Mathematics IV
(Probability and Statistics)

MATH 403

Lecture 2

Dr. Phoebe Edward Nashed

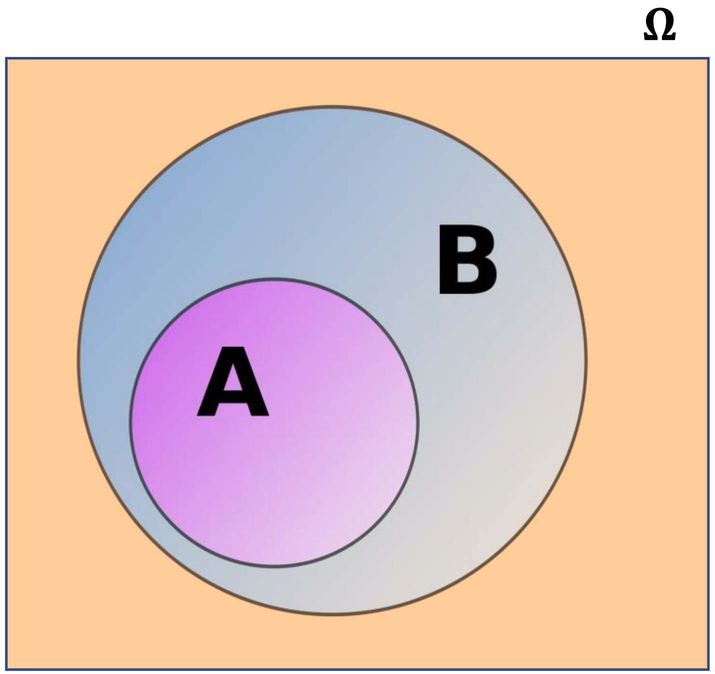
Lecture 2 - Outline

- Set Theory
- Set Operations
  - Union, Intersection, Difference, Complement
- Probability
  - Random Experiments, Sample Space, Events
  - Tree Diagram
  - Some Axioms of Probability
  - De Morgan's laws

Set Theory

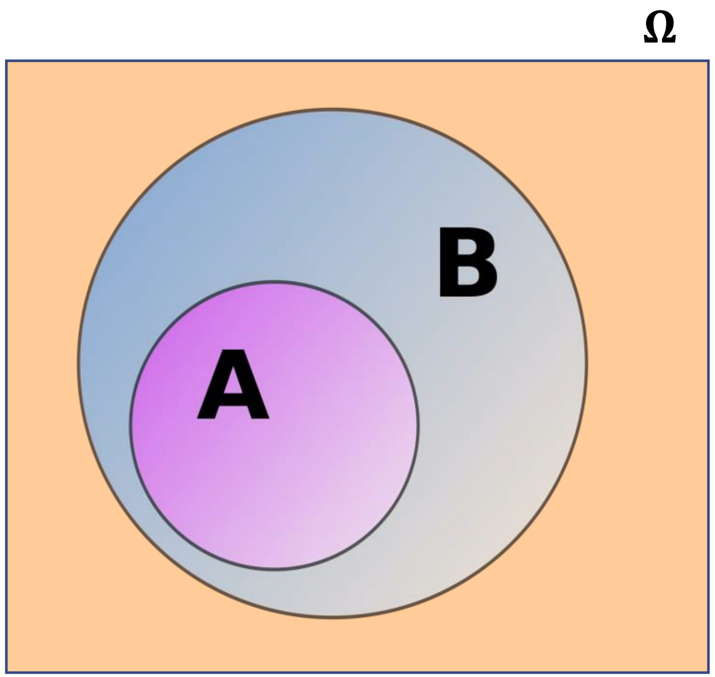
- Any collection of objects is called a set; the objects forming the set are called the **elements** of the set or **members** of the set. When we write the set $A = \{x : a \leq x \leq b\}$, we mean that *A* is a set of all real number between the closed interval [a, b].

- **Subsets**: If every element of a set *A* belongs to the set *B* then, *A* is called a **subset of or equal to** *B* and we write $A \subseteq B$ or B is a **superset or equal to** A; i.e. $B \supseteq A$.



Set Theory

- **Universal Set**: It will be assumed that all sets under investigation are subsets of some fixed set called the **universal** set and denoted by Ω.
- **Empty Set (Null Set)**: The set contains no elements called the **empty** set or a **null** set denoted by Φ. Thus for any set A we have, $\Phi \subset A \subset \Omega$.



