^{2}\right)} + \frac{p}{\left(\frac{1}{2}\left(1-K\right)+e^{2}\right)}
\end{aligned} \tag{C103}$$

$$\begin{aligned}
\left(\frac{1}{e+d} + \frac{e}{\left(\frac{1}{2}\left(1+K\right)-e^{2}\right)}\right)\cdot\Delta\frac{N}{\pi\rho J} &= e\lambda_{KR} + eW + e\lambda_{SC} \\
\left(\frac{1}{e+d} + \frac{d}{\left(\frac{1}{2}\left(1-K\right)+e^{2}\right)}\right)\cdot\Delta\frac{N}{\pi\rho J} &= d\lambda_{KR} + 2d\lambda + \lambda_{SC}d + Ud \\
\left(\frac{1}{p} + \frac{p}{\left(\frac{1}{2}\left(1+K\right)-e^{2}\right)} + \frac{p}{\left(\frac{1}{2}\left(1-K\right)+e\right)}\right)\cdot\Delta\frac{N}{\pi\rho J} &= 2p\lambda_{KR} + 2p\lambda \\
d^{2} &= K - e^{2} \\
p^{2} &= \frac{1}{2}\left(1-K\right)
\end{aligned} \tag{C104}$$

This leads to:

$$\left(1 + \frac{\frac{1}{2}\left(1-K\right)}{\left(\frac{1}{2}\left(1+K\right)-e^{2}\right)} + \frac{\frac{1}{2}\left(1-K\right)}{\left(\frac{1}{2}\left(1-K\right)+e\right)}\right)\cdot\Delta\frac{N}{\pi\rho J} = \left(1-K\right)\left(\lambda_{KR}+\lambda\right) \tag{C105}$$

Lets look for solutions with particle hole symmetry which means that $e^2 = d^2 = \frac{K}{2}$. Then we get

$$\begin{aligned}
\left(\frac{1}{2} + 2d^{2}\right)\Delta\frac{N}{\pi\rho J} &= d^{2}\lambda_{KR} + Wd^{2} + d^{2}\lambda_{SC} \\
\left(\frac{1}{2} + 2d^{2}\right)\cdot\Delta\frac{N}{\pi\rho J} &= d^{2}\lambda_{GR} + 2d^{2}\lambda + \lambda_{SC}d + Ud^{2} \\
\left(1 + 4p^{2}\right)\cdot\Delta\frac{N}{\pi\rho J} &= 2p^{2}\lambda_{KR} + 2p^{2}\lambda
\end{aligned} \tag{C106}$$

We get that

$$W = \lambda + U \tag{C107}$$

Furthermore:

$$\frac{\lambda}{z^{2}\Delta} = \cot\left(\frac{\pi}{N}\right) \tag{C108}$$

This leads to $\lambda = 0$ and $W = U$
This leads to:

$$\begin{aligned}
\left(1 + 4d^{2}\right)\Delta\frac{N}{\pi\rho J} &= 2d^{2}\left(\lambda_{KR} + W + \lambda_{SC}\right) \\
d^{2} + p^{2} &= \frac{1}{2} \\
\left(1 + 4p^{2}\right)\cdot\Delta\frac{N}{\pi\rho J} &= 2p^{2}\lambda_{KR}
\end{aligned}$$

Introducing the notation $\delta = d^2$ and $\varrho = p^2$ we get and $R = \frac{(\lambda_{KR}+W+\lambda_{SC})}{\lambda_{KR}}$

$$\frac{\left(1+4\delta\right)}{\left(1+4\left(\frac{1}{2}-\delta\right)\right)} = R\frac{\delta}{\frac{1}{2}-\delta} \tag{C109}$$

This means that:

$$4\left(R-1\right)\delta^{2}+\left(1-3R\right)\delta+\frac{1}{2}=0$$

And:

$$\delta=\frac{2\left(1-3R\right)\pm\sqrt{\left(1-3R\right)^{2}-8\left(1-R\right)}}{8\left(R-1\right)}$$

Furthermore

$$\Delta=z^{-2}D\exp\left(-\frac{z^{-2}}{\rho J}\right) \tag{C110}$$

We get the same z and the same Δ. Again in the symmetric case the magnetic susceptibility and heat capacity stay the same.

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