Since $\{\mathbf{a},\mathbf{b}\}$ is a basis for the two-dimensional vector space $ker_{\mathbf{R}}(\mathcal{A})$, there exist positive reals $\lambda_i > 0$ such that $\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3 + \lambda_4\omega_4$ is orthogonal to $ker(\mathcal{A})$. In particular, it is orthogonal to $\mathbf{v}-\mathbf{v}'$, which shows that $\omega_i \cdot \mathbf{v} \leq \omega_i \cdot \mathbf{v}'$ for at least one i. This implies that $\mathbf{x}^{\mathbf{v}'}$ lies in $in_{\prec_i}(I) = in_{\omega_i}(I_{\mathcal{A}})$, which is a contradiction. ■

Exercises:

(1) Let $n = d+1$ where $d = dim(\mathcal{A})$. Show that there are precisely three isomorphism types of $\mathcal{A}$-graded algebras and that all three are coherent.

(2) List all $\mathcal{A}$-graded ideals for $\mathcal{A} = \{1,2,3\}$.

(3) List all $\mathcal{A}$-graded monomial ideals for $\mathcal{A} = \{1,3,4,7\}$.

(4) Give an algorithm for constructing all $\mathcal{A}$-graded monomial ideals, for an arbitrary configuration $\mathcal{A} \subset \mathrm{N}^d \backslash \{0\}$.

(5) List all polyhedral subdivisions of the configuration in Example 10.11. Does each of them correspond to an $\mathcal{A}$-graded ideal ?

(6) Does there exist a non-coherent $\mathcal{A}$-graded algebra in the case where $\mathcal{A}$ is the vertex set of a regular d-dimensional cube ?

(7) Extend Lemma 10.14 to include all polyhedral subdivisions (not just triangulations) of a unimodular configuration.

Notes:

Arnold (1989) expressed the number of isomorphism classes of $\mathcal{A}$-graded algebras for $\mathcal{A} = \{1, p_2, p_3\}$ in terms of the continued fraction expansion for the rational number p_3/p_2. The extension to the case $\mathcal{A} = \{p_1, p_2, p_3\}$ was given in (Korkina, Post & Roelofs 1995). Our proof of Theorem 10.2 is based on this article.

All results in this chapter are new and not published elsewhere (with the exception of Theorem 10.2, of course). The first example of an infinite family of pairwise non-isomorphic AGA's was constructed by D. Eisenbud (unpublished) for $d = 1, n = 7$. The $n = 4$ example in Theorem 10.4 was found afterwards. Peeva's Theorem 10.13 constitutes a counterexample to a conjecture which I stated after Theorem 10.10 had been found. Theorem 10.15 is an extension of the results in (De Loera 1995a).

CHAPTER 11

Canonical Subalgebra Bases

Toric ideals arise naturally in the study of canonical subalgebra bases. It is the objective of this chapter to explain this connection and to develop an intrinsic Gröbner basis theory for subalgebras of the polynomial ring. The basic idea is to degenerate the algebra generators into monomials and thereby the algebra relations to binomials. Geometrically speaking, we wish to deform an arbitrary parametrically presented variety X into a toric variety. As an application we shall see how this can be accomplished if X is a Grassmann variety.

Let R be a finitely generated subalgebra of the polynomial ring $k[\mathbf{t}] = k[t_1,\ldots,t_d]$. Fix a term order $\prec$ on $k[\mathbf{t}]$. The *initial algebra* $in_\prec(R)$ is the k-vector space spanned by $\{in_\prec(f) \,:\, f \in R\}$. A *canonical basis* is a subset $\mathcal{C}$ of R such that $in_\prec(R)$ is generated as a k-algebra by the set of monomials $\{in_\prec(f) \,:\, f \in \mathcal{C}\}$. Canonical bases for subalgebras are similar to Gröbner bases with regard to their reduction properties.

Algorithm 11.1. *(The subduction algorithm for a canonical basis $\mathcal{C}$)*
Input: A canonical basis $\mathcal{C}$ for a subalgebra $R \subset k[\mathbf{t}]$. A polynomial $f \in k[\mathbf{t}]$.
Output: An expression of f as a polynomial in the elements of $\mathcal{C}$, provided $f \in R$.
While f is not a constant in k do

1. Find $f_1, f_2, \ldots, f_r \in \mathcal{C}$, exponents $i_1, i_2, \ldots, i_r \in \mathrm{N}$ and $c \in k^*$ such that

$$in_\prec(f) \quad = \quad c \cdot in_\prec(f_1)^{i_1} \cdot in_\prec(f_2)^{i_2} \cdots in_\prec(f_r)^{i_r}. \qquad (11.1)$$

2. If no representation (11.1) exists, then output "f does not lie in R" and STOP.

3. Otherwise, output $p := c \cdot f_1^{i_1} f_2^{i_2} \cdots f_r^{i_r}$, and replace f by $f - p$.

Output the constant f.

A nice example of a canonical subalgebra basis is the set of elementary symmetric polynomials $\sigma_1,\ldots,\sigma_d$ in $t_1,\ldots,t_d$. Indeed, the familiar algorithm for expressing symmetric polynomials in terms of $\sigma_1,\ldots,\sigma_d$ is precisely Algorithm 11.1 in this case.

The main difference between Gröbner bases for ideals and canonical bases for subalgebras is that the initial algebra $in_\prec(R)$ need not be finitely generated. If $in_\prec(R)$ is not finitely generated, then there is no finite canonical basis for R with respect to $\prec$. The following example appears in (Göbel 1995).

Example 11.2. *(The invariants of the alternating group have an infinite canonical basis)* Let $d = 3$ and let $R = k[t_1,t_2,t_3]^{A_3}$ be the subalgebra of polynomials which are invariant under the cyclic permutation $t_1 \mapsto t_2, t_2 \mapsto t_3, t_3 \mapsto t_1$. It has four minimal generators:

$$R \quad = \quad k\big[\,t_1+t_2+t_3,\; t_1t_2+t_1t_3+t_2t_3,\; t_1t_2t_3,\; (t_1-t_2)(t_1-t_3)(t_2-t_3)\,\big].$$

Let $\prec$ be the lexicographic term order with $t_1 \succ t_2 \succ t_3$. If f is any invariant in R and $in_\prec(f) = t_1^{i_1} t_2^{i_2} t_3^{i_3}$, then it is easy to see that either $i_1 \geq i_2 \geq i_3$ or $i_1 > i_3 \geq i_2$. Suppose that $in_\prec(R)$ is finitely generated. Among the generators consider the subset $\{t_1^{a_1} t_3^{b_1}, t_1^{a_2} t_3^{b_2}, \cdots, t_1^{a_s} t_3^{b_s}\}$ of those generators which do not contain the variable t_2. There exists a constant $C > 1$ such that $a_i \geq C \cdot b_i$ for $i = 1, \ldots, s$. Choose any integer $d > \frac{1}{C-1}$, so that $d + 1 < C \cdot d$. We consider the A_3-invariant polynomial

$$g \quad := \quad \underline{t_1^{d+1} t_3^d} + t_1^d t_2^{d+1} + t_2^d t_3^{d+1} \quad \in \quad R.$$

The underlined initial term must lie in the semigroup generated by the $t_1^{a_i} t_3^{b_i}$. This implies that the vector $(d+1, d)$ lies in the planar convex cone spanned by the vectors (a_i, b_i). Hence it also satisfies the linear inequality $d+1 \geq C \cdot d$, a contradiction. This shows that $in_\prec(R)$ is not finitely generated. ∎

It is an important open problem to find good criteria which guarantee finite generation for $in_\prec(R)$. In what follows we consider mainly the case where $in_\prec(R)$ is finitely generated. (Most of our discussion about canonical bases, however, would extend to the infinite case.)

