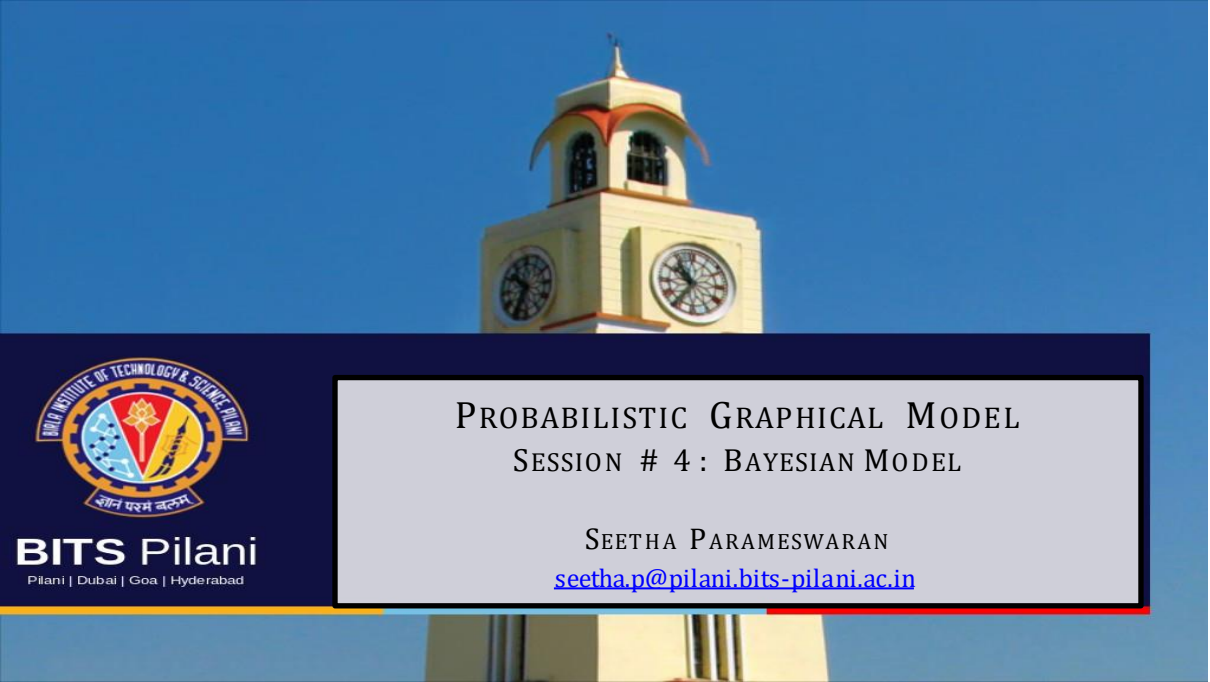

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Probabilistic Graphical Model

Session # 4 : Bayesian Model

Seetha Parameswaran
seetha.p@pilani.bits-pilani.ac.in

The instructor is gratefully acknowledging the authors who made their course materials freely available online.

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BAYESIAN NETWORK

DEFINITION (GLOBAL SEMANTICS)

A Bayesian Network is a directed acyclic graph G whose nodes represent the random variables $\{X_1, X_2, \ldots X_n\}$ and represents a joint distribution via the chain rule for the Bayesian Networks.

$$P(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | Pa(X_i)) \quad (1)$$

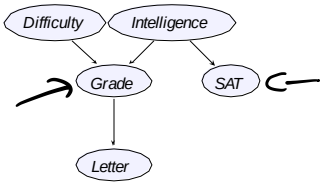
- Each node is associated with a CPD.

$$CPD(X_i) = P(X_i | Pa(X_i))$$

BAYESIAN NETWORK IS LEGAL

A BN is a legal distribution; if

- $P \geq 0$
  - P is a product of CPDs.
  - CPDs are non negative.
- $\sum P = 1$



