1 Exercise 1

Are $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{i}{2} \end{bmatrix}^T$ and $\begin{bmatrix} -\frac{i}{\sqrt{6}} & i \end{bmatrix}^T$ valid quantum states?

- $|\psi_1\rangle = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{i}{2} \end{bmatrix}^T = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{i}{2} \end{bmatrix} = \frac{\sqrt{3}}{2} |0\rangle + (-\frac{i}{2}) |1\rangle$
  $|\alpha|^2 + |\beta|^2 = (\frac{\sqrt{3}}{2})^2 + (\frac{i}{2})^2 = \frac{3}{4} + \frac{i^2}{4} = \frac{4}{4} = 1$
  So $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{i}{2} \end{bmatrix}^T$ is a valid quantum state.

- $|\psi_2\rangle = \begin{bmatrix} -\frac{i}{\sqrt{6}} & i \end{bmatrix}^T = \begin{bmatrix} -\frac{i}{\sqrt{6}} \\ i \end{bmatrix} = (-\frac{i}{\sqrt{6}}) |0\rangle + i |1\rangle$
  $|\alpha|^2 + |\beta|^2 = (\frac{i}{\sqrt{6}})^2 + i^2 = \frac{1}{6} + 1 = \frac{7}{6}$
  So $\begin{bmatrix} -\frac{i}{\sqrt{6}} & i \end{bmatrix}^T$ is not a valid quantum state.

2 Exercise 2

Are $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ unitary operators?

- $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
  $A\dagger = A^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
  $A\dagger A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 1 \times 1 & 1 \times 1 + 1 \times (-1) \\ 1 \times 1 + 1 \times (-1) & 1 \times 1 + (-1) \times (-1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq \mathbb{I}$
  Thus $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is not a unitary operator.

- $B = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
  $B\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$
  $B\dagger B = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 0 \times 0 & 1 \times 0 + 0 \times i \\ 0 \times 1 + (-i) \times 0 & 0 \times 0 + (-i) \times i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{I}$
  So $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ is an unitary operator.

3 Exercise 3

Check the following equalities for Pauli matrices:

- $XY = -YX$
  $XY = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0\times 0 + 1\times i & 0\times(-i) + 1\times 0 \\ 1\times 0 + 0\times i & 1\times(-i) + 0\times 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$
  $-YX = -\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0\times 0 + (-i)\times 1 & 0\times 1 + (-i)\times 0 \\ i\times 0 + 0\times 1 & i\times 1 + 0\times 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$
  So $XY = -YX$.

- $XZ = -ZX$
  $XZ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0\times 1 + 1\times 0 & 0\times 0 + 1\times(-1) \\ 1\times 1 + 0\times 0 & 1\times 0 + 0\times(-1) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
  $-ZX = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 1\times 0 + 0\times 1 & 1\times 1 + 0\times 0 \\ 0\times 0 + (-1)\times 1 & 0\times 1 + (-1)\times 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
  So $XZ = -ZX$.

- $YZ = -ZY$
  $YZ = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0\times 1 + (-i)\times 0 & 0\times 0 + (-i)\times(-1) \\ i\times 1 + 0\times 0 & i\times 0 + 0\times(-1) \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$
  $-ZY = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -\begin{bmatrix} 1\times 0 + 0\times i & 1\times(-i) + 0\times 0 \\ 0\times 0 + (-1)\times i & 0\times(-i) + (-1)\times 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$
  So $YZ = -ZY$.

4 Exercise 4

Check if $|1\rangle$ is an eigenvector of $2Y$. If yes, what is the corresponding eigenvalue?
$2Y\,|1\rangle = 2\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 0\times 0 + (-i)\times 1 \\ i\times 0 + 0\times 1 \end{bmatrix} = 2\begin{bmatrix} -i \\ 0 \end{bmatrix}$
$|1\rangle$ is not an eigenvector of $2Y$.

5 Exercise 5

Check if the matrices $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -i \\ i & 3 \end{bmatrix}$ are Hermitian.

- $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$
  $A\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$
  Since $A = A\dagger$ then the matrix $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is Hermitian.

- $B = \begin{bmatrix} 1 & -i \\ i & 3 \end{bmatrix}$
  $B\dagger = \begin{bmatrix} 1 & -i \\ i & 3 \end{bmatrix}\dagger = \begin{bmatrix} 1 & i \\ -i & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & -i \\ i & 3 \end{bmatrix}$
  Since $B = B\dagger$ then the matrix $\begin{bmatrix} 1 & -i \\ i & 3 \end{bmatrix}$ is Hermitian.

