Optimal policies for battery operation and design via stochastic optimal control of jump diffusions

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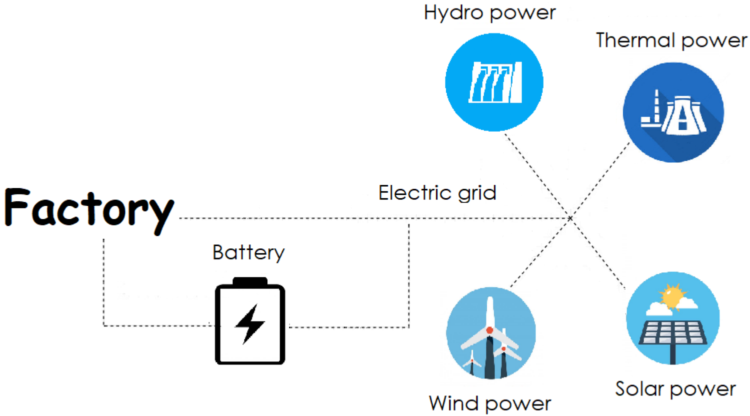
Introduction to the project



An individual factory connected to the Uruguayan power system.

The general framework

- $K(t)$ - the electricity spot price,
- $D(t)$ - the demand,
- $A(t)$ - the battery charge,
- $P_A(t)$ - the power supplied by the battery,
- $P_E(t)$ - the power supplied by the grid

Given the system

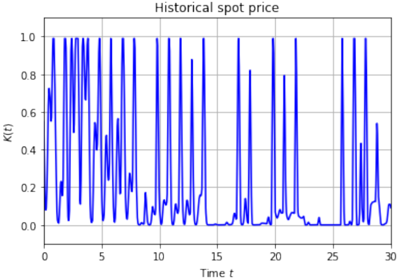
$$\begin{cases} \text{Price dynamics } dK(t), \\ \text{Demand dynamics } dD(t), \\ \text{Battery dynamics } dA(t) \end{cases} \tag{1}$$

maximize its performance, measured by the cost of energy

$$\min_{P_A} \left(\int_0^T K(s) P_E(s) ds \right) \tag{2}$$

Price model

The historical electricity spot price of the Uruguayan power system:



We propose the price model $\forall t \in [0, T]$:

$$\begin{cases} dK(t) = a(p_K(t), K(t))dt + \sigma(K(t))dW(t) + dP(t), \\ K(0) = K_0 \end{cases} \quad (3)$$

where $W(t)$ is a Wiener process and the continuous-time jump process $P(t)$, the forecast $p_K(t)$, the drift term $a(p_K(t), K(t))$ and the diffusion term $\sigma(K(t))$ are to be determined.

Price model: forecast

We assume, that there exists a forecast, available to the user of the model. This work focuses on the stochastic optimal control and not the statistical modelling. Thus, we resort to building a forecast that simply looks reasonable. We propose to capture the general trend by the moving average method. We let the data pass through the moving average filter several times and take the obtained curve as the forecast $p_K(t)$:

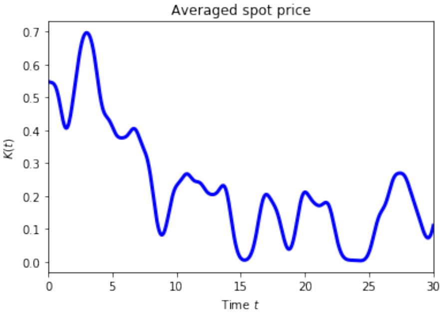


Figure: The forecast of the spot price obtained by the moving average method.

Price model: drift

The drift term is chosen to be

$$a(p_K(t), K(t)) = -\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - E[Y(t)], \quad (4)$$

where $Y(t)$ is a stochastic process, such that $E[dP(t)] = E[Y(t)dt]$.
The desired centering property, for all t:

$$E[K(t)] = p_K(t), \quad (5)$$
$$E[K(0)] = p_K(0). \quad (6)$$

Price model: drift

Let's first subtract $dp_K(t)$ and look at the deviation of $K(t)$ from the forecast:

$$
\begin{aligned}
d(K(t) - p_K(t)) = (-\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - E[Y(t)])dt \\
-\dot{p}_K(t) + \sigma(K(t))dW(t) + dP(t) \qquad (7)
\end{aligned}
$$

Taking expectation and dividing by dt we get

$$
\frac{E[d(K(t) - p_K(t))]}{dt} = E[-\theta_K(t)(K(t) - p_K(t))] \qquad (8)
$$

Then by Grönwall's lemma we have:

$$
E[K(t) - p_K(t)] \leq e^{-\theta_K(t)t} E[K(0) - p_K(0)] = 0, \qquad (9)
$$

$$
E[p_K(t) - K(t)] \leq e^{-\theta_K(t)t} E[p_K(0) - K(0)] = 0 \qquad (10)
$$

so that $E[K(t)] = p_K(t)$, as intended.

