Data mining

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Appendix A

Complex numbers

A broad range of problems can be solved within real numbers, however it is easy to find some that are not solvable in $\mathbb{R}$. For example the equation $x^2 + 1 = 0$ has no solution in the real number. Because of this the real number set is extended, trying to develop a coherent framework in which this problems can be treated. Following this procedure a new variable i is defined, such that:

$$i := \sqrt{-1} \notin \mathbb{R}$$

This quantity is called the imaginary unit and it is used to define a new kind of numbers or complex numbers, defined in standard form as:

$$z := \underbrace{a}_{\text{Real part, } \Re z} + \underbrace{bi}_{\text{Imaginary part, } \Im z}$$

Where $a \wedge b \in \mathbb{R}$. This create a new set of numbers $\mathbb{C}$ such that $z \in \mathbb{C}$ and $\mathbb{R} \subset \mathbb{C}$. In fact any real number is a complex number where $b = 0$.

A.1 Argand plane

Complex numbers can be seen as ordered pairs of reals and they can be naturally plotted on the complex or argand plane. The horizontal direction represente the real axis and on the vertical the imaginary one.

A.2 Operations

A.2.1 Addition

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. The addition is defined as:

$$z + w = (a + c) + (b + d)i$$

A.2.2 Subtraction

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. The subtraction is defined as:

$$z - w = (a - c) + (b - d)i$$

A.2.3 Multiplication

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. Remembering that $i^2 = -1$, the multiplication of two complex number is:

$$\begin{aligned}
z \cdot w &= (a + bi)(c + di) = \\
&= ac + adi + bci + bdi^2 = \\
&= ac + (ad + bc)i - bd = \\
&= (ac - bd) + (ad + bc)i
\end{aligned}$$

A.2.4 Complex conjugate

Let $z \in \mathbb{C}$ be a complex number such that $z = a + bi$, where $a, b \in \mathbb{R}$. The complex conjugate is defined as:

$$z^* = a - bi$$

So we take the opposite of the imaginary part.

A.2.5 Division

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. The complex conjugate can be used to define a division operation that brings the result in standard form. The operation is similar to the rationalization of a fraction: the nominator and the denominator are multiplied by the complex conjugate of the denominator. This is because the product of a complex number and its conjugate is always real. So the division id defined as:

$$\begin{aligned}
\frac{z}{w} &= \frac{a + bi}{c + di} = \\
&= \frac{a + bi}{c + di} \frac{c - di}{c - di} = \\
&= \frac{ac - adi + bci + bd}{c^2 + d^2} = \\
&= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \\
&= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i
\end{aligned}$$

A.3 Polar form

A.3.1 Complex numbers as vectors

Complex numbers can be plotted as points in the Argand plane, using as coordinates the real and the imaginary parts. In this way a complex number can be seen as a vector of modulus:

$$\rho = |z| = \sqrt{a^2 + b^2}$$

Due to Pitagora's theorem. Complex number are added and subctrated as such.

A.3.2 Definition

The polar form is useful to have a simple interpretation of multiplication and division and it is defined as:

$$z := \rho(\cos\theta + i\sin\theta)$$

The variable used for this repreesntation are the modulus ρ and the argument θ, the angle between the positive direction of the real axis and the vector itself. The modulus of a complex number is always positive. Complex numbers in polar form are periodic with the argument θ with periodicity $2k\pi$, $\forall k \in \mathbb{Z}$.

A.3.3 Conversion between polar form and standard form

Any complex number writtein in standard form can be writtein in polar form, where:

$$\begin{cases} \theta = \arctan\frac{b}{a} \\ \rho = \sqrt{a^2 + b^2} \end{cases}$$

