PAGE No.
DATE / /

ASSIGNMENT-4 (HW)

Q.1.a) Let P be a real square matrix such that $P=P'$ & $P^2=P$
① Can P have complex eigen values? If so construct, else justify.
② What are eigen values of P?

① Let $\langle \lambda, x \rangle$ be eval, evec pair of P

$$\Rightarrow Px = \lambda x$$

Multiply b.s. by $\bar{x}'$

$$\Rightarrow \bar{x}'Px = \bar{x}'\lambda x = \lambda(\bar{x}'x)$$

$$(\bar{x}'x) = [\bar{x}_1\ \bar{x}_2\ \bar{x}_3 \dots \bar{x}_n]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = x_1\bar{x}_1 + x_2\bar{x}_2 + \dots + x_n\bar{x}_n$$

$$= \sum_1^n x_i\bar{x}_i \Rightarrow \text{Real.}$$

$$\therefore \lambda = \frac{\bar{x}'Px}{\bar{x}'x} \quad \text{— ①}$$

Take complement on b.s.

$$\bar{\lambda} = \overline{\left(\frac{\bar{x}'Px}{\bar{x}'x}\right)} = \frac{x'\bar{P}\bar{x}}{\bar{x}'x}$$

$\bar{P} = P$ ($\because$ P is a real matrix

$$\therefore \bar{\lambda} = \frac{x'P\bar{x}}{\bar{x}'x}$$

$$\bar{\lambda}' = \frac{(x'P\bar{x})'}{\bar{x}'x} = \frac{\bar{x}'P'x''}{\bar{x}'x}$$

$$\therefore \bar{\lambda} = \frac{\bar{x}'Px}{\bar{x}'x} \quad \text{— } (\because P=P' \text{ \& } \bar{\lambda} \text{ being a scalar } \bar{\lambda}^T = \bar{\lambda})$$

$\Rightarrow \bar{\lambda} = \lambda$ — From ①

$\Rightarrow a+ib = a-ib$

$\Rightarrow 2ib = 0$

$\Rightarrow b=0$

$\Rightarrow \lambda$ must be real

$\therefore$ P cannot have complex eigen values.

PAGE No.

DATE / /

(ii) $P^2 = P$ & $P' = P$

Now, let (λ, x) be eval, evec pair of P

$\Rightarrow Px = \lambda x$

Pre-Multiply b.s. by P

$P(Px) = P(\lambda x)$

$\Rightarrow (PP)x = \lambda(Px)$

$\Rightarrow P^2 x = \lambda(Px)$

Subs. $P^2 = P$ & $Px = \lambda x$

$\Rightarrow Px = \lambda(\lambda x)$

But $Px = \lambda x$

$\Rightarrow \lambda x = \lambda^2 x$

$\Rightarrow (\lambda^2 - \lambda)x = 0$

But since (λ, x) evec, eval pair, $x \neq 0$

$\Rightarrow \lambda^2 - \lambda = 0$

$\Rightarrow \lambda(\lambda - 1) = 0$

$\Rightarrow \lambda = 0, 1$

$\therefore$ P has eigen values either 0 or 1

2) Given the matrix $A = \begin{bmatrix} 1 & 2 & r \\ c & 1 & 7 \\ c & 1 & 7 \end{bmatrix}$ where $c, r \in \mathbb{R}$ $r \in [5.5, 6.5]$

and $\lambda_1 = 3$ is one eigen value. Is it possible to compute other eigen values? If so, compute them and give reasons for your answer.

$\because$ A has 2 rows same, the determinant of the matrix must be zero.

$\therefore \det(A) = 0$

But we know $\det(A) = \prod_{1}^{n} \lambda_i$ $i \in [1, n]$ – $n =$ cols of A

$\therefore \lambda_1 \lambda_2 \lambda_3 = 0$

But $\lambda_1 = 3 \Rightarrow \lambda_2 \lambda_3 = 0$ —— (1)

Also trace $(A) = \lambda_1 + \lambda_2 + \lambda_3$

$$1 + 1 + 7 = 3 + \lambda_2 + \lambda_3$$

$$\Rightarrow \lambda_2 + \lambda_3 = 9 - 3 = 6 \quad — ②$$

Now, from ① & ②,

$$\lambda_2 = 0, \lambda_3 = 6 \quad \text{or} \quad \lambda_2 = 6, \lambda_3 = 0$$

$\therefore$ The eigen values of given matrix are $[0, 3, 6]$

Q.3] Construct examples of matrices for which the defect is +ve, -ve & zero.

We know defect Δ = Algebraic multiplicity - Geometric multiplicity

Algebraic multiplicity: How many times a root is rep. (ev)

Geometric multiplicity: How many evecs are deducible for that eval

$$AM \geq GM \Rightarrow AM - GM \geq 0 \Rightarrow \Delta \geq 0$$

$\therefore$ Defect of a matrix can never be zero.

Consider $M_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\lambda_1 = \lambda_2 = \lambda_3 = 1$

Also eigen vectors are $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$

$$\Rightarrow AM = GM = 3$$

$$\therefore \Delta = AM - GM = 3 - 3 = 0$$

Consider $M_{2\times 2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\lambda_1 = 0 \; \lambda_2 = 0$

Consider $\lambda = 0$, we cannot find ~~the~~ two eigen vectors corresponding to this eigen value; only one we can find which is $[1 \;\; 0]$

$$\therefore AM = 2 \quad GM = 1 \Rightarrow \Delta = 2 - 1 = 1 \geq 0.$$

PAGE No.
DATE / /

Q.4] How do you check if a matrix is +ve semi-definite? What can you say about ⟨eval, evec⟩ of symmetric matrices?

If for a given matrix A, (if its symmetric)

$$Q(x) = x^T A x$$

If $Q(x) \geq 0 \ \forall x$, then A is +ve semi-definite

$Q(x) \leq 0 \ \forall x$, then A is -ve semi-definite.

Consider a ~~symmetric~~ matrix A, and an ⟨eval evec⟩ pair

$\Rightarrow$ ~~$Ax = A$~~

~~Also~~ Then we can say

$$Ax = \lambda x$$

$$x'Ax = \lambda x'x \quad \text{— Premultiply by } x'$$

$$\therefore Q(x) = \lambda x'x$$

Let $x = [x_1 \ x_2 \ x_3 \ \dots\dots \ x_n]'$

Then $x'x = x_1^2 + x_2^2 + x_3^2 + \dots\dots + x_n^2$

$\Rightarrow x'x > 0$

$\therefore Q(x) = \lambda$ (Positive qty)

$\Rightarrow Q(x) \geq 0$ iff $\lambda \geq 0$

$\Rightarrow$ ⟨eval, evec⟩ of symmetric matrix A must have all eval ≥ 0 for A to be +ve semi-definite.

$\Rightarrow$ Similarly, ⟨eval, evec⟩ pairs of symmetric A must have all it's eval ≤ 0 for A to be -ve semidefinate.

$\Rightarrow$ There must be some eigen values +ve & some eigen values -ve for the symmetric matrix A to be indefinite.