Assignment 11

February 13, 2022

1 Q3

Let $P(n)$ be the statement that $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ for a positive integer n.

- a. What is the statement $P(1)$
- b. Show that $P(1)$ is true and complete basis step of the proof
- c. What is the inductive hypothesis?
- d. What do you need to prove in the inductive step?
- e. Complete the inductive step, identifying where you use inductive hypothesis
- f. Explain why these steps show this formula is true $\forall n > 0$.

Assignment 11

Q.3) $P(n)$: $1^2+2^2+\cdots+n^2 = n(n+1)(2n+1)/6$

(a) $P(1)$: $1^2 = \frac{1\cdot(1+1)(2\cdot1+1)}{6}$

(b) $P(1)$: LHS $=1^2=1$ RHS $= 1\cdot(1+1)\cdot(2\cdot1+1)/6 = 1\cdot2\cdot3/6 = 1$

LHS = RHS

$\Rightarrow$ Basis step is true.

(c) Inductive Hypothesis: $P(k)$ is true

i.e. $1^2+2^2+\cdots+k^2 = k(k+1)(2k+1)/6$

(d) To prove: $P(k+1)$ is true

i.e. $1^2+2^2+\cdots+k^2+(k+1)^2 = (k+1)(k+2)\cdot(2\cdot(k+1)+1)/6$

(e) $P(k+1) = 1^2+2^2+\cdots+k^2+(k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ — from (c)

$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$

$= (k+1)\frac{(\cancel{k}\ 2k^2+k+6k+6)}{6}$

$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(2k^2+3k+4k+6)}{6}$

$= \frac{(k+1)(k(2k+3)+2(2k+3))}{6}$

$P(k+1) = \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)\cdot[(k+1)+1][2(k+1)+1]}{6}$

$\Rightarrow$ Inductive hypothesis holds good!

(f) $P(1)$ is true $\Rightarrow \because P(k)\Rightarrow P(k+1)$ & $P(1)$ is true $\Rightarrow P(2)$ is true

$\because P(k)\Rightarrow P(k+1)$ & $P(2)$ is true $\Rightarrow P(3)$ is true

$\vdots$

$\because P(k)\Rightarrow P(k+1)$ & $P(n)$ is true $\Rightarrow P(n+1)$ is true

$\therefore P(n)$ is true $\forall n \geq 1$

2 Q5

Prove that $1^2 + 3^2 + ... + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ whenever n is a non-negative integer

Q.5] Prove that $1^2 + 3^2 + \dots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3} \quad \forall n \geq 0$

Basis $P(0)$: ~~LHS = 0~~ ~~RHS = 0~~

$$LHS = [2(0)+1]^2 = 1^2 = 1$$

$$RHS = \frac{(0+1)(0+1)(0+3)}{3} = 1$$

$LHS = RHS \Rightarrow$ Basis ✓

Assume $P(k)$ is true

i.e. $1^2 + 3^2 + \dots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$ — ①

To prove $P(k+1)$ is true

$$P(k+1) = 1^2 + 3^2 + \dots + (2k+1)^2 + (2k+3)^2$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2 \quad \text{— from ①}$$

$$= (2k+3)\left[\frac{(k+1)(2k+1)}{3} + 2k+3\right]$$

$$= (2k+3)\left[\frac{2k^2 + 3k + 1 + 6k + 9}{3}\right]$$

$$= \frac{(2k+3)(2k^2 + 9k + 10)}{3}$$

$$= \frac{(2k+3)(2k^2 + 4k + 5k + 10)}{3}$$

$$= \frac{(2k+3)(2k(k+2) + 5(k+2))}{3}$$

$$= \frac{(k+2)(2k+3)(2k+5)}{3}$$

$$= \frac{[(k+1)+1][2(k+1)+1][2(k+1)+3]}{3}$$

$\Rightarrow P(k+1)$ is true if $P(k)$ is true

$\because$ Both basis & inductive steps are satisfied

$\Rightarrow P(n)$ is true $\forall n \geq 0$

3 Q18

Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

- a. What is the statement $P(2)$?
- b. Show that $P(2)$ is true, completing the basis step of the proof.
- c. What is the inductive hypothesis?
- d. What do you need to prove in the inductive step?
- e. Complete the inductive step.
- f. Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

18] $P(n): \quad n! < n^n \quad \forall\, n > 1$

(a) $P(2): \quad 2! < 2^2$

(b) $LHS = 2! = 2 \quad RHS = 2^2 = 4 \qquad 2 < 4 \Rightarrow$ ~~Induc~~ Basis step holds good.

