

Notes: Week of Jul 21

Elisha Shmalo

July 25, 2025

1 Derivation of Equations of Motion

1.1 Hamiltonian Dynamics

Our basis:

$$S_j^x = s_j, S_j^y = f(s_j) \cos(\phi_j), S_j^z = f(s_j) \sin(\phi_j)$$

With $f(x) := \sqrt{1 - x^2}$.

The hamiltonian is

$$H = \sum_j (-\vec{J} \circ S_{j+1}) \cdot \vec{S}_j$$

The only terms in H involving j values are

$$\begin{aligned} (H)_j = & -J_x(s_j s_{j-1} + s_j s_{j+1}) \\ & -J_y f(s_j) \cos \phi_j [f(s_{j-1}) \cos \phi_{j-1} + f(s_{j+1}) \cos \phi_{j+1}] \\ & -J_z f(s_j) \sin \phi_j [f(s_{j-1}) \sin \phi_{j-1} + f(s_{j+1}) \sin \phi_{j+1}] \end{aligned}$$

The cononical position is

$$s_j$$

The cononical momentum is

$$\phi_j$$

Thus

$$\begin{aligned} \dot{s}_j &= \frac{\partial H}{\partial \phi_j} \\ \dot{\phi}_j &= -\frac{\partial H}{\partial s_j} \end{aligned}$$

and so we have

$$\begin{aligned}\frac{ds_j}{dt} = & + J_y f(s_j) \sin(\phi_j) [f(s_{j+1}) \cos(\phi_{j+1}) + f(s_{j-1}) \cos(\phi_{j-1})] \\ & - J_z f(s_j) \cos(\phi_j) [f(s_{j+1}) \sin(\phi_{j+1}) + f(s_{j-1}) \sin(\phi_{j-1})]\end{aligned}$$

$$\begin{aligned}\frac{d\phi_j}{dt} = & -f'(s_j) [-J_y \cos \phi_j (f(s_{j+1}) \cos \phi_{j+1} + f(s_{j-1}) \cos \phi_{j-1}) \\ & - J_z \sin \phi_j (f(s_{j+1}) \sin \phi_{j+1} + f(s_{j-1}) \sin \phi_{j-1})] \\ & + J_x (s_{j+1} + s_{j-1})\end{aligned}$$

1.2 If $J_y = J_z = J$

$$\begin{aligned}\frac{ds_j}{dt} = & J f(s_j) [f(s_{j+1}) (\sin \phi_j \cos \phi_{j+1} - \cos \phi_j \sin \phi_{j+1}) \\ & + f(s_{j-1}) (\sin \phi_j \cos \phi_{j-1} - \cos \phi_j \sin \phi_{j-1})]\end{aligned}$$

$$\begin{aligned}\frac{d\phi_j}{dt} = & J f'(s_j) [f(s_{j+1}) (\cos \phi_j \cos \phi_{j+1} + \sin \phi_j \sin \phi_{j+1}) \\ & + f(s_{j-1}) (\cos \phi_j \cos \phi_{j-1} + \sin \phi_j \sin \phi_{j-1})] \\ & + J_x (s_{j+1} + s_{j-1})\end{aligned}$$

which become

$$\begin{aligned}\frac{ds_j}{dt} = & J f(s_j) [f(s_{j+1}) \sin(\phi_j - \phi_{j+1}) \\ & + f(s_{j-1}) \sin(\phi_j - \phi_{j-1})]\end{aligned}$$

$$\begin{aligned}\frac{d\phi_j}{dt} = & J f'(s_j) [f(s_{j+1}) \cos(\phi_j - \phi_{j+1}) \\ & + f(s_{j-1}) \cos(\phi_j - \phi_{j-1})] \\ & + J_x (s_{j+1} + s_{j-1})\end{aligned}$$

1.3 Second order Expansion of functions

1.3.1 $f(x)$ and $f'(x)$

Let

$$f(x) = \sqrt{1 - x^2}$$

We expand $f(\delta x)$ using a Taylor expansion:

$$f(\delta x) \approx 1 + 0 - 1/2\delta x^2$$

The derivative:

$$f'(x) = \frac{d}{dx} \left(\sqrt{1-x^2} \right) = \frac{-x}{\sqrt{1-x^2}}$$

Then:

$$f'(0) = 0$$

$$f''(0) = -1$$

$$f'''(0) = 0$$

Thus:

$$f'(\delta x) \approx -\delta x$$

even up to second order.

