

Notes: Week of June 23

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1 Analytic derivatation of Lyop-Exp

1.1 Hamiltonian Dynamics

Our basis:

$$S_j^x = s_j, S_j^y = f(s_j) \cos(\phi_j), S_j^z = f(s_j) \sin(\phi_j)$$

With $f(x) := \sqrt{1 - x^2}$.

The hamiltonian is

$$H = \sum_j (-\vec{J} \circ S_{j+1}) \cdot \vec{S}_j$$

The only terms in H involving j values are

$$\begin{aligned} (H)_j = & -J_x(s_j s_{j-1} + s_j s_{j+1}) \\ & -J_y f(s_j) \cos \phi_j [f(s_{j-1}) \cos \phi_{j-1} + f(s_{j+1}) \cos \phi_{j+1}] \\ & -J_z f(s_j) \sin \phi_j [f(s_{j-1}) \sin \phi_{j-1} + f(s_{j+1}) \sin \phi_{j+1}] \end{aligned}$$

The cononical position is

$$s_j$$

The cononical momentum is

$$\phi_j$$

Thus

$$\begin{aligned} \dot{s}_j &= \frac{\partial H}{\partial \phi_j} \\ \dot{\phi}_j &= -\frac{\partial H}{\partial s_j} \end{aligned}$$

and so we have

$$\begin{aligned} \frac{ds_j}{dt} = & +J_y f(s_j) \sin(\phi_j) [f(s_{j+1}) \cos(\phi_{j+1}) + f(s_{j-1}) \cos(\phi_{j-1})] \\ & -J_z f(s_j) \cos(\phi_j) [f(s_{j+1}) \sin(\phi_{j+1}) + f(s_{j-1}) \sin(\phi_{j-1})] \end{aligned}$$

$$\begin{aligned} \frac{d\phi_j}{dt} = & -f'(s_j) [-J_y \cos \phi_j (f(s_{j+1}) \cos \phi_{j+1} + f(s_{j-1}) \cos \phi_{j-1}) \\ & -J_z \sin \phi_j (f(s_{j+1}) \sin \phi_{j+1} + f(s_{j-1}) \sin \phi_{j-1})] \\ & -J_x (s_{j+1} + s_{j-1}) \end{aligned}$$

1.2 First order Expansion of functions

1.2.1 $f(x)$ and $f'(x)$

Let

$$f(x) = \sqrt{1 - x^2}$$

We expand $f(x_0 + \delta x)$ to leading order in δx using a Taylor expansion:

$$f(x_0 + \delta x) \approx f(x_0) + f'(x_0) \delta x$$

The derivative:

$$f'(x) = \frac{d}{dx} (\sqrt{1 - x^2}) = \frac{-x}{\sqrt{1 - x^2}}$$

Then:

$$f'(x_0) = \frac{-x_0}{\sqrt{1 - x_0^2}}$$

Thus:

$$f(x_0 + \delta x) \approx \sqrt{1 - x_0^2} - \frac{x_0}{\sqrt{1 - x_0^2}} \delta x$$

For $x_0 = 0$,

$$\boxed{f(x_0 + \delta x) \approx 1}$$

Simiarly

$$\boxed{f'(x_0 + \delta x) \approx -\delta x}$$

1.2.2 $\cos\phi_j$ and $\sin\phi_j$

Let

$$\phi_j = \phi_j^0 + \delta\phi_j, \quad \text{where} \quad \phi_j^0 = \frac{2\pi}{N}j$$

Using the expansion:

$$\cos(a + \epsilon) \approx \cos a - \sin a \cdot \epsilon$$

We have

$$\boxed{\cos \phi_j \approx \cos\left(\frac{2\pi}{N}j\right) - \sin\left(\frac{2\pi}{N}j\right)\delta\phi_j}$$

For $N = 4$

$$\cos \phi_j \approx \begin{cases} \mp \delta\phi_j, & j \text{ odd} \\ \pm 1, & j \text{ even} \end{cases}$$

Simiarly using the expansion:

$$\sin(a + \epsilon) \approx \sin a + \cos a \cdot \epsilon$$

We have

$$\boxed{\sin \phi_j \approx \sin\left(\frac{2\pi}{N}j\right) + \cos\left(\frac{2\pi}{N}j\right)\delta\phi_j}$$