Fix a set of polynomials $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ in $k[\mathbf{t}] = k[t_1, \ldots, t_d]$, let $R = k[\mathcal{F}]$ be the subalgebra they generate, and fix a term order $\prec$ on $k[\mathbf{t}]$. Suppose $in_\prec(f_i) = \mathbf{t}^{\mathbf{a}_i}$ and let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \subset \mathbf{N}^d$. We shall give a criterion for deciding whether $\mathcal{F}$ is a canonical basis for R with respect to $\prec$. To this end we introduce the new polynomial ring $k[\mathbf{x}] = k[x_1, x_2, \ldots, x_n]$. Consider the k-algebra epimorphism from $k[\mathbf{x}]$ onto R defined by $x_i \mapsto f_i$, and let I denote its kernel. Similarly, consider the map from $k[\mathbf{x}]$ onto $in_\prec(R)$ defined by $x_i \mapsto in_\prec(f_i) = \mathbf{t}^{\mathbf{a}_i}$. The kernel of this map is the toric ideal $I_\mathcal{A}$.

Let $\omega \in \mathbf{R}^d$ be any weight vector which represents the term order $\prec$ for the polynomials in $\mathcal{F}$. If we consider $\mathcal{A}$ as a $d \times n$-matrix, with transpose $\mathcal{A}^T$, then $\mathcal{A}^T\omega$ is a vector in $\mathbf{R}^n$. We can use it as the weight vector for forming an initial ideal of $I \subset k[\mathbf{x}]$. However, the initial ideal $in_{\mathcal{A}^T\omega}(I)$ is usually not a monomial ideal. This is explained by the fact that the vector $\mathcal{A}^T\omega$ is *not* a generic vector in $\mathbf{R}^n$, even if ω is generic in $\mathbf{R}^d$.

Lemma 11.3. *For any set $\mathcal{F} \subset k[\mathbf{t}]$, the initial ideal $in_{\mathcal{A}^T\omega}(I)$ is contained in the toric ideal $I_\mathcal{A}$.*

Proof. Let $p(\mathbf{x}) = \sum c_\mathbf{u} \mathbf{x}^\mathbf{u}$ be an element of the ideal I. This means that

$$p(f_1(\mathbf{t}), \ldots, f_n(\mathbf{t})) \quad = \quad \sum c_\mathbf{u} f_1(\mathbf{t})^{u_1} \cdots f_n(\mathbf{t})^{u_n} \quad \in \quad k[\mathbf{t}]$$

is the zero polynomial. When expanding this sum, the terms of highest ω-order must cancel. The ω-order of $f_1(\mathbf{t})^{u_1} \cdots f_n(\mathbf{t})^{u_n}$ equals the ω-order of

$$in_\omega(f_1)(\mathbf{t})^{u_1} \cdots in_\omega(f_n)(\mathbf{t})^{u_n} \;=\; \mathbf{t}^{u_1 \mathbf{a}_1} \cdots \mathbf{t}^{u_n \mathbf{a}_n},$$

which is the inner product of $\mathcal{A}^T\omega$ with $\mathbf{u} = (u_1, \ldots, u_n)$. Therefore the sum of the highest terms in the expansion $\sum c_\mathbf{u} f_1(\mathbf{t})^{u_1} \cdots f_n(\mathbf{t})^{u_n}$ equals

$$\big[in_{\mathcal{A}^T\omega}(p)\big]\big(in_\omega(f_1), \ldots, in_\omega(f_n)\big) = \big[in_{\mathcal{A}^T\omega}(p)\big]\big(\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}\big) = 0. \qquad (11.2)$$

We conclude that $in_{\mathcal{A}^T\omega}(p)$ lies in $I_\mathcal{A}$. ∎

The reverse inclusion to Lemma 11.3 is our criterion for canonical bases.

Theorem 11.4. *The set $\mathcal{F} \subset k[\mathbf{t}]$ is a canonical basis if and only if $in_{\mathcal{A}^T\omega}(I) = I_\mathcal{A}$.*

Proof: We first show the "only-if" direction. Suppose $\mathcal{F}$ is a canonical basis, and let q be any binomial in $I_\mathcal{A}$. We consider the polynomial $q(f_1, \ldots, f_n)$ in $k[\mathbf{t}]$. By the canonical basis property and using Algorithm 11.1, we can find $r \in k[\mathbf{x}]$ such that $q(f_1, \ldots, f_n) = r(f_1, \ldots, f_n)$ and $q = in_{\mathcal{A}^T\omega}(q - r)$. This proves $q \in in_{\mathcal{A}^T\omega}(I)$ as desired.

The proof of the "if" direction is by contradiction. Suppose that $in_{\mathcal{A}^T\omega}(I) = I_\mathcal{A}$ but $\mathcal{F}$ is not a canonical basis. Then there exists $p \in k[\mathbf{x}]$ such that

$$in_\omega(p(f_1, \ldots, f_n)) \quad \notin \quad k[in_\omega(f_1), \ldots, in_\omega(f_n)]. \qquad (11.3)$$

We may assume that p is minimal with respect to the partial term order defined by $\mathcal{A}^T\omega$. In order for (11.3) to hold, it must be the case that the terms $\mathbf{t}^\mathbf{u}$ of highest order in the expansion of $p(f_1(\mathbf{t}), \ldots, f_n(\mathbf{t}))$ all cancel. As in the proof of Lemma 11.3, this implies (11.2) and therefore $in_{\mathcal{A}^T\omega}(p) \in I_\mathcal{A} = in_{\mathcal{A}^T\omega}(I)$. There exists a polynomial $q \in I$ such that $in_{\mathcal{A}^T\omega}(p) = in_{\mathcal{A}^T\omega}(q)$. The initial form of $p - q$ with respect to $\mathcal{A}^T\omega$ is therefore smaller than that of p. However, since $q \in I$, we have $p(f_1, \ldots, f_n) = (p - q)(f_1, \ldots, f_n)$, so that $p - q$ shares the property (11.3) with p. This is a contradiction to the minimality in our choice of p, and the proof is complete. ∎

In order to apply the criterion in Theorem 11.4 one has to compute generators for the toric ideal $I_\mathcal{A}$, but one need *not* do this for I. Instead one uses Algorithm 11.1.

