Notes: Week of Jul 21

Elisha Shmalo

August 7, 2025

1 Derivation of Equations of Motion

1.1 Hamiltonian Dynamics

Our basis:

$$S_j^x = s_j, S_j^y = f(s_j)\cos(\phi_j), S_j^z = f(s_j)\sin(\phi_j)$$

With $f(x) := \sqrt{1-x^2}$.
The hamiltonian is

$$H = \sum_j (-\vec{J} \circ \vec{S_{j+1}}) \cdot \vec{S_j}$$

The only terms in H involving j values are

$$\begin{aligned}(H)_j = & - J_x(s_j s_{j-1} + s_j s_{j+1}) \\ & - J_y f(s_j)\cos\phi_j[f(s_{j-1})\cos\phi_{j-1} + f(s_{j+1})\cos\phi_{j+1}] \\ & - J_z f(s_j)\sin\phi_j[f(s_{j-1})\sin\phi_{j-1} + f(s_{j+1})\sin\phi_{j+1}]\end{aligned}$$

The cononical position is

$$s_j$$

The cononical momentum is

$$\phi_j$$

Thus

$$\dot{s_j} = \frac{\partial H}{\partial \phi_j}$$
$$\dot{\phi_j} = -\frac{\partial H}{\partial s_j}$$

and so we have

$$\frac{ds_j}{dt} = + J_y f(s_j)\sin(\phi_j)[f(s_{j+1})\cos(\phi_{j+1}) + f(s_{j-1})\cos(\phi_{j-1})] \\ - J_z f(s_j)\cos(\phi_j)[f(s_{j+1})\sin(\phi_{j+1}) + f(s_{j-1})\sin(\phi_{j-1})]$$

$$\frac{d\phi_j}{dt} = -f'(s_j)[-J_y\cos\phi_j(f(s_{j+1})\cos\phi_{j+1} + f(s_{j-1})\cos\phi_{j-1}) \\ - J_z\sin\phi_j(f(s_{j+1})\sin\phi_{j+1} + f(s_{j-1})\sin\phi_{j-1})] \\ + J_x(s_{j+1} + s_{j-1})$$

1.2 If $J_y = J_z = J$

$$\frac{ds_j}{dt} = Jf(s_j)[f(s_{j+1})(\sin\phi_j\cos\phi_{j+1} - \cos\phi_j\sin\phi_{j+1}) \\ + f(s_{j-1})(\sin\phi_j\cos\phi_{j-1} - \cos\phi_j\sin\phi_{j-1})]$$

$$\frac{d\phi_j}{dt} = Jf'(s_j)[f(s_{j+1})(\cos\phi_j\cos\phi_{j+1} + \sin\phi_j\sin\phi_{j+1}) \\ + f(s_{j-1})(\cos\phi_j\cos\phi_{j-1} + \sin\phi_j\sin\phi_{j-1})] \\ + J_x(s_{j+1} + s_{j-1})$$

which become

$$\frac{ds_j}{dt} = Jf(s_j)[f(s_{j+1})\sin(\phi_j - \phi_{j+1}) \\ + f(s_{j-1})\sin(\phi_j - \phi_{j-1})]$$

$$\frac{d\phi_j}{dt} = Jf'(s_j)[f(s_{j+1})\cos(\phi_j - \phi_{j+1}) \\ + f(s_{j-1})\cos(\phi_j - \phi_{j-1})] \\ + J_x(s_{j+1} + s_{j-1})$$

1.3 Second order Expansion of functions

1.3.1 $f(x)$ and $f'(x)$

Let

$$f(x) = \sqrt{1 - x^2}$$

We expand $f(\delta x)$ using a Taylor expansion:

$$\boxed{f(\delta x) \approx 1 + 0 - 1/2\delta x^2}$$

The derivative:

$$f'(x) = \frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{-x}{\sqrt{1-x^2}}$$

Then:

$$f'(0) = 0$$

$$f''(0) = -1$$

$$f'''(0) = 0$$

Thus:

$$\boxed{f'(\delta x) \approx -\delta x}$$

even up to second order.