And the invers operation:

$$\begin{cases} a = \rho\cos\theta \\ b = \rho\sin\theta \end{cases}$$

A.3.4 Operations

A.3.4.1 Multiplication

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = \rho_z(\cos\theta_z) + i\sin\theta_z)$ and $w = \rho_w(\cos\theta_w + i\sin\theta_w)$. The multiplication between w and z is:

$$\begin{aligned} zw &= \rho_z\rho_w(\cos\theta_z + i\sin\theta_z)(\cos\theta_w + i\sin\theta_w) = \\ &= \rho_z\rho_w[\cos\theta_z\cos\theta_w - \sin\theta_z\sin\theta_w + i(\sin\theta_z\cos\theta_w + \cos\theta_z\sin\theta_w)] \end{aligned}$$

Using now the addition formulas for cosine and sine:

$$zw = \rho_z\rho_q[\cos(\theta_z + \theta_w) + i\sin(\theta_z + \theta_w)]$$

A.3.4.2 Division

Let $z, w \in \mathbb{C}$ be two complex numbers such that $z = \rho_z(\cos\theta_z) + i\sin\theta_z)$ and $w = \rho_w(\cos\theta_w + i\sin\theta_w)$. In a similar way as the multiplication, the division will be:

$$\frac{z}{w} = \frac{r_z}{r_w}[\cos(\theta_z - \theta_w) + i\sin(\theta_z - \theta_q)]$$

A.3.4.3 Power

According to the de Moivre theorem, for every $n \in \mathbb{N}$ positive integer and $z \in \mathbb{C}, z = \rho(\cos\theta + i\sin\theta)$:

$$z^n - \rho^n(\cos n\theta + i\sin n\theta)$$

A.3.4.4 N-th root

For every $n \in \mathbb{N}$ positive integer and $z \in \mathbb{C}, z = \rho(\cos\theta + i\sin\theta)$:

$$\sqrt[n]{z} = \rho^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Where k is an integer. Note that k and $k = k + n$ produce identical solutiion, so k can be limited to the set $\{0, 1, \ldots, n-1\}$. In conclusion there are n distinct roots, each with modulus $r^{\frac{1}{n}}$, that lie on the circle of radius equal to the modulus equally spaced on the Argand plane, creating a regular polygon.

A.4 Complex valued functions

Real function can be extended to complex valued function. Taken f from an interval $A \subset \mathbb{R}$ to $\mathbb{C}$ the function can be written as:

$$f(x) = u(x) + v(x)i$$

Where u and v are reale valued functions. The limit of a complex valued function exists if the limits of the real and the complex component exist.

A.4.1 Derivative

The derivative of a complex valued function is obtained differentiating its real and imaginary parts:

$$f'(x) = u'(x) + v'(x)$$

The properties of the derivatives can be extended to this case: if f and g are two complex valued functions differentiable at some point x_0 in the domain of both functions, $f \pm g$, fg and $\frac{f}{g}$ $(g(x_0) \neq 0)$ are differentiable and the values of these functions are, as in the real case:

$$(f \pm g)'(x_0) = f'(x_0) + g'(x_0)$$

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$$\frac{f}{g}'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$$

A.5 Complex exponential

Due to its properties and applications it is desirable to extend the exponential function to the complex field. A complex exponential function is in the form e^{a+bi}. From the case $a = 0$:

$$e^{ti} = \cos t + i \sin t$$

If $a, b \neq 0$;

$$\begin{aligned} e^{a+bi} &= e^a e^{bi} = \\ &= e^a(\cos b + i \sin b) \end{aligned}$$

A.5.1 Properties

Not only the product of two complex exponentials meets the classical properties of the real exponentials, also the derivatives maintains them. Let $t \in \mathbb{R}$ adn $y(t) = e^{(a+bi)t} = e^{at}(\cos tb + i \sin b)$, its derivative with respecto to t is:

$$\begin{aligned} \frac{dy(t)}{dt} &= \frac{de^{(a+bi)t}}{dt} = \\ &= (a+bi)e^{(a+bi)t} \end{aligned}$$

It can be demonstrated that given $z \in \mathbb{C}$, $\frac{de^z}{dz} = e^z$.

A.5.2 Roots of a complex number

The complex exponential allows to write the n roots of a complex number $z = r(\cos\theta + i \sin\theta)$ as:

$$w_k = r^{\frac{1}{n}} e^{i\frac{\theta + 2kn}{n}}$$

Where $k \in \{0, 1, \ldots, n-1\}$.