Corollary 11.5. *Let $\{p_1, \ldots, p_s\}$ be generators of the toric ideal $I_\mathcal{A}$. Then $\mathcal{F}$ is a canonical basis if and only if Algorithm 11.1 reduces $p_i(f_1, \ldots, f_n)$ to a constant for all $i \in \{1, \ldots, s\}$.*

Proof: The only-if direction is obvious: if Algorithm 11.1 gets stuck with a non-constant polynomial while reducing $p_i(f_1, \ldots, f_n)$, then $\mathcal{F}$ fails to be a canonical basis. For the if-direction we assume that Algorithm 11.1 does reduce $p_i(f_1, \ldots, f_n)$ to a constant. The output generated by that reduction gives a polynomial $q_i(\mathbf{x}) = \sum_{i=1}^t c_i \mathbf{x}^{\mathbf{u}_i}$ whose terms form a strictly descending sequence in the partial term order $\succ$ defined by $\mathcal{A}^T\omega$ on $k[\mathbf{x}]$:

$$in_{\mathcal{A}^T\omega}(p_i) \;\succ\; \mathbf{x}^{\mathbf{u}_1} \;\succ\; \mathbf{x}^{\mathbf{u}_2} \;\succ\; \cdots \;\succ\; \mathbf{x}^{\mathbf{u}_t}. \qquad (11.4)$$

By construction, we have $p_i(f_1, \ldots, f_n) = q_i(f_1, \ldots, f_n)$, and hence $p_i - q_i \in I$. Property (11.4) implies $p_i = in_{\mathcal{A}^T\omega}(p_i - q_i) \in in_{\mathcal{A}^T\omega}(I)$. This holds for all $i \in \{1, \ldots, s\}$, and, in view of Lemma 11.3, we conclude

$$I_\mathcal{A} \quad = \quad \langle p_1, p_2, \ldots, p_s \rangle \quad = \quad in_{\mathcal{A}^T\omega}(I).$$

Using Theorem 11.4, this completes the proof. ∎

The initial ideal $in_{\mathcal{A}^T\omega}(I)$ occurring in Theorem 11.4 is not a monomial ideal (yet). It is natural to ask how its different Gröbner bases enter the overall picture.

Corollary 11.6. *Using notation as above, suppose that $\mathcal{F}$ is a canonical basis.*
(1) Every reduced Gröbner basis $\mathcal{G}$ of $I_\mathcal{A}$ lifts to a reduced Gröbner basis $\mathcal{H}$ of I, i.e., the elements of $\mathcal{G}$ are the initial forms (with respect to $\mathcal{A}^T\omega$) of the elements of $\mathcal{H}$.
(2) Every regular triangulation of $\mathcal{A}$ is an initial complex of the ideal I.
(3) The state polytope of $I_\mathcal{A}$ is a face of the state polytope of the homogenization of I.

Proof: Let $\mathcal{G}$ be the reduced Gröbner basis of $I_\mathcal{A}$ with respect to a term order $\prec$, and let $\mathcal{H}$ be the reduced Gröbner basis of I with respect to $\prec_{\mathcal{A}^T\omega}$. By Proposition 1.8, we have

$$in_\prec(I_\mathcal{A}) \quad = \quad in_\prec(in_{\mathcal{A}^T\omega}(I)) \quad = \quad in_{\prec_{\mathcal{A}^T\omega}}(I). \tag{11.5}$$

This implies $in_\prec(in_{\mathcal{A}^T\omega}(\mathcal{H})) = in_\prec(\mathcal{G})$. Since all trailing terms of elements in $\mathcal{H}$ are $\prec$-standard, we conclude that $in_{\mathcal{A}^T\omega}(\mathcal{H}) = \mathcal{G}$. This proves (1).

Part (2) follows directly from (11.5) and Theorem 8.3. Let I_{homog} denote the homogenization of I. We extend $\mathcal{A}^T\omega$ to a partial term order for I_{homog} by making the homogenizing variable reverse lexicographically smallest. For part (3) we use the following consequence of Lemma 2.6:

$$State(I_\mathcal{A}) \quad = \quad State(in_{\mathcal{A}^T\omega}(I_{homog})) \quad = \quad face_{\mathcal{A}^T\omega}(State(I_{homog})).$$

Here we had to replace I by I_{homog} because the Gröbner fan of I need not be complete (in which case the polytope $State(I)$ is not defined). However, the toric ideal $I_\mathcal{A}$ is positively graded since the columns of $\mathcal{A}$ are non-zero non-negative vectors, so that $State(I_\mathcal{A})$ is always a well-defined polytope. ∎

Corollary 11.5 gives rise to a simple completion algorithm for computing a canonical basis from any finite generating set $\mathcal{F}$, provided $in_\prec(k[\mathcal{F}])$ is finitely generated. Namely, if there exists a minimal generator p_i of $I_\mathcal{A}$ such that Algorithm 11.1 reduces $p_i(f_1,\ldots,f_n)$ to a non-constant polynomial $q_i(f_1,\ldots,f_n)$, then simply add $q_i(f_1,\ldots,f_n)$ to the set $\mathcal{F}$ and proceed. Just as in the case of the Buchberger algorithm for ideals, this completion procedure can be made more efficient by autoreductions and other more clever strategies.

Example 11.7. *(The algebra generated by the 2×2-minors of a generic 3×3-matrix)* Consider a 3×3-matrix of indeterminates (t_{ij}) and let $\mathcal{F}$ be the set of its 2×2-minors,

$$\begin{aligned} f_1 &= \underline{t_{11}t_{22}} - t_{12}t_{21}, & f_2 &= \underline{t_{11}t_{23}} - t_{13}t_{21}, & f_3 &= \underline{t_{12}t_{23}} - t_{13}t_{22},\\ f_4 &= \underline{t_{11}t_{32}} - t_{12}t_{31}, & f_5 &= \underline{t_{11}t_{33}} - t_{13}t_{31}, & f_6 &= \underline{t_{12}t_{33}} - t_{13}t_{32},\\ f_7 &= \underline{t_{21}t_{32}} - t_{22}t_{31}, & f_8 &= \underline{t_{21}t_{33}} - t_{23}t_{31}, & f_9 &= \underline{t_{22}t_{33}} - t_{23}t_{32}. \end{aligned} \tag{11.6}$$

We fix a term order $\prec$ which selects the main diagonal term to be the initial term, for each minor of (t_{ij}). The ideal of algebraic relations among the underlined initial terms has only two generators:

$$I_\mathcal{A} \quad = \quad \langle x_4x_8 - x_5x_7,\ x_2x_6 - x_3x_5\rangle.$$

Under the substitution $x_i \mapsto f_i(\mathbf{t})$ we get

$$f_{10} := f_4f_8 - f_5f_7 \;=\; t_{31}\cdot det(t_{ij}) \quad \text{and} \quad f_{11} := f_2f_6 - f_3f_5 \;=\; t_{13}\cdot det(t_{ij}),$$

where $det(t_{ij})$ is the determinant of the given 3×3-matrix. Neither of the initial terms

$$in_\prec(f_{10}) \;=\; t_{11}t_{22}t_{31}t_{33} \quad \text{or} \quad in_\prec(f_{11}) \;=\; t_{11}t_{13}t_{22}t_{33}$$

lies in the subalgebra generated by the nine underlined monomials in (11.6). We enlarge the generating set to $\mathcal{F}' := \mathcal{F}\cup\{f_{10},f_{11}\}$. The corresponding toric ideal remains unchanged:

$$I_{\mathcal{A}'} \quad = \quad \langle x_4x_8 - x_5x_7,\ x_2x_6 - x_3x_5\rangle,$$

since the variable t_{31} (resp. t_{13}) occurs only in $in_\prec(f_{10})$ (resp. $in_\prec(f_{11})$) but in no other initial term. This implies that $\mathcal{F}'$ is a canonical basis for its subalgebra $k[\mathcal{F}] = k[\mathcal{F}']$. ∎

Example 11.7 stands in a certain contrast to the next result which concerns maximal minors. Consider a matrix of indeterminates (t_{ij}) of format $r\times s$, where $r\le s$. Let R be the subalgebra of $k[\mathbf{t}] = k[t_{11},\ldots,t_{rs}]$ generated by the $r\times r$-minors of (t_{ij}). Its projective spectrum $Proj(R)$ is the *Grassmann variety $Grass_{r,s}$* of r-dimensional linear subspaces in an s-dimensional vector space, presented in its usual *Plücker embedding*. A term order on $k[\mathbf{t}]$ is called *diagonal* if the main diagonal term is the initial term for each $r\times r$-minor.