Bayesian Network is Legal

A BN is a legal distribution; if

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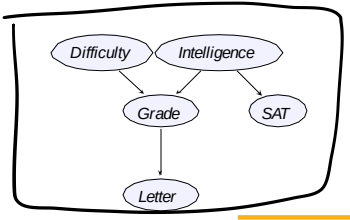

$$\sum P = P(I, D, G, S, L)$$

$$= \sum_{D,I,G,S,L} P(I)P(D)P(G|I,D)P(S|I)P(L|G)$$

$$= \sum_{D,I,G,S} P(I)P(D)P(G|I,D)P(S|I) \sum_{L} P(L|G)$$

$$= \sum_{D,I,G} P(I)P(D)P(G|I,D) \sum_{S} P(S|I)$$

$$= \sum_{D,I} P(I)P(D) \sum_{G} P(G|I,D)$$

$$= \left(\sum_{I} P(I)\right)\left(\sum_{D} P(D)\right) = 1$$

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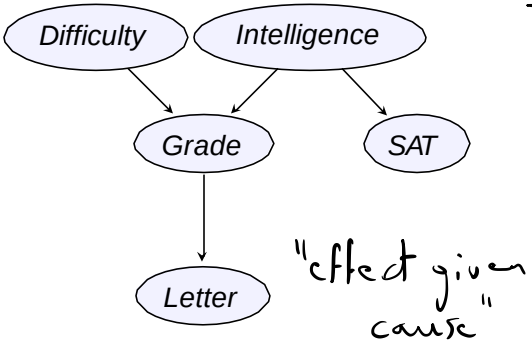
Reasoning Patterns

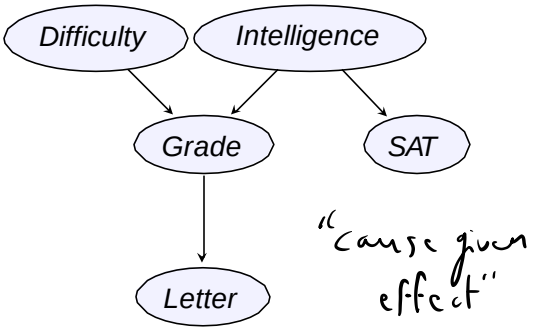
1. Causal reasoning
2. Evidential reasoning
3. Intercausal reasoning

CAUSAL REASONING



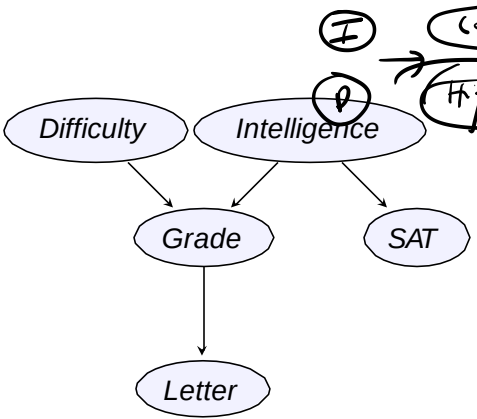
- How likely will a student get a strong recommendation?
  $P(l^1) = ?$
- Given that the student is not so intelligent, what is chance that he gets a strong letter?
  $P(l^1 | i^0) = ?$
- What if the course is easy?
  $P(l^1 | i^0, d^0) = ?$
- Queries that predict the effects of various factors or features are called causal reasoning.

EVIDENTIAL REASONING



- Given that a student gets C grade for a course, comment on his intelligence.
  $P(i^1|g^3) = ?$
- Given that the student got a weak letter, comment on his intelligence.
  $P(i^1|l^0) = ?$
- $P(i^1|l^0, g^3) = ?$
- Queries that reason from effects to causes are called evidential reasoning.

INTERCAUSAL REASONING



- Given that a student gets C grade for a course, and a high SAT score, comment on his intelligence.
  $P(i^1 | g^3, s^1) = ?$
- Does this give any idea regarding the difficulty of the course?
  $P(d^1 | g^3, s^1) = ?$
- Explained away the poor grade via difficulty of the class.
- Explaining away is an instance of intercausal reasoning, where different causes of the same effect can interact.

Causal Reasoning Example

In the absence of any other information

$P(B_{Student}(l^1)) = 0.502$

How do we calculate this? → From CPDs in the Bayesian network.