Price model: the diffusion

The diffusion coefficient $\sigma(K(t))$ has the form[1]

$$\sigma(K(t)) = \sqrt{2\theta_{K_t}\alpha_K K(t)(1-K(t))} \tag{11}$$

where

$$\theta_{K_t} = \max\left(\theta_{K_0}, \frac{\alpha_K \theta_{K_0} + |\dot{p}_{Kt}|}{\min(p_{Kt}, 1-p_{Kt})}\right) \tag{12}$$

and

- $\theta_{K_0} = 25$
- $\alpha_K = 0.003$

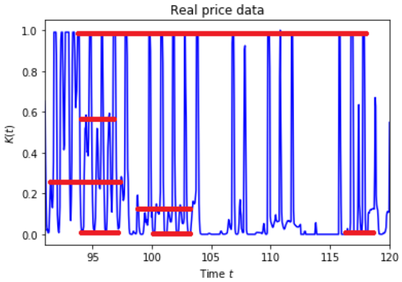
[1] R.Caballero, A.Kebaier, M.Scavino, R.Tempone (2020). "Quantifying Uncertainty with a Derivative Tracking SDE Model and Application to Wind Power Forecast Data". arXiv:2006.15907v2

Price model: jumps

The jump process $Y(t)$ is a compound Poisson process

$$Y(t) = \sum_{i=1}^{N(t)} Z_i(t) \tag{13}$$

with constant intensity of jumps λ. There is a finite set of values to which the process can jump: $L_Z = \{0, 0.1, 0.2, 0.4, 0.6, 0.8, 1\}$.



Price model: jumps

The probabilities of jumping from one level to the other are given by the following matrix P:

$$P = \begin{bmatrix} 0 & 0 & 8/27 & 0 & 5/27 & 1/27 & 13/27 \\ 2/6 & 0 & 1/6 & 0 & 1/6 & 0 & 2/6 \\ 2/4 & 2/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1/6 & 2/6 & 1/6 & 1/6 & 0 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 17/21 & 2/21 & 0 & 1/21 & 1/21 & 0 & 0 \end{bmatrix} \tag{14}$$

where each row P_i represents the distribution of the jumps given that the current value of the process is $K \in \Omega_i$,
$\Omega = \{[0, 0.05], [0.05, 0.15], [0.15, 0.3], [0.3, 0.5], [0.5, 0.7], [0.7, 0.9], [0.9, 1]\}$
The expected value $E[Y(t)]$:

$$E[Y(t)] = E[N(t)]E[Z(t)] = \lambda E[Z(t)] \tag{15}$$

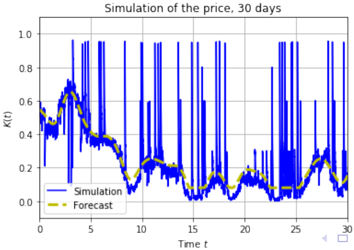
The price model becomes

$$\begin{cases} dK(t) = (-\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - \lambda E[Z(t)])dt \\ +\sqrt{2\theta_{K_t}\alpha_K K(t)(1-K(t))}dW(t) + Y(t)dt \\ K(0) = K_0 \end{cases} \tag{16}$$

where

$$\theta_{K_t} = \max\left(\theta_{K_0}, \frac{\alpha_K \theta_{K_0} + |\dot{p}_{Kt}|}{\min(p_{Kt}, 1 - p_{Kt})}\right), \quad \theta_{K_0} = 25, \quad \alpha_K = 0.003 \tag{17}$$

The price simulation:



Demand model

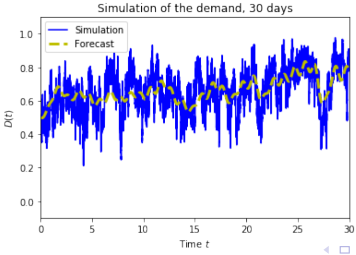
The demand model reads

$$\begin{cases} dD(t) = (-\theta_D(t)(D(t) - p_D(t)) + \dot{p}_D(t))dt \\ +\sqrt{2\theta_{D_t}\alpha_D D(t)(1-D(t))}dW(t) \\ D(0) = D_0 \end{cases} \quad (18)$$

where

$$\theta_{D_t} = \max\left(\theta_{D_0}, \frac{\alpha_D \theta_{D_0} + |\dot{p}_{Dt}|}{\min(p_{Dt}, 1 - p_{Dt})}\right), \quad \theta_{D_0} = 10, \qquad \alpha_D = 0.05 \quad (19)$$

The demand simulation:



Battery model

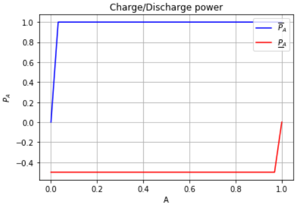
The battery dynamics[1] is given by

$$dA(t) = -P_A(t)dt, \quad A(0) = A_0, \tag{20}$$

$$\underline{A} < A < \overline{A}, \;\; \underline{P}_A < P_A < \overline{P}_A \tag{21}$$

The flux of charge and discharge of the battery depend on the current level of charge and are described by the following relations:

$$\overline{P}_A(A) = \min\left(\frac{A(t)-\underline{A}}{\Delta t}, (P_A)_{max}\right), \;\; \underline{P}_A(A) = \min\left(\frac{A(t)-\overline{A}}{\Delta t}, (P_A)_{min}\right)$$



[1] R.Caballero. "Stochastic Optimal Control of Renewable Energy", MS thesis (2019).

