Notes: Week of June 23

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1 Analytic derivatation of Lyop-Exp

1.1 Hamiltonian Dynamics

Our basis:

$$S_j^x = s_j, S_j^y = f(s_j)\cos(\phi_j), S_j^z = f(s_j)\sin(\phi_j)$$

With $f(x) := \sqrt{1-x^2}$.
The hamiltonian is

$$H = \sum_j (-\vec{J} \circ \vec{S_{j+1}}) \cdot \vec{S_j}$$

The only terms in H involving j values are

$$\begin{aligned}(H)_j = &- J_x(s_j s_{j-1} + s_j s_{j+1}) \\ &- J_y f(s_j)\cos\phi_j[f(s_{j-1})\cos\phi_{j-1} + f(s_{j+1})\cos\phi_{j+1}] \\ &- J_z f(s_j)\sin\phi_j[f(s_{j-1})\sin\phi_{j-1} + f(s_{j+1})\sin\phi_{j+1}]\end{aligned}$$

The cononical position is

$$s_j$$

The cononical momentum is

$$\phi_j$$

Thus

$$\dot{s_j} = \frac{\partial H}{\partial \phi_j}$$
$$\dot{\phi_j} = -\frac{\partial H}{\partial s_j}$$

and so we have

$$\begin{aligned}\frac{ds_j}{dt} =& + J_y f(s_j)\sin(\phi_j)[f(s_{j+1})\cos(\phi_{j+1}) + f(s_{j-1})\cos(\phi_{j-1})] \\ & - J_z f(s_j)\cos(\phi_j)[f(s_{j+1})\sin(\phi_{j+1}) + f(s_{j-1})\sin(\phi_{j-1})]\end{aligned}$$

$$\begin{aligned}\frac{d\phi_j}{dt} = -f'(s_j)[& - J_y \cos\phi_j (f(s_{j+1})\cos\phi_{j+1} + f(s_{j-1})\cos\phi_{j-1}) \\ & - J_z \sin\phi_j (f(s_{j+1})\sin\phi_{j+1} + f(s_{j-1})\sin\phi_{j-1})] \\ & - J_x (s_{j+1} + s_{j-1})\end{aligned}$$

1.2 First order Expansion of functions

1.2.1 $f(x)$ and $f'(x)$

Let

$$f(x) = \sqrt{1 - x^2}$$

We expand $f(x_0 + \delta x)$ to leading order in δx using a Taylor expansion:

$$f(x_0 + \delta x) \approx f(x_0) + f'(x_0)\,\delta x$$

The derivative:

$$f'(x) = \frac{d}{dx}\left(\sqrt{1 - x^2}\right) = \frac{-x}{\sqrt{1 - x^2}}$$

Then:

$$f'(x_0) = \frac{-x_0}{\sqrt{1 - x_0^2}}$$

Thus:

$$f(x_0 + \delta x) \approx \sqrt{1 - x_0^2} - \frac{x_0}{\sqrt{1 - x_0^2}}\,\delta x$$

For $x_0 = 0$,

$$\boxed{f(x_0 + \delta x) \approx 1}$$

Simiarly

$$\boxed{f'(x_0 + \delta x) \approx -\delta x}$$

1.2.2 $cos\phi_j$ and $sin\phi_j$

Let

$$\phi_j = \phi_j^0 + \delta\phi_j, \quad \text{where} \quad \phi_j^0 = \frac{2\pi}{N} j$$

Using the expansion:

$$\cos(a+\epsilon) \approx \cos a - \sin a \cdot \epsilon$$

We have

$$\boxed{\cos\phi_j \approx \cos(\frac{2\pi}{N} j) - \sin(\frac{2\pi}{N} j)\delta\phi_j}$$

For $N = 4$

$$\cos\phi_j \approx \begin{cases} \mp\delta\phi_j, & j \text{odd} \\ \pm 1, & j \text{even} \end{cases}$$

Simiarly using the expansion:

$$\sin(a+\epsilon) \approx \sin a + \cos a \cdot \epsilon$$

We have

$$\boxed{\sin\phi_j \approx \sin(\frac{2\pi}{N} j) + \cos(\frac{2\pi}{N} j)\delta\phi_j}$$

For $N = 4$

$$\sin\phi_j \begin{cases} \pm 1, & j \text{odd} \\ \pm\delta\phi_j, & j \text{even} \end{cases}$$