Theorem 11.8. *The set of $r\times r$-minors of an $r\times s$-matrix of indeterminates is a canonical basis for the subalgebra they generate, with respect to any diagonal term order on $k[\mathbf{t}]$.*

Proof: See Theorem 3.2.9 in (Sturmfels 1993a). ∎

We associate a new variable $[i_1i_2\cdots i_r]$ to the $r\times r$-minor with column indices $i_1<i_2<\cdots<i_r$. Thus the polynomial ring $k[\mathbf{x}]$ is generated by these $\binom{s}{r}$ *brackets*. Let $I_{r,s}$ denote the ideal in $k[\mathbf{x}] = k[\cdots,[i_1\cdots i_r],\cdots]$ generated by the algebraic relations among the $r\times r$-minors. The ideal $I_{r,s}$ is called the *Grassmann-Plücker ideal*. Theorem 11.8 is a consequence of the classical *straightening algorithm* for $I_{r,s}$.

Let $\mathbf{e}_{ij}$ denote the unit vector in $\mathbf{N}^{r\times s}$ corresponding to the variable t_{ij}. The vector configuration associated with the diagonal initial monomials $t_{1i_1}t_{2i_2}\cdots t_{ri_r}$ equals

$$\mathcal{A}_{r,s} \quad = \quad \{\mathbf{e}_{1i_1} + \mathbf{e}_{2i_2} + \cdots + \mathbf{e}_{ri_r} \;:\; 1\le i_1<i_2<\cdots<i_r\le s\}. \tag{11.7}$$

The toric ideal $I_{\mathcal{A}_{r,s}}$ is the kernel of the map

$$k[\cdots,[i_1i_2\cdots i_r],\cdots] \;\to\; k[t_{11},t_{12},\ldots,t_{rs}],\quad [i_1i_2\cdots i_r] \mapsto t_{1i_1}t_{2i_2}\cdots t_{ri_r}. \tag{11.8}$$

Example 11.9. *(The Grassmann variety of lines in projective 4-space)*
In the special case $r = 2$ and $s = 5$ we shall prove Theorem 11.8 by applying

the criterion in Theorem 11.4. The algebra of interest is generated by the ten 2×2-minors of the matrix

$$\mathbf{t} \quad = \quad \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} \end{pmatrix}$$

A diagonal term order is given, for instance, by the following weight matrix

$$\omega \quad = \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix}.$$

The ten diagonal initial terms generate the toric algebra of the configuration (11.7):

$$k[\mathcal{A}_{2,5}] = k[\, t_{11}t_{22},\ t_{11}t_{23},\ t_{11}t_{24},\ t_{11}t_{25},\ t_{12}t_{23},\ t_{12}t_{24},\ t_{12}t_{25},\ t_{13}t_{24},\ t_{13}t_{25},\ t_{14}t_{25}\,].$$

We consider the map (11.8) from the free polynomial ring

$$k[\mathbf{x}] \quad = \quad k[\,[12],[13],[14],[15],[23],[24],[25],[34],[35],[45]\,].$$

onto $k[\mathcal{A}_{2,5}]$. Its kernel is the toric ideal $I_{\mathcal{A}_{2,5}}$. This ideal is generated by the five binomials

$$[14][23]-[13][24],\ [15][23]-[13][25],[15][24]-[14][25],$$
$$[15][34]-[14][35],\ [25][34]-[24][35].$$

To establish Theorem 11.8, we must verify the inclusion $I_{\mathcal{A}} \subseteq in_{\mathcal{A}^T\omega}(I_{2,5})$. The induced weight vector $\mathcal{A}^T\omega$ has the entry $i+j^2$ in the coordinate indexed by $[ij]$. Each of the following five *Plücker relations* lies in the kernel $I_{2,5}$ of the canonical epimorphism $k[\mathbf{x}] \to R$:

$$\begin{aligned} &\underline{[14][23]-[13][24]} + [12][34], \\ &\underline{[15][23]-[13][25]} + [12][35], \\ &\underline{[15][24]-[14][25]} + [12][45], \\ &\underline{[15][34]-[14][35]} + [13][45], \\ &\underline{[25][34]-[24][35]} + [23][45]. \end{aligned} \tag{11.9}$$

The underlined initial forms selected by the weight $\mathcal{A}^T\omega$ coincide with the generators. This completes the proof of Theorem 11.8 in the special case $r=2$ and $s=5$. ■

Theorem 11.8 and Corollary 11.6 have the following geometric implications. By a *toric deformation* we mean a flat deformation using a one-parameter subgroup of the torus $(k^*)^n$.

Proposition 11.10.
(a) *There exists a toric deformation taking the Grassmann variety $Grass_{r,s}$ into the projective toric variety defined by the configuration $\mathcal{A}_{r,s}$.*
(b) *Every initial ideal of the toric ideal $I_{\mathcal{A}_{r,s}}$ is an initial ideal of the Grassmann-Plücker ideal $I_{r,s}$.*

(c) *Every regular triangulation of $\mathcal{A}_{r,s}$ is an initial complex of the Grassmann variety.*
(d) *The state polytope of $\mathcal{A}_{r,s}$ is a face of the state polytope of the Grassmann variety.*

Remark 11.11. *(Digression into algebraic combinatorics)*
The classical straightening algorithm for the Grassmann-Plücker ideal is a special case of the Gröbner bases arising from Proposition 11.10 (b). The best route to seeing this is the following detour through the land of algebraic combinatorics: The set $\mathcal{A}_{r,s}$ is (affinely isomorphic to) the vertex set of the *order polytope* of the product of two chains $[r] \times [s-r]$. The *distributive lattice* $\mathcal{L} = J([r]\times[s-r])$ is (isomorphic to) the poset of brackets in the coordinatewise order. We depict this lattice for $r=2, s=5$ (Example 11.9):

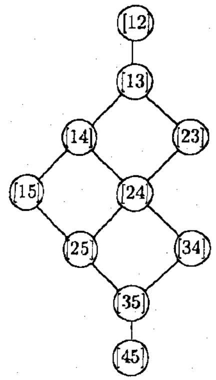

Figure 11-1. The distributive lattice $J([2]\times[3])$.

The toric ideal $I_{\mathcal{A}}$ is generated by the binomials $x \cdot y - (x \vee y)\cdot(x \wedge y)$ where $x, y \in \mathcal{L}$ stand for brackets and $\vee$ and $\wedge$ are the lattice operations. We define a term order on $k[\mathbf{x}] = k[\mathcal{L}]$ as follows: first sort the variables by any linear extension of $\mathcal{L}$ and then sort monomials by the reverse lexicographic order. The initial ideal of $I_{\mathcal{A}}$ coincides with the initial ideal of the Grassmann-Plücker ideal $I_{r,s}$. It is generated by all products $x \cdot y$ of incomparable elements in $\mathcal{L}$. The initial complex is the *chain complex* of $\mathcal{L}$, which is known to be a regular triangulation of the order polytope of $[r]\times[s-r]$. ■

Not every Gröbner bases of $I_{r,s}$ arises from a Gröbner basis of $I_{\mathcal{A}_{r,s}}$, i.e., the converse of Proposition 11.10 (b) does not hold. We will demonstrate this in the case $r=2$.