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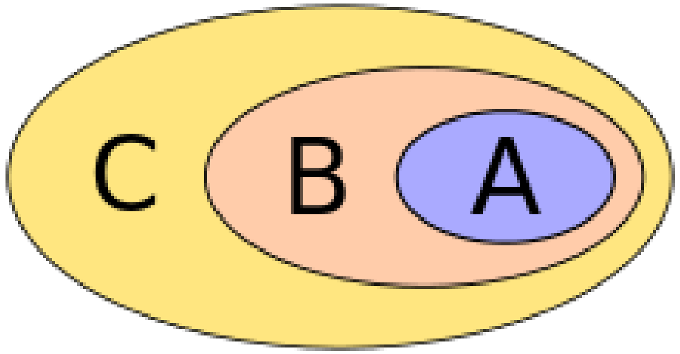
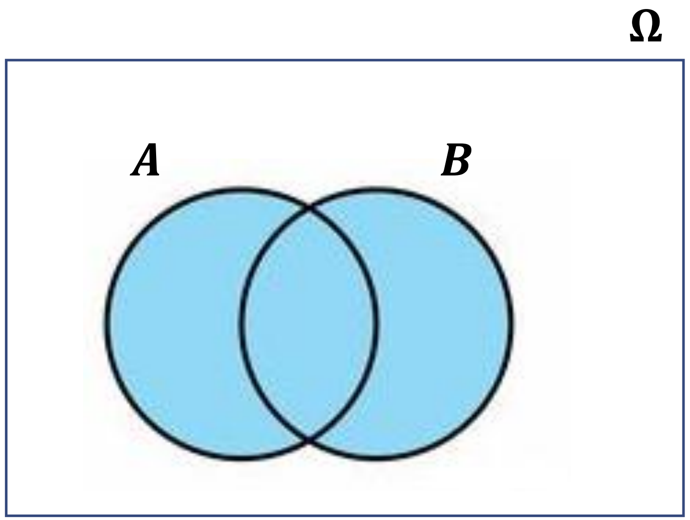
Thorem

- If A, B and C are any three sets, then:
  - $A \subseteq A$
  - $If A \subseteq B$ and $B \subseteq A$, then $A = B$
  - $If A \subset B$ and $B \subset C$, then $A \subset C$



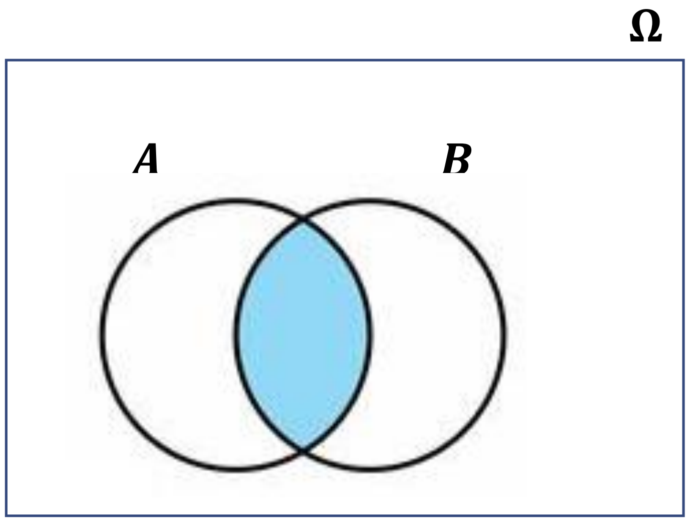
Set Operations: Union

- The union of two sets A and B is the set of elements which belongs to A or B and denoted by $A \cup B$, also by **Venn diagram** see below figure.



