# Optimal policies for battery operation and design via stochastic optimal control of jump diffusions

Thesis/Dissertation by

Eliza Rezvanova

In Partial Fulfillment of the Requirements

For the Degree of

Masters of Science/Doctor of Philosophy

King Abdullah University of Science and Technology

Thuwal, Kingdom of Saudi Arabia

April, 2021

# EXAMINATION COMMITTEE PAGE

The thesis/dissertation of Eliza Rezvanova is approved by the examination committee

Committee Chairperson: Raul Tempone
Committee Members: Daniele Boffi, David Bolin

## ABSTRACT

To operate a production plant, one requires considerable amounts of power. With a wide range of energy sources, the price of electricity changes rapidly throughout the year, and so does the cost of satisfying the electricity demand. Battery technology allows storing energy while the electric power is lower, saving us from purchasing at higher prices. Thus, adding batteries to run plants can significantly reduce production costs. This thesis proposes a method to determine the optimal battery regime and its maximum capacity, minimizing the production plant's energy expenditures. We use stochastic differential equations to model the dynamics of the system. In this way, our spot price mimics the Uruguayan energy system's historical data: a diffusion process represents the electricity demand and a jump-diffusion process - the spot price. We formulate a corresponding stochastic optimal control problem to determine the battery's optimal operation policy and its optimal storage capacity. To solve our stochastic optimal control problem, we obtain the value function by solving the Hamilton-Jacobi-Bellman partial differential equation associated with the system. We discretize the Hamilton-Jacobi-Bellman partial differential equation using finite differences and a time splitting operator technique, providing a stability analysis. Finally, we solve a one-dimensional minimization problem to determine the battery's optimal capacity.

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# Chapter 1

# Introduction

A country's energy system consists of power plants that use various sources of energy. For example, there are hydro power stations, fossil-fuel stations, wind and solar farms, that supply energy to the grid in Uruguay. All these energy sources have different cost of energy production and as a result the electricity prices vary significantly throughout the day. For an individual manufacturer who uses the grid as a primary energy source, this means that to satisfy their demand in electricity, they have to buy it from the grid at available prices. However, we can reduce the cost of energy for the factory if we install the battery to store energy when the prices are low, so that we can use the battery instead of the grid when the prices are high. The work in this thesis is devoted to the problem of finding the optimal policy of operating the battery for energy storage.

Thesis has the following structure. In **Chapter 1** we introduce the problem and pose its actuality. **Chapter 2** presents the main mathematical concepts used in this work. We state deterministic and stochastic optimal control problem, and evoke Dynamic Programming principle to establish the connection between the value function and and the Hamilton-Jacobi-Bellman equation. In **Chapter 3** we develop the mathematical model which describes the cost of energy required to operate the system, and contains the price dynamics as a jump-diffusion process, the demand dynamics as the diffusion process and the deterministic battery dynamics. For this system we pose

the deterministic and stochastic optimal control problems. In **Chapter 4** we give the description of the numerical methods, which we used to solve the HJB equation. We describe the finite-difference scheme and show its monotonicity, stability and consistency leading to the convergence of the approximate solution to the true one. We employ the operator-splitting method, which simplifies the programming of the HJB equation. Finally, we show the regression model to solve a one-dimensional optimization problem of finding the optimum battery size. In **Chapter 5** we present the results, showing the optimal control computation for the battery of the optimum size.

# Chapter 2

# Theoretical background

In this chapter we give a brief introduction into the topics of probability theory, stochastic calculus and deterministic and stochastic optimal control.

- Section 2.1 gives the background on necessary concepts in stochastic calculus and closely follows "Stochastic numerics for applications" course lecture slides [1] in the description of probabilities, stochastic processes and infinitesimal generators; we follow [2] and [3] in the introduction to stochastic integrals and differential equations.
- Section 2.2 provides the basics of deterministic optimal control theory and is based on [4] and [5].
- Section 2.3 covers stochastic optimal control theory with Dynamic Programming principle and Hamilton-Jacobi-Bellman equation derivation, following AMCS 336 course lecture notes [2] and slides [6].

## 2.1 Stochastic calculus

In this section we will provide the basic definitions, properties and results in probability theory and stochastic differential equations, which are fundamental for stochastic optimal control theory.

### 2.1.1 Probabilities background

This section gives the basic and most important definitions of the probability theory that will be useful to us.

A probability space is a triple $(\Omega, \mathcal{F}, P)$, where $\Omega$ is the set of outcomes, $\mathcal{F}$ is the set of events and $P : \mathcal{F} \to [0, 1]$ is a set function that assigns probabilities to events satisfying the following definitions.

**Definition 1.** If $\Omega$ is a given non-empty set, then a *$\sigma$-algebra* $\mathcal{F}$ on $\Omega$ is a collection $\mathcal{F}$ of subsets of $\Omega$ such that:

1. $\Omega \in \mathcal{F}$;

2. $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$, where $F^c = \Omega - F$ is the complement set of $F$ in $\Omega$; and

3. $F_1, F_2, ... \in \mathcal{F} \to \bigcup_{i=1}^{+\infty} F_i \in \mathcal{F}$.

**Definition 2.** A probability measure on $(\Omega, \mathcal{F})$ is a set function $P : \mathcal{F} \to [0, 1]$ such that:

1. $P(\varnothing) = 0$, $P(\Omega) = 1$; and

2. If $A_1, A_2, ... \in \mathcal{F}$ are mutually disjoint sets then

$$P\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} P(A_i) \tag{2.1}$$

**Definition 3.** A random variable $X$, on the probability space $(\Omega, \mathcal{F}, P)$, is a function $X : \Omega \to \mathbb{R}^d$, such that

$$X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}, \tag{2.2}$$

for all open subsets $A$ of $\mathbb{R}^d$.

**Definition 4.** Two events $A, B \in \mathcal{F}$ are said to be independent if

$$P(A \bigcap B) = P(A)P(B) \tag{2.3}$$

Two random variables $X, Y$ in $\mathbb{R}^d$ are independent if

$$X^{-1}(A) \text{ and } Y^{-1}(B) \text{ are independent for all open sets } A, B \subseteq \mathbb{R}^d \tag{2.4}$$

### 2.1.2 Stochastic processes

In this section we will introduce the notion of stochastic processes and give examples most relevant to this work.

**Definition 5.** A stochastic process $X : [t_0, t_1] \times \Omega \to \mathbb{R}^d$ in the probability space $(\Omega, \mathcal{F}, P)$ is a function such that $X(t, \cdot)$ is a d-dimensional random variable in $(\Omega, \mathcal{F}, P)$ for all $t \in (t_0, t_1)$.

We are particularly interested in stochastic processes with the Markov property, because the future state of such processes depends only on its state at the present and does not depend on the past states. This property gives an advantage when analyzing the control of such processes. Let's give the formal definition of Markov property and time-continuous Markov processes.

**Definition 6.** $X_t$ is a continuous-time finite space Markov process (Markov chain) if for any finite set of indices $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq s \leq t$ and corresponding states $i_1, i_2, ..., t_n, i, j \in \mathcal{S}$ such that $P(X(t_1) = i_1, ..., X(t_n) = i_n, X(s) = i) > 0$, we have

$$P(X(t) = j | X(s) = i, X(t_n) = i_n, ..., X(t_1) = i_1) = P(X(t) = j | X(s) = i) \tag{2.5}$$

This is called the *Markov property*.

The function $P_{ij}(t) = P(X(t) = j|X(0) = i)$ for all $i, j \in \mathcal{S}$ in (2.5) is called the *transition function* of the process $X(t)$.

We introduce two types of processes with the Markov property - the Wiener and Compound Poisson processes.

## Wiener Process

**Definition 7.** A real-valued stochastic process $W(t)$ is called a Wiener process if

1. $W(0) = 0$, and the mapping $t \to W(t)$ is continuous with probability 1,

2. For any choice of non-negative real numbers $0 \leq t_1 \leq t_2 \leq ... \leq t_{n-1} \leq t_n$ the random variables $W(t_1), W(t_2) - W(t_1), ..., W(t_n) - W(t_{n-1})$ are independent,

3. $\Delta W = W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$ for all $t \geq s \geq 0$

## Poisson Process

The Poisson process represents a number of events $N(t)$ that have occurred up to time $t$.

**Definition 8.** A real-valued stochastic process $N(t)$ is called a Poisson process if

1. $N(0) = 0$

2. $N$ has independent increments

3. Let $I = (s, t] \subseteq \mathbb{R}$. For $k = 1, 2, ...$

$$P(N(t) - N(s) = k) = e^{-\lambda|I|} \frac{(\lambda|I|)^k}{k!} \tag{2.6}$$

The realization of a Poisson process is a non-negative right-continuous integer-valued and non-decreasing step function. $\lambda$ in (2.6) is called the rate (or intensity) of jumps. The Poisson process can be viewed as a collection of Poisson distributed

random variables $N(t)$ - number of events in the time interval $[0, t]$, or as a collection of exponentially distributed inter-arrival times $Z_k = T_k - T_{k-1}$, where $T_i$ - is the time of the $i$th arrival.

### Compound Poisson Process

**Definition 9.** A compound Poisson process with rate $\lambda > 0$ and jump size distribution $G$ is a continuous-time stochastic process $\{Y(t)\}$ given by

$$Y(t) = \sum_{j=1}^{N(t)} Z_j \tag{2.7}$$

where $N(t)$ is a Poisson counting process with rate $\lambda > 0$ and the random variables (jump sizes) $Z_j$ are independent identically distributed with distribution $G$.

## 2.1.3 Itô integral

Here we give the definition of the stochastic integral

$$\int_0^T f(s)dW(s) \tag{2.8}$$

of a function $f : [0, T] \to \mathbb{R}$, which is Lipschitz continuous. Consider an arbitrary partition of the interval $[0, T]$

$$0 = t_0 < t_1 < t_2 < ... < t_N = T \tag{2.9}$$

and compute the Forward Euler approximation of the integral (2.8)

$$\sum_{i=0}^{N-1} f(t_i)(W(t_{i+1}) - W(t_i)) \tag{2.10}$$

The integral with respect to the Wiener process is defined as the the limit in the mean square sense of the sum in (2.10) as $\Delta t_{max} = \max_i(t_{i+1} - t_i)$ goes to 0:

$$\sum_{i=0}^{N-1} f(t_i)(W(t_{i+1}) - W(t_i)) \xrightarrow[\Delta t_{max}\to 0]{\text{m.s.}} \int_0^T f(s)dW(s) \tag{2.11}$$

### 2.1.4 Stochastic differential equation

In this section we will define what we understand as a stochastic differential equation. It is convenient to start with an ordinary differential equation.

Consider a deterministic system, the state of which, $x(t)$, evolves according to the following ODE initial-value problem:

$$\begin{cases} dx(t) = g(t, x)dt, \quad \forall t \in [0, T] \\ x(0) = x_0 \end{cases} \tag{2.12}$$

The Picard–Lindelöf theorem states that (2.12) has a unique solution, given that the function $g(t, x)$ is Lipschitz continuous in the spatial variable, that is, by definition, there exists a positive real constant $C$, such that

$$|g(t, x) - g(t, y)| \leq C|x - y| \tag{2.13}$$

for all $x$ and $y$ in $\mathbb{R}$.

A stochastic differential equation introduces noise through the Wiener process and has the form

$$\begin{cases} dX(t) = g(t, X(t))dt + \sigma(t, X(t))dW(t), \quad \forall t \in [0, T] \\ X(0) = X_0 \end{cases} \tag{2.14}$$

for given functions $g(t,x)$ and $\sigma(t,x)$.