Causal Reasoning Example

From the CPDs we have

$$P(l^1) = P(l^1/g^1)P(g^1) + P(l^1/g^2)P(g^2) + P(l^1/g^3)P(g^3)$$

$$P(g^1) = P(g^1/i^0,d^0)P(i^0,d^0) + P(g^1/i^0,d^1)P(i^0,d^1) + P(g^1/i^1,d^0)P(i^1,d^0) + P(g^1/i^1,d^1)P(i^1,d^1)$$

Causal Reasoning Example

$P(i^0, d^0) = P(i^0)P(d^0)$

We have $P(g^1) = (0.3)(0.42) + (0.05)(0.28) + (0.9)(0.18) + (0.5)(0.12) = 0.362$

$P(g^2) = (0.4)(0.42) + (0.25)(0.28) + (0.08)(0.18) + (0.3)(0.12) = 0.2884$

$P(g^3) = (0.3)(0.42) + (0.7)(0.28) + (0.02)(0.18) + (0.2)(0.12) = 0.3496$

Causal Reasoning Example

$$P(L) = (0.9)(0.362) + (0.6)(0.2874) + (0.01)(0.349)$$

$$\approx 0.502$$

If we know that George is not so intelligent, what is the probability that he gets a strong letter of recommendation?

$$P(L \mid \neg I) \rightarrow ?$$

Causal Reasoning Example

$$P(l^1, i^0) = P(l^1, i^0, d^0) + P(l^1, i^0, d^1) \quad \text{[Marginalization]}$$

$$P(l^1, i^0, d^0) = P(l^1 | g^1) P(g^1 | i^0, d^0) P(i^0, d^0) + P(l^1 | g^2) P(g^2 | i^0, d^0) P(i^0, d^0)$$
$$+ P(l^1 | g^3) P(g^3 | i^0, d^0) P(i^0, d^0)$$

Substituting for the various probabilities we have

$$P(l^1, i^0, d^0) = 0.21546$$

Causal Reasoning Example

Similarly $P(l^1, i^0, d^1) = 0.056$

$P(l^1, i^0) = P(l^1, i^0, d^0) + P(l^1, i^0, d^1) = 0.27146$

$P(l^1 | i^0) = \frac{P(l^1, i^0)}{P(i^0)} = \frac{0.27146}{0.7} \approx 0.389$

The probability that the student gets a good letter of recommendation goes down given low intelligence

Evidential Reasoning Example

Suppose a student received a grade C → g^3

What is the probability of high intelligence now?

$P(i^1/g^3) = 0.079$ (cause → effect)

"cause given effect"

$P(d^1) = 0.40$ but $P(d^1/g^3) = 0.62$

$P(i^1/g^3, s^1) = 0.578$] grade is poor, but SAT score is high, s^1 = high intelligence is favoured

Intercausal Reasoning

$P(i^1 / g^3) = 0.079$

If we discover that the subject is hard $\rightarrow d^1$

$P(i^1 / g^3, d^1) = 0.11 \rightarrow$ this is a partial explanation for the student's low grade (the student can be intelligent and have still got a poor grade because the subject was hard)

Intercausal Reasoning

If the student gets a B grade (g^2):

$$P(i^1 / g^2) = 0.175$$

$$P(i^1 / g^2, d^1) = 0.34$$

We have explained away the poor grade using the difficulty of the class.

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DEPENDENCY IN BN

- A node depends directly only on its parents.
- If the student's grade is known, the quality of his recommendation letter is not influenced by information about any other variable. L is conditionally independent of all other nodes in the network given its parent G.

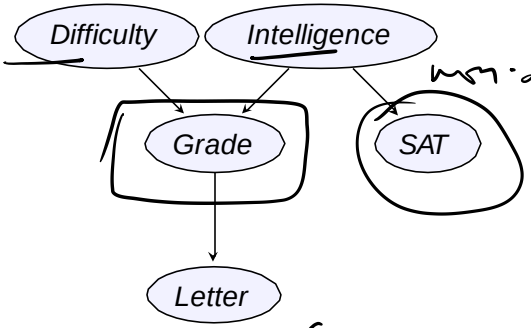
$(L \perp I, D, S \mid G)$

$(L \perp I, D \mid G)$

- The student's SAT score depends only on his intelligence. S is conditionally independent of all other nodes in the network given its parent I.

$$(S \perp D, G, L \mid I)$$

DEPENDENCY IN BN



- Given the parents, a node can depend on its descendants.

