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Assignment - 5

Q. Fibonacci sequence is defined as $V_n = V_{n-1} + V_{n-2}$. Can this problem be written as an eigen value problem and solved for V_n directly? If so find formula for V_n.

Consider a fibonacci sequence for n^{th} term

$$x_n = x_{n-1} + x_{n-2}$$

Also $$x_{n-1} = x_{n-1} + 0 \cdot x_{n-2}$$

$$\Rightarrow \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix}$$

$$\Rightarrow X_n = A X_{n-1} \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \ \& \ X_n = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

Now once we have this recurrence relation we can say

$$X_2 = A X_1$$
$$X_3 = A^2 X_1$$
$$X_4 = A^3 X_1 \text{ and so on.}$$

Since we know $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ our only unknown is A^n for computation of X_n.

Let us find evec, eval pairs for A and diagonalize it.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$
$$\Rightarrow (1-\lambda)(-\lambda) - 1 \cdot 1 = 0$$
$$\Rightarrow \lambda^2 - \lambda - 1 = 0$$
$$\Rightarrow \lambda = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Consider $\lambda = (1+\sqrt{5})/2$

$$(A - \lambda I)x = 0 \Rightarrow \begin{bmatrix} 1 - \left(\frac{1-\sqrt{5}}{2}\right) & 1 \\ 1 & 0 - \left(\frac{1-\sqrt{5}}{2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1 \left(\frac{\sqrt{5}-1}{2}\right) \Rightarrow$

$$\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{evec} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

Similarly, other $\text{evec} = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$

Now, $D = P^{-1} A P$

or $A = PDP^{-1}$

and we can compute $A^m = PD^m P^{-1}$

where $P = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$P^{-1} = \frac{1}{\det P} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\left(\frac{\sqrt{5}+\sqrt{5}}{2}\right)} \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$\therefore P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & \sqrt{5}-1/2\sqrt{5} \\ -1/\sqrt{5} & \sqrt{5}+1/2\sqrt{5} \end{bmatrix}$$

$$\therefore A^m = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^m \begin{bmatrix} 1/\sqrt{5} & \sqrt{5}-1/2\sqrt{5} \\ -1/\sqrt{5} & \sqrt{5}+1/2\sqrt{5} \end{bmatrix}$$

$$\therefore X_m = A^m \cdot X_0$$

To find the m^{th} fibonacci number, we can use the above equations and we will not have to compute all the $X_2, X_3, X_4, \ldots, X_{m-1}$ to find out the value of X_m