Data mining

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October 30, 2021

Contents

Chapter 1

Introduction

Appendices

Appendix A

Complex numbers

A broad range of problems can be solved within real numbers, however it is easy to find some that are not solvable in $\mathbb{R}$. For example the equation $x^2 + 1 = 0$ has no solution in the real number. Because of this the real number set is extended, trying to develop a coherent framework in which this problems can be treated. Following this procedure a new variable i is defined, such that:

$$i := \sqrt{-1} \notin \mathbb{R}$$

This quantity is called the imaginary unit and it is used to define a new kind of numbers or complex numbers, defined in standard form as:

$$z := \underbrace{a}_{\text{Real part, } \Re z} + \underbrace{bi}_{\text{Imaginary part, } \Im z}$$

Where $a \wedge b \in \mathbb{R}$. This create a new set of numbers $\mathbb{C}$ such that $z \in \mathbb{C}$ and $\mathbb{R} \subset \mathbb{C}$. In fact any real number is a complex number where $b = 0$.

A.1 Argand plane

Complex numbers can be seen as ordered pairs of reals and they can be naturally plotted on the complex or argand plane. The horizontal direction represente the real axis and on the vertical the imaginary one.

A.2 Operations

A.2.1 Addition

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. The addition is defined as:

$$z + w = (a + c) + (b + d)i$$

A.2.2 Subtraction

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. The subtraction is defined as:

$$z - w = (a - c) + (b - d)i$$

A.2.3 Multiplication

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. Remembering that $i^2 = -1$, the multiplication of two complex number is:

$$\begin{aligned} z \cdot w &= (a + bi)(c + di) = \\ &= ac + adi + bci + bdi^2 = \\ &= ac + (ad + bc)i - bd = \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

A.2.4 Complex conjugate

Let $z \in \mathbb{C}$ be a complex number such that $z = a + bi$, where $a, b \in \mathbb{R}$. The complex conjugate is defined as:

$$z^* = a - bi$$

So we take the opposite of the imaginary part.

A.2.5 Division

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. The complex conjugate can be used to define a division operation that brings the result in standard form. The operation is similar to the rationalization of a fraction: the nominator and the denominator are multiplied by the complex conjugate of the denominator. This is because the product of a complex number and its conjugate is always real. So the division id defined as:

$$\begin{aligned} \frac{z}{w} &= \frac{a + bi}{c + di} = \\ &= \frac{a + bi}{c + di} \frac{c - di}{c - di} = \\ &= \frac{ac - adi + bci + bd}{c^2 + d^2} = \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i \end{aligned}$$

A.3 Polar form

A.3.1 Complex numbers as vectors

Complex numbers can be plotted as points in the Argand plane, using as coordinates the real and the imaginary parts. In this way a complex number can be seen as a vector of modulus:

$$\rho = |z| = \sqrt{a^2 + b^2}$$

Due to Pitagora's theorem. Complex number are added and subctrated as such.

A.3.2 Definition

The polar form is useful to have a simple interpretation of multiplication and division and it is defined as:

$$z := \rho(\cos\theta + i\sin\theta)$$

The variable used for this repreesntation are the modulus ρ and the argument θ, the angle between the positive direction of the real axis and the vector itself. The modulus of a complex number is always positive. Complex numbers in polar form are periodic with the argument θ with periodicity $2k\pi$, $\forall k \in \mathbb{Z}$.

A.3.3 Conversion between polar form and standard form

Any complex number writtein in standard form can be writtein in polar form, where:

$$\begin{cases} \theta = \arctan\frac{b}{a} \\ \rho = \sqrt{a^2 + b^2} \end{cases}$$

And the invers operation:

$$\begin{cases} a = \rho\cos\theta \\ b = \rho\sin\theta \end{cases}$$

A.3.4 Operations

A.3.4.1 Multiplication

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = \rho_z(\cos\theta_z) + i\sin\theta_z)$ and $w = \rho_w(\cos\theta_w + i\sin\theta_w)$. The multiplication between w and z is:

$$\begin{aligned} zw &= \rho_z\rho_w(\cos\theta_z + i\sin\theta_z)(\cos\theta_w + i\sin\theta_w) = \\ &= \rho_z\rho_w[\cos\theta_z\cos\theta_w - \sin\theta_z\sin\theta_w + i(\sin\theta_z\cos\theta_w + \cos\theta_z\sin\theta_w)] \end{aligned}$$

