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Assignment-3

Q.1 Let $B = (b_1, b_2, \ldots, b_{r-1}, b_r, b_{r+1}, \ldots b_n)$ be a non-singular matrix. If column b_r is replaced by a and the resulting matrix is called B_a along with $a = \sum^n y_i b_i$, then state the necessary and sufficient condition for B_a to be non-singular.

Consider,

$$B = [b_1 \; b_2 \; \ldots \; b_{r-1} \; b_r \; b_{r+1} \; \ldots \; b_n]$$

$\because B$ is non-singular $\Rightarrow \det(B) \neq 0 \Rightarrow \det(B^T) \neq 0$
$\Rightarrow$ Column-space of B has linearly indep. vectors.

$$\left.\begin{array}{l}\therefore \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_r b_r + \ldots + \alpha_n b_n = 0 \\ \text{has only one sol}^n \; \alpha_i = 0 \; \forall i \in [1, n]\end{array}\right\} \text{— } 1$$

Given $b_r = \sum_1^n y_i b_i$

So we can write $B_a = [b_1 \; b_2 \; \ldots \; b_{r-1} \; \sum_1^n y_i b_i \; \ldots \; \cancel{b_{n}} b_n]$

Let's assume B_a to be a non-singular matrix, then similar to 1, we can write

$$\beta_1 b_1 + \beta_2 b_2 + \ldots + \beta_r \sum_1^n y_i b_i + \ldots + \beta_n b_n = 0$$
— (for all $\beta_i = 0 \because B_a$ non-si

$$\therefore \beta_1 b_1 + \beta_2 b_2 + \ldots \beta_r [y_1 b_1 + y_2 b_2 + \ldots + y_r b_r + \ldots + y_n b_n] + \ldots + \beta_n b_n = 0$$

Regrouping, we get

$$(\beta_1 + \beta_r y_1) b_1 + (\beta_2 + \beta_r y_2) b_2 + \ldots + \beta_r y_r b_r + \ldots + \cancel{(\beta_n + \beta_r b_n)} \ldots + (\beta_n + \beta_r y_n) b_n = 0$$

We know that $(b_1, b_2, \ldots, b_n)$ are linearly indep.

$$\therefore \beta_1 + \beta_r y_1 = 0$$
$$\beta_2 + \beta_r y_2 = 0 \; \ldots \; \beta_r y_r = 0 \; \ldots \; \beta_n + \beta_r y_n = 0$$

i.e. $\beta_i + b_r y_i = 0 \quad \forall i \in [1,n] - \{r\}$
& $\beta_r y_r = 0$

Now this means in $\beta_r y_r = 0$, $\beta_r = 0$ or $y_r = 0$ — ($\beta_r = 0$ ∵ B_a is non-sing)
Now, we get $y_r = 0$ or $y_r \neq 0$ ∵ $\beta_r = 0$ n
Let $y_r = 0 \Rightarrow B_a = [b_1 \; b_2 \; b_3 \; \dots\dots \; \sum_i y_i b_i \; \forall i \in [1,n]-\{r\} \dots\, b_n]$

$\Rightarrow$ Column ~~B~~r becomes a linear combination of all the other columns
$\Rightarrow$ $\det(B_a) = 0$ which contradicts our previous assumption i.e. $\det(B_a) \neq 0$
$\Rightarrow$ $y_r \neq 0$ is a necessary condition for B_a to be non-singular.

If $y_r \neq 0$; then B_a's column r becomes a new independent column $\Rightarrow B_a$ is non-singular.
Hence B_a is non-singular iff $y_r \neq 0$

Q.2] Let V be a finite dimensional vector space over $\mathbb{R}$. If S is a set of elements in V such that Span(S) = V, what is relation between S and basis of V?

If we consider $\mathcal{Q}$ to be a set of ~~all the b set of all~~ all the sets of basis vector of V, then we can say
$$\{S\} \subseteq \mathcal{Q} \quad \text{or} \quad S \in \mathcal{Q}.$$

Simply stated, S is one set/combination of vectors which can span V. There can be others but S is one of the possible bases of V.