For $N = 4$

$$\sin \phi_j \approx \begin{cases} \pm 1, & j \text{ odd} \\ \pm \delta\phi_j, & j \text{ even} \end{cases}$$

1.2.3 $\cos(\phi_{j+1} - \phi_j)$

In the notes I have:

$$“\cos(\phi_{j+1} - \phi_j) \approx 1 + O(\delta\phi^2),” N = 4$$

Let

$$\phi_j = \phi_j^0 + \delta\phi_j, \quad \text{where} \quad \phi_j^0 = \frac{2\pi}{N}j$$

with $N \in \mathbb{Z}$. Then:

$$\phi_{j+1} - \phi_j = (\phi_{j+1}^0 - \phi_j^0) + (\delta\phi_{j+1} - \delta\phi_j) = \frac{2\pi}{N} + (\delta\phi_{j+1} - \delta\phi_j)$$

We expand the cosine to first order in $\delta\phi_j$ and $\delta\phi_{j+1}$:

$$\cos(\phi_{j+1} - \phi_j) = \cos\left(\frac{2\pi}{N} + \delta\phi_{j+1} - \delta\phi_j\right)$$

Using the expansion:

$$\cos(a + \epsilon) \approx \cos a - \sin a \cdot \epsilon$$

with $a = \frac{2\pi}{N}$ and $\epsilon = \delta\phi_{j+1} - \delta\phi_j$, we obtain:

$$\cos(\phi_{j+1} - \phi_j) \approx \cos\left(\frac{2\pi}{N}\right) - \sin\left(\frac{2\pi}{N}\right)(\delta\phi_{j+1} - \delta\phi_j) + O(\delta\phi^2)$$

If $N = 4$,

$$\cos(\phi_{j+1} - \phi_j) \approx -(\delta\phi_{j+1} - \delta\phi_j)$$

1.2.4 $\sin(\phi_{j+1} - \phi_j)$

Let

$$\phi_j = \phi_j^0 + \delta\phi_j, \quad \text{where} \quad \phi_j^0 = \frac{2\pi}{N}j$$

Then:

$$\phi_{j+1} - \phi_j = \frac{2\pi}{N} + (\delta\phi_{j+1} - \delta\phi_j)$$

We expand the sine to first order in $\delta\phi_j$ and $\delta\phi_{j+1}$:

$$\sin(\phi_{j+1} - \phi_j) = \sin\left(\frac{2\pi}{N} + \delta\phi_{j+1} - \delta\phi_j\right)$$

Using the Taylor expansion:

$$\sin(a + \epsilon) \approx \sin a + \cos a \cdot \epsilon$$

with $a = \frac{2\pi}{N}$ and $\epsilon = \delta\phi_{j+1} - \delta\phi_j$, we obtain:

$$\sin(\phi_{j+1} - \phi_j) \approx \sin\left(\frac{2\pi}{N}\right) + \cos\left(\frac{2\pi}{N}\right)(\delta\phi_{j+1} - \delta\phi_j) + O(\delta\phi^2)$$

If $N = 4$, Then

$$\sin(\phi_{j+1} - \phi_j) \approx 1 + O(\delta\phi^2)$$

1.3 First order approximation of differential equations

Let

$$s_j = s_j^0 + \delta s_j = \delta s_j \quad \phi_j = \phi_j^0 + \delta \phi_j$$

Recall the equations of motion in our chosen basis

$$\begin{aligned} \frac{ds_j}{dt} = \frac{d\delta s_j}{dt} = & + J_y f(s_j) \sin(\phi_j) [f(s_{j+1}) \cos(\phi_{j+1}) + f(s_{j-1}) \cos(\phi_{j-1})] \\ & - J_z f(s_j) \cos(\phi_j) [f(s_{j+1}) \sin(\phi_{j+1}) + f(s_{j-1}) \sin(\phi_{j-1})] \end{aligned}$$

$$\begin{aligned} \frac{d\phi_j}{dt} = \frac{d\delta \phi_j}{dt} = & -f'(s_j) [-J_y \cos \phi_j (f(s_{j+1}) \cos \phi_{j+1} + f(s_{j-1}) \cos \phi_{j-1}) \\ & - J_z \sin \phi_j (f(s_{j+1}) \sin \phi_{j+1} + f(s_{j-1}) \sin \phi_{j-1})] \\ & - J_x (s_{j+1} + s_{j-1}) \end{aligned}$$

