Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

Cluster Programme - M.Tech. in Data Science and Engg.

II Semester 2019-20

Course Number	DSECL ZC416
Course Name	Mathematical Foundation for Data Science
Nature of Exam	Open Book
Weightage for grading	30%
Duration	90 minutes
Date of Exam	21/06/2020 (10:00 a.m - 11:30 a.m)

# Pages	2
# Questions	5

Answer Key and Marking Scheme

(1) Marks are indicated in red font at the right side corner
(2) Alternate approaches would be considered and awarded marks

Q1a) Justifications are given below.

(1) E^{-1} is well defined as its rank $= n$ due to rank nullity theorem.
$F = M(I - M)^{-1}$
$G = (I - M)^{-1}M$
Now consider $(I - M)^{-1}F(I - M) = (I - M)^{-1}M = G$
From this it can be concluded that F and G are similar. (1)

(2) Consider $M(I - M) = M - M^2 = (I - M)M$
Multiplying the above equation by $(I - M)$ on the left and by $(I - M)^{-1}$ on the right, we obtain $(I-M)^{-1}M = M(I-M)^{-1}$. Note that $E = I-M$ is invertible due to its rank being n. This is same as saying $F = G$ (1)

Q1b) The professor is correct. It is true that both U and W can be potentially correct, we list an example each for $\mathbb{R}^2$ over $\mathbb{R}$

(1) Student U is correct case
Let $S = \{\begin{bmatrix}1\\0\end{bmatrix}\}$.
By adding new elements, we obtain an updated set $S_1 = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$.
Let $T = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$.
By removing certain vectors, we obtain an updated set $T_1 = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$
Here S_1 and T_1 match exactly. (2)

(2) Student W is correct case

Let $S = \{\begin{bmatrix}1\\0\end{bmatrix}\}$.

By adding new elements, we obtain an updated set $S_1 = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$.

Let $T = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\}$.

By removing certain vectors, we obtain an updated set $T_1 = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$

Here S_1 and T_1 do not match. (2)

Q2a) Given that $T(B) = BA$.
(i) First we verify whether $T(B + C) = T(B) + T(C)$
$T(B + C) = (B + C)A = BA + CA = T(B) + T(C)$ (1)

(ii) Now we verify whether $T(\alpha B) = \alpha T(B)$
$T(\alpha B) = (\alpha B)A = \alpha(BA) = \alpha T(B)$ (1)
Since T satisfies the 2 basic properties, it is a linear transformation

Q2b) We need to express $c = (1, 3, -7)$ as a linear combination of $a = (1, 0, -1)$ and $b = (-2, 1, 0)$. To achieve that, we need to solve $\alpha a + \beta b = c$ for α and β.
Solving yields $\alpha = 7$ and $\beta = 3$. (1)
Thus, $T(c) = T(\alpha a + \beta b) = 7T(a) + 3T(b) = (-7, -13, 18)$. (0.5)
Since α and β are unique, the value of $T(c)$ is unique. (0.5)

Q2c) We have to prove that $\dim(\mathbb{R}^4) = \dim(\text{Kernel}(T)) + \dim(\text{Range}(T))$. The given linear transform can be written as $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix}1 & 1 & 0 & 0\\0 & 2 & -1 & 0\\1 & 1 & 3 & 0\end{bmatrix}$$

(0.5)

By performing rref calculation we can see that :

$$\text{rref}(A) = \begin{bmatrix}1 & 1 & 0 & 0\\0 & 2 & -1 & 0\\0 & 0 & 3 & 0\end{bmatrix}$$

Hence rank = 3 and nullity = 1 as the equation $Ax = 0$ has solutions of the form $(0, 0, 0, 1)$. (1)
Hence, $\dim(\text{Range}(T)) + \dim(\text{Kernel}(T)) = 3 + 1 = 4 = \dim(\mathbb{R}^4)$. (0.5)

Q3a) The given set of equations are

$$0.1036x_1 + 0.2122x_2 = 0.7381$$
$$0.2081x_1 + 0.4247x_2 = 0.9327$$

Without partial pivoting: Choosing first equation as pivot equation, the multiplier $m = \frac{0.2081}{0.1036} = 2.009$. (0.5)
Multiplying the first equation by m and subtracting from the second equation, we get $-0.0016x_2 = -0.5503$ and thus $x_2 = 343.9$. (0.5)
By using the value of x_2, we get $x_1 = -697.3$. (1)

With partial pivoting: We choose the second equation as the pivot equation and following the calculation as done above yields,
$m = \frac{0.1036}{0.2081} = 0.4978$. (0.5)
$x_2 = 342.2$. (0.5)
$x_1 = -693.9$. (1)

Q3b) Refer the tables below. Both are awarded marks.