1.3.2 Now $\cos(\phi_j - \phi_{j\pm 1})$ and $\sin \phi_j - \phi_{j\pm 1}$

Let

$$\phi_j = \phi_j^0 + \delta\phi_j, \quad \text{where} \quad \phi_j^0 = \frac{2\pi}{N} j$$

Let

$$x = \phi_j^0 - \phi_{j+1}^0, y = \phi_j^0 - \phi_{j-1}^0$$

Then

$$\cos(\phi_j - \phi_{j+1}) = \cos(x + \delta x)$$

and similar for the others.
Note

$$\cos(x) = \cos(y), \sin(x) = -\sin(y)$$

Now for the exapnsions:

$$\boxed{\cos(x + \delta x) \approx \cos(x) - \sin(x)\delta x - 1/2\cos(x)\delta x^2}$$

and

$$\boxed{\sin(x + \delta x) \approx \sin(x) + \cos(x)\delta x - 1/2\sin(x)\delta x^2}$$

1.4 Second order approximation of diffrential equations

1.4.1 First ds_j/dt

Recall

$$\begin{aligned}\frac{ds_j}{dt} =&J f(s_j)[f(s_{j+1})\sin(\phi_j - \phi_{j+1})\\ &+ f(s_{j-1})\sin(\phi_j - \phi_{j-1})]\end{aligned}$$

Using the exapnsions

$$\begin{aligned}\frac{ds_j}{dt} \approx&J(1-1/2\delta s_j^2)[(1-1/2\delta s_{j+1}^2)(\sin(x)+\cos(x)\delta x - 1/2\sin(x)\delta x^2)\\ &+(1-1/2\delta s_{j-1}^2)(\sin(y)+\cos(y)\delta y - 1/2\sin(y)\delta y^2)]\end{aligned}$$

Only keeping up to order two inside the brackets and grouping like terms gives us

$$\frac{ds_j}{dt} \approx J(1-\delta s_j^2)[\cos(x)(\delta x+\delta y) - 1/2\sin(x)(\delta x^2 - \delta y^2 + \delta s_{j+1}^2 - \delta s_{j-1}^2)]$$

Again we are only keeping up to second order in δ.... So the terms with $-\delta s_j^2$ don't survive and we are left with

$$\frac{ds_j}{dt} \approx J[\cos(x)(\delta x+\delta y) - 1/2\sin(x)(\delta x^2 - \delta y^2 + \delta s_{j+1}^2 - \delta s_{j-1}^2)]$$

Which is

$$\begin{aligned}\frac{ds_j}{dt} \approx\ & J[\cos(\frac{2\pi}{N})(2\delta\phi_j - \delta\phi_{j+1} - \delta\phi_{j-1})\\ &+1/2\sin(\frac{2\pi}{N})(\delta\phi_{j+1}^2 - \delta\phi_{j-1}^2 + 2\delta\phi_j(\delta\phi_{j+1} - \delta\phi_{j-1}) + \delta s_{j+1}^2 - \delta s_{j-1}^2)]\end{aligned}$$

1.4.2 Next $d\phi_j/dt$

Recall

$$\begin{aligned}\frac{d\phi_j}{dt} =\ & Jf'(s_j)[f(s_{j+1})\cos(\phi_j - \phi_{j+1})\\ &+ f(s_{j-1})\cos(\phi_j - \phi_{j-1})]\\ &+ J_x(s_{j+1} + s_{j-1})\end{aligned}$$

using the expansions to second order

$$\frac{d\phi_j}{dt} \approx -J\delta s_j[(1-1/2\delta s_{j+1}^2)(\cos(x)-\sin(x)\delta x - 1/2\cos(x)\delta x^2)$$
$$+(1-1/2\delta s_{j+1}^2)(\cos(y)-\sin(y)\delta y - 1/2\cos(y)\delta y^2)]$$
$$+J_x(s_{j+1}+s_{j-1})$$

When we consider that we are only keeping things up to second order a lot of terms don't survive

$$\frac{d\phi_j}{dt} \approx -J\delta s_j[(\cos(x)-\sin(x)\delta x)$$
$$+(\cos(y)-\sin(y)\delta y)]$$
$$+J_x(s_{j+1}+s_{j-1})$$

