Worksheet 9

Problem 1: Divisibility by 17

Question: Does 17 divide each of these numbers?

a) 68

Solution:

$$\frac{68}{17} = 4$$

Since the division results in an integer, 17 divides 68.

b) 84

Solution:

$$\frac{84}{17} \approx 4.941\ldots$$

Since the division does not result in an integer, 17 does not divide 84.

c) 357

Solution:

$$\frac{357}{17} = 21$$

Since the division results in an integer, 17 divides 357.

d) 1001

Solution:

$$\frac{1001}{17} \approx 58.882\ldots$$

Since the division does not result in an integer, 17 does not divide 1001.

Problem 9: Quotient and Remainder

Find the quotient and remainder for each division.

a) 19 divided by 7

Solution:

$$19 = 7 \times 2 + 5$$

Quotient: 2, **Remainder:** 5.

b) -111 divided by 11

Solution: We seek integers q and r such that:

$$-111 = 11q + r \quad \text{with} \quad 0 \leq r < 11$$

$$q = -11, \quad r = 10$$

$$-111 = 11 \times (-11) + 10$$

Quotient: -11, **Remainder:** 10.

c) 789 divided by 23

Solution:

$$789 = 23 \times 34 + 7$$

Quotient: 34, **Remainder:** 7.

d) 1001 divided by 13

Solution:

$$1001 = 13 \times 77 + 0$$

Quotient: 77, **Remainder:** 0.

e) 0 divided by 19

Solution:

$$0 = 19 \times 0 + 0$$

Quotient: 0, **Remainder:** 0.

f) 3 divided by 5

Solution:

$$3 = 5 \times 0 + 3$$

Quotient: 0, **Remainder:** 3.

g) -1 divided by 3

Solution: We seek integers q and r such that:

$$-1 = 3q + r \quad \text{with} \quad 0 \leq r < 3$$

$$q = -1, \quad r = 2$$

$$-1 = 3 \times (-1) + 2$$

Quotient: -1, **Remainder:** 2.

h) 4 divided by 1

Solution:

$$4 = 1 \times 4 + 0$$

Quotient: 4, **Remainder:** 0.

Problem 11: Time on a 12-hour Clock

a) 80 hours after it reads 11 : 00

Solution:

Adding 80 hours to 11 : 00:

$$11 + 80 = 91 \equiv 7 \mod 12$$

Answer: 7 : 00

b) 40 hours before it reads 12 : 00

Solution:

Subtracting 4 hours from 12 : 00:

$$12 - 40 = -28 \equiv 8 \mod 12$$

Answer: 8 : 00

c) 100 hours after it reads 6 : 00

Solution:

Adding 4 hours to 6 : 00:

$$6 + 100 = 106 \equiv 10 \mod 12$$

Answer: 10 : 00

Problem 13: Modular Arithmetic

Given $a \equiv 4 \pmod{13}$ and $b \equiv 9 \pmod{13}$. Find the integer c with $0 \leq c \leq 12$ such that:

a) $c \equiv 9a \pmod{13}$

Solution:

$$c \equiv 9a \equiv 9 \times 4 = 36 \equiv 36 - 2 \times 13 = 10 \pmod{13}$$

Answer: $c = 10$

b) $c \equiv 11b \pmod{13}$

Solution:

$$c \equiv 11b \equiv 11 \times 9 = 99 \equiv 99 - 7 \times 13 = 8 \pmod{13}$$

Answer: $c = 8$

c) $c \equiv a + b \pmod{13}$

Solution:

$$c \equiv a + b \equiv 4 + 9 = 13 \equiv 0 \pmod{13}$$

Answer: $c = 0$

d) $c \equiv 2a + 3b \pmod{13}$

Solution:

$$c \equiv 2a + 3b \equiv 2 \times 4 + 3 \times 9 = 8 + 27 = 35 \equiv 35 - 2 \times 13 = 9 \pmod{13}$$

Answer: $c = 9$

e) $c \equiv a^2 + b^2 \pmod{13}$

Solution:

$$c \equiv a^2 + b^2 \equiv 4^2 + 9^2 = 16 + 81 = 97 \equiv 97 - 7 \times 13 = 6 \pmod{13}$$

