Notes: Week of Jul 14

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1 Analytic derivatation of Lyop-Exp

1.1 Hamiltonian Dynamics

Our basis:

$$S_j^x = s_j, S_j^y = f(s_j)\cos(\phi_j), S_j^z = f(s_j)\sin(\phi_j)$$

With $f(x) := \sqrt{1-x^2}$.
The hamiltonian is

$$H = \sum_j (-\vec{J} \circ \vec{S_{j+1}}) \cdot \vec{S_j}$$

The only terms in H involving j values are

$$\begin{aligned}(H)_j = &- J_x(s_j s_{j-1} + s_j s_{j+1}) \\ &- J_y f(s_j)\cos\phi_j[f(s_{j-1})\cos\phi_{j-1} + f(s_{j+1})\cos\phi_{j+1}] \\ &- J_z f(s_j)\sin\phi_j[f(s_{j-1})\sin\phi_{j-1} + f(s_{j+1})\sin\phi_{j+1}]\end{aligned}$$

The cononical position is

$$s_j$$

The cononical momentum is

$$\phi_j$$

Thus

$$\begin{aligned}\dot{s_j} &= \frac{\partial H}{\partial \phi_j} \\ \dot{\phi_j} &= -\frac{\partial H}{\partial s_j}\end{aligned}$$

and so we have

$$\begin{aligned}\frac{ds_j}{dt} = &+ J_y f(s_j)\sin(\phi_j)[f(s_{j+1})\cos(\phi_{j+1}) + f(s_{j-1})\cos(\phi_{j-1})] \\ &- J_z f(s_j)\cos(\phi_j)[f(s_{j+1})\sin(\phi_{j+1}) + f(s_{j-1})\sin(\phi_{j-1})]\end{aligned}$$

$$\begin{aligned}\frac{d\phi_j}{dt} = -f'(s_j)[&- J_y \cos\phi_j (f(s_{j+1})\cos\phi_{j+1} + f(s_{j-1})\cos\phi_{j-1}) \\ &- J_z \sin\phi_j (f(s_{j+1})\sin\phi_{j+1} + f(s_{j-1})\sin\phi_{j-1})] \\ &+ J_x(s_{j+1} + s_{j-1})\end{aligned}$$

1.2 First order Expansion of functions

1.2.1 $f(x)$ and $f'(x)$

Let

$$f(x) = \sqrt{1-x^2}$$

We expand $f(x_0 + \delta x)$ to leading order in δx using a Taylor expansion:

$$f(x_0 + \delta x) \approx f(x_0) + f'(x_0)\,\delta x$$

The derivative:

$$f'(x) = \frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{-x}{\sqrt{1-x^2}}$$

Then:

$$f'(x_0) = \frac{-x_0}{\sqrt{1-x_0^2}}$$

Thus:

$$f(x_0 + \delta x) \approx \sqrt{1-x_0^2} - \frac{x_0}{\sqrt{1-x_0^2}}\,\delta x$$

For $x_0 = 0$,

$$\boxed{f(x_0 + \delta x) \approx 1}$$

Simiarly

$$\boxed{f'(x_0 + \delta x) \approx -\delta x}$$

1.2.2 $cos\phi_j$ and $sin\phi_j$

Let

$$\phi_j = \phi_j^0 + \delta\phi_j, \quad \text{where} \quad \phi_j^0 = \frac{2\pi}{N}j$$

Using the expansion:

$$\cos(a+\epsilon) \approx \cos a - \sin a \cdot \epsilon$$

We have

$$\boxed{\cos\phi_j \approx \cos(\phi_j^0) - \sin(\phi_j^0)\delta\phi_j}$$

Simiarly using the expansion:

$$\sin(a+\epsilon) \approx \sin a + \cos a \cdot \epsilon$$

We have

$$\boxed{\sin\phi_j \approx \sin(\phi_j^0) + \cos(\phi_j^0)\delta\phi_j}$$

1.2.3 $cos(\phi_{j+1} - \phi_j)$

Let

$$\phi_j = \phi_j^0 + \delta\phi_j, \quad \text{where} \quad \phi_j^0 = \frac{2\pi}{N}j$$

with $N \in \mathbb{Z}$. Then:

$$\phi_{j+1} - \phi_j = \left(\phi_{j+1}^0 - \phi_j^0\right) + (\delta\phi_{j+1} - \delta\phi_j) = \frac{2\pi}{N} + (\delta\phi_{j+1} - \delta\phi_j)$$