Computation	Forward Elimination	Back Substitution	Total
Divisions	$\dfrac{n(n-1)}{2}$	n	$\dfrac{n(n+1)}{2}$
Multiplications	$\dfrac{n^3-n}{3}$	$\dfrac{n(n-1)}{2}$	$\dfrac{2n^3+3n^2-5n}{6}$
Additions	$\dfrac{n^3-n}{3}$	$\dfrac{n(n-1)}{2}$	$\dfrac{2n^3+3n^2-5n}{6}$

Computation	Gauss Jordan
Divisions	n^2
Multiplications	$\dfrac{2n^3+3n^2-5n}{6}$
Additions	$\dfrac{2n^3+3n^2-5n}{6}$

	Gauss Elimination	Gauss Jordan method
No of mult. / div.	$\dfrac{n(n^2+3n-1)}{3}$	$\dfrac{n^3}{3}+\dfrac{3n^2}{2}-\dfrac{5n}{6}$
No of Add. / subt.	$\dfrac{n(n-1)(2n+5)}{6}$	$\dfrac{n^3}{3}+\dfrac{n^2}{2}-\dfrac{5n}{6}$

$(0.5 \times 4 = 2)$

Q4a) $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 3 & 7 \\ 1 & -1 & -4 \\ 1 & -1 & 2 \end{pmatrix}$.

Applying Gram-Schmidt process gives

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

(1.5)

On normalizing the vectors, we get

$$Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

(0.5)

$$Q^T A = Q^T(QR) = IR = R.$$

(1)

$$R = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

(1)

Q4b) Rewriting the equations

$$\begin{array}{rcrcrcr} 4x & + & 2y & - & 2z & = & 10 \\ 0x & + & 4y & + & 2z & = & 9 \\ x & + & 0y & + & 4z & = & 8. \end{array}$$

(0.5)

Decomposition

$$\begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{4} & 0 & 1 \end{pmatrix} = I + L + U = I + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Gauss Jacobi method, $C = -I^{-1}(U + L)$. (0.5)

$$C = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ -\frac{1}{4} & 0 & 0 \end{pmatrix}.$$

(0.5)

It could be observed that the row sum and column sum norm are 1, $||C||_{\text{frob}} < 1$, Gauss Jacobi method converges. (0.5)

OR

Spectral method:

$$I - A = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & 0 \end{pmatrix}.$$

(0.5)

Spectral radius $\rho =$ maximum of eigenvalues in absolute value.
Since the eigenvalues are $0.1474 + 0.4361i$, $0.1474 - 0.4361i$ and -0.2949, we observe that $\rho = .4604 < 1$. (1)
Iteration converges as $\rho < 1$. (0.5)

Q5a) Let the number of units of B_1 be x_1 and the number of units of $B2$ be x_2.

Max $Z = 30x_1 + 22.5x_2$ subject to

$$\begin{array}{rcrcl} 2x_1 & + & x_2 & \leq & 1200 \\ x_1 & + & x_2 & \leq & 1000 \\ & & x_2 & \geq & 250 \\ & & x_1, x_2 & \geq & 0. \end{array}$$

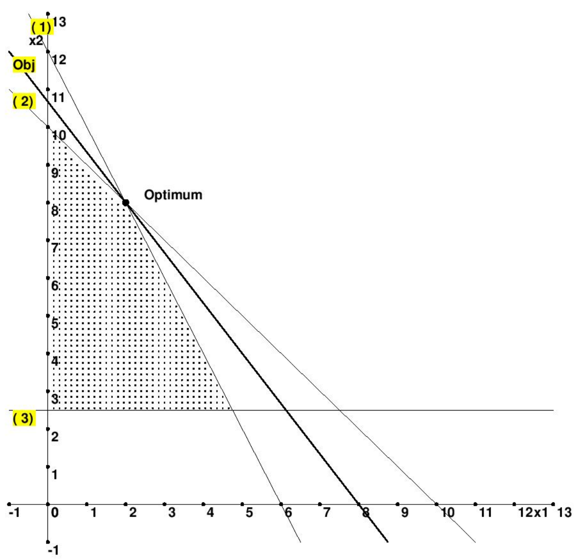
(2)

Q5b) $x_1 = 200$ and $x_2 = 800$ and objective value $Z = 24000$. (1)
(1 mark for graph)

LINEAR PROGRAMMING -- GRAPHICAL SOLUTION

Title: T1

Summary of Optimal Solution:
Objective Value = 2.400
x1 = 2.000
x2 = 8.000



OR

Scaling x_i in 100's be number of pieces of B_i, Max $Z = 0.30x_1 + 0.225x_2$

$$\begin{array}{rcrcl} 2x_1 & + & x_2 & \leq & 12 \\ x_1 & + & x_2 & \leq & 10 \\ & & x_2 & \geq & 2.5 \\ & & x_1, x_2 & \geq & 0. \end{array}$$

(1)
(1 mark for graph)

Q5c) Max $Z = c_1x_1 + c_2x_2$.
Optimality range is $1 \leq \frac{c_1}{c_2} \leq 2$ OR $\frac{1}{2} \leq \frac{c_2}{c_1} \leq 1$. (1)

Q5d) $x_1 \geq \alpha$ Scenario 1: $\alpha > 475 \rightarrow$ No solution (0.5 marks)
Scenario 2: $\alpha < 475 \rightarrow$ Solution exists. (0.5 marks)