Appendix B

Partial derivatives

B.1 First order derivatives

The concept of derivative can be used to explore function of $n \geq 2$ variables. Let $f : \mathbb{R}^2 \supseteq A \to \mathbb{R}$, where A is an open set of $\mathbb{R}^2$ a function of two variables: $f(x, y)$. The partial derivative of $f(x, y)$ with respect to x in the point (x_0, y_0) is defined as:

$$\frac{\partial f(x_0, y_0)}{\partial x} := \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

With $h \in \mathbb{R}$ when the limit exists. Equivalently the partial derivative of $f(x, y)$ with respect to y in (x_0, y_0) is:

$$\frac{\partial f(x_0, y_0)}{\partial y} := \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

With $h \in \mathbb{R}$ when the limit exists. That is the derivative of $f(x, y)$ with respect to a variable is computed as if other variables are held constant. The existence of the partial derivative with respect to one variable does not imply the existence of the partial derivatives along any other direction. The derivative along a general direction $\vec{v}$ is called directional derivative and is defined as:

$$D_{\vec{v}} f(x_0, y_0) := \lim_{t \to 0} \frac{f((x_0, y_0) + t\vec{v}) - f(x_0, y_0)}{t}$$

With $t \in \mathbb{R}$ where the limit exists.

B.1.1 Differentiability

The concept of differentiability is introduced because the existence of the derivative along one direction does not imply the existence of directional derivatives along different directions. Let $\mathbb{R}^2 \supseteq A \to \mathbb{R}$, with A an open set of $\mathbb{R}^2$, a function of two variables $f(x, y)$ is differentiable if the partial derivatives exist in (x_0, y_0) and:

$$\lim_{*h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f_x(x_0, y_0)h - f_y(x_0, y_0)k}{\sqrt{h^2 + k^2}} = 0$$

Where f_x and f_y are the partial derivative with respect to x or y.

B.1.2 Tangent plane

The tangent plane of $f(x, y)$ in the point (x_0, y_0) has the following form:

$$g(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

B.1.3 Determine if a function is differentiable

A function if differentiable in a point if the following condition holds true. Let $f : \mathbb{R}^2 \supseteq A \to \mathbb{R}$ with A an open set of $\mathbb{R}^2$. If in a neighbourhood of (x_o, y_0) all the partial derivatives of $f(x, y)$ exists and are continuous in (x_0, y_0) then $f(x, y)$ are differentiable in (x_0, y_0). If a function has all the partial derivatives in a point and they are continuous, the function is differentiable. That means that exists the tangent plane in that point.

B.2 Higher order derivatives

Let $f : \mathbb{R}^2 \supseteq A \to \mathbb{R}$, with A an open set of $\mathbb{R}^2$, a function of two variables $f(x, y)$. Supposing that the partial derivatives exist in a neighbourhood I of $(x_0, y_0($, the two functions $g(x, y) = \frac{\partial f(x,y)}{dx} : \mathbb{R}^2 \supseteq I \to \mathbb{R}$ and $h(x, y) = \frac{\partial f(x,y)}{dy} : \mathbb{R}^2 \supseteq I \to \mathbb{R}$ can be seen as the analogous of f and there is a possibility of taking the partial derivatives of g and h in a point (x_0, y_0). This means applying the g and h the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. The second order derivatives are defined as:

$$\begin{array}{ll} \frac{\partial^2 f}{\partial x^2} : \frac{\partial}{\partial x} g = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} & \frac{\partial^2 f}{\partial y \partial x} : \frac{\partial}{\partial y} g = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\ \frac{\partial^2 f}{\partial x \partial y} : \frac{\partial}{\partial x} h = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} & \frac{\partial^2 f}{\partial y^2} : \frac{\partial}{\partial y} h = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} \end{array}$$

When the partial derivative is taken two times in the same direction the second partial derivatives are named pures, when is taken along a different direction with respect to the first time they are named mixed.