We expand the cosine to first order in $\delta\phi_j$ and $\delta\phi_{j+1}$:

$$\cos(\phi_{j+1} - \phi_j) = \cos\left(\frac{2\pi}{N} + \delta\phi_{j+1} - \delta\phi_j\right)$$

Using the expansion:

$$\cos(a+\epsilon) \approx \cos a - \sin a \cdot \epsilon$$

with $a = \frac{2\pi}{N}$ and $\epsilon = \delta\phi_{j+1} - \delta\phi_j$, we obtain:

$$\boxed{\cos(\phi_{j+1} - \phi_j) \approx \cos\left(\frac{2\pi}{N}\right) - \sin\left(\frac{2\pi}{N}\right)(\delta\phi_{j+1} - \delta\phi_j) + O(\delta\phi^2)}$$

1.2.4 $sin(\phi_{j+1} - \phi_j)$

Just like before

$$\boxed{\sin(\phi_{j+1} - \phi_j) \approx \sin\left(\frac{2\pi}{N}\right) + \cos\left(\frac{2\pi}{N}\right)(\delta\phi_{j+1} - \delta\phi_j) + O(\delta\phi^2)}$$

1.3 First order approximation of diffrential equations

Let

$$s_j = s_j^0 + \delta s_j = \delta s_j \quad \phi_j = \phi_j^0 + \delta\phi_j$$

Recall the equations of motion in our chosen basis

$$\begin{aligned}\frac{ds_j}{dt} = \frac{d\delta s_j}{dt} = &+ J_y f(s_j)\sin(\phi_j)[f(s_{j+1})\cos(\phi_{j+1}) + f(s_{j-1})\cos(\phi_{j-1})] \\ &- J_z f(s_j)\cos(\phi_j)[f(s_{j+1})\sin(\phi_{j+1}) + f(s_{j-1})\sin(\phi_{j-1})]\end{aligned}$$

$$\begin{aligned}\frac{d\phi_j}{dt} = \frac{d\delta\phi_j}{dt} = -f'(s_j)[&- J_y \cos\phi_j(f(s_{j+1})\cos\phi_{j+1} + f(s_{j-1})\cos\phi_{j-1}) \\ &- J_z \sin\phi_j(f(s_{j+1})\sin\phi_{j+1} + f(s_{j-1})\sin\phi_{j-1})] \\ &+ J_x(s_{j+1} + s_{j-1})\end{aligned}$$

Applying first order approx to $\frac{d\delta s_j}{dt}$

$$\begin{aligned}\frac{d\delta s_j}{dt} \approx &+ J_y(\sin\phi_j^0 + \delta\phi_j\cos\phi_j^0)[\cos\phi_{j+1}^0 - \delta\phi_{j+1}\sin\phi_{j+1}^0 + \cos\phi_{j-1}^0 - \delta\phi_{j+1}\sin\phi_{j-1}^0] \\ &- J_z(\cos\phi_j^0 - \delta\phi_j\sin\phi_j^0)[\sin\phi_{j+1}^0 + \delta\phi_{j+1}\cos\phi_{j+1}^0 + \sin\phi_{j-1}^0 + \delta\phi_{j-1}\cos\phi_{j-1}^0]\end{aligned}$$