(2.14) should be understood as a shorthand for the integral equation:

$$X(t) = X_0 + \int_0^t g(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s), \quad \forall t \in [0,T] \tag{2.15}$$

We call $g(t, X(t))$ the *drift term* and $\sigma(t, X(t))$ the *diffusion term.*

It can be proven, that if $g(t,x)$ and $\sigma(t,x)$ satisfy certain conditions, then there exists a continuous solution $X(t)$, which is $L^2$-bounded. Sufficient conditions are:

$$\max\{|g(t,x) - g(t,y)|, |\sigma(t,x) - \sigma(t,y)|\} \leq C|x-y| \tag{2.16}$$

$$\max\{|g(t,x) - g(s,x)| + |\sigma(t,x) - \sigma(s,x)|\} \leq C(1+|x|)\sqrt{|s-t|} \tag{2.17}$$

for some constant $C$.

(2.14) can also be thought of as a system of equations for $X(t)$ in $\mathbb{R}^n$, $g, \sigma : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, if $W(t)$ is a vector of $n$ pairwise independent Wiener processes.

### 2.1.5 Itô's formula

In the following two subsections we will introduce the Itô's formula and the infinitesimal generator of a stochastic process, which will be useful in the derivation of the Hamilton-Jacobi-Bellman equation. In subsection 2.3.3 we will derive the Hamilton-Jacobi-Bellman equation using infinitesimal generator in a general form, and show its particular form for the Wiener and the compound Poisson processes.

The Itô's formula gives a way to express the differential of the function of a stochastic process.

*Itô's formula.* Let $X(t)$ be a system of $n$ independent Itô processes defined by

(2.14) and let $y : \mathbb{R}^n \times [0,T] \to \mathbb{R} \in C^2$. Then

$$dy(X,t) = \frac{\partial y(X,t)}{\partial t}dt + \sum_i \frac{\partial y(X,t)}{\partial x_i}dX_i + \frac{1}{2}\sum_i \frac{\partial^2 y(X,t)}{\partial x_i^2}(dX_i)^2 \tag{2.18}$$

where $dW_i dt = dt dW_i = 0, dt^2 = 0$.

If we expand $(dX_i)^2$ in (2.1.5), we get

$$\begin{aligned} dy(X,t) = \frac{\partial y(X,t)}{\partial t}dt + \sum_i \left( g(X,a_t)\frac{\partial y(X,t)}{\partial x_i} + \frac{1}{2}\sigma^2(X,a_t)\frac{\partial^2 y(X,t)}{\partial x_i^2} \right) dt \\ + \sum_i \sigma(X,a_t)\frac{\partial y(X,t)}{\partial x_i}dW_i \end{aligned} \tag{2.19}$$

### 2.1.6 Infinitesimal generators

The *infinitesimal generator* serves as a good instrument to describe the evolution of a Markov process $X_t = X(t)$ and is defined to act on a function of this process $y(X_t) : \mathbb{R}^n \to \mathbb{R}$:

$$\mathcal{L}y(X_t) = \lim_{dt \to 0} \frac{\mathbb{E}[y(X_{t+dt}) - y(X_t)|X_t = x]}{dt} \tag{2.20}$$

### Diffusion generator

Consider the following Ito SDE:

$$\begin{cases} dX_t = g(X_t, a_t)dt + \sigma(X_t, a_t)dW_t, \quad t \in [0,T] \\ X_0 = x_0 \end{cases} \tag{2.21}$$

where $X_t = X(t)$, $a_t = a(t)$, $W_t = W(t)$. Notice that $y(X_{t+dt}) - y(X_t) = \int_t^{t+dt} dy(X_s)$ in (2.20), and by Ito's lemma we can write:

$$\int_t^{t+dt} dy(X_s) = \int_t^{t+dt} \frac{\partial y(X_s)}{\partial x} dX_s + \int_t^{t+dt} \frac{1}{2}\sigma^2(X_s, a_s)\frac{\partial^2 y(X_s)}{\partial x^2} ds$$
$$= \int_t^{t+dt} \left( g(X_s, a_s)\frac{\partial y(X_s)}{\partial x} + \frac{1}{2}\sigma^2(X_s, a_s)\frac{\partial^2 y(X_s)}{\partial x^2} \right) ds + \int_t^{t+dt} \sigma(X_s, a_s)\frac{\partial y(X_s)}{\partial x} dW_s \tag{2.22}$$

Thus, substituting (2.22) into (2.20) and recalling that the expectation of the Wiener process is zero $\mathbb{E}[\int_t^{t+dt} \sigma(X_s, a_s)\partial_x y(X_s) dW_s] = 0$ we arrive at the expression for the generator of the diffusion process:

$$\mathcal{L}y(X_t) = g(X_t, a_t)\frac{\partial y(X_t)}{\partial x} + \frac{1}{2}\sigma^2(X_t, a_t)\frac{\partial^2 y(X_t)}{\partial x^2} \tag{2.23}$$

## Generator of the compound Poisson process

Consider now the jump process:

$$Y(t) = \sum_{j=1}^{N(t)} Z_j \tag{2.24}$$

Let $\mathcal{S} = \{s_0, s_1, ..., s_N\}$ be the state space of jumps $Z$ and $p_Z(z_i) = P(Z = z_i)$ be the probability mass function of jumps $Z$. Then the generator for the Compound Poisson with rate $\lambda$ and finite-dimensional jump distribution has the form:

$$\mathcal{L}y(X_t) = \lambda \sum_{z \in \mathcal{S}} (y(X_t + z_i) - y(X_t)) p_Z(z_i) \tag{2.25}$$

We recall the definition of the generator

$$\mathcal{L}y(X_t) = \lim_{dt \to 0} \frac{\mathbb{E}[y(X_{t+dt}) - y(X_t)|X_t = x]}{dt} \tag{2.26}$$

Let us condition on the number of jumps that $X$ takes in the interval $t, t + dt$:

$$\mathbb{E}[y(X_{t+dt})|X_t = x] = \sum_{k \geq 0} \mathbb{E}[y(X_{t+dt})|X_t = x \text{ and } k \text{ jumps }]P(k \text{ jumps on } (t, t + dt)) \tag{2.27}$$

Since $X$ is compound Poisson with rate $\lambda$ we can further express $P(k$ jumps on $(t, t + dt))$:

$$P(0 \text{ jumps on } (t, t + dt)) = 1 - \lambda dt + \mathcal{O}(dt^2), \tag{2.28}$$

$$P(1 \text{ jumps on } (t, t + dt)) = \lambda dt + \mathcal{O}(dt^2), \tag{2.29}$$

$$P(k \text{ jumps on } (t, t + dt)) = \mathcal{O}(dt^k), \quad k \geq 1 \tag{2.30}$$

We denote by $Z$ the jump size distributed according to $\mu$ and rewrite (2.27):

$$\mathbb{E}[y(X_{t+dt})|X_t = x] = y(X_t)(1 - \lambda dt) + \mathbb{E}[y(X_t + Z)|X_t = x]\lambda dt + \mathcal{O}(dt^2) \tag{2.31}$$

This yields

$$\frac{\mathbb{E}[y(X_{t+dt})|X_t = x] - y(X_t)}{dt} = \lambda(\mathbb{E}[y(X_t + Z)] - y(X_t)) + \mathcal{O}(dt^2) \tag{2.32}$$

Taking the limit $dt \to 0$ in (2.32) and recalling that

$$\mathbb{E}[y(X_t + Z)] = \sum_{z_i \in \mathcal{S}} y(X_t + z_i)p_Z(z_i) \tag{2.33}$$

we arrive at (2.25).

## 2.2 Deterministic optimal control

In this section we will state the deterministic optimal control problem, dynamic programming principle and establish the connection between the value function and Hamilton-Jacobi-Bellman equation.

### 2.2.1 Problem statement

Consider a system governed by the following differential equation:

$$\begin{cases} dx(t) = g(x(t), a(t))dt, \quad t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{2.34}$$

where the *state trajectory* of the system $x(t) \in \mathbb{R}^n$ is controlled by the parameter $a : [0, T] \to \Gamma$, called the *control*, and $g : \mathbb{R}^n \times \Gamma \to \mathbb{R}^n$ is a given function.

Suppose we have constraints on the state variable $x(t)$ and control $a(t)$, which originate from the physical framework of the system

$$x(t) \in X(t), \quad \forall t \in [0, T], \tag{2.35}$$

$$a(t) \in A(t), \quad \forall t \in [0, T]. \tag{2.36}$$

A control $a(t) \in A(t)$ such that $x(t) \in X(t)$ solves (2.34) is called an *admissible control*. We will denote the set of admissible controls by $\mathcal{A}$:

$$\mathcal{A}[0, T] = \{a(t) \in A(t) : x(t) \in X(t), \text{ s.t. (2.34) holds } \forall t \in [0, T]\} \tag{2.37}$$

The cost of operating the system from time $t$ to $T$ in general form is defined by

$$J(x,t) = \int_t^T f(x(s), a(s))ds + h(x(T)) \tag{2.38}$$

where $f : \mathbb{R}^n \times \Gamma \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are given functions. We call $\int_t^T f(x(s), a(s))ds$ the *running cost* of the system and $h(x(T))$ the *terminal cost*. Then the optimal control problem is formulated in the following way: minimize the cost functional $J(x,t)$ over all admissible controls $a(t) \in \mathcal{A}[0,T]$, that is

$$u(x,t) = \min_{a(t)\in\mathcal{A}[t,T]} J(x,t) = \min_{a(t)\in\mathcal{A}[t,T]} \int_t^T f(x(s), a(s))ds + h(x(T)) \tag{2.39}$$

$u(x,t)$ defined by (2.39) is called the *value function*. If $a^*(t)$ is an admissible control that satisfies (2.39), it is called an *optimal control* and the corresponding $x^*(t)$, called an *optimal trajectory*.

### 2.2.2 Dynamic programming and Hamilton-Jacobi-Bellman equation

The dynamic programming principle uses the idea that if $a^*(t)$ is an optimal trajectory on time interval $[0,T]$, then it is also optimal on all subintervals $[t,T] \subset [0,T]$. This suggests that if we start from time T, where the value function is known $u(T) = h(x(T))$, we will be able to recover the optimal trajectory to go from each point $(x(T-\delta t), T-\delta t)$ to the point $(x(T), T)$; and therefore we will be able to recover recursively the optimal path from each point $(x(t), t)$ to the point $(x(t+\delta t), t+\delta t)$. The *dynamic programming relation* has the following form:

$$u(x(t),t) = \min_{a(t)\in\mathcal{A}[t,t+\delta t]} \left( \int_t^{t+\delta t} f(x(s), a(s))ds + u(x(t+\delta t), t+\delta t) \right) \tag{2.40}$$

It is easy to show why (2.40) holds. Lets put $h(x(T)) = 0$ for simplicity and rewrite (2.39), splitting the integral into two integrals over adjacent intervals:

$$u(x(t),t) = \min_{a(t)\in\mathcal{A}[t,T]} \left( \int_t^{t+\delta t} f(x(s),a(s))ds + \int_{t+\delta t}^{T} f(x(s),a(s))ds \right) \tag{2.41}$$

We subdivide the time interval $[t,T]$ into two adjacent intervals $[t,t+\delta t]$ and $[t+\delta t,T]$; we also split the minimization intervals and write

$$u(x(t),t) = \min_{a(t)\in\mathcal{A}[t,t+\delta t]} \left( \min_{a(t)\in\mathcal{A}[t+\delta t,T]} \left( \int_t^{t+\delta t} f(x(s),a(s))ds + \int_{t+\delta t}^{T} f(x(s),a(s))ds \right) \right) \tag{2.42}$$

$$= \min_{a(t)\in\mathcal{A}[t,t+\delta t]} \left( \int_t^{t+\delta t} f(x(s),a(s))ds + \min_{a(t)\in\mathcal{A}[t+\delta t,T]} \int_{t+\delta t}^{T} f(x(s),a(s))ds \right) \tag{2.43}$$

$$= \min_{a(t)\in\mathcal{A}[t,t+\delta t]} \left( \int_t^{t+\delta t} f(x(s),a(s))ds + u(x(t+\delta t),t+\delta t) \right), \tag{2.44}$$

which is the expression in (2.40).

