Mathematics

for Computer Science Students (Math 403)

WorkSheet No. (2) - Solution

Problem 1:

Given that $P(A) = 0.9$, $P(B) = 0.8$ and $P(A \cap B) = 0.75$. **Find**

(i) $P(A \cup B)$ (ii) $P(A - B)$ (iii) $P(A^c \cap B^c)$

(iv) $P(A^c \cup B^c)$ (v) $P(A \cap B^c)$ (vi) $P(A^c \cap B)$

Solution:

(i) $\boldsymbol{P(A \cup B)} = P(A) + P(B) - P(A \cap B) = 0.9 + 0.8 - 0.75 = 0.95$

(ii) $\boldsymbol{P(A - B)} = P(A) - P(A \cap B) = 0.9 - 0.75 = 0.15$

(iii) $\boldsymbol{P(A^c \cap B^c)} = P(A \cup B)^c = 1 - P(A \cup B) = 1 - 0.95 = 0.05$

(iv) $\boldsymbol{P(A^c \cup B^c)} = P(A \cap B)^c = 1 - P(A \cap B) = 1 - 0.75 = 0.25$

(v) $\boldsymbol{P(A \cap B^c)} = P(A - B) = 0.15$

(vi) $\boldsymbol{P(A^c \cap B)} = P(B - A) = P(B) - P(A \cap B) = 0.8 - 0.75 = 0.05$

Problem 2:

Two dice are rolled. **Find** the probability that:

(i) The sum 8 appears.

(ii) The sum 7 or 11 comes up.

Solution:

$$\boldsymbol{S} = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

(i) Let

A: event that the sum of outcomes is 8. $\boldsymbol{A} = \{(2,6),(6,2),(3,5),(5,3),(4,4)\}$

B: event that the sum of outcomes is 7. $\boldsymbol{B} = \{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$

C: event that the sum of outcomes is 11. $\boldsymbol{C} = \{(6,5),(5,6)\}$

$$\therefore P(A) = \frac{5}{36}, \quad P(B) = \frac{6}{36} = \frac{1}{6} \quad \text{and} \quad P(C) = \frac{2}{36} = \frac{1}{18}$$

(ii) $P(B \cup C) = P(B) + P(C) - P(B \cap C)$

Since, B and C are mutually exclusive events.

$$\therefore P(B \cap C) = 0$$

$$P(B \cup C) = P(B) + P(C) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}$$

Problem 3:

A fair coin is tossed three times. **Find** the probability of getting:

(i) At least one head.

(ii) At most one head.

(iii) No tail.

(iv) One head and two tails.

Solution:

Number of the sample space elements $= 2^3 = 8$

$$\boldsymbol{S} = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

(i) $At\ least\ one\ head\ appears = \{\boldsymbol{HHH}, \boldsymbol{HHT}, \boldsymbol{HTH}, \boldsymbol{THH}, \boldsymbol{HTT}, \boldsymbol{THT}, \boldsymbol{TTH}\}$

$$P(At\ least\ one\ head\ appears) = \frac{7}{8}$$

(ii) $At\ most\ one\ head\ appears = \{\boldsymbol{HTT}, \boldsymbol{THT}, \boldsymbol{TTH}, \boldsymbol{TTT}\}$

$$P(At\ most\ one\ head\ appears) = \frac{4}{8} = \frac{1}{2}$$

(iii) $No\ tail\ appears = \{\boldsymbol{HHH}\}$

$$P(No\ tail\ appears) = \frac{1}{8}$$

(iv) $One\ head\ and\ two\ tails\ appear = \{\boldsymbol{HTT}, \boldsymbol{THT}, \boldsymbol{TTH}\}$

$$P(One\ head\ and\ two\ tails\ appear) = \frac{3}{8}$$

Problem 4:

Three horses **A**, **B** and **C** in a race. **A** is twice as likely to win as **B** while **B** is triple as likely to win as **C**. **Find** the probability that:

(i) **B** wins

(ii) Either **A** or **B** wins

(iii) Neither **C** wins nor **A** loses

Solution:

Let $P(\boldsymbol{C}\ wins) = p$, $P(\boldsymbol{B}\ wins) = 3p$ and $P(\boldsymbol{A}\ wins) = 2(3p) = 6p$

