Lens potential: Exact Solution, Multipole Expansion, & Their First & Second Derivatives

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February 17, 2015

1 General

$$\hat{M}_i = \frac{M_i}{\Sigma_{\text{crit}}} \tag{1}$$

2 Exact Solution

$$\phi(\vec{x}) = \sum_{i=1}^{N} \frac{\hat{M}_i}{\pi} \ln|\vec{x} - \vec{x}_i| \tag{2}$$

$$\phi(r,\theta) = \frac{1}{2\pi} \sum_{i=1}^{N} \hat{M}_i \ln\left[r^2 + r_i^2 - 2rr_i\cos(\theta - \theta_i)\right] \tag{3}$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{\pi} \sum_{i=1}^{N} \hat{M}_i \left[\frac{r\cos\theta - r_i\cos\theta_i}{r^2 + r_i^2 - 2rr_i\cos(\theta - \theta_i)}\right] \tag{4}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{\pi} \sum_{i=1}^{N} \hat{M}_i \left[\frac{r\sin\theta - r_i\sin\theta_i}{r^2 + r_i^2 - 2rr_i\cos(\theta - \theta_i)}\right] \tag{5}$$

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{\partial^2\phi}{\partial y^2} = -\frac{1}{\pi} \sum_{i=1}^{N} \hat{M}_i \left[\frac{(r\cos\theta - r_i\cos\theta_i)^2 - (r\sin\theta - r_i\sin\theta_i)^2}{\left[r^2 + r_i^2 - 2rr_i\cos(\theta - \theta_i)\right]^2}\right] \tag{6}$$

$$\frac{\partial^2\phi}{\partial x\partial y} = -\frac{2}{\pi} \sum_{i=1}^{N} \hat{M}_i \left[\frac{(r\cos\theta - r_i\cos\theta_i)(r\sin\theta - r_i\sin\theta_i)}{\left[r^2 + r_i^2 - 2rr_i\cos(\theta - \theta_i)\right]^2}\right] \tag{7}$$

3 Multipole Expansion

3.1 Multipole moments

$$J_0^c = \frac{1}{2\pi} \sum_{i=1}^{N} \hat{M}_i \tag{8}$$

$$J_m^c = \frac{1}{2\pi} \sum_{i=1}^{N} \hat{M}_i r_i^{-m} \cos m\theta_i \tag{9}$$

$$J_m^s = \frac{1}{2\pi} \sum_{i=1}^{N} \hat{M}_i r_i^{-m} \sin m\theta_i \tag{10}$$

3.2 Potential

$$\phi(r,\theta) = 2J_0^c - 2\sum_{m=1}^{\infty} \frac{r^m}{m} \left[J_m^c \cos m\theta + J_m^s \sin m\theta\right] \tag{11}$$

3.3 Derivatives wrt r and θ

$$\frac{\partial \phi}{\partial r} = -2\sum_{m=1}^{\infty} r^{m-1}\left[J_m^c \cos m\theta + J_m^s \sin m\theta\right] \tag{12}$$

$$\frac{\partial \phi}{\partial \theta} = -2\sum_{m=1}^{\infty} r^{m}\left[-J_m^c \sin m\theta + J_m^s \cos m\theta\right] \tag{13}$$

$$\frac{\partial^2 \phi}{\partial r^2} = -2\sum_{m=1}^{\infty} (m-1)\, r^{m-2}\left[J_m^c \cos m\theta + J_m^s \sin m\theta\right] \tag{14}$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = +2\sum_{m=1}^{\infty} m r^{m}\left[J_m^c \cos m\theta + J_m^s \sin m\theta\right] \tag{15}$$

$$\frac{\partial^2 \phi}{\partial r \partial \theta} = -2\sum_{m=1}^{\infty} m r^{m-1}\left[-J_m^c \sin m\theta + J_m^s \cos m\theta\right] \tag{16}$$

3.4 Derivatives wrt x and y in terms of derivatives wrt r and θ

$$\begin{bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial \phi}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} \dfrac{\partial^2 \phi}{\partial x^2} = -\dfrac{\partial^2 \phi}{\partial y^2} \\ \dfrac{\partial^2 \phi}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 \phi}{\partial r^2} \\ \dfrac{1}{r}\left(\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} - \dfrac{\partial^2 \phi}{\partial r \partial \theta}\right) \end{bmatrix} \tag{18}$$

3.5 Derivatives wrt x and y

$$\frac{\partial \phi}{\partial x} = -2\sum_{m=1}^{\infty} r^{m-1}\left[J_m^c \cos(m-1)\,\theta + J_m^s \sin(m-1)\,\theta\right] \tag{19}$$

$$\frac{\partial \phi}{\partial y} = -2\sum_{m=1}^{\infty} r^{m-1}\left[-J_m^c \sin(m-1)\,\theta + J_m^s \cos(m-1)\,\theta\right] \tag{20}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial^2 \phi}{\partial y^2} = -2\sum_{m=1}^{\infty} (m-1)\, r^{m-2}\left[J_m^c \cos(m-2)\,\theta + J_m^s \sin(m-2)\,\theta\right] \tag{21}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = -2\sum_{m=1}^{\infty} (m-1)\, r^{m-2}\left[-J_m^c \sin(m-2)\,\theta + J_m^s \cos(m-2)\,\theta\right] \tag{22}$$