We can further approximate for sufficiently small $\delta t$

$$\int_t^{t+\delta t} f(x(s),a(s))ds = f(x,a)\delta t \tag{2.45}$$

With (2.45) we can rewrite (2.40) as

$$u(x(t),t) = \min_{a(t)\in\mathcal{A}[t,t+\delta t]} \left( f(x,a)\delta t + u(x(t+\delta t),t+\delta t) \right) \tag{2.46}$$

Let's expand $u(x(t+\delta t), t+\delta t)$ in Taylor series around $(x(t), t)$:

$$u(x(t+\delta t), t+\delta t) = u(x(t), t) + \left(\frac{x(t+\delta t)-x(t)}{\delta t}\delta t\right)^T \frac{\partial u(x(t),t)}{\partial x} + \frac{\partial u(x(t),t)}{\partial t}\delta t + \left(\frac{x(t+\delta t)-x(t)}{\delta t}\delta t\right)^T \frac{\partial^2 u(x(t),t)}{\partial x^2}\frac{x(t+\delta t)-x(t)}{\delta t}\delta t + \left(\frac{x(t+\delta t)-x(t)}{\delta t}\delta t\right)^T \frac{\partial^2 u(x(t),t)}{\partial x \partial t}\delta t + \frac{\partial^2 u(x(t),t)}{\partial t^2}\delta t^2 + O(\delta t^3) \tag{2.47}$$

We will drop the terms of order $O(\delta t^2)$ in the Taylor expansion (2.47) (in assumption that the second derivatives exist and are bounded) and rewrite (2.46) for small $\delta t$ as

$$u(x(t), t) = \min_{a(t)\in\mathcal{A}[t,t+\delta t]} \Bigg( f(x,a)\delta t + u(x(t),t) \tag{2.48}$$

$$+ \left(\frac{\partial u(x(t),t)}{\partial x}\frac{x(t+\delta t)-x(t)}{\delta t} + \frac{\partial u(x(t),t)}{\partial t}\right)\delta t\Bigg) \tag{2.49}$$

We subtract $u(x(t), t)$ on both sides and divide by $\delta t$:

$$0 = \min_{a(t)\in\mathcal{A}[t,t+\delta t]} \left( f(x,a) + \frac{\partial u(x(t),t)}{\partial x}\frac{x(t+\delta t)-x(t)}{\delta t} + \frac{\partial u(x(t),t)}{\partial t}\right) \tag{2.50}$$

We now put $\delta t \to 0$ in (2.50) so that $\frac{x(t+\delta t)-x(t)}{\delta t} \to \frac{dx(t)}{dt} = g(x(t), a(t))$ and get our final result:

$$0 = \frac{\partial u(x,t)}{\partial t} + \min_{a(t)} \left( f(x,a) + g^T(x,a)\frac{\partial u(x,t)}{\partial x}\right) \tag{2.51}$$

We define the Hamiltonian as

$$H\left(x, \frac{\partial u(x,t)}{\partial x}\right) = \min_{a(t)} \left( f(x,a) + g^T(x,a)\frac{\partial u(x,t)}{\partial x}\right) \tag{2.52}$$

and rewrite (2.51) in the form

$$0 = \frac{\partial u(x,t)}{\partial t} + H\left(x, \frac{\partial u(x,t)}{\partial x}\right) \tag{2.53}$$

Equation (2.53) together with the final time condition $u(x,T) = 0$ is called *Hamilton-Jacobi-Bellman* equation. It is significant to us because it is solved by the value function $u(x,t)$. Once we have the value function we can recover the optimal control $a^*(t)$ and the optimal trajectory $x^*(t)$ for any initial condition $x(0)$.

## 2.3 Stochastic optimal control

In this section we will introduce the stochastic optimal control problem and derive the Hamilton-Jacobi-Bellman equation for stochastic optimal control problem.

### 2.3.1 Problem statement

Suppose the system's trajectory $X(t)$ is described by the Ito SDE:

$$\begin{cases} dX(t) = g(X(t), a(t))dt + \sigma(X(t), a(t))dW(t), \quad t \in [0,T] \\ X(0) = X_0 \end{cases} \tag{2.54}$$

where the *state trajectory* of the system $X(t) \in \mathbb{R}^n$ is a stochastic process, $a(t) \in \Gamma$ is a Markov control function, $g$, $\sigma : \mathbb{R}^n \times \Gamma \to \mathbb{R}$ are given functions and $W(t) = (W_1(t), ..., W_n(t))$ are pairwise independent Wiener processes.

We will define the expected cost of operating the system:

$$J(x,t) = \mathbb{E}\left[\int_t^T f(X(s), a(s))ds + h(X(T))|X(t) = x\right] \tag{2.55}$$

where $f : \mathbb{R}^n \times \Gamma \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are given functions. As in deterministic case,

we call $\int_t^T f(X(s), a(s))ds$ the *running cost* of the system and h(X(T)) the *terminal cost*. Then the optimal control problem is formulated in the following way: minimize the cost functional $J(x,t)$ over all admissible controls $a(t) \in \mathcal{A}[0,T]$, that is

$$u(x,t) = \min_{a(t)\in\mathcal{A}[t,T]} J(x,t) = \min_{a(t)\in\mathcal{A}[t,T]} \mathbb{E}\left[\int_t^T f(X(s),a(s))ds + h(X(T))|X(t) = x\right]. \tag{2.56}$$

### 2.3.2 Dynamic programming principle

As we did with deterministic case, we start with the dynamic programming relation

$$u(X_t,t) = \min_{a\in\mathcal{A}[t,t+\delta t]} \mathbb{E}\left[\int_t^{t+\delta t} f(X_s,a_s)ds + u(X_{t+\delta t}, t+\delta t)|X_t\right] \tag{2.57}$$

where we use the notation $X_t = X(t)$, $a_t = a(t)$. To show that (2.57) is true, we will show that on the one hand

$$u(X_t,t) \geq \min_{a\in\mathcal{A}[t,t+\delta t]} \mathbb{E}\left[\int_t^{t+\delta t} f(X_s,a_s)ds + u(X_{t+\delta t}, t+\delta t)|X_t\right] \tag{2.58}$$

while on the other hand

$$u(X_t,t) \leq \min_{a\in\mathcal{A}[t,t+\delta t]} \mathbb{E}\left[\int_t^{t+\delta t} f(X_s,a_s)ds + u(X_{t+\delta t}, t+\delta t)|X_t\right] \tag{2.59}$$

which gives the desired result.

Suppose that $a^*$ is the optimal control path on the interval $[t,T]$. We split the

interval $[t, T]$ into two parts and rewrite (2.56) as

$$\begin{aligned} u(X_t, t) &= \min_{a \in \mathcal{A}[t,T]} \mathbb{E}\left[\int_t^T f(X_s, a_s) ds + h(X_T) | X_t\right] \\ &= \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^*) ds | X_t\right] + \mathbb{E}\left[\int_{t+\delta t}^T f(X_s, a_s^*) ds + h(X_T) | X_t\right] \end{aligned} \tag{2.60}$$

We use the property of conditional expectation $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$ to get:

$$u(X_t, t) = \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^*) ds | X_t\right] + \mathbb{E}\left[\mathbb{E}\left[\int_{t+\delta t}^T f(X_s, a_s^*) ds + h(X_T) | X_{t+\delta t}\right] | X_t\right] \tag{2.61}$$

Notice that the inner expectation in the second term of (2.61) is exactly the value function $u(X_{t+\delta t}, t + \delta t)$, so that (2.61) becomes:

$$\begin{aligned} u(X_t, t) &= \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^*) ds | X_t\right] + \mathbb{E}\left[u(X_{t+\delta t}, t + \delta t) | X_t\right] \\ &\geq \min_{a \in \mathcal{A}[t,t+\delta t]} \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s) ds + u(X_{t+\delta t}, t + \delta t) | X_t\right] \end{aligned} \tag{2.62}$$

which gives (2.58).

Now choose an arbitrary control $a^+$ on the interval $[t, t + \delta t]$ and optimal control $a^*$ on the interval $[t + \delta t, T]$, and denote this control $a' = (a^+, a^*)$. By definition of the value function (2.56) we get:

$$\begin{aligned} u(X_t, t) &\leq \mathbb{E}\left[\int_t^T f(X_s, a'_s) ds + h(X_T) | X_t\right] \\ &= \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^+) ds + h(X_T) | X_t\right] + \mathbb{E}\left[\int_{t+\delta t}^T f(X_s, a_s^*) ds + h(X_T) | X_t\right] \\ &= \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^+) ds + h(X_T) | X_t\right] \\ &\quad + \mathbb{E}\left[\mathbb{E}\left[\int_{t+\delta t}^T f(X_s, a_s^*) ds + h(X_T) | X_{t+\delta t}\right] | X_t\right] \end{aligned} \tag{2.63}$$

Again, noticing that $\mathbb{E}\left[\int_{t+\delta t}^{T} f(X_s, a_s^*)ds + h(X_T)|X_{t+\delta t}\right] = u(X_{t+\delta t}, t+\delta t)$, we can write:

$$u(X_t, t) \leq \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^+)ds + h(X_T)|X_t\right] + \mathbb{E}\left[u(X_{t+\delta t}, t+\delta t)|X_t\right] \tag{2.64}$$

Taking the minimum over all controls $a^+$ gives:

$$u(X_t, t) \leq \min_{a^+ \in \mathcal{A}[t,t+\delta t]} \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^+)ds + u(X_{t+\delta t}, t+\delta t)|X_t\right] \tag{2.65}$$

which is exactly the expression in (2.59).

### 2.3.3 Hamilton-Jacobi-Bellman equation

We will now use the dynamic programming relation to derive the Hamilton-Jacobi-Bellman equation. We will do it in two steps. First we will show that an optimal control $a^*$ yields

$$0 = f(x, a_t^*)dt + \left(\frac{\partial u}{\partial t} + \mathcal{L}u\right)(x, t)dt \tag{2.66}$$

Second, we will show that an arbitrary control $a^+$ implies

$$0 \leq f(x, a_t^+)dt + \left(\frac{\partial u}{\partial t} + \mathcal{L}u\right)(x, t)dt \tag{2.67}$$

*Step 1.* Suppose $a_t^*$ is the optimal control, satisfying (2.56). Then the dynamic programming relation (2.57) holds, which we will restate here for convenience

$$u(X_t, t) = \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^*)ds + u(X_{t+\delta t}, t+\delta t)|X_t\right] \tag{2.68}$$

Notice that $u(X_{t+\delta t}, t+\delta t) - u(X_t, t) = \int_t^{t+\delta t} du(X_t, t)$ and by Itô's formula

$$du(X_t,t) = \left(\frac{\partial u}{\partial t}(X_t,t) + \mathcal{L}u(X_t,t)\right)dt + \sigma(X_t,t)\frac{\partial u}{\partial x}(X_t,t)dW_t \tag{2.69}$$

We use the property of the Itô integral $\mathbb{E}[\int_t^{t+\delta t} dW_t] = 0$ and substitute (2.69) into (2.73) to get

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^*)ds + u(X_{t+\delta t}, t+\delta t) - u(X_t,t)|X_t\right] \\ &= \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^*)ds + \int_t^{t+\delta t}\left(\frac{\partial u}{\partial s} + \mathcal{L}u\right)(X_s,s)ds|X_t\right] \end{aligned} \tag{2.70}$$