Since $P(S) = 1$, then $p + 3p + 6p = 1 \quad \rightarrow \quad \boxed{\boldsymbol{p = 1/10}}$

$\therefore P(\boldsymbol{A}\ wins) = 6/10$, $P(\boldsymbol{B}\ wins) = 3/10$ and $P(\boldsymbol{C}\ wins) = 1/10$

(i) $P(\boldsymbol{B}\ wins) = 3p = 3/10$

Since, the three events are mutually exclusive events.

$$\therefore\ P(A \cap B) = P(A \cap C) = P(B \cap C) = 0$$

(ii) $P(Either\ \boldsymbol{A}\ or\ \boldsymbol{B}\ wins) = P(\boldsymbol{A} \cup \boldsymbol{B}) = P(A) + P(B) - P(A \cap B) = 9/10$

(iii) $P(Neither\ \boldsymbol{C}\ wins\ nor\ \boldsymbol{A}\ loses) = P(\boldsymbol{C}\ loses\ and\ \boldsymbol{A}\ wins)$

$$= P(\boldsymbol{C}^c \cap \boldsymbol{A}) = P(A) - P(A \cap C) = P(A) = \frac{6}{10}$$

Problem 5:

A pair of dice I and II is thrown. Let

A: event that the outcome on die I is more than 5.

B: event that the outcome on die II is more than 5.

C: event that the outcome on die I and die II are equal.

Find

(i) $P(A \cup B)$ (ii) $P(B \cup C)$ (iii) $P(A \cup C)$ (iv) $P(A - B)$

(v) $P(A - C)$ (vi) $P(A^c)$ (vii) $P(B^c)$

Solution:

$$\boldsymbol{S} = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$$\boldsymbol{A} = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$$\boldsymbol{B} = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6)\}$$

$$\boldsymbol{C} = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$

$$\boldsymbol{A} \cap \boldsymbol{B} = \boldsymbol{B} \cap \boldsymbol{C} = \boldsymbol{A} \cap \boldsymbol{C} = \{(6,6)\} \quad \text{and} \quad P(\boldsymbol{A}) = P(\boldsymbol{B}) = P(\boldsymbol{C}) = \frac{1}{6}$$

(i) $P(\boldsymbol{A} \cup \boldsymbol{B}) = P(A) + P(B) - P(A \cap B) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$

(ii) $P(\boldsymbol{B} \cup \boldsymbol{C}) = P(B) + P(C) - P(B \cap C) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$

(iii) $P(\boldsymbol{A} \cup \boldsymbol{C}) = P(A) + P(C) - P(A \cap C) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$

(iv) $P(\boldsymbol{A} - \boldsymbol{B}) = P(A) - P(A \cap B) = \frac{1}{6} - \frac{1}{36} = \frac{5}{36}$

(v) $P(\boldsymbol{A}-\boldsymbol{C}) = P(A) - P(A \cap C) = \frac{1}{6} - \frac{1}{36} = \frac{5}{36}$

(vi) $P(\boldsymbol{A}^{\boldsymbol{c}}) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$

(vii) $P(\boldsymbol{B}^{\boldsymbol{c}}) = 1 - P(B) = 1 - \frac{1}{6} = \frac{5}{6}$

Problem 6:

Consider families with three children and assume that all eight possible distributions as

$$S = \{ BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG \}$$

where B is for boy and G for girl. Let

E: event that a family has at most two girls.

F: event that a family has children of both genders.

Find

(i)	$P(E \cup F)$	(ii)	$P(E - F)$	(iii)	$P(E^c \cap F^c)$
(iv)	$P(E^c \cup F^c)$	(v)	$P(E \cap F^c)$	(vi)	$P(E^c \cap F)$

Solution:

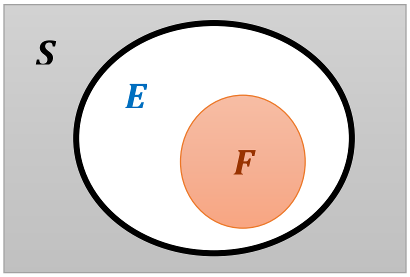
$\boldsymbol{E} = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB\}$

$\boldsymbol{F} = \{BBG, BGB, GBB, BGG, GBG, GGB\}$

$\boldsymbol{E} \cap \boldsymbol{F} = \{BBG, BGB, BGG, GBB, GBG, GGB\} = \boldsymbol{F}$