Which, only keeping first oder of $\delta\phi$ becomes

$$\begin{aligned}\frac{d\delta s_j}{dt} \approx &+ J_y[\sin\phi_j^0\cos\phi_{j+1}^0 - \delta\phi_{j+1}\sin\phi_j^0\sin\phi_{j+1}^0 + \sin\phi_j^0\cos\phi_{j-1}^0 - \delta\phi_{j+1}\sin\phi_j^0\sin\phi_{j-1}^0 \\ &+ \delta\phi_j\cos\phi_j^0\cos\phi_{j+1}^0 + \delta\phi_j\cos\phi_j^0\cos\phi_{j-1}^0] \\ &- J_z[\cos\phi_j^0\sin\phi_{j+1}^0 + \delta\phi_{j+1}\cos\phi_j^0\cos\phi_{j+1}^0 + \cos\phi_j^0\sin\phi_{j-1}^0 + \delta\phi_{j-1}\cos\phi_j^0\cos\phi_{j-1}^0 \\ &- \delta\phi_j\sin\phi_j^0\sin\phi_{j+1}^0 - \delta\phi_j\sin\phi_j^0\sin\phi_{j-1}^0]\end{aligned}$$

1.3.1 For $J_y = J_z = J$

$$\begin{aligned}\frac{d\delta s_j}{dt} \approx J[&\sin(\phi_j^0 - \phi_{j+1}^0) + \sin(\phi_j^0 - \phi_{j+1}^0) - \cos(\phi_j^0 - \phi_{j+1}^0)\delta\phi_{j+1} \\ &- \cos(\phi_j^0 - \phi_{j-1}^0)\delta\phi_{j-1} + \cos(\phi_j^0 - \phi_{j+1}^0)\delta\phi_j + \cos(\phi_j^0 - \phi_{j+1}^0)\delta\phi_j]\end{aligned}$$

Recalling that $\phi_j^0 = \frac{2\pi}{N}j$ this becomes

$$\frac{d\delta s_j}{dt} \approx J\cos(\frac{2\pi}{N})[2\delta\phi_j - \delta\phi_{j+1} - \delta\phi_{j-1}]$$

Now we can actually make a lot of headway for $\frac{d\phi_j}{dt}$ with no approximations (just by using $J_y = J_z = J$):

$$\begin{aligned}\frac{d\phi_j}{dt} = f'(s_j)J[f(s_{j+1})(\cos\phi_j\cos\phi_{j+1} + \sin\phi_j\sin\phi_{j+1})\\ + f(s_{j-1})(\cos\phi_j\cos\phi_{j-1} + \sin\phi_j\sin\phi_{j-1})]\\ + J_x(s_{j+1} + s_{j-1})\end{aligned}$$

Using the well known trig identities this becomes

$$\begin{aligned}\frac{d\phi_j}{dt} = f'(s_j)J[f(s_{j+1})\cos(\phi_j - \phi_{j+1}) + f(s_{j-1})\cos(\phi_j - \phi_{j-1})]\\ + J_x(s_{j+1} + s_{j-1})\end{aligned}$$

Now finally remember that we are using $\phi_j - \phi_{j\pm 1} = \mp\frac{2\pi}{N}$ and apply a first order approximation to get

$$\frac{d\phi_j}{dt} \approx -J\delta s_j 2\cos(\frac{2\pi}{N}) + J_x(\delta s_{j+1} + \delta s_{j-1})$$

In conclusion if we just choose $J_y = J_z = J$ we get

$$\boxed{\frac{d\delta s_j}{dt} \approx J\cos(\frac{2\pi}{N})[2\delta\phi_j - \delta\phi_{j+1} - \delta\phi_{j-1}]}$$

and

$$\boxed{\frac{d\phi_j}{dt} \approx -J\delta s_j 2\cos(\frac{2\pi}{N}) + J_x(\delta s_{j+1} + \delta s_{j-1})}$$

I propose we don't do random J_y, and we just do random J_x.

2 Harmonic Oscillators to Model Spin

Consider a series RLC circuit driven by an AC voltage source:

$$V_{\text{drive}}(t) = V_0\cos(\omega_d t)$$

The circuit contains:

- Inductor with inductance L
- Resistor with resistance R
- Capacitor with capacitance C

2.1 Equation of Motion

$$V_L + V_R + V_C = V_{\text{drive}}(t)$$

$$\boxed{L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V_0\cos(\omega_d t)}$$

2.2 Solution Structure

The general solution is:

$$Q(t) = Q_{\text{hom}}(t) + Q_{\text{part}}(t)$$

The homogeneous solution describes transient behavior and decays over time due to resistance with time scale.

$$\tau = -\frac{2L}{R}$$

.