Deterministic model

We aim to minimize the cost of energy

$$u(t, A) = \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} J(t, A)$$

$$= \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} \left(\int_t^T K(s) P_E(s) ds - A(T) \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \right) \tag{22}$$

given the price $K(t)$, demand $D(t)$ and battery dynamics $A(t)$:

$$\begin{cases} dA_t = -P_A(t) dt, \quad \forall t \in [0, T], \\ K(t) = p_K(t), \quad \forall t \in [0, T], \\ D(t) = p_D(t), \quad \forall t \in [0, T], \\ P_E(t) + P_A(t) = D(t), \quad \forall t \in [0, T], \\ A(0) = 0, \quad \underline{A} < A < \overline{A} \\ \underline{P}_A < P_A < \overline{P}_A \\ T = 365 \end{cases} \tag{23}$$

Deterministic model

The cost function consists of the following terms:

- $\int_t^T K(s)P_E(s)ds$ - the cost of electricity from the grid
- $A(T)p_K(T)e^{-T\cdot Pb\cdot r/365}$ - the discounted price of energy left in the battery at final time T according to the forecast, which we can sell or use at the next production cycle,
- $K_A\overline{A}$ - the cost of initially purchasing the battery of size $\overline{A}$.

Deterministic model

Given $\overline{A}$, the function $u(t, A)$ solves the HJB equation:

$$\begin{aligned} &\partial_t u + H(\partial_A u, A) = 0 \\ 0 < t < T, \quad &u(T, A) = -A \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \end{aligned} \tag{24}$$

where the Hamiltonian is defined as

$$\begin{aligned} H &= \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} (-P_A(t)\partial_A u + K(t)P_E(t)) = \\ &= K(t)D(t) + \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} (-P_A(t)(\partial_A u + K(t))) \end{aligned} \tag{25}$$

Deterministic model

The following expression minimizes the Hamiltonian:

$$\min_{P_A(t)\in[\underline{P}_A,\overline{P}_A]}(-P_A(\partial_A u + K)) = \begin{cases} \min\{D,\overline{P}_A\}(\partial_A u + K), & \text{if } \partial_A u + K \geq 0 \\ -\underline{P}_A(\partial_A u + K), & \text{if } \partial_A u + K < 0 \end{cases}$$

Expression (17) gives the optimal control:

$$P_A^* = -1_{\{\partial_A u + K < 0\}}\underline{P}_A(A) + 1_{\{\partial_A u + K \geq 0\}}\min\{D(t),\overline{P}_A(A)\} \qquad (26)$$

With (26) we rewrite the HJB equation in the following way:

$$\begin{aligned} \partial_t u + K(t)D(t) - P_A^*(\partial_A u + K(t)) &= 0, \\ u(T,A) &= -A\cdot p_K(T)e^{-T\cdot Pb\cdot r/365} + K_A\cdot\overline{A} \end{aligned} \qquad (27)$$

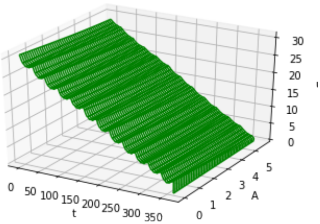
Deterministic model: numerical solution

Discretization of equation (27) reads:

$$\frac{u_i^t - u_i^{t-1}}{\Delta t} - [f_A^+ u_A^{t-} + f_A^- u_A^{t+}] + K^t D^t = 0, \tag{28}$$

$$f_A^+ = \max\{P_A^t, 0\},\ f_A^- = \min\{P_A^t, 0\},\ u_A^{t-} = \frac{u_i^t - u_{i-1}^t}{\Delta A},\ u_A^{t+} = \frac{u_{i+1}^t - u_i^t}{\Delta A} \tag{29}$$

The plot shows the solution to (28) with $\Delta t = 2^{-5}$, $\Delta a = 2^{-3}$

