Probabilistic Graphical Model

Session # 3 : Bayesian Model

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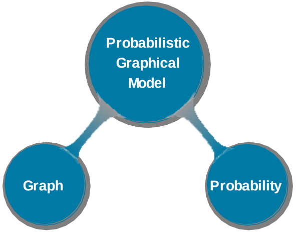
PROBABILISTIC GRAPHICAL MODELS

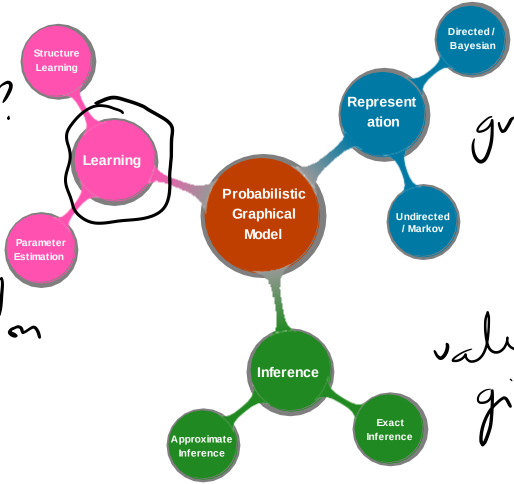
- Probabilistic Graphical Model is a model that is standalone, where probability distributions and its semantics represent uncertainty about state of world.



Components of Probabilistic Graphical Model

how are nodes related ?

values of probability distribution



graph structure

values of variables given other variables

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STUDENT EXAMPLE

- Model the difficulty of a course, intelligence of students, Grade the students score in a particular course.
- Let D represent the difficulty of a course.

$$Domain\ of\ D = \{easy, hard\} = \{d^0, d^1\}$$
$$P(D) = \{0.6, 0.4\}$$

- Let I represent the intelligence of a student.

$$Domain\ of\ I = \{low, high\} = \{i^0, i^1\}$$
$$P(I) = \{0.7, 0.3\}$$

STUDENT EXAMPLE

- Let G represent the grade a student gets for a course.

$$Domain\ of\ G = \{A, B, C\} = \{g^1, g^1, g^2\}$$

- How do we represent Joint distribution of the 3 random variables? How many parameters are required?
- $P(I, D, G)$ denotes the probabilities of all combinations of the values of the 3 random variables.
- These $\underline{2 * 2 * 3} = 12$ parameters can be represented using a Joint Distribution.

STUDENT EXAMPLE - JOINT DISTRIBUTION

I	D	G	$P(I, D, G)$
i^0	d^0	g^1	0.126
		g^2	0.168
		g^3	0.126
i^0	d^1	g^1	0.009
		g^2	0.045
		g^3	0.126
i^1	d^0	g^1	0.252
		g^2	0.0224
		g^3	0.0056
i^1	d^1	g^1	0.060
		g^2	0.036
		g^3	0.024

(i^0, d^0, g^1)

(i^0, d^0, g^2)

$= 1$

What is the sum of the joint distribution?

$$\sum P(I, D, G) = 1 \quad (1)$$

Operations on Joint Distribution

1. Conditioning
2. Renormalization
3. Marginalization

1. Conditioning on Joint Distribution

- Suppose a student score 'A' grade.
- Observation: $G = g^1$.
- This conditioning gives a reduced Joint distribution.
- Conditioning reduces Joint distribution.

I	D	G	$P(I, D, g^1)$
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.060

What is sum of the distribution now?

$$\sum P(I, D, g^1) \neq 1 \qquad (2)$$

2. Renormalization of Conditioned JD

I	D	G	$P(I, D, g^1)$
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.060
			$\boxed{0.447}$

$\xrightarrow{\text{normalize}}$

I	D	G	$P(I, D \mid g^1)$
i^0	d^0	g^1	$0.126/0.447 = 0.282$
i^0	d^1	g^1	$0.009/0.447 = 0.020$
i^1	d^0	g^1	$0.252/0.447 = 0.564$
i^1	d^1	g^1	$0.060/0.447 = 0.134$

1

$P(i^0, d^0 / g^1) = 0.282$

$$P(I, D, g^1) \xrightarrow{\text{normalize}} \boxed{P(I, D \mid g^1)} \quad (3)$$