1.3.2 Now $\cos(\phi_j - \phi_{j\pm 1})$ and $\sin \phi_j - \phi_{j\pm 1}$

Let

$$\phi_j = \phi_j^0 + \delta\phi_j, \quad \text{where} \quad \phi_j^0 = \frac{2\pi}{N}j$$

Let

$$x = \phi_j^0 - \phi_{j+1}^0, y = \phi_j^0 - \phi_{j-1}^0$$

Then

$$\cos(\phi_j - \phi_{j+1}) = \cos(x + \delta x)$$

and similar for the others.

Note

$$\cos(x) = \cos(y), \sin(x) = -\sin(y)$$

Now for the expansions:

$$\cos(x + \delta x) \approx \cos(x) - \sin(x)\delta x - 1/2 \cos(x)\delta x^2$$

and

$$\sin(x + \delta x) \approx \sin(x) + \cos(x)\delta x - 1/2 \sin(x)\delta x^2$$

1.4 Second order approximation of differential equations

1.4.1 First ds_j/dt

Recall

$$\begin{aligned}\frac{ds_j}{dt} = & Jf(s_j)[f(s_{j+1})\sin(\phi_j - \phi_{j+1}) \\ & + f(s_{j-1})\sin(\phi_j - \phi_{j-1})]\end{aligned}$$

Using the expansions

$$\begin{aligned}\frac{ds_j}{dt} \approx & J(1 - 1/2\delta s_j^2)[(1 - 1/2\delta s_{j+1}^2)(\sin(x) + \cos(x)\delta x - 1/2\sin(x)\delta x^2) \\ & + (1 - 1/2\delta s_{j-1}^2)(\sin(y) + \cos(y)\delta y - 1/2\sin(y)\delta y^2)]\end{aligned}$$

Only keeping up to order two inside the brackets and grouping like terms gives us

$$\frac{ds_j}{dt} \approx J(1 - \delta s_j^2)[\cos(x)(\delta x + \delta y) - 1/2\sin(x)(\delta x^2 - \delta y^2 + \delta s_{j+1}^2 - \delta s_{j-1}^2)]$$

Again we are only keeping up to second order in δ So the terms with $-\delta s_j^2$ don't survive and we are left with

$$\frac{ds_j}{dt} \approx J[\cos(x)(\delta x + \delta y) - 1/2\sin(x)(\delta x^2 - \delta y^2 + \delta s_{j+1}^2 - \delta s_{j-1}^2)]$$

Which is

$$\begin{aligned}\frac{ds_j}{dt} \approx & J[\cos(\frac{2\pi}{N})(2\delta\phi_j - \delta\phi_{j+1} - \delta\phi_{j-1}) \\ & + 1/2\sin(\frac{2\pi}{N})(\delta\phi_{j+1}^2 - \delta\phi_{j-1}^2 + 2\delta\phi_j(\delta\phi_{j+1} - \delta\phi_{j-1}) + \delta s_{j+1}^2 - \delta s_{j-1}^2)]\end{aligned}$$

1.4.2 Next $d\phi_j/dt$

Recall

$$\begin{aligned}\frac{d\phi_j}{dt} = & Jf'(s_j)[f(s_{j+1})\cos(\phi_j - \phi_{j+1}) \\ & + f(s_{j-1})\cos(\phi_j - \phi_{j-1})] \\ & + J_x(s_{j+1} + s_{j-1})\end{aligned}$$

using the expansions to second order

$$\begin{aligned}\frac{d\phi_j}{dt} \approx & -J\delta s_j[(1 - 1/2\delta s_{j+1}^2)(\cos(x) - \sin(x)\delta x - 1/2\cos(x)\delta x^2) \\ & + (1 - 1/2\delta s_{j+1}^2)(\cos(y) - \sin(y)\delta y - 1/2\cos(y)\delta y^2)] \\ & + J_x(s_{j+1} + s_{j-1})\end{aligned}$$

When we consider that we are only keeping things up to second order a lot of terms don't survive

$$\begin{aligned}\frac{d\phi_j}{dt} \approx & -J\delta s_j[(\cos(x) - \sin(x)\delta x) \\ & + (\cos(y) - \sin(y)\delta y)] \\ & + J_x(s_{j+1} + s_{j-1})\end{aligned}$$