1.2.3 $cos(\phi_{j+1} - \phi_j)$

In the notes I have:

$$\text{``}cos(\phi_{j+1} - \phi_j) \approx 1 + O(\delta\phi^2),'' \; N = 4$$

Let

$$\phi_j = \phi_j^0 + \delta\phi_j, \quad \text{where} \quad \phi_j^0 = \frac{2\pi}{N} j$$

with $N \in \mathbb{Z}$. Then:

$$\phi_{j+1} - \phi_j = \left(\phi_{j+1}^0 - \phi_j^0\right) + (\delta\phi_{j+1} - \delta\phi_j) = \frac{2\pi}{N} + (\delta\phi_{j+1} - \delta\phi_j)$$

We expand the cosine to first order in $\delta\phi_j$ and $\delta\phi_{j+1}$:

$$\cos(\phi_{j+1} - \phi_j) = \cos\left(\frac{2\pi}{N} + \delta\phi_{j+1} - \delta\phi_j\right)$$

Using the expansion:

$$\cos(a+\epsilon) \approx \cos a - \sin a \cdot \epsilon$$

with $a = \frac{2\pi}{N}$ and $\epsilon = \delta\phi_{j+1} - \delta\phi_j$, we obtain:

$$\boxed{\cos(\phi_{j+1} - \phi_j) \approx \cos\left(\frac{2\pi}{N}\right) - \sin\left(\frac{2\pi}{N}\right)(\delta\phi_{j+1} - \delta\phi_j) + O(\delta\phi^2)}$$

If $N = 4$,

$$\cos(\phi_{j+1} - \phi_j) \approx -(\delta\phi_{j+1} - \delta\phi_j)$$

1.2.4 $sin(\phi_{j+1} - \phi_j)$

Let

$$\phi_j = \phi_j^0 + \delta\phi_j, \quad \text{where} \quad \phi_j^0 = \frac{2\pi}{N} j$$

Then:

$$\phi_{j+1} - \phi_j = \frac{2\pi}{N} + (\delta\phi_{j+1} - \delta\phi_j)$$

We expand the sine to first order in $\delta\phi_j$ and $\delta\phi_{j+1}$:

$$\sin(\phi_{j+1} - \phi_j) = \sin\left(\frac{2\pi}{N} + \delta\phi_{j+1} - \delta\phi_j\right)$$

Using the Taylor expansion:

$$\sin(a+\epsilon) \approx \sin a + \cos a \cdot \epsilon$$

with $a = \frac{2\pi}{N}$ and $\epsilon = \delta\phi_{j+1} - \delta\phi_j$, we obtain:

$$\boxed{\sin(\phi_{j+1} - \phi_j) \approx \sin\left(\frac{2\pi}{N}\right) + \cos\left(\frac{2\pi}{N}\right)(\delta\phi_{j+1} - \delta\phi_j) + O(\delta\phi^2)}$$

If $N = 4$, Then

$$\sin(\phi_{j+1} - \phi_j) \approx 1 + O(\delta\phi^2)$$

1.3 First order approximation of diffrential equations

Let

$$s_j = s_j^0 + \delta s_j = \delta s_j \quad \phi_j = \phi_j^0 + \delta\phi_j$$

Recall the equations of motion in our chosen basis

$$
\begin{aligned}
\frac{ds_j}{dt} = \frac{d\delta s_j}{dt} &= + J_y f(s_j)\sin(\phi_j)[f(s_{j+1})\cos(\phi_{j+1}) + f(s_{j-1})\cos(\phi_{j-1})] \\
&\quad - J_z f(s_j)\cos(\phi_j)[f(s_{j+1})\sin(\phi_{j+1}) + f(s_{j-1})\sin(\phi_{j-1})]
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\phi_j}{dt} = \frac{d\delta\phi_j}{dt} = -f'(s_j)[&- J_y \cos\phi_j (f(s_{j+1})\cos\phi_{j+1} + f(s_{j-1})\cos\phi_{j-1}) \\
&- J_z \sin\phi_j (f(s_{j+1})\sin\phi_{j+1} + f(s_{j-1})\sin\phi_{j-1})] \\
&- J_x(s_{j+1} + s_{j-1})
\end{aligned}
$$