Answer: $c = 6$

f) $c \equiv a^3 - b^3 \pmod{13}$

Solution:

$$c \equiv a^3 - b^3 \equiv 4^3 - 9^3 = 64 - 729 = -665$$

To find $-665 \pmod{13}$:

$$665 \div 13 = 51 \times 13 = 663, \quad \text{remainder } 2$$

Thus:

$$-665 \equiv -663 - 2 \equiv 0 - 2 \equiv 11 \pmod{13}$$

Answer: $c = 11$

Problem 24: Finding Integers with Given Congruences

a) Find integer a such that:

$$a \equiv 43 \pmod{23} \quad \text{and} \quad -22 \leq a \leq 0$$

Solution: First, find $43 \mod 23$:

$$43 \div 23 = 1 \quad \text{remainder } 20 \quad \Rightarrow \quad 43 \equiv 20 \pmod{23}$$

We need to find $a \equiv 20 \pmod{23}$ within $-22 \leq a \leq 0$.

Possible values:

$$a = 20 - 23 = -3$$

Since $-22 \leq -3 \leq 0$, the solution is:

$$a = -3$$

b) Find integer a such that:

$$a \equiv 17 \pmod{29} \quad \text{and} \quad -14 \leq a \leq 14$$

Solution: We need to find a such that:

$$a = 17 + 29k$$

and

$$-14 \leq 17 + 29k \leq 14$$

Solving for k:

$$-14 \leq 17 + 29k \leq 14 \quad \Rightarrow \quad -31 \leq 29k \leq -3 \quad \Rightarrow \quad -\frac{31}{29} \leq k \leq -\frac{3}{29}$$

The only integer k in this range is $k = -1$.

Thus:

$$a = 17 + 29 \times (-1) = 17 - 29 = -12$$

Answer: $a = -12$

c) Find integer a such that:

$$a \equiv -11 \pmod{21} \quad \text{and} \quad 90 \leq a \leq 110$$

Solution: First, express the congruence as:

$$a \equiv -11 \pmod{21} \quad \Rightarrow \quad a \equiv 10 \pmod{21}$$

We need to find $a = 10 + 21k$ within $90 \leq a \leq 110$.

Solving for k:

$$90 \leq 10 + 21k \leq 110 \quad \Rightarrow \quad 80 \leq 21k \leq 100 \quad \Rightarrow \quad \frac{80}{21} \leq k \leq \frac{100}{21}$$

$$3.8095 \leq k \leq 4.7619$$

The integer k satisfying this is $k = 4$.

Thus:

$$a = 10 + 21 \times 4 = 10 + 84 = 94$$

Answer: $a = 94$

Problem 45: Addition and Multiplication Tables for $\mathbb{Z}_5$

Addition Table for $\mathbb{Z}_5$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Multiplication Table for $\mathbb{Z}_5$

$\cdot_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Binary and Decimal Conversions with Steps

Problem 1: Convert the Decimal Expansion of Each of These Integers to Binary

a) Convert 231 to binary:

$$
\begin{aligned}
231 \div 2 &= 115 \quad \text{remainder } 1 \\
115 \div 2 &= 57 \quad \text{remainder } 1 \\
57 \div 2 &= 28 \quad \text{remainder } 1 \\
28 \div 2 &= 14 \quad \text{remainder } 0 \\
14 \div 2 &= 7 \quad \text{remainder } 0 \\
7 \div 2 &= 3 \quad \text{remainder } 1 \\
3 \div 2 &= 1 \quad \text{remainder } 1 \\
1 \div 2 &= 0 \quad \text{remainder } 1
\end{aligned}
$$

Reading the remainders from the bottom up, we get:

$$231_{10} = 11100111_2$$

b) Convert 4532 to binary:

$$
\begin{aligned}
4532 \div 2 &= 2266 \quad \text{remainder } 0 \\
2266 \div 2 &= 1133 \quad \text{remainder } 0 \\
1133 \div 2 &= 566 \quad \text{remainder } 1 \\
566 \div 2 &= 283 \quad \text{remainder } 0 \\
283 \div 2 &= 141 \quad \text{remainder } 1 \\
141 \div 2 &= 70 \quad \text{remainder } 1 \\
70 \div 2 &= 35 \quad \text{remainder } 0 \\
35 \div 2 &= 17 \quad \text{remainder } 1 \\
17 \div 2 &= 8 \quad \text{remainder } 1 \\
8 \div 2 &= 4 \quad \text{remainder } 0 \\
4 \div 2 &= 2 \quad \text{remainder } 0 \\
2 \div 2 &= 1 \quad \text{remainder } 0 \\
1 \div 2 &= 0 \quad \text{remainder } 1
\end{aligned}
$$