B.2.1 Schwartz's theorem

Let $f(x, y)$ be a function defined in $\mathbb{R}^2$ and I a neighbourhood of (x_0, y_0) and $\partial x \partial y f$ and $\partial y \partial x f$ be continuous in I, then:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

All of this can be extended to higher order partial derivatives and to functions from $\mathbb{R}^{\ltimes}$ to $\mathbb{R}$ with an increasing number of combinations of derivatives. This theorem is useful form many reasons, one of which the fact that if the four order mixed partial derivatives are continuous at (x_0, y_0) the order of the first order partial derivatives can be rearranged as preferred.

Appendix C

Differential operators

Given the concept of partial derivatives new differential operators can be defined to be applied to scalar functions or vectorial fields.

C.1 Gradient, divergence and curl

C.1.1 Gradient

Let $f : \mathbb{R}^3 \supseteq A \rightarrow \mathbb{R}$, with A an open set of $\mathbb{R}^2$, a function of three variables $f(x, y, z)$. Be the function derivable. The gradient of f in Cartesian coordinates is defined by:

$$grad f = \nabla f := \frac{\partial f(x,y,z)}{\partial x}\hat{\imath} + \frac{\partial f(x,y,z)}{\partial y}\hat{\jmath} + \frac{partial f(x,y,z)}{\partial z}\hat{k}$$

The gradient is then the vector that takes as components along the axes directions the first order partial derivatives. The gradient is the vector of major increment of the function with respect to the variations in the variables.

C.1.2 Divergence

Suppose now to have a derivable vectorial field $\vec{V} : \mathbb{R}^3 \supseteq A \rightarrow \mathbb{R}^3$, with A an open set of $\mathbb{R}^3$. This vectorial field is defined by means of its components along the axis directions: $\vec{V} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$. It associates a vector to each point of A. The gradient, if it exists and if it is derivable is a vectorial field. The divergence of $\vec{V}$ in Cartesian coordinates is:

$$div\vec{V} = \nabla \cdot \vec{V} := \frac{\partial v_1(x,y,z)}{\partial x} + \frac{\partial v_2(x,y,z)}{\partial y} + \frac{\partial v_3(x,y,z)}{\partial z}$$

This gives informations on where a vectorial fields has source or sink, or, when the vectorial field represent a fluid flux, if the fluid is incompressible or solenoidal.

C.1.3 Curl

Suppose now to have a derivable vectorial field $\vec{V} : \mathbb{R}^3 \supseteq A \rightarrow \mathbb{R}^3$, with A an open set of $\mathbb{R}^3$. The application of the curl to $\vec{V}$ is:

$$rot\vec{V} = \nabla \times \vec{V} := \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

The application of the curl to a vectorial field gives information on if the field rotates around a point and the verse of that rotation. The output of the curl is the modulus of the rotation, and the direction is linked by means of the right hand rule to the verse of rotation.

C.1.4 Properties

- $\nabla \times \nabla f = \vec{0}$
- $\nabla \cdot \nabla \times \vec{V} = 0$
- $\nabla \cdot f = \nabla^2 f$

C.1.5 Laplacian

The Laplacian is the last operator and it is defines as:

$$\nabla^2 f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f$$

C.2 Hessian matrix: maxima and minima

A matrix of second partial derivatives can be build to study functions from $\mathbb{R}^n$ to $\mathbb{R}$. The presence of possible extrema at one point is linked to the one of a null gradient in that point. Let $f : \mathbb{R}^2 \supseteq A \to R$, with A an open set of $\mathbb{R}^2$ a function $f(x, y)$. If (x_0, y_0) is a local extremum if exists $\nabla f = \vec{0}$. To find local extrema are considered points in which $\nabla f = \vec{0}$. Let $f : \mathbb{R}^2 \supseteq A \to R$, with A an open set of $\mathbb{R}^2$ and $f \in C^2$, where c^2 meaning that all partial second derivatives exist and are continuous. Then the hessian matrix is defined as:

$$H_f(x_0, y_0) = \begin{pmatrix} \partial_{xx} f(x_0, y_0) & \partial_{yx} f(x_0, y_0) \\ \partial_{xy} f(x_0, y_0) & \partial_{yy} f(x_0, y_0) \end{pmatrix}$$