Since $E \cap F = F$, then $F \subset E$ and $E \cup F = E$.

$$\therefore P(\boldsymbol{E}) = \frac{\mathbf{7}}{\mathbf{8}} \quad \text{and} \quad P(\boldsymbol{F}) = P(\boldsymbol{E} \cap \boldsymbol{F}) = \frac{\mathbf{6}}{\mathbf{8}}$$



(i) $P(\boldsymbol{E} \cup \boldsymbol{F}) = P(E) = \frac{7}{8}$

(ii) $P(\boldsymbol{E} - \boldsymbol{F}) = P(E) - P(E \cap F) = P(E) - P(F) = \frac{7}{8} - \frac{6}{8} = \frac{1}{8}$

(iii) $P(\boldsymbol{E^c} \cap \boldsymbol{F^c}) = 1 - P(E \cup F) = 1 - P(E) = 1 - \frac{7}{8} = \frac{1}{8}$

(iv) $P(\boldsymbol{E^c} \cup \boldsymbol{F^c}) = 1 - P(E \cap F) = 1 - P(F) = 1 - \frac{6}{8} = \frac{1}{4}$

(v) $P(\boldsymbol{E} \cap \boldsymbol{F^c}) = P(E - F) = \frac{1}{8}$

(vi) $P(\boldsymbol{E^c} \cap \boldsymbol{F}) = P(F - E) = P(F) - P(E \cap F) = P(F) - P(F) = 0$

Since $F \subset E$, then $F - E = \varphi$.

Problem 7:

How many different ways can the gold, silver, and bronze medals be awarded in an Olympic event with 12 athletes competing?

Solution:

$$\text{Number of ways} = 12 * 11 * 10 = {}^{12}\boldsymbol{P}_3 = 1320$$

Problem 8:

How many different salads can be made from 5 different greens?

Solution:

$$\text{Number of salads} = {}^{5}\boldsymbol{C}_5 + {}^{5}\boldsymbol{C}_4 + {}^{5}\boldsymbol{C}_3 + {}^{5}\boldsymbol{C}_2 + {}^{5}\boldsymbol{C}_1 = 31$$

Problem 9:

A student is taking a probability test in which 7 questions out of 10 must be answered. In **how many** ways can the student answer the exam if

(i) Any 7 questions may be selected?

(ii) The first 2 questions must be selected?

(iii) The student must choose 3 questions from the first 5 and 4 questions from the last 5?

Solution:

(i) $$\text{Number of ways} = {}^{10}\boldsymbol{C}_7 = \frac{10!}{7! * 3!} = 120$$

(ii) $$\text{Number of ways} = {}^{2}\boldsymbol{C}_2\ {}^{8}\boldsymbol{C}_5 = \frac{8!}{5! * 3!} = 56$$

(iii) $$\text{Number of ways} = {}^{5}\boldsymbol{C}_3\ {}^{5}\boldsymbol{C}_4 = \left(\frac{5!}{3! * 2!}\right)\left(\frac{5!}{4! * 1!}\right) = 50$$

Problem 10:

You are going on a road trip with 4 friends in a car that fits 5 people. **How many** different ways can everyone sit if you must drive the whole way?

Solution:

$$\text{Number of ways} = 1 * 4 * 3 * 2 * 1 = 1 * 4! = 24$$

Problem 11:

From a set containing 10 digits and 26 letters, how many available passwords can you create if the password consists of

(i) 6 different letters?

(ii) any 6 characters including digits and letters?

Solution:

(i) $\text{Number of available passwords} = {}^{26}\boldsymbol{P}_6$

(ii) $\text{Number of available passwords} = (36)^6$

Problem 12:

Four letters {A, B, C, D}. **How many** words can be formed from any 3 letters?

Solution:

Case (1): There is repetition & order matters.

$$\text{Number of words} = n^r = 4^3 = 64$$

Case (2): There is **no** repetition & order matters.

$$\text{Number of words} = {}^{n}\boldsymbol{P}_r = \frac{n!}{(n-r)!} = \frac{4!}{1!} = 24$$

Problem 13:

From 7 consonants and 5 vowels. **How many** 7 letters words can be formed from chosen 4 consonants and 3 vowels if:

(i) Any letter can be repeated.

(ii) No letter can be repeated.