Which simplifies to

$$\frac{d\phi_j}{dt} \approx -J\delta s_j[2\cos(x)-\sin(x)(\delta x-\delta y)]$$
$$+J_x(s_{j+1}+s_{j-1})$$

which is

$$\frac{d\phi_j}{dt} \approx -J\delta s_j[2\cos(\frac{2\pi}{N})+\sin(\frac{2\pi}{N})(\delta\phi_{j-1}-\delta\phi_{j+1})]$$
$$+J_x(s_{j+1}+s_{j-1})$$

Thus we have that the equations of motion to second order are

$$\frac{ds_j}{dt} \approx J[\cos(\frac{2\pi}{N})(2\delta\phi_j-\delta\phi_{j+1}-\delta\phi_{j-1})$$
$$+1/2\sin(\frac{2\pi}{N})(\delta\phi_{j+1}^2-\delta\phi_{j-1}^2+2\delta\phi_j(\delta\phi_{j+1}-\delta\phi_{j-1})+\delta s_{j+1}^2-\delta s_{j-1}^2)]$$

$$\frac{d\phi_j}{dt} \approx -J\delta s_j[2\cos(\frac{2\pi}{N})+\sin(\frac{2\pi}{N})(\delta\phi_{j-1}-\delta\phi_{j+1})]$$
$$+J_x(s_{j+1}+s_{j-1})$$

2 Seeking Solitons

Try

$$
\begin{aligned}
s_j &= f(z), z = aj - ut \\
\phi_j &= \frac{2\pi}{N} + g(z)
\end{aligned}
$$

Such that

$$
\begin{aligned}
\delta s_j &= f(z), z = aj - ut \\
\delta \phi_j &= g(z)
\end{aligned}
$$

Note that I have chosen a to be my latice constant.
Rewriting the second order approximation for the equations of motion:

$$
\begin{aligned}
-uf'(z) \approx J[\cos(\frac{2\pi}{N})(2g(z) - g(z+a) - g(z-a)) \\
+1/2\sin(\frac{2\pi}{N})(g(z+a)^2 - g(z-a)^2 + 2g(z)(g(z+a) - g(z-a)) + f(z+a)^2 - f(z-a)^2)]
\end{aligned}
$$

$$
\begin{aligned}
-ug'(z) \approx -Jf(z)[2\cos(\frac{2\pi}{N}) + \sin(\frac{2\pi}{N})(g(z-a) - g(z+a))] \\
+ J_x(f(z+a) + f(z-a))
\end{aligned}
$$

If we take the continuam limit and a linear exapnsions

$$
\begin{aligned}
f(z \pm a) &\approx f(z) \pm af'(z) \\
g(z \pm a) &\approx g(z) \pm ag'(z)
\end{aligned}
$$

Then we get the follwing relations

$$
\begin{aligned}
&2f(z) - f(z+a) - f(z-a) \approx 0 \\
&f(z-a) - f(z+a) \approx -2af' \\
&f(z-a) + f(z+a) \approx 2f
\end{aligned}
$$

(where I've written $f(z) = f$ for brevity). The same is true for $g(z)$.
Using these (and defining $\theta = \frac{2\pi}{N}$), our diffrential equations become

$$
\begin{aligned}
-uf' &\approx J[0 + 1/2\sin(\theta)(4agg' + 4agg' + 4aff')] \\
&= 2J\sin(\theta)[2agg' + aff']
\end{aligned}
$$

$$-ug' \approx -Jf[2\cos(\theta) - 2a\sin(\theta)g'] + 2J_x f$$

Rearanging these become

$$\begin{aligned} f' &\approx -4Ja\sin(\theta)\frac{gg'}{u + 2Ja\sin(\theta)f} \\ g' &\approx \frac{2(J\cos(\theta) - J_x)}{u + 2Ja\sin(\theta)f} f \end{aligned}$$

It may be easier to work with

$$\begin{aligned} -uf' &\approx J[0 + 1/2\sin(\theta)(4agg' + 4agg' + 4aff')] \\ &= 2aJ\sin(\theta)[2gg' + ff'] \end{aligned}$$

$$-ug' \approx -Jf[2\cos(\theta) - 2a\sin(\theta)g'] + 2J_x f$$

directly.