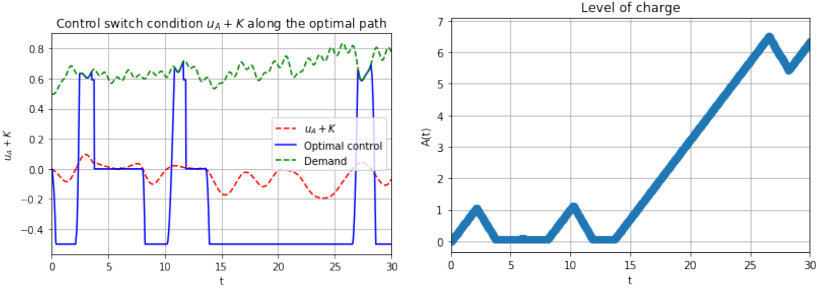


Deterministic model: the optimal control

We compute the optimal controls $P_A^*(t)$ starting from $A(0) = 0$ using the expression

$$P_A^* = -1_{\{\partial_A u + K < 0\}} \underline{P}_A(A) + 1_{\{\partial_A u + K \geq 0\}} \min\{D(t), \overline{P}_A(A)\} \qquad (30)$$

The optimal control $P_A^*(t)$ that minimizes the cost of energy and the level of charge $A(t)$ that corresponds to it:



The solution for (28),(29) with obtained control $P_A^*(t)$ is $u(0,0) = 30.4$

Deterministic model: convergence

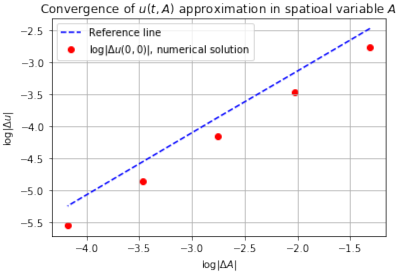
Convergence framework:

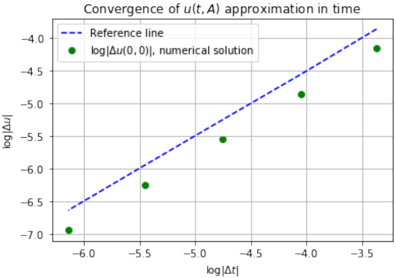
$$\text{Monotonicity} + \text{Stability} + \text{Consistency} \Rightarrow \text{Convergence} \tag{31}$$

CFL condition:

$$\Delta T \max_{a \in A} \frac{|P_A^*|}{\bar{A} \Delta A} \leq 1 \tag{32}$$

Convergence estimation:





Deterministic model with battery use penalization

We add an L^1 penalization of the battery use to the value function:

$$u(t,A) = \min_{P_A(t)\in[\underline{P}_A,\overline{P}_A]} J(t,A) = \min_{P_A(t)\in[\underline{P}_A,\overline{P}_A]} \left(\int_t^T K(s)P_E(s)ds + \int_t^T C_A|P_A(s)|ds - A(T)\cdot p_K(T)e^{-T\cdot Pb\cdot r/365} + K_A \cdot \overline{A} \right) \tag{33}$$

given the deterministic electricity price $K(t)$, demand $D(t)$ and battery dynamics $A(t)$:

$$\begin{cases} dA_t = -P_A(t)dt, \quad \forall t \in [0,T], \\ K(t) = p_K(t), \quad \forall t \in [0,T], \\ D(t) = p_D(t), \quad \forall t \in [0,T], \\ P_E(t) + P_A(t) = D(t), \quad \forall t \in [0,T], \\ A(0) = 0, \quad \underline{A} < A < \overline{A} \\ \underline{P}_A < P_A < \overline{P}_A \\ T = 365 \end{cases} \tag{34}$$

Deterministic model with battery use penalization

Given $\overline{A}$, the function $u(t, A)$ solves the HJB equation:

$$\partial_t u + H(\partial_A u, A) = 0$$
$$0 < t < T, \quad u(T, A) = -A \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \tag{35}$$

where the Hamiltonian is defined as

$$H = \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} (-P_A(t)\partial_A u + K(t)P_E(t)) =$$
$$= K(t)D(t) + \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} (-P_A(t)(\partial_A u + K(t)) + C_A|P_A|) \tag{36}$$

Deterministic model with battery use penalization

The explicit expression for the Hamiltonian:

$$H = \begin{cases} KD - \underline{P}_A(C_A + \partial_A u + K), & \text{if } \partial_A u + K < -C_A \\ KD - \min\{D, \overline{P}_A\}(-C_A + \partial_A u + K), & \text{if } \partial_A u + K \geq C_A \\ KD, & \text{if } -C_A \leq \partial_A u + K < C_A \end{cases} \tag{37}$$

The optimal control then takes the form

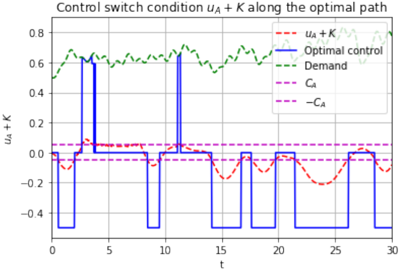
$$P_A^* = -1_{\{\partial_A u + K < -C_A\}} \underline{P}_A + 1_{\{\partial_A u + K \geq C_A\}} \min\{D, \overline{P}_A\} + 0 \cdot 1_{\{-C_A \leq \partial_A u + K < C_A\}} \tag{38}$$