As in

$$Q_{\text{hom}}(t) \sim e^{-\frac{Rt}{2L}}$$

I seek a steady state solution, so I feel like we should try something like:

$$Q_{\text{part}}(t) = A\cos(\omega_d t) + B\sin(\omega_d t)$$

2.3 Determaning Particular Solution Ansatz

Compute the first and second derivatives:

$$\dot{Q}_{\text{part}}(t) = -A\omega_d\sin(\omega_d t) + B\omega_d\cos(\omega_d t)$$

$$\ddot{Q}_{\text{part}}(t) = -A\omega_d^2\cos(\omega_d t) - B\omega_d^2\sin(\omega_d t)$$

Substitute into the differential equation:

$$\begin{aligned}
&L\ddot{Q}_{\text{part}} + R\dot{Q}_{\text{part}} + \frac{1}{C}Q_{\text{part}} = V_0\cos(\omega_d t)\\
&= L(-A\omega_d^2\cos(\omega_d t) - B\omega_d^2\sin(\omega_d t))\\
&\quad + R(-A\omega_d\sin(\omega_d t) + B\omega_d\cos(\omega_d t))\\
&\quad + \frac{1}{C}(A\cos(\omega_d t) + B\sin(\omega_d t))
\end{aligned}$$

Group terms:

$$\text{Coefficient of } \cos(\omega_d t): \quad -LA\omega_d^2 + RB\omega_d + \frac{A}{C}$$

$$\text{Coefficient of } \sin(\omega_d t): \quad -LB\omega_d^2 - RA\omega_d + \frac{B}{C}$$

Set the equation equal to the driving term $V_0 \cos(\omega_d t)$, and match coefficients:

$$\begin{cases} -LA\omega_d^2 + RB\omega_d + \frac{A}{C} = V_0 \\ -LB\omega_d^2 - RA\omega_d + \frac{B}{C} = 0 \end{cases}$$

Solve the System

We now solve this linear system for A and B.

Define:

$$X := \left(\frac{1}{C} - L\omega_d^2\right), \quad Y := R\omega_d$$

Then the system becomes:

$$\begin{cases} AX + BY = V_0 \\ BX - AY = 0 \end{cases}$$

Solve the second equation for B:

$$BX = AY \Rightarrow B = \frac{AY}{X}$$

Substitute into the first equation:

$$AX + \left(\frac{AY}{X}\right) Y = V_0 \Rightarrow A\left(X + \frac{Y^2}{X}\right) = V_0 \Rightarrow A = \frac{V_0 X}{X^2 + Y^2}$$

Then:

$$B = \frac{AY}{X} = \frac{V_0 Y}{X^2 + Y^2}$$

Final Particular Solution

Thus, the particular solution is:

$$Q_{\text{part}}(t) = \frac{V_0 X}{X^2 + Y^2} \cos(\omega_d t) + \frac{V_0 Y}{X^2 + Y^2} \sin(\omega_d t)$$

$$\text{where } X = \left(\frac{1}{C} - L\omega_d^2\right), \quad Y = R\omega_d$$

This can also be written in amplitude-phase form:

$$Q_{\text{part}}(t) = Q_p \cos(\omega_d t - \delta)$$

with:

$$Q_p = \frac{V_0}{\sqrt{X^2 + Y^2}} = \frac{V_0}{\sqrt{\left(\frac{1}{C} - L\omega_d^2\right)^2 + (R\omega_d)^2}}$$

$$\tan\delta = \frac{Y}{X} = \frac{R\omega_d}{\frac{1}{C} - L\omega_d^2}$$

Voltage Across the Capacitor

The voltage across the capacitor is:

$$V_C(t) = \frac{Q_{\text{part}}(t)}{C} = \frac{Q_p}{C}\cos(\omega_d t - \delta)$$

Final expression:

$$\boxed{V_C(t) = \frac{V_0}{\sqrt{(1 - LC\omega_d^2)^2 + (RC\omega_d)^2}}\cos(\omega_d t - \delta)} \quad \text{with} \quad \tan\delta = \frac{R\omega_d}{\frac{1}{C} - L\omega_d^2}$$