2.1 Considering effect of control

When we add in the control push, since the solitons appear near $\lambda = 0$, we should look for results where the first order terms of the evolution. The result is

$$\begin{aligned} -uf' &\approx 2aJ\sin(\theta)[2gg' + ff'] \\ -ug' &\approx 2aJf\sin(\theta)g' \end{aligned}$$

The second of these gives

$$g'(u + 2aJsin(\theta)f) = 0$$

implying

$$g' = 0 \text{ or } f = \frac{-u}{2aJsin(\theta)}$$

Thus either g or f is constant. In the case that g is constant the first equation gives

$$-uf' = 2aJsin(\theta)ff'$$

Which implies that f is constant implying (from $g'(u + 2aJsin(\theta)f) = 0$) that g is constant.

I would say that constant solutions for f and g will not give rise to the solitons we are seeing. I good idea would be to go back and expand f and g beyond the linear order for the equations of motion.

2.2 Going Back

Recall

$$-uf'(z) \approx J[\cos(\theta)(2g(z) - g(z+a) - g(z-a))$$
$$+1/2\sin(\theta)(g(z+a)^2 - g(z-a)^2 + 2g(z)(g(z+a) - g(z-a)) + f(z+a)^2 - f(z-a)^2)]$$

$$-ug'(z) \approx -Jf(z)[2\cos(\theta) + \sin(\theta)(g(z-a) - g(z+a))]$$
$$+ J_x(f(z+a) + f(z-a))$$

we are going to be ignoring first order contributions because we are seeking solutions near the phase transition. Thus we focus on

$$-uf'(z) \approx \frac{J\sin(\theta)}{2}(g(z+a)^2 - g(z-a)^2 + 2g(z)(g(z+a) - g(z-a)) + f(z+a)^2 - f(z-a)^2)]$$

$$-ug'(z) \approx Jf(z)\sin(\theta)(g(z+a) - g(z-a))$$

expanding to second order gives

$$f(z \pm a) \approx f(z) \pm af'(z) + \frac{a^2}{2}f''(z)$$

and same for g.

As before, write $f(z)$ as f just for convinence. So

$$f(z \pm a) \approx f \pm af' + \frac{a^2}{2}f''$$

and same for g.

Note that terms like $f(z+1) - f(z-1)$ are still simply $2af'$, while terms like $f(z+1) + f(z-1)$ are now $2f + a^2f''$.

plugging these into our equations of motion gives

$$-uf' \approx \frac{J\sin(\theta)}{2}[(2ag'(2g + a^2g'') + 2g(2ag') + 2af'(2f + a^2f''))]$$

$$-ug' \approx Jf\sin(\theta)(2ag')$$

From the second equation, which hasn't changed, we see that g is constant or

$$f = \frac{-u}{2Ja\sin(\theta)}$$

is constant.

Now since the spiral states all have $s_j = 0$ constant, it might be benificial to look at the $f = \frac{-u}{2Ja\sin(\theta)}$ constant case.

2.2.1 Examining f constant case

In such a case the first equation becomes

$$0 = \frac{J\sin\theta}{2}[2ag'(2g + a^2g'') + 4agg']$$

Now we already chose f constant, so lets assume g not constant $\implies g' \neq 0$. This gives

$$0 = 2a(2g + a^2g'') + 4ag$$

which is linear and thus doesn't give a soliton. We are forced to consider $g' = 0$.

2.2.2 Considering constant g

In this case the second equation vanishes and the first equation gives

$$-uf' = Ja\sin\theta f'(2f + a^2f'').$$

Unless we take $f' = 0 \implies f$ is const, this gives.

$$-u = Ja\sin\theta(2f + a^2f'')$$

which is once again linear. It seems we are once again forced into f and g constant (might not be a bad idea).

3 Harmonic Oscillators to Model Spin

Consider a series RLC circuit driven by an AC voltage source:

$$V_{\text{drive}}(t) = V_0\cos(\omega_d t)$$

The circuit contains:

- Inductor with inductance L
- Resistor with resistance R
- Capacitor with capacitance C

3.1 Equation of Motion

$$V_L + V_R + V_C = V_{\text{drive}}(t)$$

$$\boxed{L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V_0\cos(\omega_d t)}$$

3.2 Solution Structure

The general solution is:

$$Q(t) = Q_{\text{hom}}(t) + Q_{\text{part}}(t)$$

The homogeneous solution describes transient behavior and decays over time due to resistance with time scale.

$$\tau = -\frac{2L}{R}$$

.