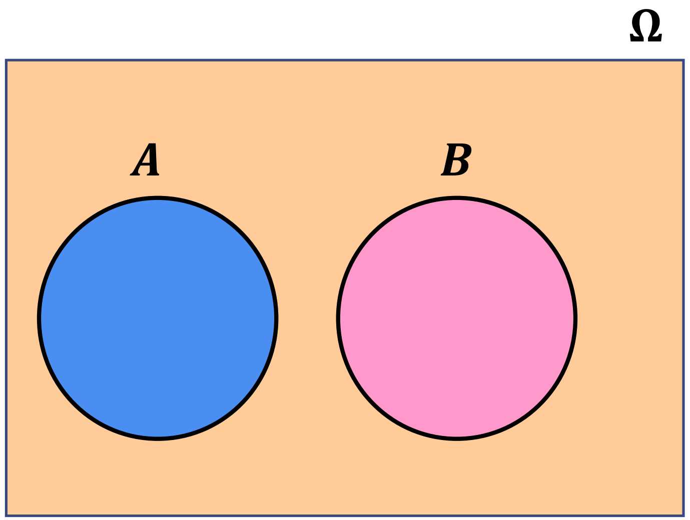
Set Operations: Intersection

- The Intersection of two sets A and B is the set of elements which belongs to A and B in the same time and denoted by $A \cap B$, also by **Venn diagram** see below figure.



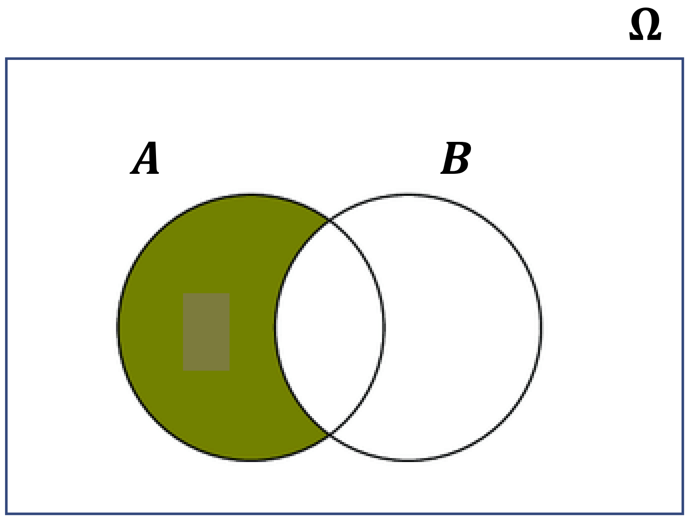
Mutually Exclusive Events

❖ If $A \cap B = \Phi$ then, A and B said to be **disjoint sets** or **mutually exclusive** see below figure.

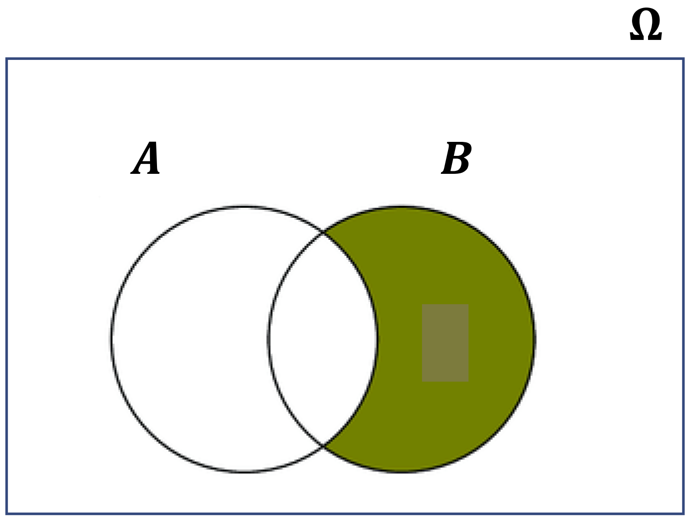


Set Operations: Difference

- The difference of two sets *A* and *B* is the set of elements which belongs to *A* but not to *B* and denoted by $A - B$.
- Also, the set of elements which belongs to *B* but not to *A* denoted by $B - A$, by **Venn diagram** (see below figures).