Reading the remainders from the bottom up, we get:

$$4532_{10} = 1000110110100_2$$

c) Convert 97644 to binary:

$$\begin{aligned}
97644 \div 2 &= 48822 \quad \text{remainder } 0 \\
48822 \div 2 &= 24411 \quad \text{remainder } 0 \\
24411 \div 2 &= 12205 \quad \text{remainder } 1 \\
12205 \div 2 &= 6102 \quad \text{remainder } 1 \\
6102 \div 2 &= 3051 \quad \text{remainder } 0 \\
3051 \div 2 &= 1525 \quad \text{remainder } 1 \\
1525 \div 2 &= 762 \quad \text{remainder } 1 \\
762 \div 2 &= 381 \quad \text{remainder } 0 \\
381 \div 2 &= 190 \quad \text{remainder } 1 \\
190 \div 2 &= 95 \quad \text{remainder } 0 \\
95 \div 2 &= 47 \quad \text{remainder } 1 \\
47 \div 2 &= 23 \quad \text{remainder } 1 \\
23 \div 2 &= 11 \quad \text{remainder } 1 \\
11 \div 2 &= 5 \quad \text{remainder } 1 \\
5 \div 2 &= 2 \quad \text{remainder } 1 \\
2 \div 2 &= 1 \quad \text{remainder } 0 \\
1 \div 2 &= 0 \quad \text{remainder } 1
\end{aligned}$$

Reading the remainders from the bottom up, we get:

$$97644_{10} = 10111110101111 0100_2$$

Problem 3: Convert the Binary Expansion of Each of These Integers to Decimal

a) Convert $(11111)_2$ to decimal:

$$(11111)_2 = 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 16 + 8 + 4 + 2 + 1 = 31_{10}$$

b) Convert $(100000001)_2$ to decimal:

$$(100000001)_2 = 1 \cdot 2^8 + 0 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 256 + 1 = 257_{10}$$

c) Convert $(101010101)_2$ to decimal:

$$(101010101)_2 = 1 \cdot 2^8 + 0 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 341_{10}$$

d) Convert $(1101001000010000)_2$ to decimal:

$$\begin{aligned}
(1101001000010000)_2 = 1 \cdot 2^{15} + 1 \cdot 2^{14} + 0 \cdot 2^{13} + 1 \cdot 2^{12} + 0 \cdot 2^{11} \\
+ 0 \cdot 2^{10} + 1 \cdot 2^9 + 0 \cdot 2^8 + 0 \cdot 2^7 + 0 \cdot 2^6 \\
+ 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 \\
= 32768 + 16384 + 4096 + 512 + 16 \\
= 53776_{10}
\end{aligned}$$

Problem 7: Convert the Hexadecimal Expansion of Each of These Integers to Binary

a) $(80E)_{16}$:

$$8_{16} = 1000_2, \quad 0_{16} = 0000_2, \quad E_{16} = 1110_2$$
$$(80E)_{16} = 1000\ 0000\ 1110_2$$

b) $(135AB)_{16}$:

$$1_{16} = 0001_2, \quad 3_{16} = 0011_2, \quad 5_{16} = 0101_2, \quad A_{16} = 1010_2, \quad B_{16} = 1011_2$$

$$(135AB)_{16} = 0001\ 0011\ 0101\ 1010\ 1011_2$$

c) $(ABBA)_{16}$:

$$A_{16} = 1010_2, \quad B_{16} = 1011_2, \quad B_{16} = 1011_2, \quad A_{16} = 1010_2$$

$$(ABBA)_{16} = 1010\ 1011\ 1011\ 1010_2$$

d) $(DEFACED)_{16}$:

$$D_{16} = 1101_2, \quad E_{16} = 1110_2, \quad F_{16} = 1111_2, \quad A_{16} = 1010_2, \quad C_{16} = 1100_2, \quad E_{16} = 1110_2, \quad D_{16} = 1101_2$$

$$(DEFACED)_{16} = 1101\ 1110\ 1111\ 1010\ 1100\ 1110\ 1101_2$$