Corollary 11.12. *The toric ideal $I_{\mathcal{A}_{2,s}}$ has the following properties:*
(a) *The set of circuits equals the universal Gröbner basis:*

$$[i_1j_1][i_2j_2]\cdots[i_\nu j_\nu]-[i_2j_1][i_3j_2]\cdots[i_1j_\nu],$$
$$(i_1,i_2<j_1 \text{ and } i_2,i_3<j_2 \text{ and } \ldots \text{ and } i_\nu,i_1<j_\nu).$$

(b) *All initial ideals of $I_{\mathcal{A}_{2,s}}$ are square-free.*

Proof: The configuration $\{ \mathbf{e}_{1i} + \mathbf{e}_{2j} \; : \; 1 \leq i, j \leq s \}$ is isomorphic to the vertex set of the product of simplices $\Delta_{s-1} \times \Delta_{s-1}$; see Example 5.1. It is unimodular by Exercise (9) in Chapter 8. Therefore its subset $\mathcal{A}_{2,s} = \{ \mathbf{e}_{1i} + \mathbf{e}_{2j} \; : \; 1 \leq i < j \leq s \}$ is unimodular as well. Using Remark 8.10 this proves the assertion (b). To prove (a) we note that the circuits of $\Delta_{s-1} \times \Delta_{s-1}$ are identified with the circuits in the complete bipartite graph $K_{s,s}$. What is listed in (a) is the subset of circuits whose support lies in $\mathcal{A}_{2,s}$. To complete the proof we use Proposition 8.11. ∎

The Grassmann-Plücker ideal $I_{2,6}$ has initial ideals which are not square-free. For example, choose the weight vector

$$\omega = (9, 56, 82, 40, 86, 95, 55, 85, 88, 88, 39, 46, 10, 26, 62)$$

for the $\binom{6}{2} = 15$ brackets $[ij]$ in the usual lexicographic order. We get an initial monomial ideal $in_\omega(I_{2,6})$ among whose minimal generators there is $[15][23]^2[46]$. By Corollary 11.12 (b), the monomial ideal $in_\omega(I_{2,6})$ is not an initial ideal of $I_{\mathcal{A}_{2,6}}$.

We also remark that statement (b) does not hold for $r \geq 3$; for instance, $\mathcal{A}_{3,6}$ contains the vertices of a regular 3-cube as a subset, hence it has a regular triangulation one of whose simplices does not have unit volume, hence it has an initial ideal that is not square-free.

Here is an open problem: *What is the maximum degree $F(r,s)$ appearing in any reduced Gröbner basis for the Grassmann-Plücker ideal $I_{r,s}$?* In view of Proposition 11.10 (b) and Proposition 4.11, the number $F(r,s)$ is bounded below by the maximum degree of any circuit of the configuration $\mathcal{A}_{r,s}$. For instance, Corollary 11.12 (a) implies

$$F(2,s) \quad \geq \quad s-2. \tag{11.10}$$

For an explicit proof of this inequality we may consider the degree $s-2$ circuit

$$[13][24][35]\cdots[s-2,s] \;\; - \;\; [23][34][45]\cdots[1,s]. \tag{11.11}$$

To construct an initial ideal of the Grassmann-Plücker ideal which has one of the monomials in (11.11) as a minimal generator, start with the *Vandermonde weights* $[ij] \mapsto i + j^2$ and break ties using an elimination order for the variables appearing in (11.11).

Returning to our general discussion, we assume that $\mathcal{F} = \{f_1, \ldots, f_n\} \subset k[t_1, \ldots, t_d]$ is a canonical basis with respect to $\omega \in \mathbf{R}^d$. We shall present an *intrinsic* Gröbner basis theory for the "canonically presented" subalgebra $k[\mathcal{F}]$. Let J be any ideal in $k[\mathcal{F}]$. The *initial ideal* of J is the following ideal in the initial algebra $k[\mathcal{A}] = in_\omega(k[\mathcal{F}]) = k[in_\omega(\mathcal{F})]$:

$$in_\omega(J) \quad := \quad \langle in_\omega(f) \, : \, f \in J \rangle. \tag{11.12}$$

An important difference from "classical" Gröbner basis theory is this: the original ideal J lies in $k[\mathcal{F}]$, but the initial ideal $in_\omega(J)$ lies in a different ring, namely, it lies in the toric ring $k[\mathcal{A}]$. A subset $\mathcal{G}$ of J is a *Gröbner basis* with respect to ω if $in_\omega(J)$ is generated by $\{in_\omega(g) \, : \, g \in \mathcal{G}\}$. If this set minimally generates $in_\omega(J)$, then $\mathcal{G}$ is a *minimal* Gröbner basis. From now on we shall always assume that ω is a *term order for J*, that is, $in_\omega(J)$ is generated by monomials $\mathbf{t}^{\mathbf{c}}$, where $\mathbf{c} \in \mathbf{N}\mathcal{A}$. A minimal Gröbner basis $\mathcal{G}$ of J is *reduced* if none of the trailing terms $\mathbf{t}^{\mathbf{b}}$ appearing in any $g \in \mathcal{G}$ are contained in $in_\omega(J)$.

Lemma 11.13. *Every ideal $J \subset k[\mathcal{F}]$ possesses a unique finite reduced Gröbner basis.*

Proof: We first show uniqueness. Suppose $\mathcal{G}$ and $\mathcal{G}'$ are two distinct reduced Gröbner bases of J. There exist $g \in \mathcal{G}$ and $g' \in \mathcal{G}'$ such that $g \neq g'$ but $in_\omega(g) = in_\omega(g')$. The polynomial $g - g'$ lies in $J \backslash \{0\}$ but none of its terms lies in $in_\omega(J)$. This is a contradiction.

We next show existence. Since $k[\mathcal{A}]$ is Noetherian and positively graded, the monomial ideal $in_\omega(J)$ has a unique finite minimal generating set of the form $\{\mathbf{t}^{\mathbf{c}_1}, \mathbf{t}^{\mathbf{c}_2}, \ldots, \mathbf{t}^{\mathbf{c}_r}\}$. For each $i \in \{1, \ldots, r\}$ there exists an element $g_i \in J$ such that $in_\omega(g_i) = \mathbf{t}^{\mathbf{c}_i}$. The set $\{g_1, g_2, \ldots, g_r\}$ is a minimal Gröbner basis. To replace it by the reduced Gröbner basis, we successively reduce each element g_j by the complementary set $\{g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_r\}$ using Subroutine 11.14 below. This reduction process terminates because ω defines a Noetherian order on the monomials $\{ \mathbf{t}^{\mathbf{b}} \, : \, \mathbf{b} \in \mathbf{N}\mathcal{A}\} \subset k[\mathbf{t}]$. ∎

Subroutine 11.14. *(Reduction inside a canonically presented subalgebra $k[\mathcal{F}]$)*
Input: A polynomial $p \in k[\mathcal{F}]$, a finite subset $\mathcal{G} \subset k[\mathcal{F}]$, a term order ω on $k[\mathbf{t}]$.
Output: A normal form for p modulo $\mathcal{G}$ with respect to ω.