Dividing by $\delta t$, taking the limit as $\delta t \to 0$ and using Fubini's theorem to interchange taking expectation and integrating in time gives

$$\begin{aligned} 0 = \lim_{\delta t\to 0}\mathbb{E}&\left[\frac{1}{\delta t}\int_t^{t+\delta t} f(X_s, a_s^*)ds + \int_t^{t+\delta t}\left(\frac{\partial u}{\partial s} + \mathcal{L}u\right)(X_s,s)ds|X_t\right] \\ &= \mathbb{E}\left[f(X_t, a_t^*) + \left(\frac{\partial u}{\partial t} + \mathcal{L}u\right)(X_t,t)|X_t\right] \end{aligned} \tag{2.71}$$

Putting $X_t = x$ in (2.71) we get

$$0 = f(x, a_t^*) + \left(\frac{\partial u}{\partial t} + \mathcal{L}u\right)(x,t) \tag{2.72}$$

*Step 2.* Let $a_t^+$ be any control, satisfying (2.56). Then the dynamic programming relation (2.57) implies that

$$u(X_t,t) \le \mathbb{E}\left[\int_t^{t+\delta t} f(X_s, a_s^+)ds + u(X_{t+\delta t}, t+\delta t)|X_t\right] \tag{2.73}$$

Then, given that $\mathbb{E}[\int_t^{t+\delta t} dW_t] = 0$ and by Ito's formula

$$\mathbb{E}\left[du(X_t,t)\right] = \mathbb{E}\left[\left(\frac{\partial u}{\partial t}(X_t,t) + \mathcal{L}u(X_t,t)\right)dt\right] \tag{2.74}$$

we get

$$\begin{aligned} 0 &\leq \mathbb{E}\left[\int_t^{t+\delta t} f(X_s,a_s^+)ds + u(X_{t+\delta t},t+\delta t) - u(X_t,t)|X_t\right] \\ &\leq \mathbb{E}\left[\int_t^{t+\delta t} f(X_s,a_s^+)ds + \int_t^{t+\delta t}\left(\frac{\partial u}{\partial s} + \mathcal{L}u\right)(X_s,s)ds|X_t\right] \end{aligned} \tag{2.75}$$

Dividing by $\delta t$, taking the limit $\delta t \to 0$ and putting $X_t = x$ we arrive at the expression we claimed in (2.67):

$$0 \leq f(x,a_t^+) + \left(\frac{\partial u}{\partial t} + \mathcal{L}u\right)(x,t) \tag{2.76}$$

*Remark 1.* We will use the Wiener and the compound Poisson process to model the price and the demand. Let's show the exact form of the HJB equation when $X_t$ is of these types of processes.

Let $X_t$ be an $n$-dimensional diffusion process with pairwise independent $W_i(t)$, $i = 1, ..., n$ and let $u : \mathbb{R} \times [0,\infty] \to \mathbb{R} \in C^2$. Denote the gradient of $u$ as $D_x u$ and the Hessian as $D_x^2 u$. Then the generator reads

$$\mathcal{L}u(X_t,t) = g^T(X_t,a_t)D_x u(X_t,t) + \frac{1}{2}\sigma^T(X_t,a_t)D_x^2 u(X_t,t)\sigma(X_t,a_t) \tag{2.77}$$

and the HJB equation takes the following form:

$$f(x,a_t^*) + \partial_t u(x,t) + g(x,a_t^*)^T D_x u(X_t,t) + \frac{1}{2}\sigma^T(x,a_t^*)D_x^2 u(X_t,t)\sigma(x,a_t^*) = 0 \tag{2.78}$$

Further in this thesis, we will be writing the HJB equation in a slightly different form, which is given in (2.80)-(2.79). Part of (2.78) is called the Hamiltonian and is defined as

$$H(x, D_x u(x,t), D_x^2 u(x,t)) = \min_{a(t)}[f(x,a_t) + g(x,a_t)^T D_x u(x,t) + \frac{1}{2}\sigma^T(x,a_t) D_x^2 u(x,t)\sigma(x,a_t)] \tag{2.79}$$

With Hamiltonian we rewrite equation (2.78) in the form

$$\partial_t u(x,t) + H(x, D_x u(x,t), D_x^2 u(x,t)) = 0. \tag{2.80}$$

For a jump-diffusion process $X_t$, where the jumps are given by the compound Poisson process with finite-dimensional jump distribution, the generator reads

$$\mathcal{L}u(X_t) = g^T(X_t,a_t) D_x u(X_t,t) + \frac{1}{2}\sigma^T(X_t,a_t) D_x^2 u(X_t,t)\sigma(X_t,a_t) + \lambda \sum_{z_i \in \mathcal{S}} (u(X_t + z_i, t) - u(X_t,t)) p_Z(z_i) \tag{2.81}$$

and the HJB equation becomes

$$f(x,a_t^*) + \partial_t u(x,t) + g(x,a_t^*)^T D_x u(x,t) + \frac{1}{2}\sigma^T(x,a_t^*) D_x^2 u(x,t) D_x^2 u(x,t) + \lambda \sum_{z_i \in \mathcal{S}} (u(x + z_i, t) - u(x,t)) p_Z(z_i) = 0 \tag{2.82}$$

With the Hamiltonian defined as

$$H(x, D_x u(x,t), D_x^2 u(x,t)) = \min_{a(t)}[f(x,a_t) + g^T(x,a_t) D_x u(x,t) + \frac{1}{2}\sigma^T(x,a_t) D_x^2 u(x,t)\sigma(x,a_t) + \lambda \sum_{z_i \in \mathcal{S}} (u(x + z_i) - u(x)) p_Z(z_i)] \tag{2.83}$$

the HJB (2.82) reads

$$\partial_t u(x,t) + H(x, D_x u(x,t), D_x^2 u(x,t)) = 0 \tag{2.84}$$

*Remark 2.*

Consider a simpler system, similar to (2.54) but without the control $a(t)$:

$$\begin{cases} dX(t) = g(X(t))dt + \sigma(X(t))dW(t), \quad t \in [0,T] \\ X(0) = X_0 \end{cases} \tag{2.85}$$

We redefine the value function accordingly to be

$$u(x,t) = \mathbb{E}\left[\int_t^T f(X(s))ds + h(X(T))|X(t) = x\right] \tag{2.86}$$

Then the derivation of HJB equation still applies and the resulting equation is known as Kolmogorov-Backward equation:

$$0 = f(x) + \left(\frac{\partial u}{\partial t} + \mathcal{L}u\right)(x,t) \tag{2.87}$$

# Chapter 3

# Models

## 3.1 System description

Consider a production site and it's electricity consumption. To satisfy the necessary production rate determined by the market demand $D(t)$ at a certain time, the site consumes power from the electric grid at some rate $P_E(t)$ at price $K(t)$. The production site has a battery at its disposal, where the energy can be stored for later use or used to satisfy current demand needs. We will denote by $A(t)$ the current energy level in the battery and by $P_A(t)$ - the power supplied by the battery. In the following subsections we will describe the demand, price and the battery dynamics.

### 3.1.1 Spot price dynamics

We will attempt to mimic the behaviour of the spot price from Uruguayan power system [7], using the available data. The Uruguayan power system includes several power sources: hydro power-stations, thermal (fossil-fuel) power stations, wind and solar generators. At each point in time the grid uses a subset of these sources to satisfy the needs of the country. The cost of power production from these sources varies significantly, and the resulting spot price is reflected in Figure 3.1. The most expensive source of energy is the fossil-fuel station, due to the cost of the fuel. The heavy use of fossil-fuel stations results in the upward spikes we observe on the plot. The wind and solar power are the least expensive sources of energy, so when the

power is supplied mostly from these sources, we see low levels of spot price on the plot.

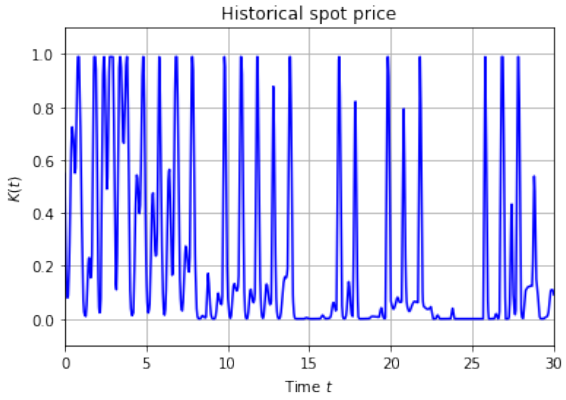


Figure 3.1: Historical spot price in January 2017, data scaled to fit $[0, 1]$.

The spikes we see in Figure 3.1 mean the process has a large derivative, which makes the HJB equation computationally too demanding, forcing us to choose an extremely small $\Delta t$. We propose to view these spikes as jumps on a larger time scale as means of regularization. Thus, we will extract the general trend from the data, which can be made as smooth as needed, and add random effects in the form of jumps and diffusion.

We will build the price $K(t)$ as a mean-reverting stochastic process, with the general trend $p_K(t)$, which we call the forecast, as its mean. We will add random effects in the form of jumps and diffusion. The model reads

$$\begin{cases} dK(t) = a(p_K(t), K(t))dt + \sigma(K(t))dW(t) + dP(t), \quad \forall t \in [0, T] \\ K(0) = K_0 \end{cases} \tag{3.1}$$

where $W(t)$ is a Wiener process and the continuous-time jump process $P(t)$, the forecast $p_K(t)$, the drift term $a(p_K(t), K(t))$ and the diffusion term $\sigma(K(t))$ are to be determined.

## Forecast

Due to the lack of data we propose to capture the general trend by the moving average method. Given the current value of the price $X_i$, the new value $\hat{X}_i$ is computed using the adjacent values with equal weights:

$$\hat{X}_i = \frac{1}{3}(X_{i-1} + X_i + X_{i+1}) \tag{3.2}$$

We let the data pass through the moving average filter several times and take the obtained curve (Figure 3.2) as the forecast $p_K(t)$.

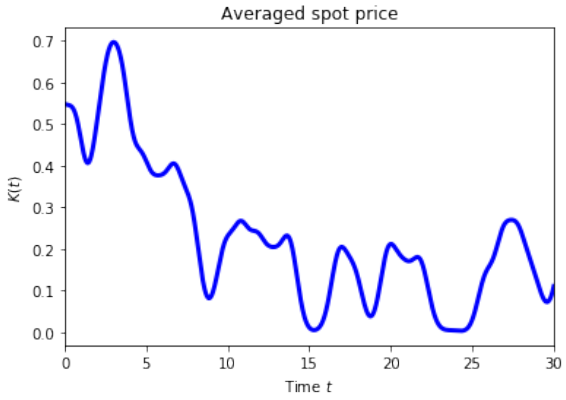


Figure 3.2: The forecast of the spot price obtained by the moving average method.

## The drift

A good starting point for designing the drift term is the drift in the Ornstein-Uhlenbeck mean-reverting process. We propose to take a similar term and perturb it with the mean of the jump process

$$a(p_K(t), K(t)) = -\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - E[Y(t)], \tag{3.3}$$

where $Y(t)$ is a stochastic process, such that $E[dP(t)] = E[Y(t)dt]$.

We will show that the process (3.1) with such drift is indeed mean-reverting, that is, that it satisfies

$$E[K(t)] = p_K(t). \tag{3.4}$$

Let's first subtract $dp_K(t)$ and look at the deviation of $K(t)$ from the forecast:

$$\begin{aligned} d(K(t) - p_K(t)) = (-\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - E[Y(t)])dt \\ -\frac{dp_K(t)}{dt}dt + \sigma(K(t))dW(t) + dP(t) \end{aligned} \tag{3.5}$$

Taking expectation and dividing by $dt$ we get

$$\frac{E[d(K(t) - p_K(t))]}{dt} = E[-\theta_K(t)(K(t) - p_K(t))] \tag{3.6}$$

Then by Grönwall's lemma we have:

$$E[K(t) - p_K(t)] \leq e^{-\theta_K(t)t}(K(0) - p_K(0)) = 0, \tag{3.7}$$

$$E[p_K(t) - K(t)] \leq e^{-\theta_K(t)t}(p_K(0) - K(0)) = 0 \tag{3.8}$$

so that $E[K(t)] = p_K(t)$, as intended.