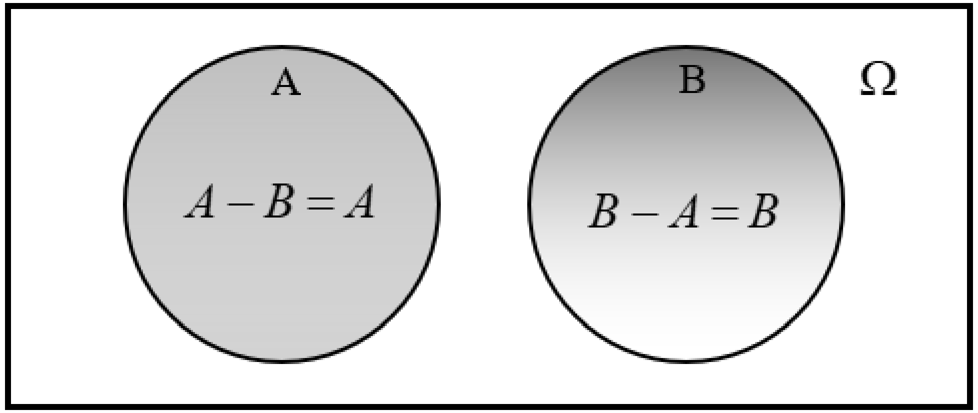
$A - B$



$B - A$

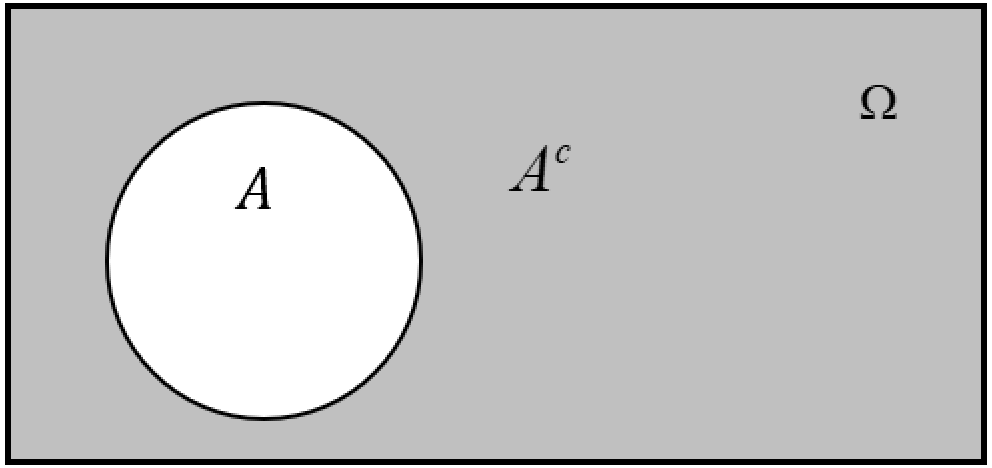
Difference in Mutually Exclusive Events

❖ If A and B are **disjoint/ mutually exclusive**, then $A - B = A$ and $B - A = B$ **(**see below figure)



Set Operations: Complement

- The complement of a set *A* is the set of all elements which do not belong to *A* and denoted by A^c *or* $\bar{A}$ also we can see from **Venn diagram** (see below figure)
- Note that:
  - $A^c = \Omega - A$
  - $(A^c)^c = A$



Some Important Definitions

- **Random Experiment**: Any process under consideration in which its results are not definitely known in advance is called random experiment. The possible results from this experiment called **outcomes**.
- **Sample Space S:** The sample space is the set of all possible outcomes from a certain random experiment.
- **An event E:** Any subset E of the sample space is known as an event.

Event Probability

- If S is a sample space of a random experiment and E is an event (any subset of the sample space), then the probability of event E, denoted by $P(E)$, is defined as

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ is the number of outcomes belonging to the event and $n(S)$ is the number of all possible outcomes in S.

Example

- Consider the experiment of tossing two dice, the sample space S of this experiment consists of 36 outcomes which are:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Example

- Consider the following events and find the probability of each:
  - Event A is the event that the sum of the two dice equal an even number:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- A={(1,1),(1,3),(1,5),
(2,2),(2,4),(2,6),
(3,1),(3,3),(3,5),
(4,2),(4,4),(4,6),
(5,1),(5,3),(5,5),
(6,2),(6,4),(6,6)}

- $P(A) = \frac{18}{36} = \frac{1}{2}$

Example

- Consider the following events and find the probability of each:
  - Event B is the event that any or both dice show(s) an odd number:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- B={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),
(2,1),(2,3),(2,5),
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),
(4,1),(4,3),(4,5),
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),
(6,1),(6,3),(6,5)}

- $P(B) = \frac{27}{36} = \frac{3}{4}$

Example

- Consider the following events and find the probability of each:
  - Event C is the event that the sum of the two numbers equals 9:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- C={(3,6),(6,3), (4,5),(5,4)}
- $P(C) = \frac{4}{36} = \frac{1}{9}$