Using now the addition formulas for cosine and sine:

$$zw = \rho_z\rho_q[\cos(\theta_z + \theta_w) + i\sin(\theta_z + \theta_w)]$$

A.3.4.2 Division

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = \rho_z(\cos\theta_z) + i\sin\theta_z)$ and $w = \rho_w(\cos\theta_w + i\sin\theta_w)$. In a similar way as the multiplication, the division will be:

$$\frac{z}{w} = \frac{r_z}{r_w}[\cos(\theta_z - \theta_w) + i\sin(\theta_z - \theta_q)]$$

A.3.4.3 Power

According to the de Moivre theorem, for every $n \in \mathbb{N}$ positive integer and $z \in \mathbb{C}, z = \rho(\cos\theta + i\sin\theta)$:

$$z^n - \rho^n(\cos n\theta + i\sin n\theta)$$

A.3.4.4 N-th root

For every $n \in \mathbb{N}$ positive integer and $z \in \mathbb{C}, z = \rho(\cos\theta + i\sin\theta)$:

$$\sqrt[n]{z} = \rho^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Where k is an integer. Note that k and $k = k + n$ produce identical solutiion, so k can be limited to the set $\{0, 1, \ldots, n-1\}$. In conclusion there are n distinct roots, each with modulus $r^{\frac{1}{n}}$, that lie on the circle of radius equal to the modulus equally spaced on the Argand plane, creating a regular polygon.

A.4 Complex valued functions

Real function can be extended to complex valued function. Taken f from an interval $A \subset \mathbb{R}$ to $\mathbb{C}$ the function can be written as:

$$f(x) = u(x) + v(x)i$$

Where u and v are reale valued functions. The limit of a complex valued function exists if the limits of the real and the complex component exist.

A.4.1 Derivative

The derivative of a complex valued function is obtained differentiating its real and imaginary parts:

$$f'(x) = u'(x) + v'(x)$$

The properties of the derivatives can be extended to this case: if f and g are two complex valued functions differentiable at some point x_0 in the domain of both functions, $f \pm g$, fg and $\frac{f}{g}$ $(g(x_0) \neq 0)$ are differentiable and the values of these functions are, as in the real case:

$$(f \pm g)'(x_0) = f'(x_0) + g'(x_0)$$

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$$\frac{f}{g}'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$$

A.5 Complex exponential

Due to its properties and applications it is desirable to extend the exponential function to the complex field. A complex exponential function is in the form e^{a+bi}. From the case $a = 0$:

$$e^{ti} = \cos t + i \sin t$$

If $a, b \neq 0$;

$$\begin{aligned} e^{a+bi} &= e^a e^{bi} = \\ &= e^a(\cos b + i \sin b) \end{aligned}$$

A.5.1 Properties

Not only the product of two complex exponentials meets the classical properties of the real exponentials, also the derivatives maintains them. Let $t \in \mathbb{R}$ adn $y(t) = e^{(a+bi)t} = e^{at}(\cos tb + i \sin b)$, its derivative with respecto to t is:

$$\begin{aligned} \frac{dy(t)}{dt} &= \frac{de^{(a+bi)t}}{dt} = \\ &= (a+bi)e^{(a+bi)t} \end{aligned}$$

It can be demonstrated that given $z \in \mathbb{C}$, $\frac{de^z}{dz} = e^z$.

A.5.2 Roots of a complex number

The complex exponential allows to write the n roots of a complex number $z = r(\cos\theta + i \sin\theta)$ as:

$$w_k = r^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}}$$

Where $k \in \{0, 1, \ldots, n-1\}$.