## The jumps

We will model the jumps $Y(t)$ as a compound Poisson process

$$Y(t) = \sum_{i=1}^{N(t)} Z_i(t) \tag{3.9}$$

with a constant intensity of jumps $\lambda$, which we will roughly estimate by counting the number of jumps in the available data.

To determine the distribution $G$ of jumps $Z_i$, we notice that there is a finite number of levels, to which the spot price jumps, as shown in Figure 3.3

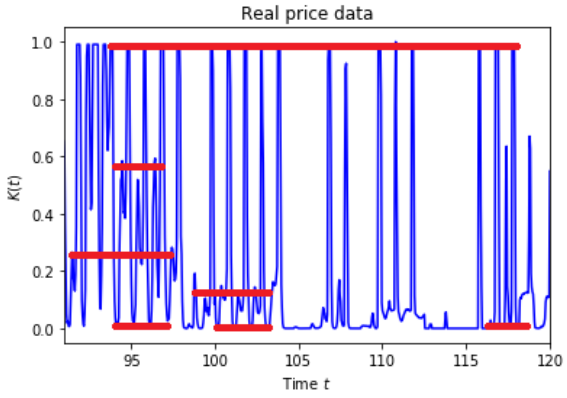


Figure 3.3: The spot price jumps to a finite number of levels.

We roughly estimate that there are six levels $L_Z = \{0.1, 0.2, 0.4, 0.6, 0.8, 0.95\}$. To evaluate the probabilities of jumping from one level to the other, we divide the interval $[0, 1]$ into six subintervals:

$$\Omega = \{[0, 0.05], [0.05, 0.15], [0.15, 0.3], [0.3, 0.5], [0.5, 0.7], [0.7, 0.9], [0.9, 1]\} \tag{3.10}$$

and count the jumps from $\Omega_i$ interval to $L_{Z_j}$ level in the data:

$$\hat{P}(\text{Jumping to } L_{Z_j} | \text{Current level } \Omega_i) = \frac{\text{Number of jumps from } \Omega_i \text{ to } L_{Z_j}}{\text{Number of jumps from } \Omega_i} \tag{3.11}$$

Using (3.11) we get the following transition probabilities matrix $P$:

$$P = \begin{bmatrix} 0 & 0 & 8/27 & 0 & 5/27 & 1/27 & 13/27 \\ 2/6 & 0 & 1/6 & 0 & 1/6 & 0 & 2/6 \\ 2/4 & 2/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1/6 & 2/6 & 1/6 & 1/6 & 0 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 17/21 & 2/21 & 0 & 1/21 & 1/21 & 0 & 0 \end{bmatrix} \tag{3.12}$$

where each row $P_i$ represents the distribution of the jumps given that the current value of the process is $K(t) \in \Omega_i$,

The expected value $E[Y(t)]$ in the drift term (3.3) can be computed as:

$$E[Y(t)] = E[N(t)]E[Z(t)] = \lambda E[Z(t)] \tag{3.13}$$

## The diffusion

The diffusion coefficient $\sigma(K(t))$ has the form

$$\sigma(K(t)) = \sqrt{2\theta_{K_t}\alpha_K K(t)(1 - K(t))} \tag{3.14}$$

where

$$\theta_{K_t} = \max\left(\theta_{K_0}, \frac{\alpha_K\theta_{K_0} + |\dot{p}_{Kt}|}{\min(p_{Kt}, 1 - p_{Kt})}\right) \tag{3.15}$$

and

- $\theta_{K_0} = 25$

- $\alpha_K = 0.003$

The values of $\theta_{K_0}$ and $\theta_{K_t}$ determine how fast the process returns to its mean. Large values of $\theta_{K_t}$ lead to faster mean-reverting behaviour. On the other hand, large $\theta_{K_t}$ increases diffusion, which in turn can be corrected by putting a smaller $\alpha_K$. $\theta_{K_0}$ and $\theta_{K_t}$ and $\alpha_K$ are calibrated to ensure that the process $K(t)$ does not leave the interval $[0, 1]$. The term $K(t)(1 - K(t))$ also tends to keep diffusion small when $K(t)$ approaches 0 or 1, which helps to avoid stepping outside the interval $[0, 1]$.

Putting together all of the above, we can write the price process (3.1) as

$$\begin{cases} dK(t) = (-\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - E[Y(t)])dt \\ +\sqrt{2\theta_{K_t}\alpha_K K(t)(1 - K(t))}dW(t) + Y(t)dt \\ K(0) = K_0 \end{cases} \tag{3.16}$$

with $\theta_{K_t}$ and $\alpha_K$ defined in (3.15)-(3.1.1).

## The simulation procedure

To simulate the Forward Euler approximation of (3.16) on the given uniform partition of the time interval $[0, T]$ with $\Delta t = 2^{-10}$, that is

$$K^t = K^{t-1} + (\dot{p}_k^{t-1} + \theta_k^{t-1}(p_k^{t-1} - K^{t-1}) - \lambda\mathbb{E}[Z_i^{t-1}|K^{t-1} \in \Omega_i])\Delta t + \tag{3.17}$$

$$\sqrt{2\theta_k^{t-1}\alpha_k K^{t-1}(1 - K^{t-1})}\Delta W + Y^{t-1}\Delta t, \tag{3.18}$$

we do the following:

- generate inter-arrival times of jumps as exponentially distributed R.V. with parameter $1/\lambda$

- generate the size of the jump at jump times according to matrix $P$ given the value at previous time point $K_{t-1}$
- generate the Wiener increment $\Delta W$

By experiment we measured that the probability of the process (3.16) taking values outside $[0, 1]$ at any time:

$$P(K(t) < 0 \cap K(t) > 1) \approx 0.001 \tag{3.19}$$

Figure 3.4 shows the simulation of $K(t)$ in blue and the forecast in yellow dotted line

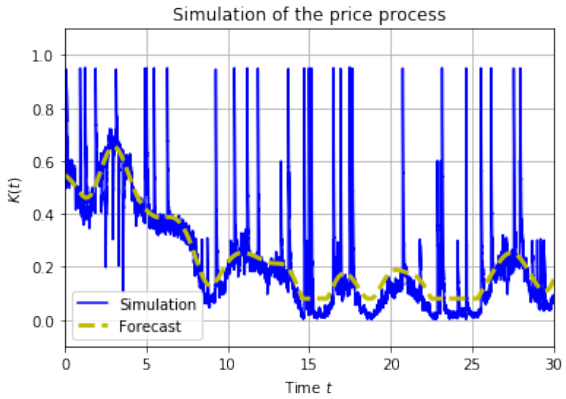


Figure 3.4: The spot price model simulation (blue) with the forecast (yellow).

### 3.1.2 Demand dynamics

To model the demand we follow a similar logic as we did modeling the spot price, with the exception of jumps. We take the forecast $p_D(t)$ computed as a moving average from the Uruguayan power system demand data [7] and add random effects using the

Wiener differential. Thus, the demand is represented by the following Ito SDE:

$$\begin{cases} dD(t) = a(p_D(t), D(t))dt + \sigma(D(t))dW(t) \\ D(0) = D_0 \end{cases} \tag{3.20}$$

with functions $a(p_D(t), D(t))$ and $\sigma(D(t))$ to be determined.

Following a similar logic we define the drift term for the demand process to be

$$a(p_D(t), D(t)) = -\theta_D(t)(D(t) - p_D(t)) + \dot{p}_D(t) \tag{3.21}$$

and the diffusion terms as:

$$\sigma(D(t)) = \sqrt{2\theta_{D_t}\alpha_D D(t)(1 - D(t))} \tag{3.22}$$

Putting (3.21) and (3.22) together we can write the demand process in the following way:

$$\begin{cases} dD(t) = (-\theta_D(t)(D(t) - p_D(t)) + \dot{p}_D(t))dt + \sqrt{2\theta_{D_t}\alpha_D D(t)(1 - D(t))}dW(t) \\ D(0) = D_0 \end{cases} \tag{3.23}$$

where

$$\theta_{D_t} = \max\left(\theta_{D_0}, \frac{\alpha_D\theta_{D_0} + |\dot{p}_{Dt}|}{\min(p_{Dt}, 1 - p_{Dt})}\right) \tag{3.24}$$

and

- $\theta_{D_0} = 10$

- $\alpha_D = 0.05$

### 3.1.3 Battery dynamics

In this model we consider a large battery, that can satisfy the needs of the factory. The level of charge $A$ in the battery at time $t + \delta t$ depends on its level at time $t$ and the flux of charge $P_A$ during the time $\delta t$. Thus, the battery dynamics is described by the following ODE:

$$\begin{cases} dA(t) = -P_A(t)dt, \\ A(0) = A_0 \end{cases} \tag{3.25}$$

The battery should be able to supply the maximum demand at any point in time, therefore we take the battery with the power of 100 kW. Since the price and demand in our model are scaled, we will also scale the battery power and capacity. Thus, we put 100 kW = 1 conventional unit (c.u.). The maximum capacity of the battery is measured in MWh, and in our model $A = 1$ c.u. represents 2.4 MWh.

The magnitude of the flux depends on whether the battery is charged or discharged: we can discharge at a rate of $(P_A)_{max} = 1$ c.u. and charge at $(P_A)_{min} = 0.5$ c.u. The flux of the battery also depends on the current level of charge. We do this to avoid charging the battery when it is full or discharging when the charge is almost 0. When the charge is low, the power supply decreases linearly until no more charge is left. When the battery is almost full the flux of charge decreases linearly until it reaches the maximum capacity of the battery. The dependence of the flux on the level of charge is described by the following relations and depicted in Figure 3.5:

$$\overline{P}_A(A) = \min\left(\frac{A(t) - \underline{A}}{\Delta t}, (P_A)_{max}\right), \tag{3.26}$$

$$\underline{P}_A(A) = \min\left(\frac{A(t) - \overline{A}}{\Delta t}, (P_A)_{min}\right) \tag{3.27}$$

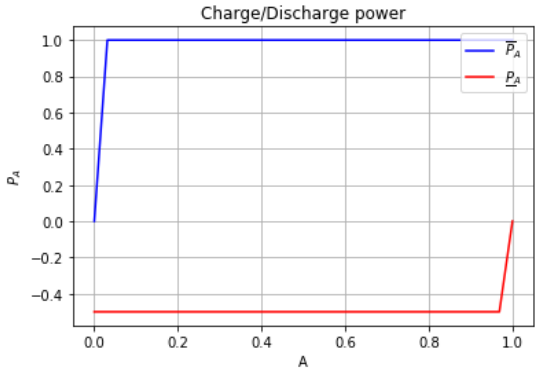


Figure 3.5: The charge and discharge rate of the battery.

The level of charge is bounded by the maximum capacity $\overline{A}$. The conditions on the level of charge and the flux of the battery are summarized below:

$$\underline{A} < A < \overline{A}, \tag{3.28}$$

$$\underline{P}_A < P_A < \overline{P}_A \tag{3.29}$$

### 3.1.4 Expected cost of energy

The expected cost of production $u(t, K, D)$ in the system without the battery at time $t$ is the overall expected cost of energy needed to satisfy the demand from time $t$ onwards:

$$u(t, K, D) = \mathbb{E}\left(\int_t^T K(s)D(s)ds\right) \tag{3.30}$$

Assuming that $K(t)$ and $D(t)$ follow the stochastic dynamics (3.16) and (3.23) respectively, the Kolmogorov-Backward equation provides a convenient way to compute the expected cost (3.30), since $u(t, K, D)$ satisfies the following PDE:

$$\begin{aligned}
&\partial_t u + KD + (\dot{p}_K - \theta_K(K - p_K) - E[Y(t)])\partial_K u + (\theta_K \alpha_K K(1-K))\partial_{KK} u \\
&+(\dot{p}_D - \theta_D(D - p_D))\partial_D u + (\theta_D \alpha_D D(1-D))\partial_{DD} u \\
&+\lambda \sum_j [u(t, K + Z_j, D, A) - u(t, K, D, A)] p_{i,j}(Z_j | K \in \Omega_i), \\
&u(T, K, D) = 0
\end{aligned} \tag{3.31}$$

Taking the values of the forecast as initial values of price $K_0 = p_K(0)$ and demand $D_0 = p_D(0)$ we get the corresponding expected cost of energy $u(0, K_0, D_0)$, that we will attempt to reduce by using the battery.