This matrix is useful to determine the nature of the extrema. Let $f \in C^2$ and (x_0, y_0) a critical point of f, then:

- If the determinant of $H_f(x_0, y_0) > 0$ and $\partial_{xx} f(x_0, y_0) > 0$ then (x_0, y_0) is a minimum.
- If the determinant of $H_f(x_0, y_0) > 0$ and $\partial_{xx} f(x_0, y_0) < 0$ then (x_0, y_0) is a maximum.
- If the determinant of $H_f(x_0, y_0) < 0$ then (x_0, y_0) is a saddle point.
- If the determinant of $H_f(x_0, y_0) = 0$ further analysis is necessary.

C.3 Jacobian matrix

With a vectorial function the concept of gradient can be extended and applied to each component of the function. Let $f : \mathbb{R}^2 \supseteq A \to \mathbb{R}^2$ be a function of two variable $f = (f_1(x, y), f_2(x, y))$ for whigh all the derivatives exist ant are continuous. The jacobian matrix is defined as:

$$J_f = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix} = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix}$$

Due to the fact that we have a function that takes as input two variables and gives as output two variables, this can be thought as a change of coordinates:

$$f = (f_1(x, y), f_2(x, y)) = (u, w) \qquad (x, y) \to (u, w)$$

The Jacobian matrix allows to determine the domain of the transformation. The change of variables is $1 \leftrightarrow 1$ (a bijective function) only if the determinant of the Jacobian matrix is not null. Also the Jacobian determinant makes possible to consistently define the change of volume in changing the coordinates. Given a transformation from (x, y) to (u, v) the change in the expression of the area with respect to the coordinates is:

$$dA = dxdy = \left| det \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

Where dx, dy, du, dv are small coordinates intervals.

C.4 Chain rule

Let $u(x, y)$ be a differentiable function of two variables that are differentiable function of two variables each $x(s, t)$ and $y(s, t)$, then the composite function is differentiable and the partial derivatives are:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

C.4.1 Gradient in polar coordinates

Suppose to have $g(x, y)$ a function of two variables in Cartesian coordinates. If (ρ, θ) are the usual polar coordinates related to (x, y) by $x = \rho \cos\theta$ and $y = \rho \sin\theta$, then by substituting these for x and y g becomes a function of ρ and θ:

$$g(x, y) = f(\rho(x, y), \theta(x, y))$$

With $\rho(x, y) = \sqrt{x^2 + y^2}$ and $\theta(x, y) = \arctan \frac{y}{x}$. The objective is to compute the gradient $\nabla g(x, y)$ and express it in terms of ρ and θ. The chain rule can be used to compute the partial derivatives of g with respect to x and y:

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} \qquad \frac{\partial g}{\partial y} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}$$

To fill these relation there is a need to compute:

$$
\begin{aligned}
\frac{\partial \rho}{\partial x} &= \frac{\partial \sqrt{x^2+y^2}}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2 y^2}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{\rho\cos\theta}{\rho} = \cos\theta \\
\frac{\partial \rho}{\partial y} &= \frac{\partial \sqrt{x^2+y^2}}{\partial y} = \frac{1}{2}\frac{2y}{\sqrt{x^2 y^2}} = \frac{y}{\sqrt{x^2+y^2}} = \frac{\rho\sin\theta}{\rho} = \sin\theta \\
\frac{\partial \theta}{\partial x} &= \frac{\partial \arctan\frac{y}{x}}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\rho\sin\theta}{\rho^2} = -\frac{\sin\theta}{\rho} \\
\frac{\partial \theta}{\partial y} &= \frac{\partial \arctan\frac{y}{x}}{\partial y} = \frac{x}{x^2+y^2} = \frac{\rho\cos\theta}{\rho^2} = \frac{\cos\theta}{\rho}
\end{aligned}
$$