non-descendant

G dependent on L

- Does *G* depend on *S*, given *I* and *D*?

$$(G \perp S | I, D)$$

- For *D*, both *I* and *S* are non descendants.

$$P(S/I,G,D) = P(S/I)$$

$$(D \perp I, S)$$

Dependency in BN

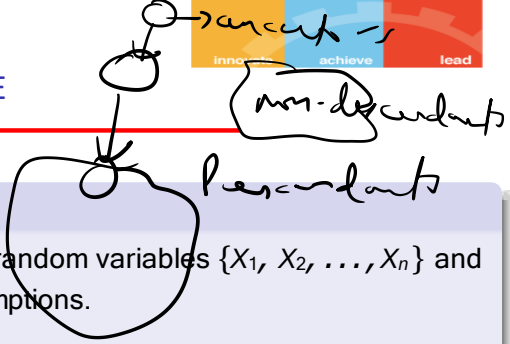
Is G independent of S given its parents I and D?

No. We have:

$P(g^1 / i^1, d^1, s^1) > P(g^1 / i^1, d^1)$

If we know that the student got a strong letter of recommendation, it increases the probability that the student had a good grade

Bayesian Network Structure



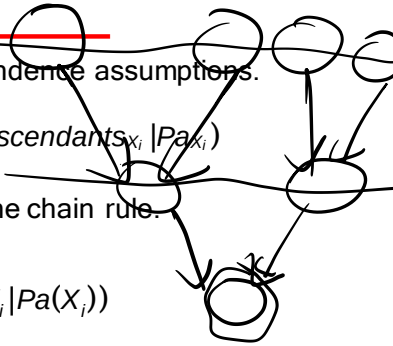
Definition (Local Semantics)

A directed acyclic graph $\mathbb{G}$ whose nodes represent random variables $\{X_1, X_2, \ldots, X_n\}$ and $\mathbb{G}$ encodes a set of conditional independence assumptions.

For each variable X_i : $(X_i \perp NonDescendants_{X_i} | Pa_{X_i})$ (2)

- Pa_{X_i} represent parents of X_i in $\mathbb{G}$.
- $NonDescendants_{X_i}$ represent the random variables that are not descendants of X_i.
- $\mathrm{I}_l(\mathbb{G})$ represents the set of conditional independence assumptions called local independencies.

BAYESIAN NETWORK SEMANTICS



LOCAL SEMANTICS BN encodes a set of conditional independence assumptions.

For each variable X_i : $(X_i \perp NonDescendants_{X_i} | Pa_{X_i})$

GLOBAL SEMANTICS BN represents a joint distribution via the chain rule:

$$P(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | Pa(X_i))$$

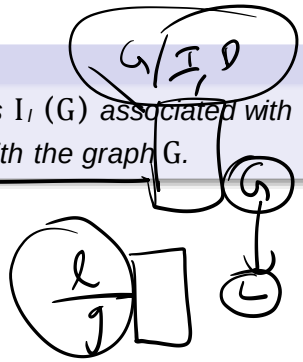
MARKOV BLANKET A node is conditionally independent of all other nodes in the Baysian Network, given its parents, children and children's parents.

For each variable X_i : $(X_i \perp other\ nodes | Pa(X_i), Ch(X_i), Pa(Ch(X_i))$

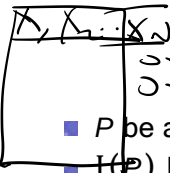
Independency Map

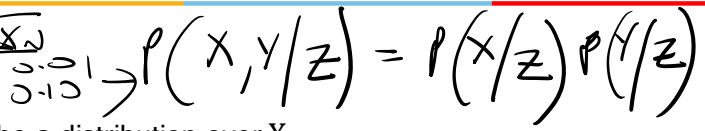
Theorem

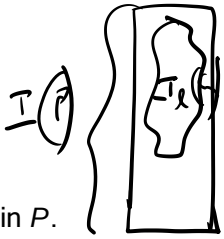
A distribution P satisfies local independencies $I_l(G)$ associated with G if and only if P is representable as a set of CPDs associated with the graph G.



