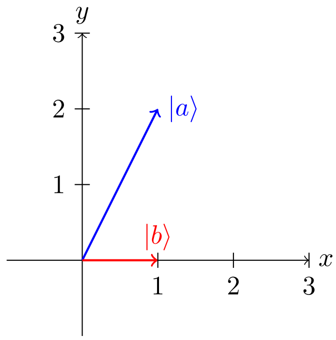
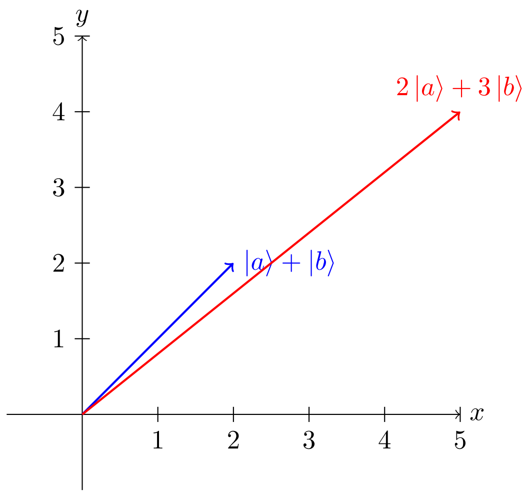
1 Exercise 1

Let $|a\rangle = \begin{bmatrix}1 & 2\end{bmatrix}^T$, $|b\rangle = \begin{bmatrix}1 & 0\end{bmatrix}^T$. Draw vectors $|a\rangle$, $|b\rangle$, $|a\rangle + |b\rangle$, $2|a\rangle + 3|b\rangle$.

- $|a\rangle = \begin{bmatrix}1\\2\end{bmatrix}$ and $|b\rangle = \begin{bmatrix}1\\0\end{bmatrix}$



- $|a\rangle + |b\rangle$ and $2|a\rangle + 3|b\rangle$
  Let's calculate the resulting vectors.
  $|a\rangle + |b\rangle = \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$
  $2|a\rangle + 3|b\rangle = 2 \times \begin{bmatrix}1\\2\end{bmatrix} + 3 \times \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix} + \begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}5\\4\end{bmatrix}$



2 Exercise 2

Calculate $\| |a\rangle \|$, $\| \frac{1}{2} |a\rangle + 2 |b\rangle \|$. $|a\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $|b\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

- Calculating $\| |a\rangle \|$.
  $\| |a\rangle \| = \sqrt{\langle a|a\rangle} = \sqrt{\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \sqrt{1 \times 1 + 2 \times 2} = \sqrt{5}$

- Calculating $\| \frac{1}{2} |a\rangle + 2 |b\rangle \|$.
  Let $|c\rangle = \frac{1}{2} |a\rangle + 2 |b\rangle$. Let's calculate $|c\rangle = \frac{1}{2} |a\rangle + 2 |b\rangle = \frac{1}{2} |a\rangle + 2 |b\rangle =$ $\frac{1}{2} \times \begin{bmatrix} 1 & 2 \end{bmatrix} + 3 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ 1 \end{bmatrix}$.
  Now $\| |c\rangle \| = \sqrt{\langle c|c\rangle} = \sqrt{\begin{bmatrix} \frac{7}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{2} \\ 1 \end{bmatrix}} = \sqrt{\frac{7}{2} \times \frac{7}{2} + 1 \times 1} = \sqrt{\frac{53}{4}}$

3 Exercise 3

Are vectors $2 |0_R\rangle$ and $|1_R\rangle$ orthogonal? Are vectors $|0_R\rangle$ and $|0\rangle$ orthogonal? Let $|0_R\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $|1_R\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \end{bmatrix}^T$.

- Let's check if vectors $2 |0_R\rangle$ and $|1_R\rangle$ are orthogonal.
  $2 \langle 0_R| = \begin{bmatrix} 2 \times \frac{1}{\sqrt{2}} & 2 \times \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{bmatrix}$
  $2 \langle 0_R|1_R\rangle = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{2}{\sqrt{2}} \times (-\frac{1}{\sqrt{2}}) + \frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 0$
  Thus, the vectors $2 |0_R\rangle$ and $|1_R\rangle$ are orthogonal.

- Let's check if vectors $|0_R\rangle$ and $|0\rangle$ are orthogonal.
  $\langle 0_R|0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{2}} \times 0 = \frac{1}{\sqrt{2}}$
  Thus, the vectors $2 |0_R\rangle$ and $|1_R\rangle$ are not orthogonal.

4 Exercise 4

Let $|a\rangle = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}^T$, $|b\rangle = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}^T$. Is basis $\{|a\rangle, |b\rangle\}$ orthonormal?

- Let's check if basis $\{|a\rangle, |b\rangle\}$ is normalized.
  $\langle a|a\rangle = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = 1$
  Since $\| |a\rangle \| = \sqrt{\langle a|a\rangle} = \sqrt{1} = 1$, then $|a\rangle$ has unit length.