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Q.3 Let $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3;\ 2x_1 + x_2;\ -x_1 - 2x_2 + 2x_3)$
then $(T: \mathbb{R}^3 \to \mathbb{R}^3)$ is a linear transformation
a. Show that T is a linear transformation
b. Find (a,b,c) such that $(a,b,c) \in$ Null space of T.

a. For a transformation to be linear, we need to prove,

$$T(v+x) = T(v) + T(x)$$
$$T(cx) = cT(x)$$

Let v & x be vectors in $\mathbb{R}^3$

$$T = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ 2x_1 + x_2 \\ -x_1 - 2x_2 + 2x_3 \end{bmatrix} \quad \text{— (I)}$$

$$T(v) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 - v_2 + 2v_3 \\ 2v_1 + v_2 \\ -v_1 - 2v_2 + 2v_3 \end{bmatrix} \quad \text{— (II)}$$

$$T(x+v) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 + v_1 \\ x_2 + v_2 \\ x_3 + v_3 \end{bmatrix} = \begin{bmatrix} x_1 + v_1 - x_2 - v_2 + 2x_3 + 2v_3 \\ 2x_1 + 2v_1 + x_2 + v_2 \\ -x_1 - v_1 - 2x_2 - 2v_2 + 2x_3 + 2v_3 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1 - x_2 + 2x_3) + (v_1 - v_2 + 2v_3) \\ (2x_1 + x_2) + (2v_1 + v_2) \\ (-x_1 - 2x_2 + 2x_3) + (-v_1 - 2v_2 + 2v_3) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x_2 + 2x_3 \\ 2x_1 + x_2 \\ -x_1 - 2x_2 + 2x_3 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 + 2v_3 \\ 2v_1 + v_2 \\ -v_1 - 2v_2 + 2v_3 \end{bmatrix} \quad \text{— (III)}$$

$$\boxed{T(x+v) = T(x) + T(v)} \text{ — From (I) \& (II)}$$

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$$T(cx) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} = \begin{bmatrix} cx_1 - cx_2 + 2cx_3 \\ 2cx_1 + cx_2 \\ -cx_1 - 2cx_2 + 2cx_3 \end{bmatrix}$$

$$= c\begin{bmatrix} x_1 - x_2 + 2x_3 \\ 2x_1 + x_2 \\ -x_1 - 2x_2 + 2x_3 \end{bmatrix}$$

$$T(cx) = c\,T(x) \quad \text{—— (IV)}$$

From (III) & (IV), T is a linear transformation.

(ii)

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 = R_1 + R_3 \qquad R_2 = R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 = R_3 + R_2 \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \dim(\text{Range}(T)) = 2 \Rightarrow \dim(NS(T)) = 3 - 2 = 1 = \text{Nullity}$

Further more,

$$a - b + 2c = 0 \Rightarrow a = b - 2c = -2c/3$$

$$\& \quad 3b - 4c = 0 \Rightarrow b = +4c/3$$

$$\therefore (a,b,c) = (-2c/3,\ 4c/3,\ c)$$

$\boxed{(a,b,c) = k(-2,\ 4,\ 3)}$ —— All such vectors will be in the null space of T

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Q.4] Construct $T: V \to W$, V & W are vector spaces over F s.t. dim(kernel space of T) = 666. Is such a transformation unique? Give Reason for your answer.

Consider $F = \mathbb{R}^{700}$

Let T - 700×700 vector.
Then $\because$ dim(kernel space T) = 666, by rank-nullity theorem,
dim(range T) = $700 - 666 = 34$

$\Rightarrow$ We can construct a T such that it's first 34 rows are a set of linearly independent vectors and remaining 666 rows could be a linear ~~trans~~ comb. of these 34 rows.
Such a transformation T can certainly be constructed!

Now in 700 dim $\mathbb{R}$ space, there can certainly be more than ~~34~~ one combination of linearly indep. vectors. For eg. any ~~one~~ combi. of 34 rows from I_{700} can form such a set

$\Rightarrow$ T is not unique. There could be many T, for any space with dimensions > 666 i.e. $\mathbb{R}^{667}, \mathbb{R}^{668}, \mathbb{R}^{669}, \ldots \mathbb{R}^{n}$
$\forall n \in (666, \infty)$