Independency Map or I-map



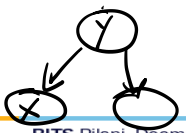




- P be a distribution over X.
- $\mathrm{I}(P)$ be the set of independence assertions $(X \perp Y | Z)$ that hold in P.
- Any independence that G asserts must also hold in P.

Definition

G is called an I-map for P if $\mathrm{I}_I(\mathrm{G}) \subseteq \mathrm{I}(P)$.



$(X \perp Y) | Z$

I-Map Example 1

X	Y	$P(X, Y)$
x^0	y^0	0.08
x^0	y^1	0.32
x^1	y^0	0.12
x^1	y^1	0.48

- Is G_ϕ: $X \perp Y$ an I-map of P ?

I-MAP EXAMPLE 1

X	Y	$P(X,Y)$
x^0	y^0	0.08
x^0	y^1	0.32
x^1	y^0	0.12
x^1	y^1	0.48

- Is $G_\phi : X \perp Y$ an I-map of P ?

G_ϕ encodes the assumption that $X \perp Y$.

- $P(x^1) = 0.48 + 0.12 = 0.60$
- $P(y^1) = 0.32 + 0.48 = 0.80$
- $P(x^1, y^1) = 0.48 = P(x^1)P(y^1)$
- Hence X and Y are independent i.e $(X \perp Y)$
- $(X \perp Y) \in I(P)$.
- G_ϕ is an I-map of P.

Similarly $P(x^0, y^0) = P(x^0)P(y^0)$
$P(x^0, y^1) = P(x^0)P(y^1)$ & $P(x^1, y^0) = P(x^1)P(y^0)$

I-MAP EXAMPLE 2

X	Y	$P(X, Y)$
x^0	y^0	0.4
x^0	y^1	0.3
x^1	y^0	0.2
x^1	y^1	0.1

$P(X, Y) \stackrel{?}{=} P(X) P(Y)$

- Is $G_\phi : X \perp Y$ an I-map of P ?

I-MAP EXAMPLE 2

X	Y	$P(X, Y)$
x^0	y^0	0.4
x^0	y^1	0.3
x^1	y^0	0.2
x^1	y^1	0.1

- Is $\mathrm{G}_\phi : X \perp Y$ an I-map of P ?

- $P(x^1) = 0.2 + 0.1 = 0.3$
- $P(y^1) = 0.3 + 0.1 = 0.4$
- $P(x^1, y^1) \neq P(x^1)P(y^1)$
- Hence X and Y are not independent.
- $(X \perp Y) \notin \mathrm{I}(P)$.
- G_ϕ is not an I-map of P.

STUDENT EXAMPLE - B *Student*

We know independence assumptions in $\mathbb{G}$

$$
\begin{aligned}
(D \perp I) &\Longrightarrow P(D|I) = P(D) \\
(L \perp I, D|G) &\Longrightarrow P(L|I,D,G) = P(L|G) \\
(S \perp D, G, L|I) &\Longrightarrow P(S|I,D,G,L) = P(S|I)
\end{aligned}
$$

$$
\begin{aligned}
P(I, D, G, S, L) &= P(I)P(D|I)P(G|I, D)P(L|I, D, G)P(S|I, D, G, L) \quad \text{(by chain rule)} \\
&= P(I)P(D)P(G|I, D)P(L|I, D, G)P(S|I, D, G, L) \\
&= P(I)P(D)P(G|I, D)P(L|G)P(S|I, D, G, L) \\
&= P(I)P(D)P(G|I, D)P(S|I)P(L|G)
\end{aligned}
$$

A distribution P factorizes $\mathbb{G}$ if P can be represented as a chain rule of CPDs

P FACTORIZES OVER $\mathbb{G}$

DEFINITION

Bayesian Network graph $\mathbb{G}$ over the variables $X_1, X_2, \ldots, X_n$ with a distribution P over the same space factorizes according to $\mathbb{G}$ if P can be expressed as a product of its CPDs.

$$P \quad \text{factorizes over } \mathbb{G} \quad \text{if} \quad P(X_1, X_2, \ldots, X_n) = \prod_i P(X_i | Pa^{\mathbb{G}}(X_i)) \tag{3}$$

THEOREM

For a Bayesian Network graph $\mathbb{G}$ over the variables $X_1, X_2, \ldots, X_n$ with a distribution P over the same space and $\mathbb{G}$ is an I-map for P, then P factorizes $\mathbb{G}$.