1.3.1 N = 4

Using the approximations we had above

$$
\begin{aligned}
\frac{d\delta s_j}{dt} &= \begin{cases} J_y(\pm 1)[\mp 1 \pm 1] - J_z(\mp\delta\phi_j)[\mp\delta\phi_{j+1} \pm \delta\phi_{j-1}], j \text{ odd} \\ J_y(\pm\delta\phi_j)[\mp\delta\phi_{j+1} \pm \delta\phi_{j-1}] - J_z(\pm 1)[\mp 1 \pm 1], j \text{ even} \end{cases} \\
&= \delta\phi_j \begin{cases} J_z[\mp\delta\phi_{j+1} \pm \delta\phi_{j-1}], j \text{ odd} \\ J_y[\mp\delta\phi_{j+1} \pm \delta\phi_{j-1}], j \text{ even} \end{cases}
\end{aligned}
$$

$$
\frac{d\delta\phi_j}{dt} = -J_x(s_{j+1} + s_{j-1}) \mp \begin{cases} J_z(\mp\phi_{j+1} \pm \phi_{j-1}), j \text{ odd} \\ J_y(\pm\phi_{j+1} \pm \phi_{j-1}), j \text{ even} \end{cases}
$$

2 Numerics for Lyop Calc

Comparison of Pushback Implementations in Benettin's Method

The goal of the `push_back` function in Benettin's algorithm is to restore the distance between two nearby trajectories, $\vec{S}_A$ (reference) and $\vec{S}_B$ (perturbed), to a fixed small value ε, without altering the direction of separation.

My Original Implementation

In the original Julia implementation, the perturbation is applied *locally* and independently to each spin:

$$
\vec{S}_i^B \leftarrow \vec{S}_i^A + \varepsilon \cdot \frac{\vec{S}_i^B - \vec{S}_i^A}{||\vec{S}_i^B - \vec{S}_i^A||}
$$

for all $i = 1, \ldots, L$.

That is, each spin is pushed away from its counterpart independently, ensuring that the local separation at each site is ε. However, this results in a total perturbation vector (in $\mathbb{R}^{3L}$) with norm approximately:

$$||\vec{S}_B - \vec{S}_A|| \approx \varepsilon \cdot \sqrt{L}$$

I tried adjusting the pushback to be

$$\vec{S}_i^B \leftarrow 1/\sqrt{L}(\vec{S}_i^A + \varepsilon \cdot \frac{\vec{S}_i^B - \vec{S}_i^A}{||\vec{S}_i^B - \vec{S}_i^A||})$$

such that the total perturbation vector has norm

$$||\vec{S}_B - \vec{S}_A|| \approx \varepsilon$$

but that still didn't fix things.

Professor Pixely's Implementation (Benettin-consistent)

In contrast, the correct implementation applies the perturbation to the entire spin chain *as a single vector* in $\mathbb{R}^{3L}$:

$$\vec{S}_B \leftarrow \vec{S}_A + \varepsilon \cdot \frac{\vec{S}_B - \vec{S}_A}{||\vec{S}_B - \vec{S}_A||}$$

This ensures that the total perturbation vector has norm exactly ε.

Note: I spoke with chatGPT and it also said that the correct implementation insured that the direction of perturbation *is preserved* across steps. This consistency is essential for correctly capturing the dominant Lyapunov exponent. But I am not sure what is meant by that.

$$\lambda = \lim_{n\to\infty} \frac{1}{n\tau} \sum_{i=1}^{n} \log\left(\frac{||\vec{S}_B^{(i)} - \vec{S}_A^{(i)}||}{\varepsilon}\right)$$

Summary of Differences

Feature	**Original (local)**	**Professor's (global)**
Normalization per spin	Yes	No
Preserves global direction	No	Yes
Total perturbation norm	$\varepsilon \cdot \sqrt{L}$	ε
Valid for Benettin method	Not quite	Yes

3 Harmonic Oscillators to Model Spin

Consider a series RLC circuit driven by an AC voltage source:

$$V_{\text{drive}}(t) = V_0 \cos(\omega_d t)$$

The circuit contains:

- Inductor with inductance L
- Resistor with resistance R
- Capacitor with capacitance C

3.1 Equation of Motion

$$V_L + V_R + V_C = V_{\text{drive}}(t)$$

$$\boxed{L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V_0 \cos(\omega_d t)}$$

3.2 Solution Structure

The general solution is:

$$Q(t) = Q_{\text{hom}}(t) + Q_{\text{part}}(t)$$

The homogeneous solution describes transient behavior and decays over time due to resistance with time scale.

$$\tau = -\frac{2L}{R}$$

.

