

Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

Cluster Programme - M.Tech. in Data Science and Engg.

II Semester 2019-20

Course Number	DSECL ZC416		
Course Name	Mathematical Foundation for Data Science		
Nature of Exam	Open Book	# Pages	2
Weightage for grading	30%	# Questions	5
Duration	90 minutes		
Date of Exam	04/07/2020 (10:00 a.m - 11:30 a.m)		

Instructions

1. All questions are compulsory
2. Answers without proper justification would not be awarded marks.

Notations

1. $\mathbb{R}^{m \times n}$ represents the set of all $m \times n$ matrices with real entries.
 2. A^T would denote the transpose of a matrix A .
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Q1a) Consider the linear system given as follows:

$$\begin{aligned}c_1x + 2y + 7z &= b_1 \\c_1x + c_1y + 4z &= b_2 \\c_1x + c_1y + c_1z &= b_3\end{aligned}$$

In this linear system we have an unknown constant named c_1 . Find out the three values of c_1 for which this linear system will fail to have exactly three pivots. (2)

b) Prof. X has two brilliant students Y and Z in his class. He introduces the concept of vector spaces, bases and dimension in his 5th session. As an exercise, he gives a vector space V of dimension n and asks Y and Z to find a basis. Y produces a set S with n elements. But Z being lazy, takes the set S , removes a vector and adds a new vector to it creating a new set T . Prof. X looks at set T and confirms to class that it is a basis. He then asks the class if the set S produced by Y could be a basis without telling them what it is. While student U says yes, student W says need not and Prof. X says that both U and W could be correct. Justify the statement of Prof. X with suitable examples of V over F , n , S and T . (4)

Q2a) Consider a transformation $T : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ such that $T(M) = M^T$. This is infact a linear transformation. Based on this, justify if the following statements are true or not. (2)

- a) $T \circ T$ is the identity transformation.
- b) The kernel of T is the zero matrix.
- c) $\text{Range } T = \mathbb{R}^{2 \times 2}$
- d) $T(M) = -M$ is impossible.

b) Assume that you are given a matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ with $(1 \leq i, j \leq n)$ and having the following interesting property:

$$a_{i1} + a_{i2} + \dots + a_{in} = 0 \quad \text{for each } i = 1, 2, \dots, n$$

Based on this information, prove that $\text{rank}(A) < n$. (2)

c) Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank r . Suppose there are right hand sides b for which $Ax = b$ has no solution, which of following expression(s) is/are correct: $r < m$, $r = m$, $r > m$.

Now, consider the linear system $A^T y = 0$. Do you think this linear system can have non-zero solutions, that is $y \neq 0$ such that $A^T y = 0$. Give justification for all your answers. (2)

Q3a) Given the following matrix

$$A = \begin{pmatrix} \sqrt{2} \cos(x) & i \sin(x) & 0 \\ i \sin(x) & 0 & -i \sin(x) \\ 0 & -i \sin(x) & -\sqrt{2} \cos(x) \end{pmatrix}$$

where $x \in \mathbb{R}$ and $0 \leq x \leq 2\pi$. Determine the value(s) of x for which A is diagonalizable. (4)

b) Let $A \in \mathbb{R}^{m \times n}$ with $m \geq n$. Assuming A has a decomposition of the form $A = PQ$ where P is orthogonal and Q is upper triangular, find out (1+1)

- the relationship between the singular values of A and the singular values of Q ?
- the minimum number of multiplications and additions necessary to express A as $U\Sigma V^T$, where U and V are orthogonal matrices, from the SVD of Q ?

Q4a) Solve the following system using Choleskey's method. (4)

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 15 \\ -x_1 + 5x_2 - 5x_3 &= -35 \\ 3x_1 - 5x_2 + 19x_3 &= 94 \end{aligned}$$

b) Let $U_{n \times n}$ be an upper triangular matrix of rank n . If any arithmetic operation takes 1μ second on a computing resource, compute the time taken to solve the system $Ux = b$, assuming it has a unique solution. What would be the time taken if $U_{n \times n}$ is lower triangular? (1+1)

Q5a) A leading company in Delhi is planning to rent houses and open spaces. The houses are in three categories namely, having three bedrooms, two bedrooms and single bedroom homes. A market survey conducted by a team indicates that a maximum of 650 three bedroom homes, 500 two bedroom homes and 300 single bedroom homes can be rented. Also, the number of three bedroom homes should be at least 60% of the number of two bedroom and single bedroom homes. Open space is proportionate to the number of home units at the rates of at least 10 sq.ft, 15 sq.ft and 18 sq.ft for three bedroom, two bedroom and single bedroom homes respectively. However, land availability limits open space to no more than 10000 sq.ft. The monthly rental income is estimated at Rs. 45000, Rs. 56250 and Rs. 90000 for single bedroom, two bedroom and three bedroom homes respectively. The open space rents for Rs. 7500/sq.ft. Formulate the above as an LPP so as to get maximal revenue. (4)

b) Convert the following problem to standard form explaining the various steps. (2)

$$\begin{aligned} \text{Minimize: } Z &= -3x_1 + x_2 + x_3 \\ \text{Subject to: } x_1 - 2x_2 + x_3 &\leq 11 \\ &-4x_1 + x_2 + 2x_3 \geq 3 \\ &2x_1 - x_3 = -1 \\ &x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

→ All the best ←