## 3.2 Deterministic model

In this section we give description of the one-dimensional optimal control of the battery, where we put the price and demand as deterministic functions.

### 3.2.1 Deterministic model description

We will first assume that the demand and the price are deterministic and follow the forecast functions we defined in subsections (3.1.1) and (3.1.2):

$$\begin{cases} K(t) = p_K(t), \\ D(t) = p_D(t) \end{cases} \tag{3.32}$$

The battery dynamics is described in subsection (3.1.3):

$$\begin{cases} dA(t) = -P_A(t)dt, \\ A(0) = A_0 \end{cases} \tag{3.33}$$

The rate of battery discharge $P_A(t)$ is the control of the system of equations (3.33). A trajectory $A(t)$ is the state trajectory corresponding to control $P_A(t)$ if it solves the system (3.33).

There is a number of constraints we have to impose on the variables to have a physically feasible system. They are the constraints on the level of charge, charge and discharge of the battery and are described in subsection (3.1.3). The demand should be satisfied by power supplied both from the battery and from the electric grid. The time period we consider is $T = 365$ days. The following system of constraints summarizes the above:

$$\underline{A} \leq A(t) \leq \overline{A}, \quad \forall t \in [0, T] \tag{3.34}$$

$$\underline{P_A} \leq P_A(t) \leq \overline{P_A}, \quad \forall t \in [0, T] \tag{3.35}$$

$$D(t) = P_E(t) + P_A(t) \tag{3.36}$$

We define the cost function $J(t, A)$ as follows:

$$J(t, A) = \int_t^T K(s)P_E(s)ds - A \cdot p_K(T)e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \tag{3.37}$$

where

- $\int_t^T K(s)P_E(s)ds$ - the cost of electricity from the grid - the running cost;
- $A \cdot p_K(T)e^{-T \cdot Pb \cdot r/365}$ - the price of energy, discounted at annual rate $r$, which is

left in the battery at final time $T$ according to the forecast, which we can sell or use at the next production cycle. The discount rate $r$ can be put equal to the inflation, for example;

- $K_A\overline{A}$ - the cost of purchasing the battery of size $\overline{A}$.

The cost function is measured in c.u. and scaled by $K_{max}D_{max}$. The current battery price is quite high comparing to the price of electricity, so the payback period $Pb$ of the battery purchase is greater that 1 year. The coefficient $K_A$ includes the current price of the battery in USD/kWh and the payback period, which the user of the model can choose based on their preference:

$$K_A = \frac{\text{Battery price}}{\text{Electricity price}_{max}\text{Payback period}} \tag{3.38}$$

Any rational economic agent pursues a goal of minimizing the cost of production. We are now ready to formulate the optimal control of the battery problem: minimize the cost functional defined in (3.37) over a set of admissible controls (3.35), given (3.32)-(3.34):

$$u(t,A) = \min_{P_A(t)\in[\underline{P}_A,\overline{P}_A]} J(t,A)$$
$$= \min_{P_A(t)\in[\underline{P}_A,\overline{P}_A]} \left[\int_t^T K(s)(D(s)-P_A(s))ds - A\cdot p_K(T)e^{-T\cdot Pb\cdot r/365} + K_A\cdot\overline{A}\right] \tag{3.39}$$

### 3.2.2 HJB equation

As we established in subsection 2.2.2, the value function $u(t, A)$ in (3.39) solves the following Hamilton-Jacobi-Bellman equation:

$$\begin{cases} \partial_t u + H(\partial_A u, A) = 0, \quad 0 < t < T \\ u(T, A) = -A \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \end{cases} \tag{3.40}$$

where the Hamiltonian is given by

$$H = KD + \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} (-P_A(t)(\partial_A u(t, A) + K)) \tag{3.41}$$

The term $-P_A(t)(\partial_A u(t, A) + K)$ is to be minimized over $P_A(t)$, which is a bang-bang type control. Consider the following cases:

$$\partial_A u + K \geq 0, \tag{3.42}$$

$$\partial_A u + K < 0 \tag{3.43}$$

In the first case,

$$\min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} (-P_A(t)(\partial_A u(t, A) + K)) = \min\{D, \overline{P}_A\}(\partial_A u(t, A) + K) \tag{3.44}$$

where we also take into account the constraint $D(t) = P_E(t) + P_A(t)$.

In the second case

$$\min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} (-P_A(t)(\partial_A u(t, A) + K)) = \underline{P}_A(\partial_A u(t, A) + K) \tag{3.45}$$

Summarizing the above gives the optimal control:

$$P_A^* = -\mathbb{1}_{\{\partial_A u + K < 0\}} \underline{P}_A(A) + \mathbb{1}_{\{\partial_A u + K \geq 0\}} \min\{D, \overline{P}_A(A)\} \tag{3.46}$$

With (3.46) we rewrite the HJB equation as follows:

$$\begin{cases} \partial_t u + KD - (-\mathbb{1}_{\{\partial_A u + K < 0\}} \underline{P}_A + \mathbb{1}_{\{\partial_A u + K \geq 0\}} \min\{D, \overline{P}_A\})(\partial_A u + K) = 0, \\ u(T, A) = -A \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \end{cases} \tag{3.47}$$

Once the solution for (3.47) is obtained we can find the optimal control (3.46) for any initial state of the battery $A(0)$.

### 3.2.3 Battery penalization

Charging and discharging the battery wears it out and it becomes less effective over time. We are interested in prolonging the life-span of the battery, so would like to find such an optimal control that takes into account the effect of the battery wearing out. To do that we introduce a term that penalizes battery use into the cost function:

$$u(t, A) = \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} (J(t, A)) =$$
$$\min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} \left( \int_t^T K(s) P_E(s) + C_A |P_A(s)| ds - A \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \right) \tag{3.48}$$

with some constant $C_A$.

Given the same system dynamics we had in subsection (3.2.1):

$$\begin{cases} dA_t = -P_A(t)dt, \quad \forall t \in [0,T] \\ A(0) = A_0, \\ K(t) = p_K(t), \quad \forall t \in [0,T] \\ D(t) = p_D(t), \quad \forall t \in [0,T] \\ \underline{A} \leq A(t) \leq \overline{A}, \quad \forall t \in [0,T] \\ \underline{P_A} \leq P_A(t) \leq \overline{P_A}, \quad \forall t \in [0,T] \\ D(t) = P_E(t) + P_A(t), \quad \forall t \in [0,T] \end{cases} \tag{3.49}$$

the value function $u(t, A)$ solves the following HJB equation:

$$\begin{aligned} \partial_t u + H(\partial_A u, P_A) &= 0, \\ u(T, A) &= -A \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \end{aligned} \tag{3.50}$$

where the Hamiltonian is defined as follows:

$$H = KD + \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} (-P_A(\partial_A u + K) + C_A |P_A|) \tag{3.51}$$

The value of the Hamiltonian depends on the sign of $P_A$:

$$H = \begin{cases} KD + \min\limits_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} - P_A(-C_A + \partial_A u + K), & P_A \geq 0, \\ KD + \min\limits_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} - P_A(C_A + \partial_A u + K), & P_A < 0. \end{cases} \tag{3.52}$$

Minimizing expression (3.52) gives:

$$H = \begin{cases} KD - \underline{P}_A(C_A + \partial_A u + K), & \text{if } (-C_A + \partial_A u + K) < 0 \\ & \text{and } (C_A + \partial_A u + K) < 0 \\ KD - \min\{D, \overline{P}_A\}(-C_A + \partial_A u + K), & \text{if } (-C_A + \partial_A u + K) \geq 0 \\ & \text{and } (C_A + \partial_A u + K) \geq 0 \\ KD, & \text{if } (-C_A + \partial_A u + K) < 0 \\ & \text{and } (C_A + \partial_A u + K) \geq 0 \end{cases} \tag{3.53}$$

Rewriting (3.54) in a more concise form gives:

$$H = \begin{cases} KD - \underline{P}_A(C_A + \partial_A u + K), & \text{if } \partial_A u + K < -C_A \\ KD - \min\{D, \overline{P}_A\}(-C_A + \partial_A u + K), & \text{if } \partial_A u + K \geq C_A \\ KD, & \text{if } -C_A \leq \partial_A u + K < C_A \end{cases} \tag{3.54}$$

The optimal control then takes the form:

$$P_A^* = -\mathbb{1}_{\{\partial_A u + K < -C_A\}}\underline{P}_A + \mathbb{1}_{\{\partial_A u + K \geq C_A\}} \min\{D, \overline{P}_A\} + 0 \cdot \mathbb{1}_{\{-C_A \leq \partial_A u + K < C_A\}} \tag{3.55}$$

## 3.3 Stochastic model

In this section we change the model in section 3.2.1 by putting the price and demand as stochastic processes and solve three-dimensional stochastic optimal control problem.

### 3.3.1 Stochastic model description

Assume now that electricity price and demand for electricity of the production site are described by the stochastic dynamics we described in subsections 3.1.1 and 3.1.2. The battery dynamics are deterministic and defined in subsection 3.1.3:

$$\begin{cases} dK(t) = (-\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - E[Y(t)])dt \\ +\sqrt{2\theta_{K_t}\alpha_K K(t)(1-K(t))}dW_K(t) + Y(t)dt \\ K(0) = K_0 \\ dD(t) = (-\theta_D(t)(D(t) - p_D(t)) + \dot{p}_D(t))dt + \sqrt{2\theta_{D_t}\alpha_D D(t)(1-D(t))}dW_D(t) \\ D(0) = D_0 \\ dA(t) = -P_A(t)dt, \\ A(0) = A_0 \end{cases} \tag{3.56}$$

where $W_K(t)$ and $W_D(t)$ are independent Wiener processes, $\theta_K(t)$ and $\theta_D(t)$ are given by:

$$\theta_{K_t} = \max\left(\theta_{K_0}, \frac{\alpha_K\theta_{K_0} + |\dot{p}_{Kt}|}{\min(p_{Kt}, 1-p_{Kt})}\right) \tag{3.57}$$

$$\theta_{D_t} = \max\left(\theta_{D_0}, \frac{\alpha_D\theta_{D_0} + |\dot{p}_{Dt}|}{\min(p_{Dt}, 1-p_{Dt})}\right) \tag{3.58}$$

and

- $\theta_{K_0} = 25$
- $\alpha_K = 0.003$
- $\theta_{D_0} = 10$

- $\alpha_D = 0.05$

The rate of battery discharge $P_A(t)$ is the control of the system of equations (3.56). A trajectory $A(t)$ is the state trajectory corresponding to control $P_A(t)$ if it solves the system (3.56).