As in

$$Q_{\text{hom}}(t) \sim e^{-\frac{Rt}{2L}}$$

I seek a steady state solution, so I feel like we should try something like:

$$Q_{\text{part}}(t) = A\cos(\omega_d t) + B\sin(\omega_d t)$$

3.3 Determaning Particular Solution Ansatz

Compute the first and second derivatives:

$$\dot{Q}_{\text{part}}(t) = -A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t)$$

$$\ddot{Q}_{\text{part}}(t) = -A\omega_d^2 \cos(\omega_d t) - B\omega_d^2 \sin(\omega_d t)$$

Substitute into the differential equation:

$$\begin{aligned}
&L\ddot{Q}_{\text{part}} + R\dot{Q}_{\text{part}} + \frac{1}{C}Q_{\text{part}} = V_0 \cos(\omega_d t) \\
&= L(-A\omega_d^2 \cos(\omega_d t) - B\omega_d^2 \sin(\omega_d t)) \\
&\quad + R(-A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t)) \\
&\quad + \frac{1}{C}(A\cos(\omega_d t) + B\sin(\omega_d t))
\end{aligned}$$

Group terms:

$$\text{Coefficient of } \cos(\omega_d t): \quad -LA\omega_d^2 + RB\omega_d + \frac{A}{C}$$

$$\text{Coefficient of } \sin(\omega_d t): \quad -LB\omega_d^2 - RA\omega_d + \frac{B}{C}$$

Set the equation equal to the driving term $V_0 \cos(\omega_d t)$, and match coefficients:

$$\begin{cases} -LA\omega_d^2 + RB\omega_d + \frac{A}{C} = V_0 \\ -LB\omega_d^2 - RA\omega_d + \frac{B}{C} = 0 \end{cases}$$

Solve the System

We now solve this linear system for A and B.

Define:

$$X := \left(\frac{1}{C} - L\omega_d^2\right), \quad Y := R\omega_d$$

Then the system becomes:

$$\begin{cases} AX + BY = V_0 \\ BX - AY = 0 \end{cases}$$

Solve the second equation for B:

$$BX = AY \Rightarrow B = \frac{AY}{X}$$

Substitute into the first equation:

$$AX + \left(\frac{AY}{X}\right) Y = V_0 \Rightarrow A\left(X + \frac{Y^2}{X}\right) = V_0 \Rightarrow A = \frac{V_0 X}{X^2 + Y^2}$$

Then:

$$B = \frac{AY}{X} = \frac{V_0 Y}{X^2 + Y^2}$$

Final Particular Solution

Thus, the particular solution is:

$$Q_{\text{part}}(t) = \frac{V_0 X}{X^2 + Y^2} \cos(\omega_d t) + \frac{V_0 Y}{X^2 + Y^2} \sin(\omega_d t)$$

$$\text{where } X = \left(\frac{1}{C} - L\omega_d^2\right), \quad Y = R\omega_d$$

This can also be written in amplitude-phase form:

$$Q_{\text{part}}(t) = Q_p \cos(\omega_d t - \delta)$$

with:

$$Q_p = \frac{V_0}{\sqrt{X^2 + Y^2}} = \frac{V_0}{\sqrt{\left(\frac{1}{C} - L\omega_d^2\right)^2 + (R\omega_d)^2}}$$

$$\tan\delta = \frac{Y}{X} = \frac{R\omega_d}{\frac{1}{C} - L\omega_d^2}$$

Voltage Across the Capacitor

The voltage across the capacitor is:

$$V_C(t) = \frac{Q_{\text{part}}(t)}{C} = \frac{Q_p}{C}\cos(\omega_d t - \delta)$$

Final expression:

$$\boxed{V_C(t) = \frac{V_0}{\sqrt{(1 - LC\omega_d^2)^2 + (RC\omega_d)^2}}\cos(\omega_d t - \delta)} \quad \text{with} \quad \tan\delta = \frac{R\omega_d}{\frac{1}{C} - L\omega_d^2}$$