$P(i^0, d^1 / g^1) = 0.020$

3. Marginalization on JD

Marginalization on JD = Summing Out

I	D	$P(I,D)$
i^0	d^0	0.282
i^0	d^1	0.020
i^1	d^0	0.564
i^1	d^1	0.134

D	$P(D)$
d^0	0.846
d^1	0.154

$$P(D = d^0) = P(I = i^0, D = d^0) + P(I = i^1, D = d^0)$$

$$P(D = D_0) = \sum_I P(I, D = D_0)$$

$$\sum_I P(I,D) = P(D) \qquad (4)$$

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FACTOR

$$P(A,B,C,D) = \frac{1}{Z}\,\phi_1(A,B)\,\phi_2(B,C)\,\phi_3(C,D)\,\phi_4(D,A)$$

normalizing

- A factor Φ is a function or a table that maps a set of random variables to a real value.

$$\Phi : Val(X_1, \ldots, X_n) \rightarrow \mathbb{R} \tag{5}$$

- The argument of the factor is called scope of the factor.

$$Scope : \{X_1, \ldots, X_n\} \tag{6}$$

- Factors are building blocks used for defining high dimensional spaces and distributions.
- Factors are used to define an exponentially large probability distribution of N random variables.

$$\hat{2^n} \rightarrow P(X_1, X_2, \ldots X_n) = \phi_1(X_1, X_2)\,\phi_2(X_2, X_3)\,\phi_3(X_3, X_4)$$

- Factors are manipulated in the same way as probability distributions.

4×3

Joint Distribution is a Factor

I	D	G	$P(I, D, G)$
i^0	d^0	g^1	0.126
		g^2	0.168
		g^3	0.126
i^0	d^1	g^1	0.009
		g^2	0.045
		g^3	0.126
i^1	d^0	g^1	0.252
		g^2	0.0224
		g^3	0.0056
i^1	d^1	g^1	0.060
		g^2	0.036
		g^3	0.024

$Scope : \{I, D, G\}$

Unnormalized Conditioned JD is a Factor

I	D	G	$P(I, D, g^1)$
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.060
			0.447

$\phi(i^0, d^0, g^1) = 0.126$

$P(i^0, d^1, g^1) = 0.009$

Scope : $\{I, D\}$

not $\{I, D, G\}$

CONDITIONAL PROBABILITY DISTRIBUTION

- CPD is a factor, which gives the conditional probability of a random variable, when other random variables are observed or known.
- For every combination of I and D, the value of G is observed.

$P(G|I,D)$

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

$\leftarrow P(G/i^0, d^0)$

- Each row sums to 1.

$$\sum P i^1, d^1 = 1$$

Operations on Factors

1. Factor Product
2. Factor Marginalization
3. Factor Reduction

1. Factor Product

- Factor product is the cross product of two factors.

$\{A, B\} \cup \{B, C\} = \{A, B, C\}$

$\phi_1(A,B) * \phi_2(B,C)$

A	B	$\Phi_1(A,B)$
a^1	b^1	0.5
a^1	b^2	0.8
a^2	b^1	0.2
a^2	b^2	0

B	C	$\Phi_2(B,C)$
b^1	c^1	0.5
b^1	c^2	0.7
b^2	c^1	0.1
b^2	c^2	0.2

A	B	C	$\Phi_3(A,B,C) = \Phi_1 * \Phi_2$
a^1	b^1	c^1	$0.5 * 0.5 = 0.25$
a^1	b^1	c^2	$0.5 * 0.7 = 0.35$
a^1	b^2	c^1	$0.8 * 0.1 = 0.08$
a^1	b^2	c^2	$0.8 * 0.2 = 0.16$
a^2	b^1	c^1	$0.2 * 0.5 = 0.25$
a^2	b^1	c^2	$0.2 * 0.7 = 0.35$
a^2	b^2	c^1	$0 * 0.1 = 0$
a^2	b^2	c^2	$0 * 0.2 = 0$

2. Factor Marginalization

- Remove one random variable.

A	B	C	$\Phi_1(A,B,C)$
a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.25
a^2	b^1	c^2	0.35
a^2	b^2	c^1	0
a^2	b^2	c^2	0

$$\phi_2(A,C) = \sum_B \phi_1(A,B,C)$$

A	C	$\Phi_2(A,C)$ marginalized on B
a^1	c^1	$0.25 + 0.08 = 0.33$
a^1	c^2	$0.35 + 0.16 = 0.51$
a^2	c^1	$0.25 + 0 = 0.25$
a^2	c^2	$0.35 + 0 = 0.35$

3. Factor Reduction

- Extract only one random variable.
- Observe $C = c^1$.