(c) Hypothesis: $\quad k! < k^k$

(d) To prove: $\cancel{(k+1)k}\ (k+1)! < (k+1)^{k+1}$

(e) We know that

$$k! < k^k \quad \text{— from (c)}$$

Multiply b.s. by $(k+1)$

$$(k+1)\cdot k! < (k+1)\cdot k^k \quad \text{— (Inequality holds} \because k > 1)$$

$$\text{Now } (k+1)^k > k^k \quad \text{— } (\because k+1 > k \text{ as } k > 1)$$

$$(k+1)\cdot(k+1)^k > (k+1)\cdot k^k \quad \text{— (Multiply b.s. by } k+1)$$

$$(k+1)^{k+1} > (k+1)\cdot k^k \quad \text{— (Ineq. maintained as } k+1 > 0)$$

Now we have $(k+1)\cdot k! < (k+1)\cdot k^k$ & $(k+1)\cdot k^k < (k+1)^{k+1}$

$$\Rightarrow (k+1)\cdot k! < (k+1)^{k+1}$$

$$\Rightarrow (k+1)! < (k+1)^{k+1}$$

$\Rightarrow P(k+1)$ is true

$\Rightarrow$ Inductive hypothesis is true.

(f) $\because P(2)$ holds good & $P(k) \rightarrow P(k+1)$, $P(3)$ holds good

$\because (P(3))$ ———— " ———— , $P(4)$ holds good

$\vdots$

$\because P(n)$ holds good & $P(n) \rightarrow P(n+1)$, $P(n+1)$ holds good

$\Rightarrow P(n)$ holds good $\forall\, n > 1$.

4 Q31

Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

31] Prove that $n^2+n \,\%\, 2 = 0 \quad \forall n > 0$

Base case $P(1)$ $\quad LHS = 1^2+1 = 2 \,\%\, 2 = 0$

$RHS = 0$

$\Rightarrow P(1)$ holds good $\Rightarrow$ Base case ✓

Assume $P(k)$ is true $\Rightarrow (k^2+k) \,\%\, 2 = 0$ —— ①

To prove: $P(k+1)$ is true.

~~LHS of P(k+1) =~~

Consider $(k+1)^2 + k+1 = k^2 + 2k + 1 + k + 1$

$= (k^2+k) + 2k + 2$

$= k(k+1) + 2(k+1)$

So, $[(k+1)^2 + (k+1)] \,\%\, 2 = [k(k+1) + 2(k+1)] \,\%\, 2$

$= [k(k+1)] \,\%\, 2 + [2(k+1)] \,\%\, 2$

$= 0 + 0$ —— From ① & since any even number when divided by 2, has 0 remainder

$= 0$

$\Rightarrow P(k+1)$ is true if $P(k)$ is true

$\Rightarrow$ Inductive step holds good

$\because$ Both base case & inductive case hold good, we can say that given statement is true for all $n > 0$

5 Q38

Prove that if $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$ are sets such that $A_j \subseteq B_j \forall j = 1, 2, ..., n$ then

$$\cup_{j=1}^{n} A_j \subseteq \cup_{j=1}^{n} B_j$$

38] P.T. if $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots B_n$ s.t. $A_j \subseteq B_j \ \forall j = 1, 2, \dots, n$

Then $\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j$

Base case : $n = 1$ $\quad \bigcup_{j=1}^{1} A_1 \subseteq \bigcup_{j=1}^{1} B_1$

i.e. $A_1 \subseteq B_1$

which is true $\because A_j \subseteq B_j \ \forall j$

$\Rightarrow$ Base case holds good.

Inductive step $\quad \bigcup_{j=1}^{k} A_j \subseteq \bigcup_{j=1}^{k} B_j$ —— ①

To prove: $\quad \bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j$

We know that $\quad A_{k+1} \subseteq B_{k+1}$

Now, $\quad \bigcup_{j=1}^{k} A_j \subseteq \bigcup_{j=1}^{k} B_j$ —— From ①

Consider

$\bigcup_{j=1}^{k} A_j \cup A_{k+1} \subseteq \bigcup_{j=1}^{k} B_j \cup A_{k+1}$ —(Taking union with A_{k+1} on both sides)

$\Rightarrow \boxed{\bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k} B_j \cup A_{k+1}}$ —— ②.

Now consider

$\cancel{\bigcup_{j=1}^{k} A_j \subseteq \bigcup_{j=1}^{k} B_j}$ $\quad A_{k+1} \subseteq B_{k+1}$

$\cancel{\bigcup_{j=1}^{k}}$ $\quad \bigcup_{j=1}^{k} B_j \cup A_{k+1} \subseteq \bigcup_{j=1}^{k} B_j \cup B_{k+1}$

—(Taking union with $\bigcup_{j=1}^{k} B_j$ on both sides)

$\Rightarrow \boxed{\bigcup_{j=1}^{k} B_j \cup A_{k+1} \subseteq \bigcup_{j=1}^{k+1} B_j}$ —— ③

38]

From ② & ③, $\because$ if $p \subseteq q$ & $q \subseteq r \Rightarrow p \subseteq r$, we can say

$$\boxed{\bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j}$$

Hence the inductive step holds good.