6 Exercise 6

Let $|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T$. Calculate the probabilities of measuring $+1$, -1, and the corresponding final states for observable Z when the state before the measurement is $|\psi\rangle$.
The eigenstates for Z with eigenvalues $+1$ and -1 are $|0\rangle$ and $|1\rangle$ accordingly.

- The probability of $+1$
  The projection is $P_{|0\rangle} = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$. From this we can calculate:
  $p(+1) = \langle\psi| P_{|0\rangle} |\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = (\frac{1}{\sqrt{2}}\times 1 + \frac{i}{\sqrt{2}}\times 0)(1\times \frac{1}{\sqrt{2}} + 0\times\frac{i}{\sqrt{2}}) = \frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}} = \frac{1}{2}$
  Since the probability of measuring $+1$ is $\frac{1}{2}$, then the final state is
  $|\psi_+\rangle = \frac{\langle 0|\psi\rangle|0\rangle}{\sqrt{\frac{1}{2}}} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}|0\rangle \times \frac{1}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\times\frac{1}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\times\sqrt{2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

- The probability of -1
  The projection $P_{|1\rangle} = |1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$. From this we can calculate:
  $p(-1) = \langle\psi| P_{|1\rangle} |\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = (\frac{1}{\sqrt{2}}\times 0 + \frac{i}{\sqrt{2}}\times 1)(0\times \frac{1}{\sqrt{2}} + 1\times\frac{i}{\sqrt{2}}) = |\frac{i}{\sqrt{2}}\times\frac{i}{\sqrt{2}}| = \frac{1}{2}$
  Since the probability of measuring -1 is $\frac{1}{2}$, then the final state is
  $|\psi_+\rangle = \frac{\langle 1|\psi\rangle|1\rangle}{\sqrt{\frac{1}{2}}} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}|1\rangle \times \frac{1}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\times\frac{1}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\times\sqrt{2} =$

$\begin{bmatrix}0\\1\end{bmatrix} = |1\rangle$

7 Exercise 7

Let $|\psi\rangle = \begin{bmatrix}-\frac{i\sqrt{3}}{2} & \frac{i}{2}\end{bmatrix}^T$. Assume that we want to measure $|\psi\rangle$ in the basis $\{\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{bmatrix}^T, \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{bmatrix}^T\}$. What are the possible outcomes and the probabilities of those outcomes?

In this case $|m_1\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{bmatrix}$ and $|m_2\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{bmatrix}$. The possible outcomes are m_1 and m_2, which are the eigenvalues associated with observable M.

- Finding the probability of $p(m_1)$
  $p(m_1) = |\langle\psi|m_1\rangle|^2 = |\begin{bmatrix}-\frac{i\sqrt{3}}{2} & \frac{i}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{bmatrix}|^2 = |-\frac{i\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{i}{2} \times \frac{i}{\sqrt{2}}|^2 = |\frac{i^2}{2\sqrt{2}} - \frac{i\sqrt{3}}{2\sqrt{2}}|^2 = |\frac{-1-i\sqrt{3}}{2\sqrt{2}}|^2 = (\frac{1+i\sqrt{3}}{2\sqrt{2}})^2 = \frac{(-1)^2+i^2\sqrt{3}^2}{8} = \frac{4}{8} = \frac{1}{2}$
- Finding the probability of $p(m_2)$
  $p(m_1) = |\langle\psi|m_2\rangle|^2 = |\begin{bmatrix}-\frac{i\sqrt{3}}{2} & \frac{i}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{bmatrix}|^2 = |-\frac{i\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{i}{2} \times (-\frac{i}{\sqrt{2}})|^2 = |\frac{-i^2}{2\sqrt{2}} - \frac{i\sqrt{3}}{2\sqrt{2}}|^2 = |\frac{1-i\sqrt{3}}{2\sqrt{2}}|^2 = (\frac{1+i\sqrt{3}}{2\sqrt{2}})^2 = \frac{1^2+i^2\sqrt{3}^2}{8} = \frac{4}{8} = \frac{1}{2}$

8 Exercise 8

Let $|-\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}^T$. Calculate the expectation value of Y for the state $|-\rangle$.

$\langle -|\, Y\, |-\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0 \times \frac{1}{\sqrt{2}} + (-i) \times (-\frac{1}{\sqrt{2}})\\ i \times \frac{1}{\sqrt{2}} + 0 \times (-\frac{1}{\sqrt{2}})\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{i}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}} \times \frac{i}{\sqrt{2}} + (-\frac{1}{\sqrt{2}})\frac{i}{\sqrt{2}} = \frac{i}{2} - \frac{i}{2} = 0$

The expectation value of Y for the state $|-\rangle$ is 0.