While the polynomial p contains a term $\mathbf{t}^{\mathbf{b}}$ which lies in the ideal $\langle in_\omega(\mathcal{G})\rangle$ do:

1. *Let $g \in \mathcal{G}$ such that $\mathbf{t}^{\mathbf{b}}$ lies in the principal ideal generated by $in_\omega(g) = \mathbf{t}^{\mathbf{d}}$.*
2. *Find an integral vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ in the fiber $\pi^{-1}(\mathbf{b} - \mathbf{d})$.*
3. *Replace p by $p - g \cdot \prod_{i=1}^n f_i^{\lambda_i}$.*

The vector λ chosen in Step 2 has the property

$$in_\omega(\prod_{i=1}^n f_i^{\lambda_i}) \;\; = \;\; \prod_{i=1}^n (in_\omega(f_i))^{\lambda_i} \;\; = \;\; \prod_{i=1}^n (\mathbf{t}^{\mathbf{a}_i})^{\lambda_i} \;\; = \;\; \mathbf{t}^{\pi(\lambda)} \;\; = \;\; \mathbf{t}^{\mathbf{b}-\mathbf{d}}.$$

Therefore the subtraction in Step 3 does indeed cancel the term $\mathbf{t}^{\mathbf{b}}$ from p. The main difficulty of Subroutine 11.14 lies in the fact that testing membership in a monomial ideal of $k[\mathcal{A}]$ amounts to solving a disjunction of integer programming feasibility problems. Incidentally, we encounter a similar difficulty already in Step 1 of Algorithm 11.1.

Remark 11.15. *In the toric ring $k[\mathcal{A}]$ we have*

$$\mathbf{t}^{\mathbf{b}} \in \langle \mathbf{t}^{\mathbf{c}_1}, \ldots, \mathbf{t}^{\mathbf{c}_r} \rangle \quad \textit{if and only if} \quad \bigcup_{i=1}^r \pi^{-1}(\mathbf{b} - \mathbf{c}_i) \;\neq\; \emptyset.$$

To generalize Buchberger's criterion, we introduce the module of syzygies

$$Syz(\mathbf{t}^{\mathbf{c}_1}, \ldots, \mathbf{t}^{\mathbf{c}_r}) \quad = \quad \{ (h_1, \ldots, h_r) \in k[\mathcal{A}]^r \, : \, h_1 \mathbf{t}^{\mathbf{c}_1} + \cdots + h_r \mathbf{t}^{\mathbf{c}_r} = 0 \}. \tag{11.13}$$

Theorem 11.16. *Let $\mathcal{G} = \{g_1, \ldots, g_r\} \subset k[\mathcal{F}]$, and let $\mathcal{H}$ be any subset of $k[\mathcal{F}]^r$ such that $\{(in_\omega(h_1), \ldots, in_\omega(h_r)) \, : \, (h_1, \ldots, h_r) \in \mathcal{H}\}$ generates the $k[\mathcal{A}]$-module $Syz(in_\omega(g_1), \ldots, in_\omega(g_r))$. Then $\mathcal{G}$ is a Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ with respect to ω if and only if, for every $h = (h_1, \ldots, h_r) \in \mathcal{H}$, the polynomial $h_1 g_1 + \cdots + h_r g_r$ reduces to zero modulo $\mathcal{G}$ via Subroutine 11.14.*

Proof: See Theorem 4.9 in (Miller 1996). ∎

Algorithm 11.17. *(Intrinsic Buchberger for a canonically presented subalgebra)*
Input: A generating set $\mathcal{S}$ of an ideal $J \subset k[\mathcal{F}]$, a term order $\omega \in \mathbf{R}^d$.
Output: The reduced Gröbner basis $\mathcal{G}$ of J with respect to ω.

1. *Let* $\mathcal{S} = \{s_1, \ldots, s_r\}$ *where* $in_\omega(s_i) = \mathbf{t}^{\mathbf{c}_i}$.
2. *Compute a finite generating set* $\mathcal{M}$ *for* $Syz(\mathbf{t}^{\mathbf{c}_1}, \ldots, \mathbf{t}^{\mathbf{c}_r}) \subset k[\mathcal{A}]^r$ *(Subroutine 11.18).*
3. *Set* newguys $:= \emptyset$.
4. *For each* $m = (m_1, \ldots, m_r) \in \mathcal{M}$ *do:*
   4.1. *Find* $h = (h_1, \ldots, h_r) \in k[\mathcal{F}]^r$ *such that* $in_\omega(h_j) = m_j$ *for* $j = 1, \ldots, r$.
   4.2. *Compute the normal form* $\overline{h}$ *of* $h_1 s_1 + \cdots + h_r s_r$ *modulo* $\mathcal{S}$ *(Subroutine 11.14).*
   4.3. newguys $:=$ newguys $\cup \{\overline{h}\}$.
5. *If* newguys $\neq \{0\}$ *then set* $\mathcal{S} := \mathcal{S} \cup$ newguys$\backslash\{0\}$, *and return to Step 1.*
6. *If* newguys $= \{0\}$ *then*
   6.1. *Compute the auto-reduction* $\mathcal{G}$ *of* $\mathcal{S}$
   (by applying Subroutine 11.14 to reduce s *modulo* $\mathcal{S}\backslash\{s\}$, *for all* $s \in \mathcal{S}$*).*
   6.2. *Output* $\mathcal{G}$.

This leaves us with the problem of how to compute generators for the syzygy module (11.13). We shall present two subroutines (11.18 and 11.21) for performing this task. We write $\mathbf{e}_i$ for the standard basis vectors in the free module $k[\mathcal{A}]^r$.

Subroutine 11.18. *(Computing the syzygies on some monomials in a toric ring)*
Input: A vector of monomials $(\mathbf{t}^{\mathbf{c}_1}, \ldots, \mathbf{t}^{\mathbf{c}_r}) \in k[\mathcal{A}]^r$.
Output: A finite generating set for $Syz(\mathbf{t}^{\mathbf{c}_1}, \ldots, \mathbf{t}^{\mathbf{c}_r}) \subset k[\mathcal{A}]^r$.

1. *Find* $\mathbf{u}_i \in \pi^{-1}(\mathbf{c}_i)$ *for* $i = 1, \ldots, r$.
2. *Let* $S \subset k[\mathbf{x}]^r$ *be any generating set for the syzygies on* $(\mathbf{x}^{\mathbf{u}_1}, \ldots, \mathbf{x}^{\mathbf{u}_r})$, *for instance, the usual S-pairs. Apply the toric homomorphism* $x_i \mapsto \mathbf{t}^{\mathbf{a}_i}$ *to* S *and output the result.*
3. *Compute a reduced Gröbner basis (in the ordinary sense) for the ideal intersection*

$$\langle \mathbf{x}^{\mathbf{u}_1}, \ldots, \mathbf{x}^{\mathbf{u}_r} \rangle \cap I_{\mathcal{A}} \quad \text{in} \quad k[\mathbf{x}] = k[x_1, \ldots, x_n]. \tag{11.14}$$

4. *For each element* $g = g(\mathbf{x})$ *in the reduced Gröbner basis of (11.14) do:*
   4.1. *Write* g *in the form* $g(\mathbf{x}) = \mathbf{x}^{\mathbf{v}} \cdot \mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{w}} \cdot \mathbf{x}^{\mathbf{u}_j}$, *where* $i, j \in \{1, \ldots, r\}$.
   4.2. *Output the syzygy* $\mathbf{t}^{\pi(\mathbf{v})} \cdot \mathbf{e}_i - \mathbf{t}^{\pi(\mathbf{w})} \cdot \mathbf{e}_j$.