Example

- Consider the following events and find the probability of each:
  - Event D is the event that the difference of the two numbers equal 2:

- D={(1,3),(3,1), (2,4), (4,2), (3,5),(5,3) (4,6), (6,4)}

- $P(D) = \frac{8}{36} = \frac{2}{9}$

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Example

- Consider the following events and find the probability of each:
  - Event E is the event that the two numbers are equal:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- E={(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)}
- $P(E) = \frac{6}{36} = \frac{1}{6}$

Example

- Consider the following events and find the probability of each:
  - Event F is the event that the two numbers are not equal:

- $F = \mathrm{E}^{c} = \Omega - E = \Omega - \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

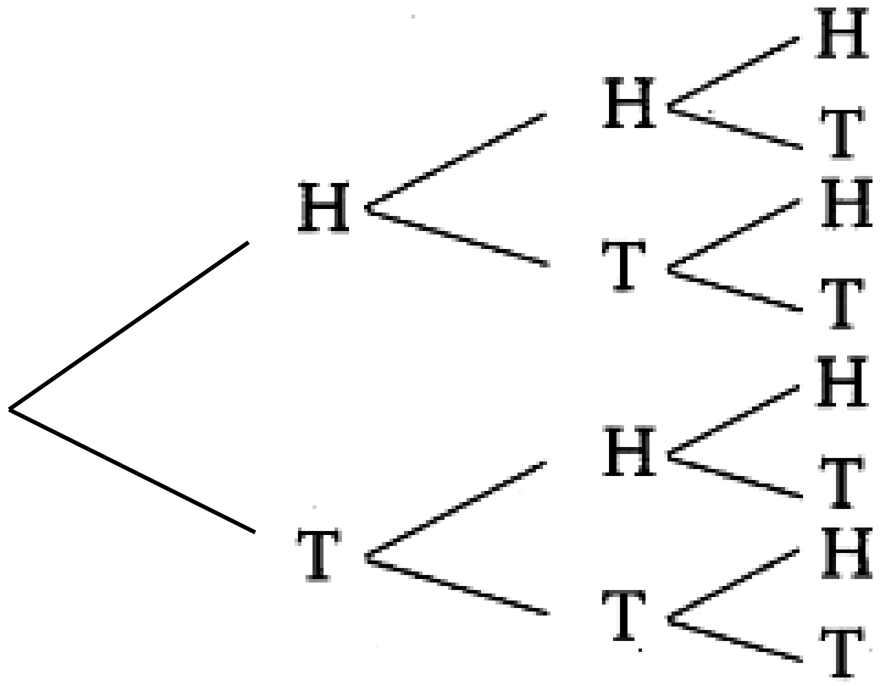
(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- $P(F) = 1 - \frac{6}{36} = 1 - \frac{1}{6} = \frac{5}{6}$