There is a number of constraints we have to impose on the variables to have a physically feasible system. They are the constraints on the level of charge, charge and discharge of the battery and are described in subsection (3.1.3). The demand should be satisfied by power supplied both from the battery and from the electric grid. The time period we consider is $T = 365$ days. The following system of constraints summarizes the above:

$$\begin{cases} \underline{A} \leq A(t) \leq \overline{A}, \quad \forall t \in [0,T] \\ \underline{P_A} \leq P_A(t) \leq \overline{P_A}, \quad \forall t \in [0,T] \\ D(t) = P_E(t) + P_A(t) \\ K(t), D(t) \in [0,1], \quad \forall t \in [0,T] \end{cases} \tag{3.59}$$

We define the expected cost of production $J(t, A, K, D)$ at time $t$ as follows:

$$J(t,A,K,D) = \mathbb{E}\left(\int_t^T K(s)P_E(s)ds\right) - A \cdot p_K(T)e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A}. \tag{3.60}$$

Any rational economic agent pursues a goal of minimizing the cost of production. We formulate the stochastic optimal control problem as follows: given the system (3.56)-(3.57), minimize the cost functional defined in (3.60) over a set of admissible

controls:

$$u(t,A,K,D) = \min_{P_A(t)\in[\underline{P}_A,\overline{P}_A]} J(t,A,K,D)$$
$$= \min_{P_A(t)\in[\underline{P}_A,\overline{P}_A]} \left[\mathbb{E}\left(\int_t^T K(s)P_E(s)ds\right) - A\cdot p_K(T)e^{-T\cdot Pb\cdot r/365} + K_A\cdot\overline{A}\right] \quad (3.61)$$

### 3.3.2 HJB equation

Section 2.3 shows that the value function $u(t,A,K,D)$ in (3.60) solves the following Hamilton-Jacobi-Bellman equation:

$$\partial_t u + H(A,K,D,\partial_A u,\partial_K u,\partial_{KK}u,\partial_D u,\partial_{DD}u) = 0, \qquad 0<t<T \quad (3.62)$$

$$u(T,A,K,D) = -A\cdot p_K(T)e^{-T\cdot Pb\cdot r/365} + K_A\cdot\overline{A} \quad (3.63)$$

where the Hamiltonian is defined as

$$H = KD + (\dot{p}_K - \theta_K(K-p_K) - E[Y(t)])\partial_K u + (\theta_K\alpha_K K(1-K))\partial_{KK}u$$
$$+(\dot{p}_D - \theta_D(D-p_D))\partial_D u + (\theta_D\alpha_D D(1-D))\partial_{DD}u$$
$$+\lambda\sum_j [u(t,K+Z_j,D,A) - u(t,K,D,A)]p_{i,j}(Z_j|K\in\Omega_i)$$
$$+\min_{P_A(t)\in[\underline{P}_A,\overline{P}_A]}(-P_A(t)(\partial_A u + K)) \quad (3.64)$$

Since the control $P_A$ does not affect the price and demand dynamics, the minimization part in the Hamiltonian can be done the same way we did it in the deterministic case (subsection 3.2.2). Optimal control is given by the expression:

$$P_A^* = -\mathbb{1}_{\{\partial_A u+K<0\}}\underline{P}_A(A) + \mathbb{1}_{\{\partial_A u+K\geq 0\}}\min\{D,\overline{P}_A(A)\} \quad (3.65)$$

With (3.65) we rewrite the HJB equation as follows:

$$\begin{aligned}
&\partial_t u + (\dot{p}_K - \theta_K(K - p_K) - E[Y(t)])\partial_K u + (\theta_K \alpha_K K(1-K))\partial_{KK} u \\
&+(\dot{p}_D - \theta_D(D - p_D))\partial_D u + (\theta_D \alpha_D D(1-D))\partial_{DD} u \\
&+\lambda \sum_j [u(t, K + Z_j, D, A) - u(t, K, D, A)] p_{i,j}(Z_j | K \in \Omega_i) \\
&+KD - P_A^*(\partial_A u + K) = 0, \qquad 0 < t < T \\
&u(T, A, K, D) = -A \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A}
\end{aligned} \tag{3.66}$$

Once we obtain the solution of (3.66), we will have $\partial_A u$ and we will be able to compute the optimal path (3.65), given any particular realization of $K(t)$ and $D(t)$ and initial battery state $A(0)$.

### 3.3.3 Optimal trajectory smoothing technique

The actual optimal control path which minimizes (3.60) often has large oscillations. An example of such control is shown in Figure 3.6. The time step in our discretization is set to 10 min, so this means that the optimal solution forces us to oscillate between charge/discharge every 10 min around $t = 0.7$ and $t = 0.9$, which may be difficult to implement in practice. We would like the controls to be smoother, so we are going to implement penalization of too large variations of optimal control.

Lets rewrite the value function (3.61) as follows:

$$\begin{aligned}
u_C(t, A, K, D) = \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} E[&\int_t^T K(\tau) P_E(\tau) d\tau - A \cdot p_K(T) \cdot e^{-T \cdot Pb \cdot r/365} \\
&+K_A \cdot \overline{A} + \int_t^T C(\frac{\partial P_A(\tau)}{\partial \tau})^2 d\tau]
\end{aligned} \tag{3.67}$$

where we add the term $(\frac{\partial P_A(\tau)}{\partial \tau})^2$ to penalize large variations of the control. In section

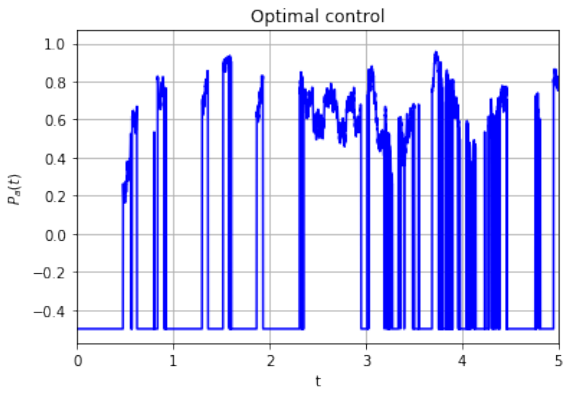


Figure 3.6: The optimal control path for given price and demand simulations.

2.3 we have shown the connection between the value function and the HJB equation in the Markovian framework, that is, when the system dynamics and the running cost in the cost functional depend only on the current state $K_\tau$, $D_\tau$. The running cost in $u_{C\neq 0}$ in (3.61) is not Markovian anymore, since it includes the derivative $\frac{\partial P_A(\tau)}{\partial \tau}$. Therefore we cannot use the same theoretical result and write down the HJB equation that would be solved by the value function $u_C$.

But if we put $C = 0$, the value function $u_0(t, A, K, D)$ in (3.67) solves the following HJB:

$$\begin{aligned} &\partial_t u + H(\partial_A u, \partial_K u, \partial_{KK} u, \partial_D u, \partial_{DD} u, A, K, D) = 0 \\ &u(T, A, K, D) = -A \cdot p_K(T) e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \end{aligned} \tag{3.68}$$

with Hamiltonian:

$$\begin{aligned} H = KD + (\dot{p}_K - \theta_K(K - p_K) - E[Y(t)])\partial_K u + (\theta_K \alpha_K K(1-K))\partial_{KK} u \\ +(\dot{p}_D - \theta_D(D - p_D))\partial_D u + (\theta_D \alpha_D D(1-D))\partial_{DD} u \\ +\lambda \sum_j [u(t, K + Z_j, D, A) - u(t, K, D, A)] p_{i,j}(Z_j | K \in \Omega_i) \\ + \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} \left( -P_A(t)(\partial_A u + K(t)) + C(\partial_t P_A(t))^2 \right) \end{aligned} \tag{3.69}$$

If we put $0 < C << 1$ in (3.69), we would get a different control path which is close to the optimal path that minimizes Hamiltonian (3.69) and the value function $u_0$. But this constant $0 < C << 1$ allows us to choose a path that is smoother (since the term $C(\partial_t P_A(t))^2$ penalizes large deviations in optimal controls) while still achieving cost close to optimal.

Putting a nonzero constant in (3.69) $0 < C << 1$ and approximating the derivative $(\partial_t P_A(t))^2 \approx \frac{(P_A(t) - P_A(t-\Delta t))^2}{(\Delta t)^2}$ we will define a penalized Hamiltonian $\hat{H}$:

$$\begin{aligned} \hat{H} = KD + (\dot{p}_K - \theta_K(K - p_K) - E[Y(t)])\partial_K u + (\theta_K \alpha_K K(1-K))\partial_{KK} u \\ +(\dot{p}_D - \theta_D(D - p_D))\partial_D u + (\theta_D \alpha_D D(1-D))\partial_{DD} u \\ +\lambda \sum_j [u(t, K + Z_j, D, A) - u(t, K, D, A)] p_{i,j}(Z_j | K \in \Omega_i) \\ + \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} \left( -P_A(t)(\partial_A u + K(t)) + \frac{C}{(\Delta t)^2}(P_A(t) - P_A(t - \Delta t))^2 \right) \end{aligned} \tag{3.70}$$

The penalized Hamiltonian is parabolic in $P_A$ and simple to minimize:

$$\begin{aligned} \hat{H} = KD + ... + \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} ( \frac{C}{(\Delta t)^2} P_A^2(t) - P_A(t)(\partial_A u + K \\ + \frac{2C}{(\Delta t)^2} P_A(t - \Delta t)) + \frac{C}{(\Delta t)^2} P_A^2(t - \Delta t)) \end{aligned} \tag{3.71}$$

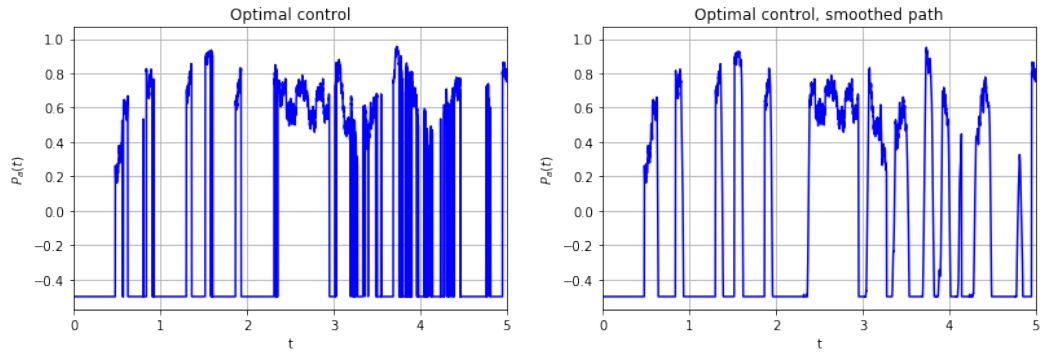


Figure 3.7: The true optimal control (left) and the smoothed control (right) paths.

The minimizer of (3.71) is either at the boundaries $[\underline{P}_A, \min\{D, \overline{P}_A\}]$ or at the vertex of the parabola given by:

$$\frac{\partial \hat{H}}{\partial \tilde{P}_A} = 0 \Rightarrow \tilde{P}_A(t) = \frac{(\Delta t)^2}{2C}(\partial_A u + K + \frac{2C}{(\Delta t)^2} P_A(t - \Delta t)) \tag{3.72}$$

which is a linear function of $P_A(t - \Delta t)$ and approaches its value as $\Delta t$ goes to 0:

$$\lim_{\Delta t \to 0} \tilde{P}_A = P_A(t) \tag{3.73}$$

Therefore, the minimizer of (3.61) reads:

$$\begin{cases} \hat{P}^*_A(t) = \tilde{P}_A(t), \text{ if } \tilde{P}_A(t) \in [\underline{P}_A, \min\{D, \overline{P}_A\}], \\ \hat{P}^*_A(t) = \underline{P}_A, \text{ if } \tilde{P}_A(t) < \underline{P}_A, \\ \hat{P}^*_A(t) = \overline{P}_A, \text{ if } \tilde{P}_A(t) > \overline{P}_A \end{cases} \tag{3.74}$$

We can choose the constant C in such a way that allows to achieve the cost $u^{**}_C(0, A_0, K_0, D_0)$ along the new "optimal" path as close to real optimal cost $u^*_0(0, A_0, K_0, D_0)$ as possible. For example, the new smoothed control in (3.74), shown in Figure (3.7) gives the cost within 5% of optimal.