Example 11.19. Here is a simple example of a syzygy module which is not generated by S-pairs. Take $d = 2, n = 3$ and $\mathcal{A} = \{(2,0),(1,1),(0,2)\}$, so that $k[\mathcal{A}] = k[t_1^2, t_1t_2, t_2^2] = k[x_1, x_2, x_3]/\langle x_1x_3 - x_2^2\rangle$. Then $Syz(t_1^2, t_1t_2)$ is minimally generated by $(t_1t_2, -t_1^2)$ and $(t_2^2, -t_1t_2)$. The first syzygy is found in Step 2 and the second is found in Step 4.2. ∎

The correctness of Subroutine 11.18 is the content of Proposition 4.10 in (Miller 1996). In Step 3.1 we are making implicitly the claim that every reduced Gröbner basis of (11.14) consists of binomials $\mathbf{x}^{\mathbf{v}} \cdot \mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{w}} \cdot \mathbf{x}^{\mathbf{u}_j}$. To prove this claim, we recall the standard algorithm for computing ideal intersections (Cox, Little & O'Shea 1992; §4.3, Theorem 11): Introduce a new variable z, form the ideal $B := \langle (1-z) \cdot \mathbf{x}^{\mathbf{u}_1}, \ldots, (1-z) \cdot \mathbf{x}^{\mathbf{u}_r} \rangle + z \cdot I_{\mathcal{A}}$ in $k[\mathbf{x}, z]$, and then compute the

elimination ideal $B \cap k[\mathbf{x}]$. Our claim follows because B is a binomial ideal and the Buchberger algorithm is "binomial-friendly". See Corollary 1.7 in (Eisenbud & Sturmfels 1996) for a more general result.

These considerations imply the following toric generalization of the familiar fact that S-pairs suffice to generate all syzygies (Cox, Little & O'Shea 1992; §4.3, Proposition 8).

Corollary 11.20. *Syzygies on monomials in* $k[\mathcal{A}]$ *are generated by pairwise syzygies:*

$$Syz\Big(\sum_{i=1}^{r} \mathbf{t}^{\mathbf{c}_i} \cdot \mathbf{e}_i\Big) \quad = \quad \sum_{1 \leq i < j \leq r} Syz\big(\mathbf{t}^{\mathbf{c}_i} \cdot \mathbf{e}_i + \mathbf{t}^{\mathbf{c}_j} \cdot \mathbf{e}_j\big).$$

Corollary 11.20 reduces the computation of the syzygy module (11.13) to the special case $r = 2$. Hence we could also use the following subroutine for Step 2 in Algorithm 11.17.

Subroutine 11.21. *(Computing syzygies on a pair of monomials in a toric ring)*
Input: Two monomials $\mathbf{t}^{\mathbf{c}}$ and $\mathbf{t}^{\mathbf{d}}$ in $k[\mathcal{A}]$.
Output: A finite generating set for $Syz(\mathbf{t}^{\mathbf{c}}, \mathbf{t}^{\mathbf{d}}) \subset k[\mathcal{A}]^2$.

1. *Form the toric ideal* $I_{\mathcal{A}\cup\{\mathbf{c}-\mathbf{d}\}} \subset k[x_1, \ldots, x_n, z]$, *where* z *is mapped to* $\mathbf{t}^{\mathbf{c}-\mathbf{d}}$.
2. *Compute the reduced Gröbner basis* $\mathcal{G}$ *for* $I_{\mathcal{A}\cup\{\mathbf{c}-\mathbf{d}\}}$ *with respect to any elimination order* $z \succ \{x_1, \ldots, x_n\}$.
3. *For each binomial in* $\mathcal{G}$ *which contains* z *linearly, such as* $\mathbf{x}^{\mathbf{u}} \cdot z - \mathbf{x}^{\mathbf{v}}$, *output the corresponding syzygy* $(\mathbf{t}^{\pi(\mathbf{u})}, -\mathbf{t}^{\pi(\mathbf{v})})$.

Proof of correctness: It follows immediately from the construction that each output pair $(\mathbf{t}^{\pi(\mathbf{u})}, -\mathbf{t}^{\pi(\mathbf{v})})$ is a syzygy of $(\mathbf{t}^{\mathbf{c}}, \mathbf{t}^{\mathbf{d}})$. Conversely, every minimal syzygy can be written as a pair $(\mathbf{t}^{\pi(\mathbf{u}')}, -\mathbf{t}^{\pi(\mathbf{v}')})$ such that $\mathbf{x}^{\mathbf{u}'} \cdot z - \mathbf{x}^{\mathbf{v}'}$ lies in $I_{\mathcal{A}\cup\{\mathbf{c}-\mathbf{d}\}}$. The Gröbner basis property of $\mathcal{G}$ implies that there exists a binomial $\mathbf{x}^{\mathbf{u}} \cdot z - \mathbf{x}^{\mathbf{v}} \in \mathcal{G}$ such that $\mathbf{x}^{\mathbf{u}}$ divides $\mathbf{x}^{\mathbf{u}'}$ and $\mathbf{x}^{\mathbf{v}'} - \mathbf{x}^{\mathbf{v}+\mathbf{u}'-\mathbf{u}}$ lies in $I_{\mathcal{A}}$. The given syzygy therefore equals $\mathbf{t}^{\pi(\mathbf{u}'-\mathbf{u})} \cdot (\mathbf{t}^{\pi(\mathbf{u})}, -\mathbf{t}^{\pi(\mathbf{v})})$. ∎

Example 11.22. The minimal number of generators of $Syz(\mathbf{t}^{\mathbf{c}}, \mathbf{t}^{\mathbf{d}})$ cannot be bounded by a function in n, d and $\mathcal{A}$. Let $d=3, n=7$ and

$$k[\mathcal{A}] = k[t_1t_2t_3, t_1^2t_2, t_1t_2^2, t_1^2t_3, t_1t_3^2, t_2^2t_3, t_2t_3^2].$$

It can be shown that the minimal generators of $Syz(t_1^{2i}t_2^{2i}t_3^{2i}, t_1^{3i+2}t_2t_3^{3i})$ include $2i+2$ syzygies of total degree $i+1$, for $i \geq 1$. ∎

Any bound must therefore involve the degrees of $\mathbf{c}$ and $\mathbf{d}$. Here is such a bound.

Theorem 11.23. *Let* $D(\cdot)$ *be defined as in Theorem 4.7. Then the total degree (in* $k[\mathbf{x}]^r$*) of any minimal generators of the syzygy module (11.13) is bounded above by*

$$max_{1 \leq i < j \leq r}\ (d+1) \cdot (n+1-d) \cdot D(\mathcal{A} \cup \{\mathbf{c}_i - \mathbf{c}_j\}).$$

Proof: This follows from Corollary 11.20, Subroutine 11.21 and Theorem 4.7. ∎

One disadvantage of the intrinsic Buchberger Algorithm 11.17 is that – at present – it is not available in any computer algebra system. However, there is an "extrinsic" method for simulating Algorithm 11.17, which is easy to run in the currently available Gröbner bases programs.

Algorithm 11.24. *(Extrinsic computation of intrinsic Gröbner bases)*
Input: Generators for an ideal J in $k[\mathcal{F}]$ and a term order ω on $k[\mathbf{t}]$.
Output: A Gröbner basis for J with respect to ω.