- Or: $P(F) = \frac{30}{36} = \frac{5}{6}$

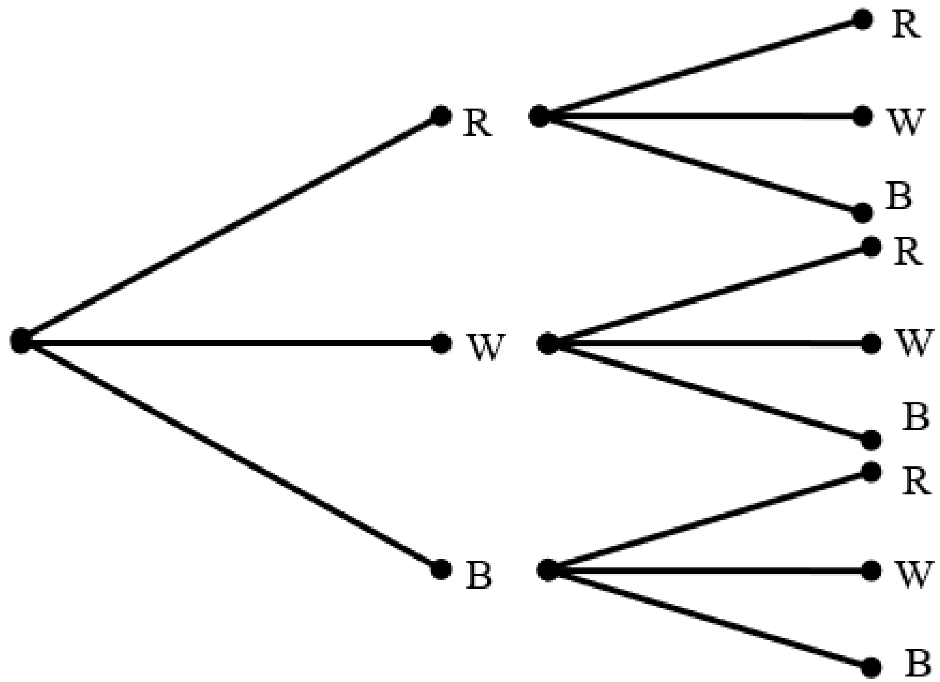
Tree Diagram

- The tree diagram is a method used to enumerate/list all the possible outcomes of a certain experiment whose outcomes are finite.
- Example: Tossing a Coin Once, Twice or Three times:



Tree Diagram - Example

- Consider an experiment of choosing **two** balls (**with replacement**) from an urn contains red balls, white balls and blue balls. Find all possible outcomes from this experiment using the tree diagram.
- Let us denote the red ball by R, the white ball by W, and the blue ball by B then:



- Therefore, all possible outcomes are:
  - {RR, RW, RB, WR, WW, WB, BR, BW, BB}

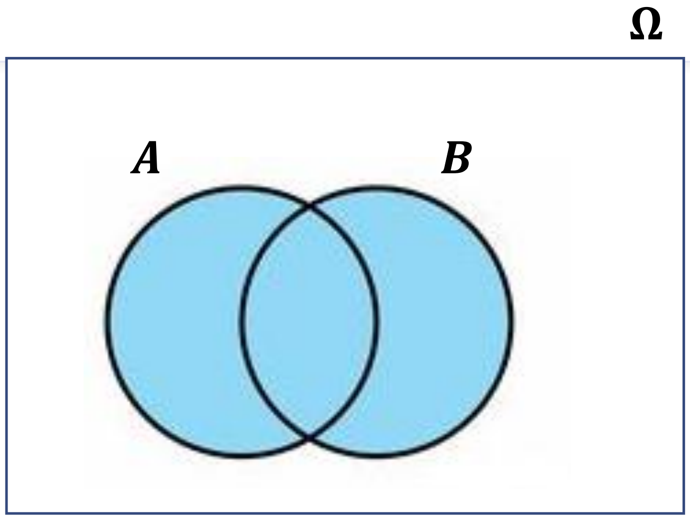
Important Remarks

- $A \cup B$ is the event that occurs if A occurs **or** B occurs (or both occurs).
- $A \cap B$ is the event that occurs if A occurs **and** B occurs.
- A^c (The complement of A) is the event that occurs if A does not occur.
- Two events A and B are called **mutually exclusive** if they are disjoint, i.e. A and B cannot occur simultaneously and, in this case, $A \cap B = \Phi$.
- A **non**-mutually exclusive events occur when $A \cap B \neq \Phi$

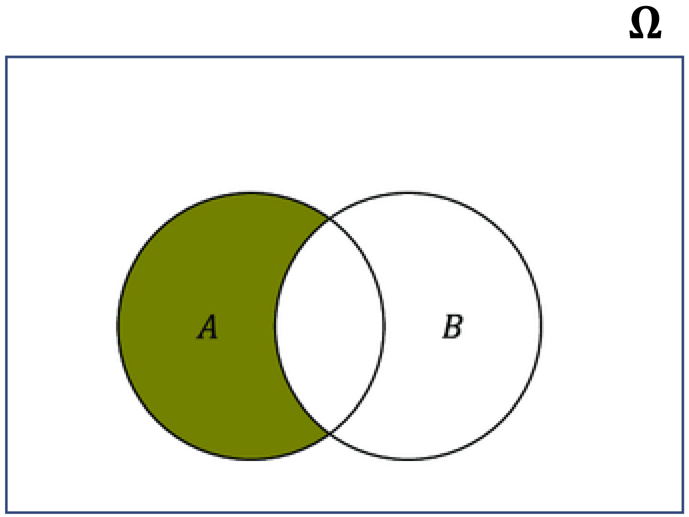
Axioms of Probability

❖ From Venn diagrams it is easy to show that:

➢ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



➢ $P(A - B) = P(A) - P(A \cap B)$

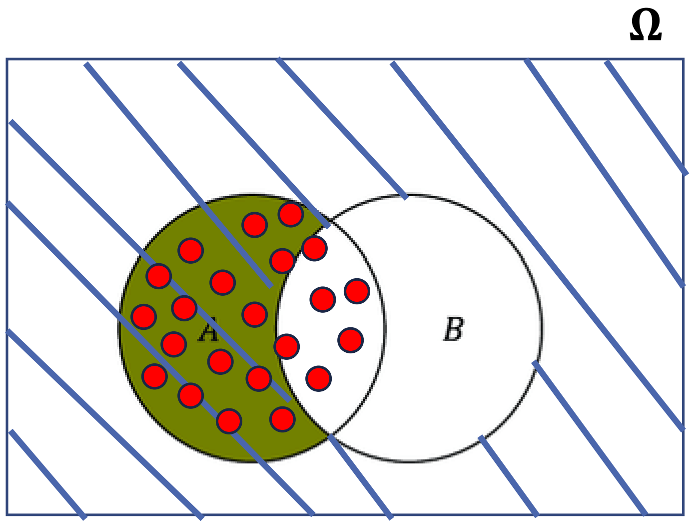


Axioms of Probability

- If A is an event, such that $A \subseteq S$, the probability of event A, denoted by $P(A)$ must satisfy the following axioms:
  - $0 \leq p(A) \leq 1$
  - $P(\Phi) = 0$, this is called the impossible event
  - $P(S) = 1$, this is called the certain event
  - $P(A^c) = 1 - P(A)$ therefore, $P(A) + P(A^c) = 1$
  - If $A \subseteq B$ then, $\mathrm{P}(A) \leq \mathrm{P}(B)$
  - Events A,B, C, . . . are exhaustive if their union equals the whole sample space, i.e., $A \cup B \cup C \cup \cdots = \Omega$,
  - If events A,B, C, . . . are exhaustive and mutually exclusive, therefore,
  $$P(A) + P(B) + P(C) + \cdots = 1$$
  - Events $E_1, E_2 \ldots, E_n$ are independent if they occur independently of each other, i.e., occurrence of one event does not affect the probabilities of others, therefore
  $$P\{E_1 \cap E_2 \ldots \cap E_n\} = P\{E_1\} \cdot P(E_2) \cdot \ \ldots \ \cdot P\{En\}$$