A	B	C	$\Phi_1(A,B,C)$
a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.25
a^2	b^1	c^2	0.35
a^2	b^2	c^1	0
a^2	b^2	c^2	0

A	B	C	$\Phi_1(A,B,\boxed{c^1})$
a^1	b^1	c^1	0.25
a^1	b^2	c^1	0.08
a^2	b^1	c^1	0.25
a^2	b^2	c^1	0

Factors and JPDF

Let us say we have 4 random variables A, B, C, D and factors defined over them as follows:

	$\phi_1[A, B]$
a^0 b^0	30
a^0 b^1	5
a^1 b^0	1
a^1 b^1	10

	$\phi_2[B, C]$
b^0 c^0	100
b^0 c^1	1
b^1 c^0	1
b^1 c^1	100

	$\phi_3[C, D]$
c^0 d^0	1
c^0 d^1	100
c^1 d^0	100
c^1 d^1	1

	$\phi_4[D, A]$
d^0 a^0	100
d^0 a^1	1
d^1 a^0	1
d^1 a^1	100

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Example JPDF using factors

$$P(A, B, C, D) = \frac{1}{Z} \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(A, D)$$

Z = partition function used to normalize the probabilities

$$Z = \sum_{A, B, C, D} \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(A, D)$$

Example JPDF using factors

What is the probability associated with the assignment $(a^0\ b^0\ c^0\ d^0)$?

$$= \phi_1(a^0\ b^0)\ \phi_2(b^0\ c^0)\ \phi_3(c^0\ d^0)\ \phi_4(d^0\ a^0)$$

$$= 30 \times 100 \times 1 \times 100 = 300000$$

After Normalization $\rightarrow$ 0.04

$f(x,y) = f_x(x)\, f_y(y)$ $\quad e^{x^2+y^2+xy}$

Example JPDF using factors

$e^{x^2+xy}\, e^{y^2}$

There is a tight connection between independence properties

$$\underline{X \perp Y \mid Z} \iff P(X,Y,Z) = \underbrace{\phi_1(X,Z)}\, \underbrace{\phi_2(Y,Z)}$$

"X is independent of Y given Z"

When Z takes a fixed value ϕ_1 and ϕ_2 are separable functions.

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INDEPENDENCE

- Independent parameters are parameters whose values are not completely determined by the values of the other parameters.
- Random variables $X = \{X_1, X_2, \ldots, X_n\}$ can be considered independent if

$$\underline{P(\{X_1, X_2, \ldots, X_n\})} = \underline{\overset{n}{P(X_1)}\overset{n}{P(X_2)} \ldots \overset{n}{P(X_n)}} \quad n^2 \text{ values} \tag{7}$$

$$P(\{X_1, X_2, \ldots, X_n\}) = \prod_{i=1}^{n} P(X_i) \tag{8}$$

- A set of random variables are independent of each other, if their joint probability distribution is equal to the product of probabilities of each individual random variable.

Another Perspective

$$P(A/B) = \frac{P(A,B)}{P(B)} \quad \text{(by definition)}$$

$$P(A,B) = \underline{P(A)\,P(B)} \quad \text{whenever } A \,\&\, B \text{ are independent}$$

$$\therefore \; P(A,B) = P(A/B)\,\cancel{P(B)} = P(A)\,\cancel{P(B)}$$

$$\Rightarrow \boxed{P(A) = P(A/B)}$$

[Knowing B does not change the probability of A]

STUDENT EXAMPLE

- A company is trying to hire a recent intelligent college graduate. The company has access to the student's SAT scores.
- The probability space is induced by Intelligence I and SAT score S.

$$I = \{high, low\} = \{i^1, i^0\}$$
$$S = \{high, low\} = \{\underline{s}^1, \underline{s}^0\}$$

Student Example - Joint Distribution

The joint distribution of $P(I, S)$ is given as

I	S	$P(I, S)$
i^0	s^0	0.665
i^0	s^1	$\boxed{0.035}$
i^1	s^0	$\boxed{0.06}$
i^1	s^1	$\boxed{0.24}$

1

Student Example - Conditional Distribution

- The student's SAT score is determined by his intelligence. This represents causality.

$$I \longrightarrow S$$

- Joint distribution $P(I, S)$ can be computed by using chain rule.

$$\boxed{P(I, S) = P(I)P(S \mid I)}$$

$P(I)$

i^0	i^1
0.7	0.3

$P(S \mid I)$

I	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

$= 1$

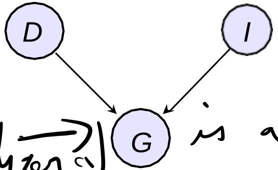
$P(s^0/i^0) = 0.95$

$P(s^1/i^0) = 0.05$

Student Example - Conditional Distribution

- The grade student score depends on her intelligence and the difficulty of the course. (by intuition)

$Pa(Grade) = \{Difficulty, Intelligence\}$



is a random variable

- Joint distribution $P(I, D, G)$ can be computed by using chain rule.

