Individualized Training of Back Muscles using Iterative Learning Control of a Compliant Balance Board – Stability analysis for the iterative learning controller

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I. Stability

Consider the following time-discrete dynamics from Section III.B of the main paper

$$K_{d,i+1} = K_{d,i} + \Delta k \left(\alpha e^{-bK_{d,i}} - 1\right)^3, \tag{1}$$

where $\alpha = \frac{a}{\sigma_\theta^*}$. We must show that the equilibrium point is globally asymptotically stable.

Since we are working with a real system, it is important to consider the bounds of the function variables, i.e. $K_d > 0$, $\Delta k > 0$, $\sigma_\theta^* > 0$, $\alpha > 1$, and $b > 0$. In this work we set $\Delta k = 400\,\mathrm{Nm}$ and $\sigma_\theta^* = 4°$. Additionally, we found that $\alpha \in [1.125, 1.375]$ and $b \in [0.0010, 0.0015]$ (95% confidence interval) for one test subject using the nonlinear least squares method.

Let $K_d^* = \frac{\ln \alpha}{b}$ be the equilibrium position of the time-discrete dynamics (1), where $\alpha > 1$, $b > 0$, and $0 < \Delta k < \Delta k_{max}$. Let $D \coloneqq \{K_{d,i} \in \mathbb{R} \mid K_{d,i} > 0\}$. Recall the definition of Lyapunov, which states that if there exists a continuously differentiable Lyapunov function $V(K_{d,i})$ that satisfies the following conditions

i) $V(K_{d,i}) > 0$, $\forall K_{d,i} \in D \setminus \{K_d^*\}$
ii) $V(K_{d,i}) = 0$, iff $K_{d,i} = K_d^*$, and
iii) $\Delta V(K_{d,i}) = V(K_{d,i+1}) - V(K_{d,i}) < 0$, $\forall K_{d,i} \in D \setminus \{K_d^*\}$,

then the equilibrium position of (1) is asymptotically stable.

Proposition 1 *Let $V(K_{d,i}) = \frac{1}{2}(K_{d,i} - K_d^*)^2$, then K_d^* is globally asymptotically stable.*

Proof: We first show that $K_d^* = \frac{\ln(\alpha)}{b}$ is the equilibrium position by setting $K_d^* = K_{d,i} = K_{d,i+1}$ and substituting in (1)

$$\begin{aligned}
K_d^* &= K_d^* + \Delta k(\alpha e^{-bK_d^*} - 1)^3 \\
0 &= \Delta k(\alpha e^{-bK_d^*} - 1)^3 \\
0 &= \alpha e^{-bK_d^*} - 1 \\
e^{-bK_d^*} &= \frac{1}{\alpha} \\
-bK_d^* &= \ln(\frac{1}{\alpha}) \\
K_d^* &= \frac{\ln(\alpha)}{b}
\end{aligned} \tag{2}$$

Next, we show that

$$V(K_{d,i}) = \frac{1}{2}(K_{d,i} - K_d^*)^2 \tag{3}$$

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satisfies the requirements of a Lyapunov function. Condition i) is always fulfilled because (3) is an upward opening parabola with vertex at $V(K_d^*) = 0$. By substituting $K_{d,i} = K_d^*$ into (3), it is evident that condition ii) is fulfilled. This leaves us with condition iii), which we will prove next.
We have

$$\begin{aligned} V(K_{d,i+1}) &= \frac{1}{2}(K_{d,i+1} - K_d^*)^2 \\ &= \frac{1}{2}(K_{d,i} + \Delta k(\alpha e^{-bK_{d,i}} - 1)^3 - K_d^*)^2 \\ &= \frac{1}{2}(K_{d,i} - K_d^*)^2 + (K_{d,i} - K_d^*)\Delta k(\alpha e^{-bK_{d,i}} - 1)^3 + \frac{1}{2}\Delta k^2(\alpha e^{-bK_{d,i}} - 1)^6 \\ &= \frac{1}{2}(K_{d,i} - K_d^*)^2 + \Delta k(\alpha e^{-bK_{d,i}} - 1)^3 \left((K_{d,i} - K_d^*) + \frac{1}{2}\Delta k(\alpha e^{-bK_{d,i}} - 1)^3\right) \end{aligned} \tag{4}$$

and, therefore,

$$\begin{aligned} \Delta V(K_{d,i}) &= V(K_{d,i+1}) - V(K_{d,i}) \\ &= \Delta k(\alpha e^{-bK_{d,i}} - 1)^3 \left((K_{d,i} - K_d^*) + \frac{1}{2}\Delta k(\alpha e^{-bK_{d,i}} - 1)^3\right). \end{aligned} \tag{5}$$

For the subsequent analysis, we define

$$g(K_{d,i}) \coloneqq \alpha e^{-bK_{d,i}} - 1 \tag{6}$$

and

$$f(K_{d,i}) \coloneqq K_{d,i} - K_d^* + \frac{1}{2}\Delta k(\alpha e^{-bK_{d,i}} - 1)^3; \tag{7}$$

thus,

$$\Delta V(K_{d,i}) = g(K_{d,i})f(K_{d,i}). \tag{8}$$

To prove that condition iii) holds, we consider both the case that $g(K_{d,i}) < 0$ and that $g(K_{d,i}) > 0$, wherein $g(K_{d,i}) = 0$ corresponds to $K_{d,i} = K_d^*$.