Appendix B

Partial derivatives

B.1 First order derivatives

The concept of derivative can be used to explore function of $n \geq 2$ variables. Let $f : \mathbb{R}^2 \supseteq A \to \mathbb{R}$, where A is an open set of $\mathbb{R}^2$ a function of two variables: $f(x, y)$. The partial derivative of $f(x, y)$ with respect to x in the point (x_0, y_0) is defined as:

$$\frac{\partial f(x_0, y_0)}{\partial x} := \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

With $h \in \mathbb{R}$ when the limit exists. Equivalently the partial derivative of $f(x, y)$ with respect to y in (x_0, y_0) is:

$$\frac{\partial f(x_0, y_0)}{\partial y} := \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

With $h \in \mathbb{R}$ when the limit exists. That is the derivative of $f(x, y)$ with respect to a variable is computed as if other variables are held constant. The existence of the partial derivative with respect to one variable does not imply the existence of the partial derivatives along any other direction. The derivative along a general direction $\vec{v}$ is called directional derivative and is defined as:

$$D_{\vec{v}} f(x_0, y_0) := \lim_{t \to 0} \frac{f((x_0, y_0) + t\vec{v}) - f(x_0, y_0)}{t}$$

With $t \in \mathbb{R}$ where the limit exists.

B.1.1 Differentiability

The concept of differentiability is introduced because the existence of the derivative along one direction does not imply the existence of directional derivatives along different directions. Let $\mathbb{R}^2 \supseteq A \to \mathbb{R}$, with A an open set of $\mathbb{R}^2$, a function of two variables $f(x, y)$ is differentiable if the partial derivatives exist in (x_0, y_0) and:

$$\lim_{*h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f_x(x_0, y_0)h - f_y(x_0, y_0)k}{\sqrt{h^2 + k^2}} = 0$$

Where f_x and f_y are the partial derivative with respect to x or y.

B.1.2 Tangent plane

The tangent plane of $f(x, y)$ in the point (x_0, y_0) has the following form:

$$g(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

B.1.3 Determine if a function is differentiable

A function if differentiable in a point if the following condition holds true. Let $f : \mathbb{R}^2 \supseteq A \to \mathbb{R}$ with A an open set of $\mathbb{R}^2$. If in a neighbourhood of (x_o, y_0) all the partial derivatives of $f(x, y)$ exists and are continuous in (x_0, y_0) then $f(x, y)$ are differentiable in (x_0, y_0). If a function has all the partial derivatives in a point and they are continuous, the function is differentiable. That means that exists the tangent plane in that point.

B.2 Higher order derivatives

Let $f : \mathbb{R}^2 \supseteq A \to \mathbb{R}$, with A an open set of $\mathbb{R}^2$, a function of two variables $f(x, y)$. Supposing that the partial derivatives exist in a neighbourhood I of $(x_0, y_0($, the two functions $g(x, y) = \frac{\partial f(x,y)}{dx} : \mathbb{R}^2 \supseteq I \to \mathbb{R}$ and $h(x, y) = \frac{\partial f(x,y)}{dy} : \mathbb{R}^2 \supseteq I \to \mathbb{R}$ can be seen as the analogous of f and there is a possibility of taking the partial derivatives of g and h in a point (x_0, y_0). This means applying the g and h the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. The second order derivatives are defined as:

$$\begin{array}{ll} \frac{\partial^2 f}{\partial x^2} : \frac{\partial}{\partial x} g = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} & \frac{\partial^2 f}{\partial y \partial x} : \frac{\partial}{\partial y} g = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\ \frac{\partial^2 f}{\partial x \partial y} : \frac{\partial}{\partial x} h = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} & \frac{\partial^2 f}{\partial y^2} : \frac{\partial}{\partial y} h = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} \end{array}$$

When the partial derivative is taken two times in the same direction the second partial derivatives are named pures, when is taken along a different direction with respect to the first time they are named mixed.

B.2.1 Schwartz's theorem

Let $f(x, y)$ be a function defined in $\mathbb{R}^2$ and I a neighbourhood of (x_0, y_0) and $\partial x \partial y f$ and $\partial y \partial x f$ be continuous in I, then:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

All of this can be extended to higher order partial derivatives and to functions from $\mathbb{R}^{\ltimes}$ to $\mathbb{R}$ with an increasing number of combinations of derivatives. This theorem is useful form many reasons, one of which the fact that if the four order mixed partial derivatives are continuous at (x_0, y_0) the order of the first order partial derivatives can be rearranged as preferred.

Appendix C

Differential operators

Given the concept of partial derivatives new differential operators can be defined to be applied to scalar functions or vectorial fields.