Solution:

(i) Number of words $= 7^7 \left[{}^{7}\boldsymbol{C}_4 \; {}^{5}\boldsymbol{C}_3 \right]$

(ii) Number of words $= 7! \left[{}^{7}\boldsymbol{C}_4 \; {}^{5}\boldsymbol{C}_3 \right]$

Problem 14:

A box contains 7 red, 9 green, 15 black and 9 yellow-colored balls. If 2 balls are selected without replacement, **what** is the probability of obtaining 1 green and 1 black?

Solution:

$$P(1 \text{ Green } \textbf{and } 1 \text{ Black}) = \frac{{}^{9}\boldsymbol{C}_1 \, {}^{15}\boldsymbol{C}_1 \, {}^{16}\boldsymbol{C}_0}{{}^{40}\boldsymbol{C}_2} = \frac{9}{52}$$

Problem 15:

Two items be chosen at random from a lot containing 12 items of which 4 are defective. Let

A: event that both items are defective.

B: event that both items are non-defective.

C: event that at least one item is defective.

Find (i) $P(A)$ (ii) $P(B)$ (iii) $P(C)$

Solution:

(i) $P(A) = \dfrac{{}^{4}\boldsymbol{C}_2 \; {}^{8}\boldsymbol{C}_0}{{}^{12}\boldsymbol{C}_2} = \dfrac{1}{11}$

(ii) $P(B) = \frac{{}^4\boldsymbol{C}_0\ {}^8\boldsymbol{C}_2}{{}^{12}\boldsymbol{C}_2} = \frac{14}{33}$

(iii) $P(C) = P(B^c) = 1 - P(B) = 1 - \frac{14}{33} = \frac{19}{33}$

OR

$$P(C) = \frac{{}^4\boldsymbol{C}_1\ {}^8\boldsymbol{C}_1}{{}^{12}\boldsymbol{C}_2} + \frac{{}^4\boldsymbol{C}_2\ {}^8\boldsymbol{C}_0}{{}^{12}\boldsymbol{C}_2} = \frac{19}{33}$$

Problem 16:

An urn contains 5 red, 10 black and 3 white marbles. If 3 marbles are drawn without replacement, **what** is the probability of obtaining:

(i) 1 black?

(ii) at most 1 white?

(iii) 3 of the same color?

(iv) at least 1 white?

Solution:

(i) $P(1\text{ Black}) = \frac{{}^{10}\boldsymbol{C}_1\ {}^8\boldsymbol{C}_2}{{}^{18}\boldsymbol{C}_3} = \frac{35}{102}$

(ii) $P(3\text{ of the same color}) = P(3\text{ are black}) + P(3\text{ are red}) + P(3\text{ are white})$

$$= \frac{{}^{10}\boldsymbol{C}_3\ {}^8\boldsymbol{C}_0}{{}^{18}\boldsymbol{C}_3} + \frac{{}^5\boldsymbol{C}_3\ {}^{13}\boldsymbol{C}_0}{{}^{18}\boldsymbol{C}_3} + \frac{{}^3\boldsymbol{C}_3\ {}^{15}\boldsymbol{C}_0}{{}^{18}\boldsymbol{C}_3} = \frac{131}{816}$$

(iii) $P(\text{at most 1 white}) = P(\text{no white}) + P(\text{1 only white})$

$$= \frac{{}^3\boldsymbol{C}_0\ {}^{15}\boldsymbol{C}_3}{{}^{18}\boldsymbol{C}_3} + \frac{{}^3\boldsymbol{C}_1\ {}^{15}\boldsymbol{C}_2}{{}^{18}\boldsymbol{C}_3} = \frac{385}{408}$$

(iv) $P(\text{at least 1 white}) = P(\text{1 white}) + P(\text{2 white}) + P(\text{3 white})$

$$= \frac{{}^3\boldsymbol{C}_1\ {}^{15}\boldsymbol{C}_2}{{}^{18}\boldsymbol{C}_3} + \frac{{}^3\boldsymbol{C}_2\ {}^{15}\boldsymbol{C}_1}{{}^{18}\boldsymbol{C}_3} + \frac{{}^3\boldsymbol{C}_3\ {}^{15}\boldsymbol{C}_0}{{}^{18}\boldsymbol{C}_3} = \frac{361}{816}$$