$P(I, D, G) = P(I)\,P(D/I)\,P(G/D,I)$

$$P(I,D,G) = \boxed{P(I)P(D)P(G \mid D,I)}$$

$P(D/I) = P(D)$

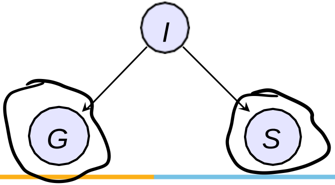
What can we say about D and I?

STUDENT EXAMPLE - CONDITIONAL INDEPENDENCE

- With 3 random variables, Intelligence I, Grade G and SAT score S, the JD has 12 entries.
- Both the SAT score and the grade are highly correlated on student's intelligence.
- If I is known, knowing $Grade = A$ no longer gives information that $S = high$.
- If I is known, knowing $S = high$ no longer gives information that $Grade = A$.

$$S \perp G \mid I$$

- The student's intelligence is the only reason why his grade and SAT score might be correlated.



$S = A + I$

$G = B + I$

Student Example

Another way to look at this situation:

$$\boxed{P(S/G, I) = P(S/I)} \longrightarrow S \perp G / I$$

If we know the student's intelligence then knowing his Grade will give no further information about his SAT score.

STUDENT EXAMPLE - CONDITIONAL INDEPENDENCE

- Joint distribution $P(I, S, G)$ can be computed by using chain rule.

$$P(I,S,G) = P(I)P(S,G \mid I)$$ ← [Bayes rule]

$$\boxed{P(S,G|I) = P(S|I)P(G \mid I)}$$ ← why is this true?

$$P(I,S,G) = \boxed{P(I)P(S \mid I)P(G \mid I)}$$

$P(I,S,G)$
$= P(I)\, P(G/I)\, P(S/G,I)$
$P(S/G,I) = P(S/I) \Rightarrow P(I)\, P(G/I)\, P(S/I)$

- 3 CPDs fully specify the JD.

$P(I)$

i^0	i^1
0.7	0.3

$P(S \mid I)$

I	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

$P(G \mid I)$

I	g^1	g^2	g^3
i^0	0.2	0.34	0.46
i^1	0.74	0.17	0.09

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STUDENT EXAMPLE

Difficulty of course D	$Val(D) = \{hard, easy\}$	$\{d^1, d^0\}$
Intelligence I	$Val(I) = \{high, low\}$	$\{i^1, i^0\}$
Grade G	$Val(G) = \{A, B, C\}$	$\{g^1, g^2, g^3\}$
SAT score S	$Val(S) = \{high, low\}$	$\{s^1, s^0\}$
Recommendation Letter L	$Val(I) = \{strong, weak\}$	$\{l^1, l^0\}$

- Joint distribution is given by

$$P(D, I, G, S, L)$$

- JD $= 2 * 3 * 2 * 2 * 2 = 48$ entries.

Student Example

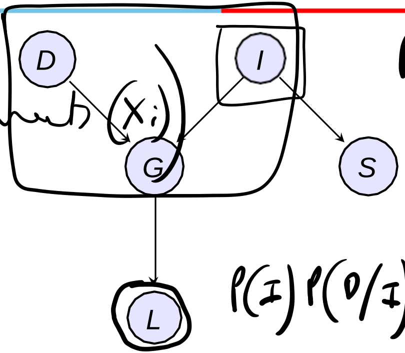
- Assume that the grade depends on *Difficulty* of the course and *Intelligence* of the student.
- The *SAT* score depends on *Intelligence* of the student
- Assume that the quality of the Recommendation *Letter* depends on *Grade*.

Student Example

$\sum P(G/I,D) = 1$ 48 entries

$2 \times 2 \times 2 \times 3 \times 2 = 48$

$\prod P(X_i / \text{Parents}(X_i))$



$P(S/I,G,D) = P(S/I)$

$P(L/G,S,I,D) = P(L/G)$

$P(L/G,S,I,D) = P(L/G)$

$P(I)P(D/I)P(G/I,D)P(S/I,G,D)P(L/G,S,I,D)$

$$P(I, D, G, S, L) = P(I)P(D)P(G \mid I, D)P(S \mid I)P(L \mid G)$$

how did we get this?