1.3.1 N = 4

Using the approximations we had above

$$\begin{aligned} \frac{d\delta s_j}{dt} &= \begin{cases} J_y(\pm 1)[\mp 1 \pm 1] - J_z(\mp \delta \phi_j)[\mp \delta \phi_{j+1} \pm \delta \phi_{j-1}], j \text{ odd} \\ J_y(\pm \delta \phi_j)[\mp \delta \phi_{j+1} \pm \delta \phi_{j-1}] - J_z(\pm 1)[\mp 1 \pm 1], j \text{ even} \end{cases} \\ &= \delta \phi_j \begin{cases} J_z[\mp \delta \phi_{j+1} \pm \delta \phi_{j-1}], j \text{ odd} \\ J_y[\mp \delta \phi_{j+1} \pm \delta \phi_{j-1}], j \text{ even} \end{cases} \end{aligned}$$

$$\frac{d\delta \phi_j}{dt} = -J_x(s_{j+1} + s_{j-1}) \mp \begin{cases} J_z(\mp \phi_{j+1} \pm \phi_{j-1}), j \text{ odd} \\ J_y(\pm \phi_{j+1} \pm \phi_{j-1}), j \text{ even} \end{cases}$$

2 Numerics for Lyop Calc

Comparison of Pushback Implementations in Benettin's Method

The goal of the `push_back` function in Benettin's algorithm is to restore the distance between two nearby trajectories, \vec{S}_A (reference) and \vec{S}_B (perturbed), to a fixed small value ε , without altering the direction of separation.

My Original Implementation

In the original Julia implementation, the perturbation is applied *locally* and independently to each spin:

$$\vec{S}_i^B \leftarrow \vec{S}_i^A + \varepsilon \cdot \frac{\vec{S}_i^B - \vec{S}_i^A}{\|\vec{S}_i^B - \vec{S}_i^A\|}$$

for all $i = 1, \dots, L$.

That is, each spin is pushed away from its counterpart independently, ensuring that the local separation at each site is ε . However, this results in a total perturbation vector (in \mathbb{R}^{3L}) with norm approximately:

$$\|\vec{S}_B - \vec{S}_A\| \approx \varepsilon \cdot \sqrt{L}$$

I tried adjusting the pushback to be

$$\vec{S}_i^B \leftarrow 1/\sqrt{L}(\vec{S}_i^A + \varepsilon \cdot \frac{\vec{S}_i^B - \vec{S}_i^A}{\|\vec{S}_i^B - \vec{S}_i^A\|})$$

such that the total perturbation vector has norm

$$\|\vec{S}_B - \vec{S}_A\| \approx \varepsilon$$

but that still didn't fix things.

Professor Pixely's Implementation (Benettin-consistent)

In contrast, the correct implementation applies the perturbation to the entire spin chain *as a single vector* in \mathbb{R}^{3L} :

$$\vec{S}_B \leftarrow \vec{S}_A + \varepsilon \cdot \frac{\vec{S}_B - \vec{S}_A}{\|\vec{S}_B - \vec{S}_A\|}$$

This ensures that the total perturbation vector has norm exactly ε .

Note: I spoke with chatGPT and it also said that the correct implementation insured that the direction of perturbation *is preserved* across steps. This consistency is essential for correctly capturing the dominant Lyapunov exponent. But I am not sure what is meant by that.

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n\tau} \sum_{i=1}^n \log \left(\frac{\|\vec{S}_B^{(i)} - \vec{S}_A^{(i)}\|}{\varepsilon} \right)$$

Summary of Differences

Feature	Original (local)	Professor's (global)
Normalization per spin	Yes	No
Preserves global direction	No	Yes
Total perturbation norm	$\varepsilon \cdot \sqrt{L}$	ε
Valid for Benettin method	Not quite	Yes

3 Harmonic Oscillators to Model Spin

Consider a series RLC circuit driven by an AC voltage source:

$$V_{\text{drive}}(t) = V_0 \cos(\omega_d t)$$

The circuit contains:

- Inductor with inductance L
- Resistor with resistance R
- Capacitor with capacitance C