$\Rightarrow$ Since base step & inductive step both hold good, we can say that the given statement holds good for all $j = 1, 2, \ldots, n$

6 Q45

Prove that a set with n elements has $\frac{n(n-1)}{2}$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

45] Consider base case

$P(2) = \frac{2(2-1)}{2} = 1$ & also if there's only 2 elems there can only be one 2-itemset $\Rightarrow$ Basecase ✓

Inductive step: Assume $P(k) = \frac{k(k-1)}{2}$ — ①

Consider $P(k+1)$

The set $\{a_1, a_2, \ldots\ldots, a_k, a_{k+1}\}$

We have listed all 2-subsets for $\{a_1, a_2, \ldots a_k\} = k(k-1)/2$ – from ①

The ones which will get added on introduction of a_{k+1} are

$\{a_1, a_{k+1}\}, \{a_2, a_{k+1}\}, \{a_3, a_{k+1}\} \ldots, \{a_k, a_{k+1}\}$

There will be k such sets

$\Rightarrow$ Total sets of 2 items $= \frac{k(k-1)}{2} + k$

$$= \frac{k^2-k}{2} + \frac{2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$$

$$= [(k+1)-1][k+1]/2$$

$\Rightarrow P(k+1)$ holds good if $P(k)$ is assumed true

$\Rightarrow$ Since base step & inductive step holds good, it means the statement holds true for all $n > 2$

7 Q69,70

Suppose there are n people in a group, each aware of a scandal no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all scandals each knows about. For example, on the first call, two people share information, so by the end of the call, each of these people knows about two scandals. The gossip problem asks for G(n), the minimum number of telephone calls that are needed for all n people to learn about all the scandals. Exercises 69, 70 deal with the gossip problem.

- 69. Find G(1), G(2), G(3), and G(4).
- 70. Use mathematical induction to prove that G(n) 2n − 4 for n 4. (Hint: In the inductive step, have a new person call a particular person at the start and at the end.)

69] $G(1) = 0$

$G(2) = 1$

$G(3) = 3$

$G(4) = 4$

Call 1: Person1, Person2

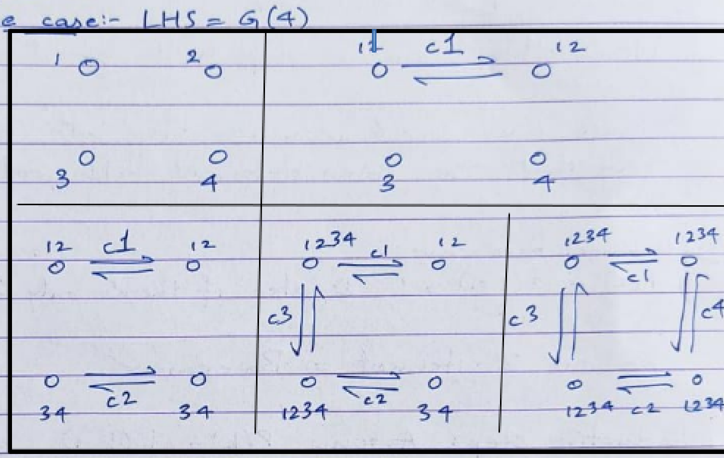
Call 2: Person3, Person4

Call 3: Person1, Person 3

Call 4: Person2, Person4

70] To prove that $G(n) \leq 2n-4 \quad \forall\, n \geq 4$

Base case:- LHS = $G(4)$



$\Rightarrow G(4) = 4 \qquad 2\cdot(4) - 4 = 4 \qquad LHS = 4;\ RHS = 4$

$LHS \leq RHS \Rightarrow$ Base case is satisfied.

Inductive step :

Let $P(k)$ be true $\Rightarrow G(k) \leq 2k-4$

If we have one more person let's say $(k+1)^{th}$ person.

Let $(k+1)$ call person r at the begining

$\Rightarrow$ r knows about $(k+1)^{th}$'s secret so, in the propogation of gossip, every person whom r talked to will know $(k+1)^{th}$'s secret.

$\because$ We know that for k people, calls are at most $2k-4$, at end of $2k-4+1 = 2k-3$ calls, everyone will know everyone's secrets except $k+1$ who only know's r's secret.

At the end of this, if $k+1$ calls anyone in the group even $k+1$ will know all secrets $\Rightarrow$ everyone knows everyone's secrets in $2k-3+1 = 2k-2$ calls

Now $2k-2 = 2(k+1) - 4 \Rightarrow G(k+1) \leq 2(k+1) - 4$

$\Rightarrow$ Inductive step proved $\Rightarrow G(n) \leq 2n-4 \quad \forall\, n \geq 4$.