$\rightarrow \mathcal{G}$ is an IMAP for $P \Rightarrow P$ factorizes $\mathcal{G}$

Proof of the theorem

Assume that $X_1, X_2 \ldots X_n$ form a topological ordering (parents of X_i have subscripts smaller than i; children of X_i have larger subscripts)

Using the chain rule

$$P(X_1, X_2, \ldots X_n) = P(X_1) P(X_2/X_1) P(X_3/X_1, X_2) \ldots P(X_i/X_1, X_2, \ldots X_{i-1}) \ldots$$

Proof of the theorem

Consider one of the factors, $P(X_i | X_1, X_2, \ldots X_{i-1})$

Since G is an I-map for P we have

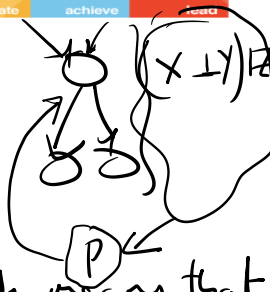
$(X_i \perp \text{NonDescendants } X_i) \mid Pa_{X_i}^G$

Now all of X_i's parents are in the set $\{X_1, X_2 \ldots X_{i-1}\}$ and none of X_i's descendants can be in this set

Proof of the theorem

$\{X_1, X_2, \ldots X_{i-1}\} = Pa(X_i) \cup Z$ where

$Z \subseteq$ Non Descendants X_i



We know that $(X \perp Z \mid Pa(X_i))$ which means that

$$P(X_i \mid Pa(X_i) \cup Z) = P(X_i \mid Pa(X_i))$$

$\pi P(X_i \mid Pa(X_i))$

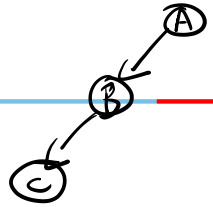
Now apply this to all the factors in the chain rule to see that P factorizes G.

BAYESIAN NETWORK - ANOTHER DEFINITION

DEFINITION

A Bayesian network is a pair $\mathrm{B} = (\mathrm{G}, \mathrm{P})$ where P factorizes over G and where P is specified as a set of CPDs associated with G.

G IS AN I-MAP OF *P*



$A \perp B$

$A \not\perp B$

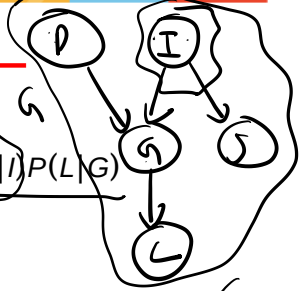
THEOREM

For a Bayesian Network graph G *over the variables* $X_1, X_2, \ldots, X_n$ *with a distribution P over the same space and if P factorizes according to* G, *then* G *is an I-map for P.*

G is an IMap for P ⇒ P factorizes according to G

P factorizes according to G ⇒ G is an IMAP for P

STUDENT EXAMPLE - B *Student*



Given: P factorizes according to G

By chain rule $P(I, D, G, S, L) = P(I)P(D)P(G|I, D)P(S|I)P(L|G)$

By definition $P(S|I, D, G, L) = \frac{P(I, D, G, S, L)}{P(I, D, G, L)}$

Marginalize over S $P(I, D, G, L) = \sum_S P(I, D, G, S, L)$

$= P(I)P(D)P(G|I, D)P(L|G) \sum_S P(S|I)$

$= P(I)P(D)P(G|I, D)P(L|G)$

$P(A)P(B|A) = P(A, B)$

$\prod P(X_i | P_a(X_i))$

$\sum P(S|I)$?

STUDENT EXAMPLE - B *Student*

$$P(S|I,D,G,L) = \frac{P(I,D,G,S,L)}{P(I,D,G,L)}$$
$$= \frac{P(I)P(D)P(G|I,D)P(S|I)P(L|G)}{P(I)P(D)P(G|I,D)P(L|G)}$$
$$= P(S|I)$$
$$(S \perp D,G,L|I)$$

$P(S/I,D,G,L) = P(S/I)$

$(S \perp D,G,L|I)$

Independence assumption holds. G is an I-map for P.

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Thank You!!!