Case 1: First, let us consider when $g(K_{d,i}) < 0$. Then,

$$\begin{aligned} \alpha e^{-bK_{d,i}} - 1 &< 0 \\ \alpha e^{-bK_{d,i}} &< 1 \\ -bK_{d,i} &< -\ln\alpha \\ K_{d,i} &> K_d^*. \end{aligned}$$

Following (8), $\Delta V(K_{d,i}) < 0$ only holds for this case if $f(K_{d,i}) > 0$, i.e.

$$K_{d,i} - K_d^* + \frac{1}{2}\Delta k(\alpha e^{-bK_{d,i}} - 1)^3 > 0. \tag{9}$$

Given that $f(K_d^*) = 0$, we need to show that $f(K_{d,i})$ is monotonously increasing for $K_{d,i} > K_d^*$, i.e.

$$f'(K_{d,i}) = 1 - \frac{3}{2}\Delta k\alpha b e^{-bK_{d,i}}(\alpha e^{-bK_{d,i}} - 1)^2 > 0. \tag{10}$$

This is most easily demonstrated by showing that all extrema of $f'(K_{d,i})$ are greater than zero for $K_{d,i} > K_d^*$. For this, we have to compute the derivative and set it to zero:

$$\begin{aligned} f''(K_{d,i}) = \frac{3}{2}\Delta k\alpha b^2 e^{-bK_{d,i}}(\alpha e^{-bK_{d,i}} - 1)^2 + 3\Delta k\alpha^2 b^2 e^{-2bK_{d,i}}(\alpha e^{-bK_{d,i}} - 1) &= 0 \\ \Leftrightarrow \quad (\alpha e^{-bK_{d,i}} - 1)^2 + 2\alpha e^{-bK_{d,i}}(\alpha e^{-bK_{d,i}} - 1) = 0 &= 0 \\ \Leftrightarrow \quad (\alpha e^{-bK_{d,i}} - 1)\left((\alpha e^{-bK_{d,i}} - 1) + 2e^{-bK_{d,i}}\right) &= 0 \\ \Leftrightarrow \quad (\alpha e^{-bK_{d,i}} - 1)\left(3\alpha e^{-bK_{d,i}} - 1\right) &= 0 \end{aligned} \tag{11}$$

If $(\alpha e^{-bK_{d,i}} - 1) = 0$, $K_{d,i} = K_d^*$, and $f'(K_d^*) = 1 > 0$. If $\left(3\alpha e^{-bK_{d,i}} - 1\right) = 0$, we have

$$K_{d,i} = K_{ex} \coloneqq \frac{\ln 3\alpha}{b} \tag{12}$$

Substituting K_{ex} back into Eq. (10) yields the following condition

$$\Delta k < \frac{9}{2b}. \tag{13}$$

There are no other extrema in the domain of interest; therefore, it only remains to verify that (10) holds as $K_{d,i}$ approaches infinity. Since

$$\lim_{K_{d,i}\to\infty} f'(K_{d,i}) = 1 \tag{14}$$

holds, (10) is fulfilled in this case, as well.

Case 2: Now we consider when $g(K_{d,i}) > 0$ and thus $K_{d,i} < K_d^*$. We need to show that

$$f(K_{d,i}) < 0. \tag{15}$$

We know that $f(K_d^*) = 0$. To show that $f(0) < 0$, we must show that

$$f(0) = -K_d^* + \frac{1}{2}\Delta k(\alpha - 1)^3 < 0. \tag{16}$$

This yields the condition

$$\Delta k < \frac{2\ln(\alpha)}{b(\alpha - 1)^3} \tag{17}$$

From our previous analysis, we know that for $K_{d,i} \in (0, K_d^*)$, there are no extrema of $f'(K_{d,i})$ and that $f'(K_d^*) > 0$. This evidence suffices to prove that $f(K_{d,i}) < 0$ for all $K_{d,i} < K_d^*$. Therefore, if Δk is selected according to (13) and (17), (15) holds, and we have proven asymptotic stability for all $K_{d,i} > 0$. ■

II. Transient behavior

We want to ensure that the controller does not generate a physically infeasible negative stiffness value $K_{d,i} < 0$.

Proposition 2 *For all $K_{d,i} > K_d^*$ and $0 < \Delta k < \Delta k_{max}$, no overshooting occurs, i.e. $K_{d,i+1} > K_d^*$.*

Proof: To prove that

$$K_{d,i+1} = K_{d,i} + \Delta k(\alpha e^{-bK_{d,i}} - 1)^3 > K_d^* \tag{18}$$

we must show that

$$(K_{d,i} - K_d^*) + \Delta k(\alpha e^{-bK_{d,i}} - 1)^3 > 0. \tag{19}$$

Let us define $h(K_{d,i}) := (K_{d,i} - K_d^*) + \Delta k(\alpha e^{-bK_{d,i}} - 1)^3$. We know that $h(K_d^*) = 0$. Therefore, we need to show that $h(K_{d,i})$ is monotonically increasing, i.e., $h'(K_{d,i}) > 0$. Taking the first derivative yields

$$h'(K_{d,i}) = 1 - 3\Delta k \alpha b e^{-bK_{d,i}}(\alpha e^{-bK_{d,i}} - 1)^2 \tag{20}$$

Similar to our proof for Proposition 1, we check for extrema in the domain $K_{d,i} > K_d^*$ by setting $h''(K_{d,i}) = 0$, which yields

$$K_{d,i} = K_{ex} = \frac{\ln 3\alpha}{b}. \tag{21}$$

Substituting this value in $h'(K_{d,i}) > 0$ yields

$$\Delta k < \frac{9}{4b}, \tag{22}$$

which is similar to the requirement outlined in (13). Furthermore, since

$$\lim_{K_{d,i}\to\infty} h'(K_{d,i}) = 1, \tag{23}$$

we have shown that $h'(K_{d,i}) > 0$ for all $K_{d,i} > K_d^*$ and thus proven Lemma 2. ■