3.1 Equation of Motion

$$V_L + V_R + V_C = V_{\text{drive}}(t)$$

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V_0 \cos(\omega_d t)$$

3.2 Solution Structure

The general solution is:

$$Q(t) = Q_{\text{hom}}(t) + Q_{\text{part}}(t)$$

The homogeneous solution describes transient behavior and decays over time due to resistance with time scale.

$$\tau = -\frac{2L}{R}$$

As in

$$Q_{\text{hom}}(t) \sim e^{-\frac{Rt}{2L}}$$

I seek a steady state solution, so I feel like we should try something like:

$$Q_{\text{part}}(t) = A \cos(\omega_d t) + B \sin(\omega_d t)$$

3.3 Determining Particular Solution Ansatz

Compute the first and second derivatives:

$$\dot{Q}_{\text{part}}(t) = -A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t)$$

$$\ddot{Q}_{\text{part}}(t) = -A\omega_d^2 \cos(\omega_d t) - B\omega_d^2 \sin(\omega_d t)$$

Substitute into the differential equation:

$$\begin{aligned} L\ddot{Q}_{\text{part}} + R\dot{Q}_{\text{part}} + \frac{1}{C}Q_{\text{part}} &= V_0 \cos(\omega_d t) \\ &= L(-A\omega_d^2 \cos(\omega_d t) - B\omega_d^2 \sin(\omega_d t)) \\ &\quad + R(-A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t)) \\ &\quad + \frac{1}{C}(A \cos(\omega_d t) + B \sin(\omega_d t)) \end{aligned}$$

Group terms:

$$\text{Coefficient of } \cos(\omega_d t) : \quad -LA\omega_d^2 + RB\omega_d + \frac{A}{C}$$

$$\text{Coefficient of } \sin(\omega_d t) : \quad -LB\omega_d^2 - RA\omega_d + \frac{B}{C}$$

Set the equation equal to the driving term $V_0 \cos(\omega_d t)$, and match coefficients:

$$\begin{cases} -LA\omega_d^2 + RB\omega_d + \frac{A}{C} = V_0 \\ -LB\omega_d^2 - RA\omega_d + \frac{B}{C} = 0 \end{cases}$$

Solve the System

We now solve this linear system for A and B .

Define:

$$X := \left(\frac{1}{C} - L\omega_d^2 \right), \quad Y := R\omega_d$$

Then the system becomes:

$$\begin{cases} AX + BY = V_0 \\ BX - AY = 0 \end{cases}$$

Solve the second equation for B :

$$BX = AY \Rightarrow B = \frac{AY}{X}$$

Substitute into the first equation:

$$AX + \left(\frac{AY}{X}\right)Y = V_0 \Rightarrow A\left(X + \frac{Y^2}{X}\right) = V_0 \Rightarrow A = \frac{V_0 X}{X^2 + Y^2}$$

Then:

$$B = \frac{AY}{X} = \frac{V_0 Y}{X^2 + Y^2}$$

Final Particular Solution

Thus, the particular solution is:

$$Q_{\text{part}}(t) = \frac{V_0 X}{X^2 + Y^2} \cos(\omega_d t) + \frac{V_0 Y}{X^2 + Y^2} \sin(\omega_d t)$$

$$\text{where } X = \left(\frac{1}{C} - L\omega_d^2\right), \quad Y = R\omega_d$$

This can also be written in amplitude-phase form:

$$Q_{\text{part}}(t) = Q_p \cos(\omega_d t - \delta)$$

with:

$$Q_p = \frac{V_0}{\sqrt{X^2 + Y^2}} = \frac{V_0}{\sqrt{\left(\frac{1}{C} - L\omega_d^2\right)^2 + (R\omega_d)^2}}$$

$$\tan \delta = \frac{Y}{X} = \frac{R\omega_d}{\frac{1}{C} - L\omega_d^2}$$

Voltage Across the Capacitor

The voltage across the capacitor is:

$$V_C(t) = \frac{Q_{\text{part}}(t)}{C} = \frac{Q_p}{C} \cos(\omega_d t - \delta)$$

Final expression:

$$\boxed{V_C(t) = \frac{V_0}{\sqrt{(1 - LC\omega_d^2)^2 + (RC\omega_d)^2}} \cos(\omega_d t - \delta)} \quad \text{with} \quad \tan \delta = \frac{R\omega_d}{\frac{1}{C} - L\omega_d^2}$$