Important Rule

- $P(A \cap B^c) = P(A - B)$

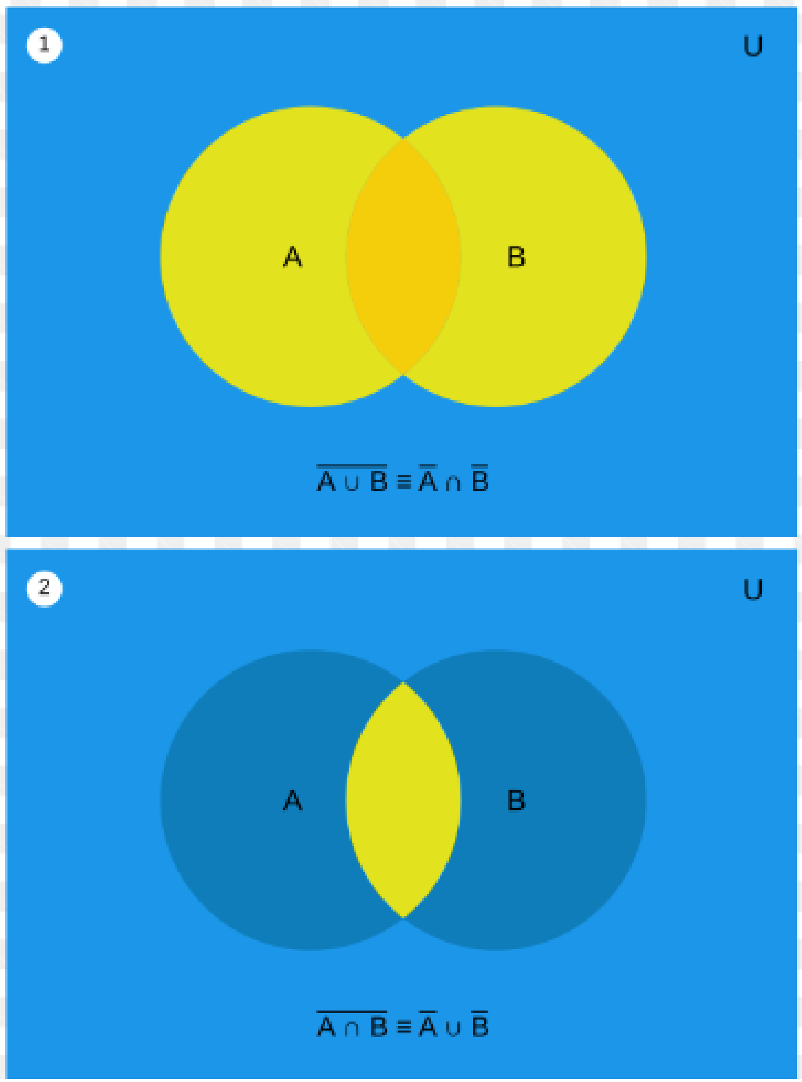


The dotted part is A
The shaded part is B^c
The intersection is the colored part: $A - B$

De Morgan's Laws:

- $(A \cup B)^c = A^c \cap B^c$

- $(A \cap B)^c = A^c \cup B^c$



Exercise 1

- In the coin tossing, if a head is equally likely to appear as a tail (fair coin), then

$$P\{H\} = P\{T\} = \frac{1}{2}$$

- If we had a biased (unfair) coin and felt that a head was twice as likely to appear as tail, find $P(H)$ and $P(T)$.
- Solution:
- $P(H) + P(T) = 1$ $(exhaustive\ and\ mutually\ exclusive$ events)
- $2P(T) + P(T) = 1 \rightarrow P(T) = \frac{1}{3}$
- $P(H) = 1 - P(T) = 1 - \frac{1}{3} = \frac{2}{3}$

Exercise 2

- In the die tossing, where all six numbers are equally likely to appear, then
- $S = \{1,2,3,4,5,6\} \quad and \quad P\{1\} = P\{2\} = P\{3\} = P\{4\} = P\{5\} = P\{6\} = \frac{1}{6}$
- Find the following Probabilities:
  a) The probability of getting an even number

$$A = \{2,4,6\}, \qquad P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

  a) The probability of getting a number divisible by 3

$$B = \{3,6\}, \qquad P(B) = \frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

Exercise 3

- A coin is tossed three times. Find the probability of getting exactly one head.
- Solution
- The sample space of this experiment is
- $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
- $P(onehead) = 3/8$

Exercise 4

- In a random experiment,

$$P(A) = \frac{1}{3} \quad , \quad P(B) = \frac{3}{5} \quad \text{and} \quad P(A - B) = \frac{1}{4}$$

- Evaluate:

i) $P(A \cup B)$ ii) $P(A^c \cup B^c)$ iii) $P(A \cap B^c)$

- Solution:

$$P(A - B) = P(A) - P(A \cap B)$$

$$P(A \cap B) = P(A) - P(A - B)$$

$$P(A \cap B) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Exercise 4

$$P(A) = \frac{1}{3} \quad , \quad P(B) = \frac{3}{5} \quad \text{and} \quad P(A - B) = \frac{1}{4}$$

$$P(A \cap B) = \frac{1}{12}$$

i) $P(A \cup B)$

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{1}{3} + \frac{3}{5} - \frac{1}{12} = \frac{51}{60}$$

Exercise 4

$$P(A) = \frac{1}{3} \quad , \quad P(B) = \frac{3}{5} \quad \text{and} \quad P(A - B) = \frac{1}{4}$$

$$P(A \cap B) = \frac{1}{12}$$

ii) $P(A^c \cup B^c)$

- $P(A^c \cup B^c) = P(A \cap B)^c$ (De Morgan's Law)
- $P(A \cap B)^c = 1 - P(A \cap B) = 1 - \frac{1}{12} = \frac{11}{12}$

Exercise 4

$$P(A) = \frac{1}{3} \quad , \quad P(B) = \frac{3}{5} \quad \text{and} \quad P(A - B) = \frac{1}{4}$$

$$P(A \cap B) = \frac{1}{12}$$

iii) $P(A \cap B^c)$

- $P(A \cap B^c) = P(A - B) = \frac{1}{4}$

Thank You ☺