So:

$$
\frac{\partial g}{\partial x} = \frac{\partial f}{\partial \rho}\cos\theta + \frac{\partial f}{\partial \theta}\frac{-\sin\theta}{\rho} \qquad \frac{\partial g}{\partial y} = \frac{\partial f}{\partial \rho}\sin\theta + \frac{\partial f}{\partial \theta}\frac{\cos\theta}{\rho}
$$

The gradient of g using Cartesian versors will be:

$$
\begin{aligned}
\nabla g = g_x\hat{e}_x + g_y\hat{e}_y &= \left(\frac{\partial f}{\partial \rho}\cos\theta + \frac{\partial f}{\partial \theta}\frac{-\sin\theta}{\rho}\right)\hat{e}_x + \left(\frac{\partial f}{\partial \rho}\sin\theta + \frac{\partial f}{\partial \theta}\frac{\cos\theta}{\rho}\right)\hat{e}_y = \\
&= \left(\frac{\partial f}{\partial \rho}\cos\theta\right)\hat{e}_x + \left(\frac{\partial f}{\partial \theta}\frac{-\sin\theta}{\rho}\right)\hat{e}_x + \left(\frac{\partial f}{\partial \rho}\sin\theta\right)\hat{e}_y + \left(\frac{\partial f}{\partial \theta}\frac{\cos\theta}{\rho}\right)\hat{e}_y = \\
&= \frac{\partial f}{\partial \rho}(\cos\theta\hat{e}_x + \sin\theta\hat{e}_y) + \frac{1}{\rho}\frac{\partial f}{\partial \theta}(-\sin\theta\hat{e}_x + \cos\theta\hat{e}_y)
\end{aligned}
$$

The unit versors $\hat{e}_\rho$ and $\hat{e}_\theta$ are introduced. They have unitary modulus and direction that change from point to point. In particolar for polar coordinates they have the components $\hat{e}^\rho = (\cos\theta, \sin\theta)$ and $\hat{e}_\theta = (-\sin\theta, \cos\theta)$. So finally the explicit gradient in polar coordinates is:

$$
\nabla g = f_\rho\hat{e}_\rho + \frac{f_\theta}{\rho}\hat{e}_\theta
$$

Appendix D

Spherical coordinates

D.1 Definition

An important change of coordinates is the one that takes the cartesian coordinates and maps them into the spherical ones (ρ, θ, ϕ). This transformation allows to simplify the treatment of systems with spherical symmetry. The relationship between cartesian and spherical coordinates can be defined as:

- $x = x_0 + \rho \cos\theta \sin\phi$
- $y = y_0 + \rho \sin\theta \sin\phi$
- $z = z_0 + \rho \cos\phi$

With conditions:

- $0 \leq \rho \leq \infty$
- $0 \leq \theta \leq 2\pi$
- $0 \leq \phi \leq \pi$

Computing all the first order partial derivatives the jacobian matrix is:

$$J_f = \begin{pmatrix} \nabla(x(\rho,\theta,\phi)) \\ \nabla(t(\rho,\theta,\phi)) \\ \nabla(z(\rho,\theta,\phi)) \end{pmatrix} = \begin{pmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{pmatrix} =$$
$$= \begin{pmatrix} \cos\theta\sin\phi & -\rho\sin\theta\cos\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\sin\theta\cos\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$$

With the jacobian determinant $-\rho^2 \sin\phi$. So for spherical coordinates:

$$dV = dxdydz = \left| det \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| d\rho d\theta d\phi = \rho^2 \sin\phi d\rho d\theta d\phi$$

D.2 The sphere volume

To compute the volume of a ball B with radius $\rho \leq R$, the most simple thing to do is to put it in spherical coordinates with the following conditions:

- $0 \leq \rho \leq R$
- $0 \leq \theta \leq 2\pi$
- $0 \leq \phi \leq \pi$

So that the integral to compute is:

$$\begin{aligned}
Vol(B) &= \int\limits_0^{\pi} \int\limits_0^{2\pi} \int\limits_0^{R} \rho^2 \sin\phi d\rho d\theta d\phi = \\
&= \int\limits_0^{\pi} \int\limits_0^{2\pi} \left[\frac{\rho^3}{3}\right]_0^R \sin\phi d\theta d\phi = \int\limits_0^{\pi} \int\limits_0^{2\pi} \frac{R^3}{3} \sin\phi d\theta d\phi = \\
&= \int\limits_0^{\pi} [\theta]_0^{2\pi} \frac{R^3}{3} \sin\phi d\phi = \int\limits_0^{\pi} \frac{2\pi R^3}{3} \sin\phi d\phi = \\
&= [-\cos\phi]_0^{\pi} \frac{2\pi R^3}{3} = \\
&= \frac{4\pi R^3}{3}
\end{aligned}$$

Appendix E

Multidimensional integrals

In many dimension the domain can have a shape that can produce effects on the integration procedure. Another difficulty introduced by solving integrals in many dimensions is the choice of order of integrations.

E.1 Definition

The integral of a function $f(x, y)$ in two dimension is the volume under the surface $z = f(x, y)$. Supposing that the function is defined over a rectangular domain $(a, b) \times (c, d)$ the domain can be divided in many smaller rectangles with dimension $\Delta x \times \Delta y$. These subdomains cover the whole original domain. In each subdomain the infimum and the supremum. The infimum of the function is the greatest element of $\mathbb{R}$ that is less than or equal to all elements of the function on the corresponding subdomain. The supremum is the least element of $\mathbb{R}$ that is less than or equal to all elements of the function on the corresponding subdomain. A specific division of the domain is denoted as P. The Darboux sums are defined as:

$$L(f, P) = \sum_{\Delta x_k \times \Delta y_k} \sup_{[\Delta x_k \times \Delta y_k]} f(x, y)(\Delta x_k \times \Delta y_k)$$

$$U(f, P) = \sum_{\Delta x_k \times \Delta y_k} \inf_{[\Delta x_k \times \Delta y_k]} f(x, y)(\Delta x_k \times \Delta y_k)$$

Where the sum is over all the subdomain labelled by k. The function $f(x, y)$ is Riemann-integrable if:

$$\sup L(f, P) = \inf U(f, P) = I$$

Varying the partition, so

$$I = \int\limits_c^d \int\limits_a^b f(x, y) dx dy$$

E.2 Properties

E.2.1 Differentiability

Let $f(x, y)$ be a continuous function from $(a, b) \times (c, d)$ then the function is integrable. This result can be extended with different kind of integrals. Also this definition can be easily extended in more dimension.

E.2.2 Order of integration

If $f(x, y)$ is continuous on $(a, b) \times (c, d)$ then:

$$\int_c^d \left(\int_a^b f(x, y) dx \right) dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

E.2.3 x-simple and y-simple domains

IN some cases the domain of the function is not defined over a rectangular domain. One case of easy integration is when the domain is x-simple or y-simple. In the case of a y-simple domain, the function is bounded on the x axis by two numerical values and on the y axis by two continuous function $y = g_1(x)$ and $y = g_2(x)$. The case of a x-simple domain is the symmetric of the y-simple one. Let $f(x, y)$ be a continuous function defined on an x-simple domain Ω:

$$\Omega = \{(x, y) \in \mathbb{R}, c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

The integral can be computed as:

$$\iint_\Omega f = \int_c^d dy \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right)$$

In the same way for an y-simple domain:

$$\Omega = \{(x, y) \in \mathbb{R}, g_1(x) \leq y \leq g_2(x), a \leq x \leq b\}$$

$$\iint_\Omega f = \int_a^b dx \left(\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right)$$

E.2.4 Change of variables

The absolute value of the jacobian determinant gives the change in the volume element when passing from a set of coordinate to another. In two dimension, given $f(x, y)$ defined on Ω and supposing to change the integral variables from x and y to u and v:

$$\iint_\Omega f(x, y) dx dy = \iint_{\Omega^*} f(x(u, v), y(u, v)) |J| du dv$$

Where Ω^* is the new region of integration in the (u, v) plane.