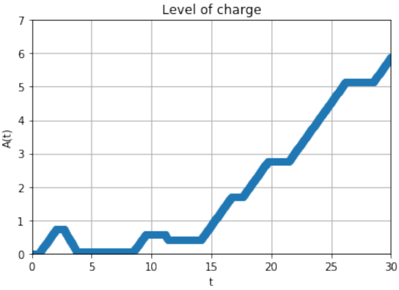
Deterministic model with battery use penalization

The optimal control:

$$P_A^* = -1_{\{\partial_A u + K < -C_A\}} \underline{P}_A + 1_{\{\partial_A u + K \geq C_A\}} \min\{D, \overline{P}_A\} \\ +0 \cdot 1_{\{-C_A \leq \partial_A u + K < C_A\}} \quad (39)$$

The optimal control $P_A^*(t)$ and the corresponding level of charge shown for 30 days:





Stochastic model

We aim to minimize the expected cost of energy

$$u(t, A, K, D) = \min_{P_A(t)} E\left(J(t, A, K, D)\right)$$

$$= \min_{P_A(t)} E\left(\int_t^T K(s)P_E(s)ds - A(T)\cdot p_K(T)e^{-T\cdot Pb\cdot r/365}\right) + K_A \cdot \overline{A} \tag{40}$$

given that $A(t)$, stochastic dynamics $K(t)$ and $D(t)$ satisfy:

$$\begin{cases} dA(t) = -P_A(t)dt \\ dK(t) = (-\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - E[Y(t)])dt \\ \qquad\qquad + \sqrt{2\theta_K(t)\alpha_K K(t)(1-K(t))}dW_K(t) + Y(t)dt \\ dD(t) = (-\theta_D(t)(D(t) - p_D(t)) + \dot{p}_D(t))dt \\ \qquad\qquad + \sqrt{2\theta_D(t)\alpha_D D(t)(1-D(t))}dW_D(t) \\ P_E(t) + P_A(t) = D(t) \;\; \forall t \in [0, T], \\ K(0) = K_0,\; D(0) = D_0,\; A(0) = 0,\;\; \underline{A} < A < \overline{A},\;\; \underline{P}_A < P_A < \overline{P}_A \end{cases} \tag{41}$$

Stochastic model

Given $\overline{A}$, the function $u(t, A, K, D)$ solves the HJB equation:

$$\begin{aligned} &\partial_t u + H(A, K, D, u, \partial_A u, \partial_K u, \partial_{KK} u, \partial_D u, \partial_{DD} u) = 0 \\ 0 < t < T, \quad &u(T, A, K, D) = -A \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \end{aligned} \tag{42}$$

where the Hamiltonian is defined as

$$\begin{aligned} H = KD &+ (\dot{p}_K - \theta_K(K - p_K) - E[Y(t)])\partial_K u + (\theta_K \alpha_K K(1 - K))\partial_{KK} u \\ &+ (\dot{p}_D - \theta_D(D - p_D))\partial_D u + (\theta_D \alpha_D D(1 - D))\partial_{DD} u \\ &+ \lambda \sum_j [u(t, K + Z_j, D, A) - u(t, K, D, A)] p_{i,j}(Z_j | K \in \Omega_i) \\ &+ \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} (-P_A(t)(\partial_A u + K)) \end{aligned} \tag{43}$$

Stochastic model

Using our results from the deterministic model we get the expression for the optimal control:

$$P_A^* = -1_{\{\partial_A u + K < 0\}} \underline{P}_A(A) + 1_{\{\partial_A u + K \geq 0\}} \min\{D, \overline{P}_A(A)\} \tag{44}$$

and rewrite the HJB equation in the following way:

$$\begin{aligned} \partial_t u + (\dot{p}_K - \theta_K(K - p_K) - E[Y(t)])\partial_K u + (\theta_K \alpha_K K(1-K))\partial_{KK} u \\ + (\dot{p}_D - \theta_D(D - p_D))\partial_D u + (\theta_D \alpha_D D(1-D))\partial_{DD} u \\ + \lambda \sum_j [u(t, K + Z_j, D, A) - u(t, K, D, A)] p_{i,j}(Z_j | K \in \Omega_i) \\ + KD - P_A^*(\partial_A u + K) = 0, \qquad 0 < t < T \\ u(T, A, K, D) = -A \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \end{aligned} \tag{45}$$

Numerical solution by operator splitting method

We rewrite (45)

$$\partial_t u + Au + Bu = 0 \tag{46}$$

where A - drift and diffusion operator, B - part of the price evolution operator, represented by the jumps. The numerical solution in time obtained with FE method is given by:

$$\frac{\overline{U}_{t+1} - \overline{U}_t}{\Delta t} + A\overline{U}_{t+1} = 0, \tag{47}$$

$$\overline{U}_t = (I + \Delta t A)\overline{U}_{t+1} \tag{48}$$

We then define $\overline{\overline{U}}_t = (I + \Delta t B)\overline{U}_t$ and

$$\overline{\overline{U}}_t = (I + \Delta t A)(I + \Delta t B)\overline{U}_{t+1} = (I + \Delta t(A + B))\overline{U}_{t+1} + O(\Delta t^2) \tag{49}$$

(49) shows that the operator splitting method gives a numerical solution of the same order as the one we had before.