Which simplifies to

$$\begin{aligned}\frac{d\phi_j}{dt} \approx & -J\delta s_j[2\cos(x) - \sin(x)(\delta x - \delta y)] \\ & + J_x(s_{j+1} + s_{j-1})\end{aligned}$$

which is

$$\begin{aligned}\frac{d\phi_j}{dt} \approx & -J\delta s_j[2\cos(\frac{2\pi}{N}) + \sin(\frac{2\pi}{N})(\delta\phi_{j-1} - \delta\phi_{j+1})] \\ & + J_x(s_{j+1} + s_{j-1})\end{aligned}$$

Thus we have that the equations of motion to second order are

$$\begin{aligned}\frac{ds_j}{dt} \approx & J[\cos(\frac{2\pi}{N})(2\delta\phi_j - \delta\phi_{j+1} - \delta\phi_{j-1}) \\ & + 1/2\sin(\frac{2\pi}{N})(\delta\phi_{j+1}^2 - \delta\phi_{j-1}^2 + 2\delta\phi_j(\delta\phi_{j+1} - \delta\phi_{j-1}) + \delta s_{j+1}^2 - \delta s_{j-1}^2)]\end{aligned}$$

$$\begin{aligned}\frac{d\phi_j}{dt} \approx & -J\delta s_j[2\cos(\frac{2\pi}{N}) + \sin(\frac{2\pi}{N})(\delta\phi_{j-1} - \delta\phi_{j+1})] \\ & + J_x(s_{j+1} + s_{j-1})\end{aligned}$$

2 Seeking Solitons

Try

$$s_j = f(z), z = j - ut$$

$$\phi_j = \frac{2\pi}{N} + g(z)$$

Such that

$$\delta s_j = f(z), z = j - ut$$

$$\delta \phi_j = g(z)$$

Rewriting the second order approximation for the equations of motion:

$$-uf'(z) \approx J[\cos(\frac{2\pi}{N})(2g(z) - g(z+1) - g(z-1))$$

$$+ 1/2 \sin(\frac{2\pi}{N})(g(z+1)^2 - g(z-1)^2 + 2g(z)(g(z+1) - g(z-1)) + f(z+1)^2 - f(z-1)^2)]$$

$$-ug'(z) \approx -Jf(z)[2\cos(\frac{2\pi}{N}) + \sin(\frac{2\pi}{N})(g(z-1) - g(z+1))]$$

$$+ J_x(f(z+1) + f(z-1))$$

If we take the continuum limit and a linear exapnsions

$$f(z \pm 1) \approx f(z) \pm f'(z)$$

$$g(z \pm 1) \approx g(z) \pm g'(z)$$

Then we get the follwing relations

$$2g(z) - g(z+1) - g(z-1) \approx 0$$

$$g(z-1) - g(z+1) \approx -2g'$$

$$f(z-1) + f(z+1) \approx 2f$$

$$g(z)(g(z+1) - g(z-1)) \approx 2gg'$$

$$g(z+1)^2 - g(z-1)^2 = (g(z+1) + g(z-1))(g(z+1) - g(z-1)) \approx 2g2g' = 4gg'$$

$$f(z+1)^2 - f(z-1)^2 \approx 4ff'$$

(where I've written $g(z) = g$ and $f(z) = f$ for brevity)

Using these (and defining $\theta = \frac{2\pi}{N}$), our differential equations become

$$-uf' \approx J[0 + 1/2 \sin(\theta)(4gg' + 4gg' + 4ff')]$$

$$= 2J \sin(\theta)[2gg' + ff']$$

$$-ug' \approx -Jf[2\cos(\theta) - 2\sin(\theta)g'] + 2J_x f$$

Rearranging these become

$$f' \approx -4J\sin(\theta)\frac{gg'}{u + 2J\sin(\theta)f}$$

$$g' \approx \frac{2(J\cos(\theta) - J_x)}{u + 2J\sin(\theta)f}f$$

3 Harmonic Oscillators to Model Spin

Consider a series RLC circuit driven by an AC voltage source:

$$V_{\text{drive}}(t) = V_0 \cos(\omega_d t)$$

The circuit contains:

- Inductor with inductance L
- Resistor with resistance R
- Capacitor with capacitance C