1. *Let I denote the kernel of the canonical epimorphism* $$\phi : k[\mathbf{x}] \to k[\mathcal{F}],\ x_i \mapsto f_i(\mathbf{t}).$$
2. *For each generator of J choose a preimage, and let $\overline{J} \subset k[\mathbf{x}]$ be the ideal they generate.*
3. *Compute the reduced Gröbner basis $\mathcal{G}$ of the ideal $I + \overline{J}$ with respect to any term order refining the weight vector $\mathcal{A}^T\omega$.*
4. *Output its image $\phi(\mathcal{G}) = \{\phi(g) : g \in \mathcal{G}\}$ in $k[\mathcal{F}]$.*

Proof of correctness: Clearly, $\phi(\mathcal{G}) \subset J$. The fact that the Gröbner basis $\mathcal{G}$ is reduced implies

$$in_\omega(\phi(g)) \quad = \quad in_{\mathcal{A}^T\omega}(g)(\mathbf{t}^{\mathbf{a}_1},\ldots,\mathbf{t}^{\mathbf{a}_n}) \qquad \text{for all} \quad g \in \mathcal{G}. \tag{11.15}$$

We must show that $\{ in_{\mathcal{A}^T\omega}(g)(\mathbf{t}^{\mathbf{a}_1},\ldots,\mathbf{t}^{\mathbf{a}_n}) \ : \ g \in \mathcal{G}\}$ generates $in_\omega(J)$. Let $h \in J \subset k[\mathcal{F}]$. By the canonical basis property of $\mathcal{F} = \{f_1,\ldots,f_n\}$, there exists $p \in k[\mathbf{x}]$ such that $h = p(f_1,\ldots,f_n)$ and $in_\omega(h) = in_{\mathcal{A}^T\omega}(p)(\mathbf{t}^{\mathbf{a}_1},\ldots,\mathbf{t}^{\mathbf{a}_n}) \neq 0$. Since $p \in I + \overline{J}$, its initial form $in_{\mathcal{A}^T\omega}(p)$ lies in the ideal $\langle in_{\mathcal{A}^T\omega}(\mathcal{G})\rangle$ in $k[\mathbf{x}]$. This implies that $in_\omega(h)$ lies in (11.15), as desired. ∎

We discuss a geometric example in the Grassmann variety of lines in projective 4-space.

Example 11.25. Consider the canonical basis in Theorem 11.8 for $r = 2, s = 5$. Here $\mathcal{F} = \{[12],[13],\ldots,[45]\}$ is the set of 2×2-minors of a 2×5-matrix of indeterminates (t_{ij}), and ω is the weight with coordinates $\omega_{ij} = j^i$. Consider the ideal $J = \langle g_1, g_2\rangle$, where

$$g_1 \ = \ ([12]-[23])\cdot([23]-[34]) \quad \text{and} \quad g_2 \ = \ ([13]+[14]+[15])\cdot([15]+[25]+[35]).$$

The corresponding subvariety of the Grassmann variety consists of all lines in P^4 which meet the following two pairs of codimension 2 subspaces: $\{x_2 = x_1 + x_3 = 0\} \cup \{x_3 = x_2 + x_4 = 0\}$ and $\{x_1 = x_3 + x_4 + x_5 = 0\} \cup \{x_5 = x_1 + x_2 + x_3 = 0\}$. We wish to compute the Gröbner basis of J intrinsically in $k[\mathcal{F}]$. The initial terms of the generators are

$$in_\omega(g_1) \ = \ t_{12}t_{13}t_{23}t_{24} \quad \text{and} \quad in_\omega(g_2) \ = \ t_{11}t_{13}t_{25}^2.$$

Using Subroutine 11.18 or 11.21, we compute the following minimal generating set for the syzygy module $Syz(t_{12}t_{13}t_{23}t_{24}, t_{11}t_{13}t_{25}^2)$:

$$\{ (t_{11}t_{13}t_{25}^2, -t_{12}t_{23}t_{13}t_{24}),\ (t_{11}^2t_{25}^2, -t_{11}t_{23}t_{12}t_{24}),\ (t_{11}t_{12}t_{25}^2, -t_{12}^2t_{23}t_{24})\}.$$

Following Step 4.1 of Algorithm 11.17 we express each of these six monomials as an initial form; for instance, $t_{11}t_{13}t_{25}^2 = in_\omega([15][35])$. In Step 4.2 we form the corresponding linear combinations in J:

$$\begin{aligned} g_3 &:= [15][35]\cdot g_1 - [23][34]\cdot g_2, \\ g_4 &:= [15]^2\cdot g_1 - [13][24]\cdot g_2, \\ g_5 &:= [15][25]\cdot g_1 - [23][24]\cdot g_2. \end{aligned}$$

The polynomial g_3 reduces to zero modulo $\{g_1, g_2\}$. The initial terms of g_4 and g_5 are in normal form modulo $\{g_1, g_2\}$. Another run through Algorithm 11.17 confirms that $\{g_1, g_2, g_4, g_5\}$ is a minimal Gröbner basis for J. However, to arrive at the reduced Gröbner basis we must further reduce g_4 and g_5 modulo $\{g_1, g_2\}$ in Step 6.1.

We remark that already for this small example the extrinsic computation is quite redundant: the reduced Gröbner basis in Step 3 of Algorithm 11.24 contains 15 elements.

Exercises:

(1) Give an example of a subalgebra $k[\mathcal{F}]$ of $k[\mathbf{x}]$ and two term orders $\prec_1$ and $\prec_2$ such that $in_{\prec_1}(k[\mathcal{F}])$ is finitely generated but $in_{\prec_2}(k[\mathcal{F}])$ is not finitely generated.

(2) Let $d = 5, n = 6$ and let $\mathcal{F}$ be the set of 2×2-minors of the matrix

$$\begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ t_2 & t_3 & t_4 & t_5 \end{pmatrix}.$$

Compute a canonical basis for the subalgebra $k[\mathcal{F}]$.

(3) Compute the state polytope of the Grassmann variety $Grass_{2,5}$. Show that the converse of Proposition 11.12 (b) holds for $r = 2, s = 5$: every initial ideal of the Grassmann-Plücker ideal $I_{2,5}$ is an initial ideal of the toric ideal $I_{\mathcal{A}_{2,5}}$.

(4) Compute the universal Gröbner basis of the toric ideal $I_{\mathcal{A}_{3,6}}$. Use your answer to give a lower bound on $F(3,6)$.

(5) Show that the intersection of two principal ideals $\langle \mathbf{t}^{\mathbf{b}}\rangle$ and $\langle \mathbf{t}^{\mathbf{c}}\rangle$ in a toric ring $k[\mathcal{A}]$ can have arbitrarily many minimal generators.

(6) The ring of symmetric polynomials $k[x_1,\ldots,x_n]^{S_n}$ is canonically presented by the set $\mathcal{F}$ of elementary symmetric functions. Implement Algorithm 11.17 in this case.

Notes:

The concept of canonical bases was introduced independently by Kapur & Madlener (1989) and Robbiano & Sweedler (1990). Further properties and applications of canonical bases were studied in (Ollivier 1991). Ollivier's results include some remarkable connections between the algebraic operation of taking integral closure and convexity properties of the initial algebra. Miller (1996) extends the theory of canonical bases to polynomial rings over general base rings. While Theorem 11.8