Stochastic model: numerical solution

Discretized operator A in equation (46) reads:

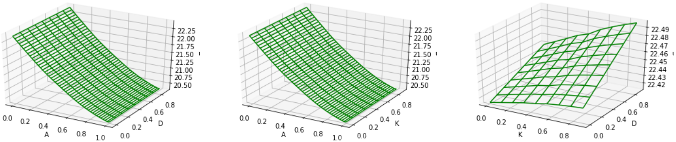
$$\frac{u^t_{i,j,s} - u^{t-1}_{i,j,s}}{\Delta t} - [f_A^+ u_A^{t-} + f_A^- u_A^{t+}] + \tag{50}$$

$$[f_K^+ u_K^{t-} + f_K^- u_K^{t+}] + \theta_K^t \alpha_K K_j (1 - K_j) \frac{u^t_{i,j-1,s} - 2u^t_{i,j,s} + u^t_{i,j+1,s}}{(\Delta K)^2} + \tag{51}$$

$$[f_D^+ u_D^{t-} + f_D^- u_D^{t+}] + \theta_D^t \alpha_D D_s (1 - D_s) \frac{u^t_{i,j,s-1} - 2u^t_{i,j,s} + u^t_{i,j,s+1}}{(\Delta D)^2} + K_j D_s = 0, \tag{52}$$

where $\mathrm{f}_K^+ = \max([\dot{p}_K^t - \theta_K^t(K_i^t - p_K^t)], 0)$, $f_K^- = \min([\dot{p}_K^t - \theta_K^t(K_i^t - p_K^t)], 0)$,
$\mathrm{u}_K^{t-} = \frac{u^t_{i,j} - u^t_{i-1,j}}{\Delta K}$, $u_K^{t+} = \frac{u^t_{i+1,j} - u^t_{i,j}}{\Delta K}$
$\mathrm{f}_D^+ = \max([\dot{p}_D^t - \theta_D^t(D_j^t - p_D^t)], 0)$, $f_D^- = \min([\dot{p}_D^t - \theta_D^t(D_j^t - p_D^t)], 0)$,
$\mathrm{u}_D^{t-} = \frac{u^t_{i,j} - u^t_{i,j-1}}{\Delta D}$, $u_D^{t+} = \frac{u^t_{i,j+1} - u^t_{i,j}}{\Delta D}$

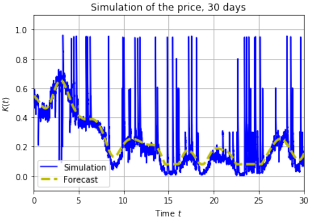
Numerical solution u(0,A,K,D) for equation (45) with $\Delta K = \Delta D = 2^{-3}$, $\Delta A \cdot \overline{A} = 2^{-3}$, $\Delta T = 2^{-10}$ is shown on the plots below. The problem I encountered before was fixed in the code.

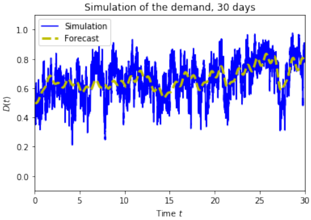


Interpretation. The plot in $K - D$ is shown for $A = 0$. The cost function $u(\cdot, \cdot, K, D)$ is expected to increase as K and D increase, since both the price of electricity and the demand volume translate to higher cost of production. The plots in $A - D$ and $A - K$ are qualitatively similar and show that $\partial_A u \leq 0$ and $\partial_D u,\ \partial_K u \geq 0$, as expected.

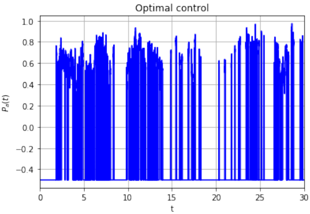
Stochastic model: simulation of the optimal control

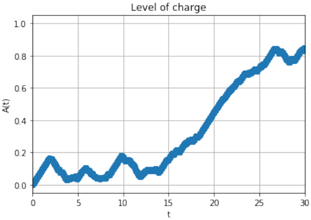
We generate price and demand processes:





and the corresponding control and the level of charge of the battery:



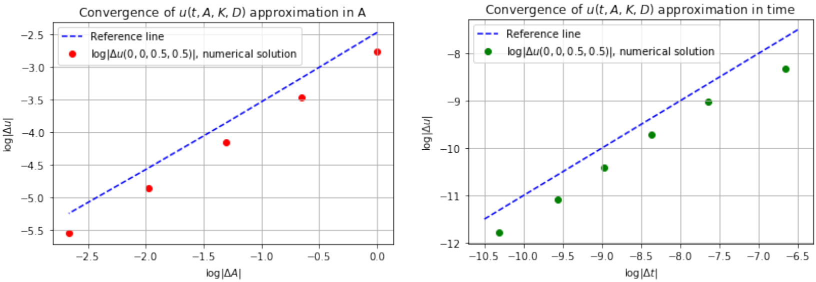


CFL condition:

$$\max_{t,A,D,K} \Delta T \left(\frac{\dot{p_k} + \theta_k (p_k - K)}{\Delta K} + \frac{\dot{p_d} + \theta_d (p_d - D)}{\Delta D} + \frac{|P_A^*|}{\bar{A} \Delta \hat{A}} + \right. \tag{53}$$

$$\left. + \frac{2\theta_K \alpha_K K(1-K)}{(\Delta K)^2} + \frac{2\theta_D \alpha_D D(1-D)}{(\Delta D)^2} \right) \leq 1 \tag{54}$$

Convergence estimation:



Comparison with no battery case

- Deterministic case. The cost of energy is a straightforward computation:

$$\hat{u} = \int_0^T p_K(s) p_D(s) ds \tag{55}$$

- Stochastic case. The expected cost of energy:

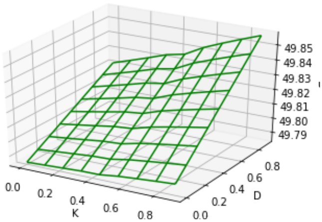
$$\hat{u}(t, K, D) = E\left(\int_t^T K(s)D(s)ds\right) \tag{56}$$

given the dynamics

$$\begin{cases} dK(t) = (-\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - E[Y(t)])dt \\ \qquad\qquad + \sqrt{2\theta_{K_t}\alpha_K K(t)(1 - K(t))} dW_K(t) + Y(t)dt \\ dD(t) = (-\theta_D(t)(D(t) - p_D(t)) + \dot{p}_D(t))dt \\ \qquad\qquad + \sqrt{2\theta_{D_t}\alpha_D D(t)(1 - D(t))} dW_D(t) \\ K(0) = K_0, \ D(0) = D_0 \end{cases} \tag{57}$$

Comparison with no battery case

- Deterministic case. $\hat{u} = 52.72$ conventional units (c.u.)
- Stochastic case. The plot shows the solution to the Kolmogorov Backward equation at time $t = 0$. $\hat{u}(t_0, K_0, D_0) = 49.81$ c.u.



Comparison of the cost of energy with and without the battery:

Case	Cost without battery	Optimal cost with battery
Deterministic	52.72	30.40
Stochastic	49.81	22.45

Optimal battery size

In this part of the project we minimize the following function:

$$\min_{\overline{A}} u_{\overline{A}}(t, A, K, D) = \min_{P_A(t),\overline{A}} E\left(J(t, A, K, D)\right) = \tag{58}$$

$$= \min_{P_A(t),\overline{A}} \{E\left(\int_t^T K(s)P_E(s)ds - A(T)\cdot p_K(T)\cdot e^{-T\cdot Pb\cdot r/365}\right) + K_A\cdot\overline{A}\} \tag{59}$$

with the battery, price and demand dynamics defined as before

$$\begin{cases} dA(t) = -P_A(t)dt \\ dK(t) = (-\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - E[Y(t)])dt \\ \qquad\qquad + \sqrt{2\theta_{K_t}\alpha_K K(t)(1-K(t))}dW_K(t) + Y(t)dt \\ dD(t) = (-\theta_D(t)(D(t) - p_D(t)) + \dot{p}_D(t))dt \\ \qquad\qquad + \sqrt{2\theta_{D_t}\alpha_D D(t)(1-D(t))}dW_D(t) \\ P_E(t) + P_A(t) = D(t) \quad \forall t \in [0, T], \\ K(0) = K_0,\ D(0) = D_0,\ A(0) = 0, \quad \underline{A} < A < \overline{A}, \quad \underline{P}_A \le P_A \le \overline{P}_A \end{cases}$$

Optimal battery size

For each fixed size of the battery $\overline{A}$, $u_{\overline{A}}(t, A, K, D)$ solves the HJB equation:

$$\partial_t u + H(A, K, D, u, \partial_A u, \partial_K u, \partial_{KK} u, \partial_D u, \partial_{DD} u) = 0 \tag{61}$$

$$0 < t < T, \quad u(T, A, K, D) = -A \cdot p_K(T) \cdot e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \tag{62}$$

with the Hamiltonian defined as

$$H = KD + (\dot{p}_K - \theta_K(K - p_K) - E[Y(t)])\partial_K u+ \tag{63}$$

$$(\theta_K \alpha_K K(1 - K))\partial_{KK} u + (\dot{p}_D - \theta_D(D - p_D))\partial_D u+ \tag{64}$$

$$(\theta_D \alpha_D D(1 - D))\partial_{DD} u + \min_{P_A \in [\underline{P}_A, \overline{P}_A]} (-P_A(\partial_A u + K))+ \tag{65}$$

$$\lambda[u(t, K + Z_j, D, A) - u(t, K, D, A)]p_{i,j}(Z_j | K \in \Omega_i) \tag{66}$$