As in

$$Q_{\text{hom}}(t) \sim e^{-\frac{Rt}{2L}}$$

I seek a steady state solution, so I feel like we should try something like:

$$Q_{\text{part}}(t) = A\cos(\omega_d t) + B\sin(\omega_d t)$$

3.3 Determaning Particular Solution Ansatz

Compute the first and second derivatives:

$$\dot{Q}_{\text{part}}(t) = -A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t)$$

$$\ddot{Q}_{\text{part}}(t) = -A\omega_d^2 \cos(\omega_d t) - B\omega_d^2 \sin(\omega_d t)$$

Substitute into the differential equation:

$$\begin{aligned}
&L\ddot{Q}_{\text{part}} + R\dot{Q}_{\text{part}} + \frac{1}{C}Q_{\text{part}} = V_0\cos(\omega_d t) \\
&= L(-A\omega_d^2 \cos(\omega_d t) - B\omega_d^2 \sin(\omega_d t)) \\
&\quad + R(-A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t)) \\
&\quad + \frac{1}{C}(A\cos(\omega_d t) + B\sin(\omega_d t))
\end{aligned}$$

Group terms:

$$\text{Coefficient of } \cos(\omega_d t): \quad -LA\omega_d^2 + RB\omega_d + \frac{A}{C}$$

$$\text{Coefficient of } \sin(\omega_d t): \quad -LB\omega_d^2 - RA\omega_d + \frac{B}{C}$$

Set the equation equal to the driving term $V_0\cos(\omega_d t)$, and match coefficients:

$$\begin{cases} -LA\omega_d^2 + RB\omega_d + \frac{A}{C} = V_0 \\ -LB\omega_d^2 - RA\omega_d + \frac{B}{C} = 0 \end{cases}$$

Solve the System

We now solve this linear system for A and B.

Define:

$$X := \left(\frac{1}{C} - L\omega_d^2\right), \quad Y := R\omega_d$$

Then the system becomes:

$$\begin{cases} AX + BY = V_0 \\ BX - AY = 0 \end{cases}$$

Solve the second equation for B:

$$BX = AY \Rightarrow B = \frac{AY}{X}$$

Substitute into the first equation:

$$AX + \left(\frac{AY}{X}\right)Y = V_0 \Rightarrow A\left(X + \frac{Y^2}{X}\right) = V_0 \Rightarrow A = \frac{V_0 X}{X^2 + Y^2}$$

Then:

$$B = \frac{AY}{X} = \frac{V_0 Y}{X^2 + Y^2}$$

Final Particular Solution

Thus, the particular solution is:

$$Q_{\text{part}}(t) = \frac{V_0 X}{X^2 + Y^2}\cos(\omega_d t) + \frac{V_0 Y}{X^2 + Y^2}\sin(\omega_d t)$$

$$\text{where } X = \left(\frac{1}{C} - L\omega_d^2\right), \quad Y = R\omega_d$$

This can also be written in amplitude-phase form:

$$Q_{\text{part}}(t) = Q_p \cos(\omega_d t - \delta)$$

with:

$$Q_p = \frac{V_0}{\sqrt{X^2 + Y^2}} = \frac{V_0}{\sqrt{\left(\frac{1}{C} - L\omega_d^2\right)^2 + (R\omega_d)^2}}$$

$$\tan\delta = \frac{Y}{X} = \frac{R\omega_d}{\frac{1}{C} - L\omega_d^2}$$

Voltage Across the Capacitor

The voltage across the capacitor is:

$$V_C(t) = \frac{Q_{\text{part}}(t)}{C} = \frac{Q_p}{C}\cos(\omega_d t - \delta)$$

Final expression:

$$\boxed{V_C(t) = \frac{V_0}{\sqrt{(1 - LC\omega_d^2)^2 + (RC\omega_d)^2}}\cos(\omega_d t - \delta)} \quad \text{with} \quad \tan\delta = \frac{R\omega_d}{\frac{1}{C} - L\omega_d^2}$$