$\langle b|b\rangle = \begin{bmatrix}-\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}-\frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{bmatrix} = (-\frac{\sqrt{3}}{2}) \times (-\frac{\sqrt{3}}{2}) + \frac{1}{2} \times \frac{1}{2} = 1$
Since $\| \, |b\rangle \, \| = \sqrt{\langle b|b\rangle} = \sqrt{1} = 1$, then $|b\rangle$ has unit length.
Basis $\{|a\rangle, \, |b\rangle\}$ is normalized.

- Let's check if basis $\{|a\rangle, \, |b\rangle\}$ is orthogonal.
  $\langle a|b\rangle = \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}-\frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{bmatrix} = \frac{1}{2} \times (-\frac{\sqrt{3}}{2}) + \frac{\sqrt{3}}{2} \times \frac{1}{2} = 0$
  Basis $\{|a\rangle, \, |b\rangle\}$ is orthogonal.

Thus, basis $\{|a\rangle, \, |b\rangle\}$ is orthonormal.

5 Exercise 5

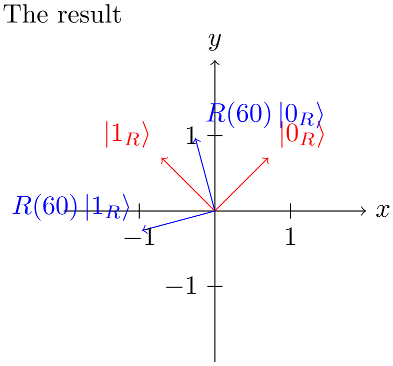
Write down $R(\alpha)$ for $\alpha = 60$ and calculate its action on the basis $\{|0_R\rangle, \, |1_R\rangle\}$.
Draw the result.
Let $R(\alpha) = \begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix}$, $|0_R\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}^T$ and $|1_R\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}^T$.

- If $\alpha = 60$ then $R(60) = \begin{bmatrix}\cos 60 & -\sin 60 \\ \sin 60 & \cos 60\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$.

- $R(60)\,|0_R\rangle = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} \times \begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} \times \frac{1}{\sqrt{2}} + (-\frac{\sqrt{3}}{2}) \times \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1-\sqrt{3}}{2\sqrt{2}} \\ \frac{\sqrt{3}+1}{2\sqrt{2}}\end{bmatrix}$

- $R(60)\,|1_R\rangle = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix}-1 \\ 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} \times \begin{bmatrix}-\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} \times (-\frac{1}{\sqrt{2}}) + (-\frac{\sqrt{3}}{2}) \times \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{2} \times (-\frac{1}{\sqrt{2}}) + \frac{1}{2} \times \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{-1-\sqrt{3}}{2\sqrt{2}} \\ \frac{-\sqrt{3}+1}{2\sqrt{2}}\end{bmatrix}$

- The result

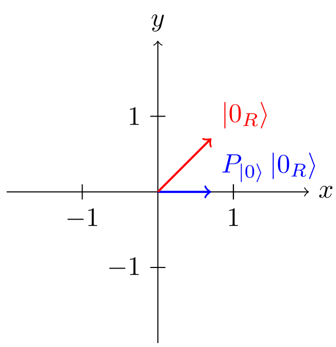


6 Exercise 6

Calculate $P_{|0\rangle}\,|0_R\rangle$ and draw the vector. Calculate $\|P_{|0\rangle}\,|0_R\rangle\|$. Finally, calculate $\frac{P_{|0\rangle}|0_R\rangle}{\|P_{|0\rangle}|0_R\rangle\|}$ and draw the resulting vector.

Let $|0_R\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}^T$ and $|0\rangle = \begin{bmatrix}1 & 0\end{bmatrix}^T$.

- $P_{|0\rangle}\,|0_R\rangle = |0\rangle\langle 0|0_R\rangle = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}|0_R\rangle = \begin{bmatrix}1\times 1 & 1\times 0\\ 0\times 1 & 0\times 0\end{bmatrix}|0_R\rangle = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}1\times\frac{1}{\sqrt{2}} + 0\times\frac{1}{\sqrt{2}}\\ 0\times\frac{1}{\sqrt{2}} + 0\times\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\end{bmatrix}$



- $\|P_{|0\rangle}\,|0_R\rangle\| = \sqrt{\langle P_{|0\rangle}\,|0_R\rangle | P_{|0\rangle}\,|0_R\rangle\rangle} = \sqrt{\begin{bmatrix}\frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\end{bmatrix}} = \sqrt{\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}} + 0\times 0} = \frac{1}{\sqrt{2}}$

- $\frac{P_{|0\rangle}|0_R\rangle}{\|P_{|0\rangle}|0_R\rangle\|} = P_{|0\rangle}\,|0_R\rangle \times \frac{1}{\|P_{|0\rangle}|0_R\rangle\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\end{bmatrix}\times\frac{1}{\frac{1}{\sqrt{2}}} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\end{bmatrix}\times\sqrt{2} = \begin{bmatrix}\frac{1}{\sqrt{2}}\times\sqrt{2}\\ 0\times\sqrt{2}\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$