Then we minimize over all possible battery sizes $\overline{A} > 0$:

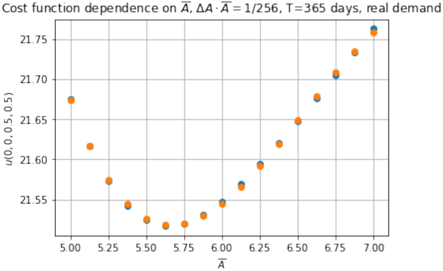
$$\overline{A}^* = \arg\min_{\overline{A}} u_{\overline{A}}(t_0, A_0, K_0, D_0) \tag{67}$$

Optimal battery size

The computations are performed on the following grids:

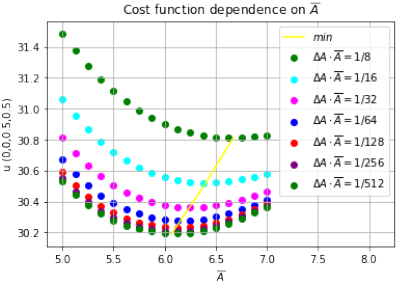
$$\Delta t = 2^{-14},\ \Delta K = \Delta D = 2^{-3},\ \Delta A \cdot \overline{A} = 2^{-3}, 2^{-4}, ..., 2^{-8} \tag{68}$$

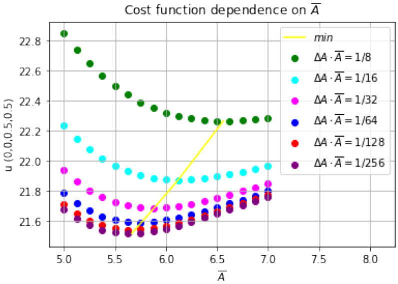
We will determine the optimal point by optimizing a polynomial of 3rd degree fitted to the computed data:



Optimal battery size

Below are the plots of $u(t_0 = 0, K_0, D_0, A_0 = 0)$ for different values of battery size $\overline{A}$ for all grids defined in (68) for deterministic (left) and stochastic (right) models.





Optimal battery size, deterministic model

The table shows the optimum value of $\overline{A}^*$ for different grids and the difference between those values for two successive grids.

$\Delta A_p \cdot \overline{A}$	$\overline{A}_p^*$	$\Delta\overline{A}_p^*$	$\Delta\overline{A}_{p-1}^*/\Delta\overline{A}_p^*$
1/8, p=1	6.67	-	-
1/16, p=2	6.40	0.268	-
1/32, p=3	6.24	0.159	1.69
1/64, p=4	6.15	0.089	1.79
1/128, p=5	6.11	0.047	1.89
1/256, p=6	6.08	0.024	2.95
1/512, p=7	6.07	0.012	2.01

As we reduce $\Delta A \cdot \overline{A}$ by 2, $\Delta\overline{A}$ decreases by a value approaching 2. Using this information we can estimate the true minimum value as $\overline{A}^* = 6.06$.

Optimal battery size, stochastic model

The table shows the optimum value of $\overline{A}^*$ for different grids and the difference between those values for two successive grids.

$\Delta A_p \cdot \overline{A}$	$\overline{A}_p^*$	$\Delta\overline{A}_p^*$	$\Delta\overline{A}_{p-1}^* / \Delta\overline{A}_p^*$
1/8, p=1	6.56	-	-
1/16, p=2	6.12	0.445	-
1/32, p=3	5.88	0.236	1.89
1/64, p=4	5.76	0.123	1.91
1/128, p=5	5.70	0.063	1.96
1/256, p=6	5.67	0.031	2.00

As we reduce $\Delta A \cdot \overline{A}$ by 2, $\Delta\overline{A}$ decreases by a value approaching 2. Using this information we can estimate the true minimum value as $\overline{A}^* = 5.63$.

Conclusions

- We addressed the problem of finding the optimal size of the battery to run the system with stochastic power consumption, spot price of electricity and the model for the battery.
- Given the battery size, we can find the optimal cost by means of HJB equation.
- This HJB equation is non-linear because of the control and has non-local terms because the price exhibits discontinuous behavior.
- To solve HJB equation we use the operator splitting technique.
- To find the optimal size of the battery we solve a one-dimensional optimization problem, whose objective function is defined by the cost-to-go function of the HJB equation.

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Thank you for your attention!