3.1 Equation of Motion

$$V_L + V_R + V_C = V_{\text{drive}}(t)$$

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V_0 \cos(\omega_d t)$$

3.2 Solution Structure

The general solution is:

$$Q(t) = Q_{\text{hom}}(t) + Q_{\text{part}}(t)$$

The homogeneous solution describes transient behavior and decays over time due to resistance with time scale.

$$\tau = -\frac{2L}{R}$$

As in

$$Q_{\text{hom}}(t) \sim e^{-\frac{Rt}{2L}}$$

I seek a steady state solution, so I feel like we should try something like:

$$Q_{\text{part}}(t) = A \cos(\omega_d t) + B \sin(\omega_d t)$$

3.3 Determining Particular Solution Ansatz

Compute the first and second derivatives:

$$\dot{Q}_{\text{part}}(t) = -A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t)$$

$$\ddot{Q}_{\text{part}}(t) = -A\omega_d^2 \cos(\omega_d t) - B\omega_d^2 \sin(\omega_d t)$$

Substitute into the differential equation:

$$\begin{aligned} L\ddot{Q}_{\text{part}} + R\dot{Q}_{\text{part}} + \frac{1}{C}Q_{\text{part}} &= V_0 \cos(\omega_d t) \\ &= L(-A\omega_d^2 \cos(\omega_d t) - B\omega_d^2 \sin(\omega_d t)) \\ &\quad + R(-A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t)) \\ &\quad + \frac{1}{C}(A \cos(\omega_d t) + B \sin(\omega_d t)) \end{aligned}$$

Group terms:

$$\text{Coefficient of } \cos(\omega_d t) : \quad -LA\omega_d^2 + RB\omega_d + \frac{A}{C}$$

$$\text{Coefficient of } \sin(\omega_d t) : \quad -LB\omega_d^2 - RA\omega_d + \frac{B}{C}$$

Set the equation equal to the driving term $V_0 \cos(\omega_d t)$, and match coefficients:

$$\begin{cases} -LA\omega_d^2 + RB\omega_d + \frac{A}{C} = V_0 \\ -LB\omega_d^2 - RA\omega_d + \frac{B}{C} = 0 \end{cases}$$

Solve the System

We now solve this linear system for A and B .

Define:

$$X := \left(\frac{1}{C} - L\omega_d^2 \right), \quad Y := R\omega_d$$

Then the system becomes:

$$\begin{cases} AX + BY = V_0 \\ BX - AY = 0 \end{cases}$$

Solve the second equation for B :

$$BX = AY \Rightarrow B = \frac{AY}{X}$$

Substitute into the first equation:

$$AX + \left(\frac{AY}{X} \right) Y = V_0 \Rightarrow A \left(X + \frac{Y^2}{X} \right) = V_0 \Rightarrow A = \frac{V_0 X}{X^2 + Y^2}$$

Then:

$$B = \frac{AY}{X} = \frac{V_0 Y}{X^2 + Y^2}$$

Final Particular Solution

Thus, the particular solution is:

$$Q_{\text{part}}(t) = \frac{V_0 X}{X^2 + Y^2} \cos(\omega_d t) + \frac{V_0 Y}{X^2 + Y^2} \sin(\omega_d t)$$

$$\text{where } X = \left(\frac{1}{C} - L\omega_d^2 \right), \quad Y = R\omega_d$$

This can also be written in amplitude-phase form:

$$Q_{\text{part}}(t) = Q_p \cos(\omega_d t - \delta)$$

with:

$$Q_p = \frac{V_0}{\sqrt{X^2 + Y^2}} = \frac{V_0}{\sqrt{\left(\frac{1}{C} - L\omega_d^2\right)^2 + (R\omega_d)^2}}$$
$$\tan \delta = \frac{Y}{X} = \frac{R\omega_d}{\frac{1}{C} - L\omega_d^2}$$

Voltage Across the Capacitor

The voltage across the capacitor is:

$$V_C(t) = \frac{Q_{\text{part}}(t)}{C} = \frac{Q_p}{C} \cos(\omega_d t - \delta)$$

Final expression:

$$\boxed{V_C(t) = \frac{V_0}{\sqrt{(1 - LC\omega_d^2)^2 + (RC\omega_d)^2}} \cos(\omega_d t - \delta)} \quad \text{with} \quad \tan \delta = \frac{R\omega_d}{\frac{1}{C} - L\omega_d^2}$$