How many parameters?

- Parameters = 1 + 1 + 8 + 2 + 3 = 15 entries

Non-redundant

STUDENT EXAMPLE - $B^{Student}$

$P(X/Pa(X))$

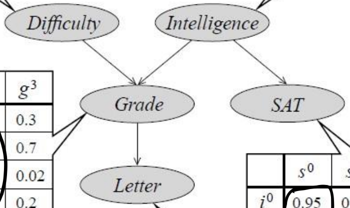
$P(\text{node}/\text{parent})$

$P(D)$ — 1

d^0	d^1
0.6	0.4

$P(I)$ — 1

i^0	i^1
0.7	0.3



$P(G/I,D)$ — 8

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

$P(S/I)$ — 2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

$P(L/G)$ — 3

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

circled values = non-redundant parameters

$1 + 1 + 8 + 2 + 3 = 15$

BAYESIAN NETWORK

- A Bayesian Network is a data structure to represent dependencies among random variables.
- Compact and natural representation.
- Represented using Directed acyclic graph (DAG) G
  - Each node is a random variable.
  - A set of directed edge connects pairs of nodes. Edges correspond to direct influence of one node on another.
- A data structure that provides the skeleton for representing a joint distribution compactly in a factorized way.
- A compact representation for a set of conditional independence assumptions about a distribution.

Bayesian Network - Topology

- Topology specifies the conditional independencies.

$$\text{Cause} = \text{Parent}(\text{Effects})$$

- A Bayesian network represents the joint distribution of all random variables.
- Network structure together with its CPDs is called a Bayesian network or local probability model.

$$P(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} P(X_i \mid Pa(X_i)) \tag{9}$$

Bayesian Network - Construction

$G \to L$

G is a parent of letter

1. Nodes
   - Determine the set of random variables that are required to model the domain.
   - Order them such that the causes precedes the effects.

$$\{X_1, \ldots, X_n\}$$

$X_i \to X_j$ if $i < j$

2. Links: For each node X_i,
   - Choose a set of parents $Pa(X_i)$.
   - For each parent, insert a link from the Parent to the node X_i.
   - Write down the conditional probability table $P(X_i \mid Pa(X_i))$.

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Restaurant Example

- Let Q represent the random variable for the quality of food.

Q	Good	Average	Bad
$P(Q)$	0.3	0.5	0.2

- Let L represent the random variable for the location of restaurant.

L	Good	Bad
$P(L)$	0.6	0.4

- Random variables Q and L are independent of each other.

RESTAURANT EXAMPLE

- Let C represent the cost of food.

$$C = \{high, low\}$$

- Cost C is dependent on the quality Q of food and the location L of the restaurant.
- Let N represent the number of people visiting the restaurant.

$$N = \{high, low\}$$

- N is affected by C which in turn is affected by Q .

Restaurant Example

- What is the size of joint distribution $P(Q, L, C, N)$?
- List all the independencies and conditionally dependencies.
- Draw the Bayesian Network.
- How many parameters are required to represent $P(Q, L, C, N)$?
- Write the expression for $P(Q, L, C, N)$.

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RESTAURANT EXAMPLE

- What is the size of joint distribution $P(Q, L, C, N)$?

$$3 * 2 * 2 * 2 = 24$$

- How many parameters are required to represent $P(Q, L, C, N)$?

$$\boxed{(3-1)+(2-1)+(6-2)+(4-1)=10} \times$$

$$2+1+6+4=13$$

- Write the expression for $P(Q, L, C, N)$.

  According to Bayesian Network ,

$$P(Q, L, C, N) = P(Q)P(L)P(C|L, Q)P(N|C, L)$$

RESTAURANT EXAMPLE

- List all the independencies and conditionally dependencies.

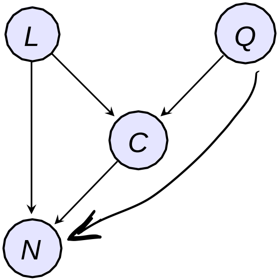
$$Q \perp L$$

$$\cancel{C | Q, L}$$

$$\cancel{N | C, L}$$

$$Q \perp N | C$$

- Draw the Bayesian Network.



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Thank You!!!