How will we choose the constant $C$? We can write the penalized value function in two terms - the energy cost $F$ and the penalization term $P$:

$$\hat{u}_C(t, A, K, D) = \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} E[F(P_A) + C \cdot P(P_A)] \tag{3.75}$$

where

$$F(P_A) = \int_t^T K(\tau) P_E(\tau) d\tau - A \cdot p_K(T) \cdot e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A}$$
$$P(P_A) = \int_t^T (\frac{\partial P_A(\tau)}{\partial \tau})^2 d\tau \tag{3.76}$$

We will generate $M$ paths for price and demand and for each fixed constant $C$ compute our penalized optimal controls. We will also compute both terms $\overline{F}(\hat{P}_A^*(C))$ and $\overline{P}(\hat{P}_A^*(C))$ along the penalized optimal path $\hat{P}_A^*$ averaging amongst all simulations:

$$\overline{F}(\hat{P}_A^*(C)) = \frac{1}{M} \sum_{i=1}^{M} F_i(\hat{P}_i{}_A^*(C)), \quad \overline{P}(\hat{P}_A^*(C)) = \frac{1}{M} \sum_{i=1}^{M} P_i(\hat{P}_i{}_A^*(C)) \tag{3.77}$$

We can then plot $\overline{F}(\hat{P}_A^*(C))$ and $\overline{P}(\hat{P}_A^*(C))$ (as shown on the next slide). $C = 0$ corresponds to the farthest point on the right of the plot, where $\overline{F}(\hat{P}_A^*(0))$ is minimum and penalization term $\overline{P}(\hat{P}_A^*(0))$ is maximum. As we increase $C \leq 0.0001$ the energy cost term $\overline{F}(\hat{P}_A^*(0))$ doesn't change much, while penalization term $\overline{P}(\hat{P}_A^*(C))$ is decreasing fast. These values of $C$ give paths close to optimum with smoother (to various degree) controls. For $C \geq 0.001$ we see the energy cost term is rapidly increasing, while penalization $\overline{P}(\hat{P}_A^*(C))$ does not change much, since penalized optimal control has already smoothed out significantly. The reasonable choice of constant $C$ will be somewhere in the middle to balance both effects (close to optimal $\overline{F}(\hat{P}_A^*(C))$ and $\overline{P}(\hat{P}_A^*(C))$ small enough to to reduce oscillations in controls).

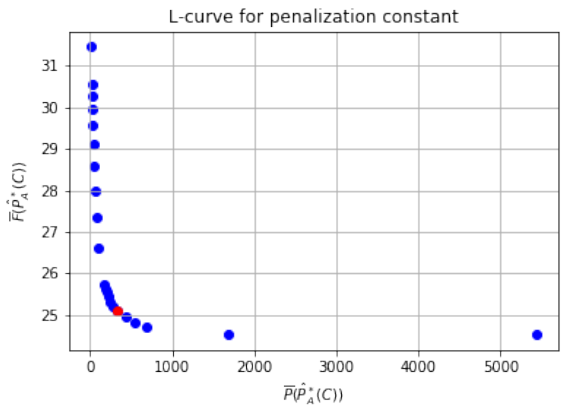


Figure 3.8: The L-curve to choose the best smoothing strategy.

We achieve good result with $C = 0.0004$, which is shown by the red dot in Figure 3.8.

## 3.4 Optimization of the battery size

In this section we will describe the strategy for how to further optimize the cost of energy by choosing the optimal maximum battery capacity. We minimize the expected cost functional not only over the set of controls $P_A$, but also over the battery sizes $\overline{A}$:

$$u(t, A, K, D) = \min_{P_A(t), \overline{A}} E\left(J(t, A, K, D)\right) =$$
$$= \min_{P_A(t), \overline{A}} \left[ E\left( \int_t^T K(s) P_E(s) ds \right) - A \cdot p_K(T) \cdot e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \right] \tag{3.78}$$

For each fixed size of the battery $\overline{A}$ we solve the optimal control problem stated in subsection 3.3: minimize the expected cost of the energy $J(t, A, K, D)$

$$u_{\overline{A}}(t, A, K, D) = \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} E\left(J(t, A, K, D)\right) =$$
$$= \min_{P_A(t) \in [\underline{P}_A, \overline{P}_A]} \left[ E\left( \int_t^T K(s) P_E(s) ds \right) - A \cdot p_K(T) \cdot e^{-T \cdot Pb \cdot r/365} + K_A \cdot \overline{A} \right] \tag{3.79}$$

given the system dynamics:

$$\begin{cases} dK(t) = (-\theta_K(t)(K(t) - p_K(t)) + \dot{p}_K(t) - E[Y(t)])dt \\ +\sqrt{2\theta_{K_t}\alpha_K K(t)(1-K(t))}dW_K(t) + Y(t)dt \\ K(0) = K_0 \\ dD(t) = (-\theta_D(t)(D(t) - p_D(t)) + \dot{p}_D(t))dt + \sqrt{2\theta_{D_t}\alpha_D D(t)(1-D(t))}dW_D(t) \\ D(0) = D_0 \\ dA(t) = -P_A(t)dt, \\ A(0) = A_0 \\ \underline{A} \leq A(t) \leq \overline{A}, \quad \forall t \in [0,T] \\ \underline{P_A} \leq P_A(t) \leq \overline{P_A}, \quad \forall t \in [0,T] \\ D(t) = P_E(t) + P_A(t) \\ K(t), D(t) \in [0,1], \quad \forall t \in [0,T] \end{cases} \tag{3.80}$$

Then we minimize over all possible battery sizes $\overline{A} > 0$:

$$\overline{A}^* = \arg\min_{\overline{A}} u_{\overline{A}}(t_0, A_0, K_0, D_0) \tag{3.81}$$

# Chapter 4

# Numerical methods

## 4.1 Finite difference method for HJB

This section provides the numerical schemes that we used to solve equations (3.47) and (3.66). Numerical schemes for Hamilton-Jacobi type equations of first-order were studied in [8] and [9]. The authors develop the framework, where the approximation scheme is proven to converge to the exact solution of the HJB equation provided that it is monotone, stable and consistent. These principles were expanded for second-order equations by [10], [11] and [12]. The proof of convergence is based on the viscosity solution method and can be applied to a wider class of fully non-linear HJB equations, and therefore is certainly valid in our case. Following the works [10] and [11] we utilize the explicit backward in time finite difference scheme with upwind discretization to approximate the first derivatives and central difference for the second derivatives. We follow chapter IX of [10] to define the finite difference scheme, give a convergence theorem and determine monotonicity conditions. We use [12] and [10] to analyze stability conditions. We turn to [13] for standard consistency analysis. We also give a description of the operator splitting technique based on [14] for time discretization and the battery size optimization procedure in this section.

### 4.1.1 Finite difference approximation

We will first consider a one dimensional HJB, which we will then generalize to the case $n > 1$. Let $x \in \mathbb{R}$ and $u : \mathbb{R} \times [0, T] \to \mathbb{R}$. Consider the HJB partial differential equation

$$\partial_t u(x,t) + H(x, \partial_x u, \partial_{xx} u) = 0, \tag{4.1}$$

$$u(x,T) = u_T(x) \tag{4.2}$$

where the Hamiltonian reads:

$$H(x, \partial_x u, \partial_{xx} u) = \min_{a \in A}[f(x,a) + \partial_x u(x,t) \cdot g(x,a) + \frac{1}{2} \cdot \partial_{xx} u(x,t)\sigma^2(x,a)] \tag{4.3}$$

Let's discretize the domain with step $\Delta t$ in time and $\Delta x$ in space:

$$\Sigma_{\Delta x}^{\Delta t} = \{t_0 < t_0 + \Delta t < ... < T - \Delta t < T, \tag{4.4}$$

$$x_0 < x_0 + \Delta x < ... < x_1 - \Delta x < x_1\} \tag{4.5}$$

We define the following functions:

$$g^+(x,a) = \max\{g(x,a), 0\} \tag{4.6}$$

$$g^-(x,a) = \max\{-g(x,a), 0\} \tag{4.7}$$

and call $g^+$ and $g^-$ the *positive* and *negative* parts of $g$.

We also define the forward and backward difference quotients for first order deriva-

tive and the central difference quotient for the second derivative:

$$\Delta_x^+ U = \frac{U(x+\Delta x,t)-U(x,t)}{\Delta x} \tag{4.8}$$

$$\Delta_x^- U = \frac{U(x,t)-U(x-\Delta x,t)}{\Delta x} \tag{4.9}$$

$$\Delta_x^2 U = \frac{U(x+\Delta x,t)-2U(x,t)+U(x-\Delta x,t)}{\Delta x^2} \tag{4.10}$$

The backward difference for the time derivative:

$$\Delta_t^- U = \frac{U(x,t)-U(x,t-\Delta t)}{\Delta t} \tag{4.11}$$

With notation (4.6)-(4.11) and partition $\Sigma_{\Delta x}^{\Delta t}$ (4.65) we discretize the HJB (4.57) as follows:

$$\Delta_t^- U^{\Delta t} + \tilde{H}(x, \Delta_x^+ U^{\Delta t}, \Delta_x^- U^{\Delta t}, \Delta_x^2 U^{\Delta t}) = 0, \tag{4.12}$$

with numerical Hamiltonian:

$$\begin{aligned}\tilde{H}(x, \Delta_x^+ U^{\Delta t}, \Delta_x^- U^{\Delta t}, \Delta_x^2 U^{\Delta t}) = \min_{a\in A}[f(x,a) + g^+(x,a)\Delta_x^+ U^{\Delta t} \\ -g^-(x,a)\Delta_x^- U^{\Delta t} + \frac{1}{2}\sigma^2 \Delta_x^2 U^{\Delta t}]\end{aligned} \tag{4.13}$$

Equation (4.13) is called an explicit backward in time finite difference scheme, since we can explicitly compute $U^{\Delta t}(x, t-\Delta t)$ given the points on the time layer $t$:

$$U^{\Delta t}(x,t-\Delta t) = U^{\Delta t}(x,t) - \Delta t \tilde{H}(x, \Delta_x^+ U^{\Delta t}, \Delta_x^- U^{\Delta t}, \Delta_x^2 U^{\Delta t}) \tag{4.14}$$

The scheme (4.14) can easily be generalized to the case of $n > 1$. Let $x \in \mathbb{R}^n$ and

$u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$. Consider the HJB partial differential equation:

$$\partial_t u(x,t) + H(x, D_x u, D_x^2 u) = 0, \tag{4.15}$$

$$u(x,T) = u_T(x) \tag{4.16}$$

where the Hamiltonian reads:

$$H(x, D_x u, D_x^2 u) = \min_{a \in A}[f(x,a) + g(x,a)^T \cdot D_x u(x,t) + \frac{1}{2}\sigma^T(x,a) \cdot D_x^2 u(x,t) \cdot \sigma(x,a)] \tag{4.17}$$

$D_x$ and $D_x^2$ in (4.15) denote the gradient and Hessian matrix of $u(x,t)$ respectively.

$g = (g_1, ..., g_n)$ is an $\mathbb{R}^n$ - valued function and we denote by $g_i^+$ and $g_i^-$ the positive and negative parts of $g_i$ by analogue with (4.6). If $e_1, ..., e_n$ is the standard basis for $\mathbb{R}^n$ we can rewrite (4.8)

$$\Delta_{x_i}^{\pm} U = \frac{U(x \pm \Delta x e_i, t) - U(x,t)}{\Delta x} \tag{4.18}$$

$$\Delta_{x_i}^2 U = \frac{U(x + \Delta x e_i, t) - 2U(x,t) + U(x - \Delta x e_i, t)}{\Delta x^2} \tag{4.19}$$

Then (4.14) is generalized as follows:

$$U^{\Delta t}(x, t - \Delta t) = U^{\Delta t}(x,t) - \Delta t \tilde{H}(x, \Delta_{x_i}^+ U^{\Delta t}, \Delta_{x_i}^- U^{\Delta t}, \Delta_{x_i}^2 U^{\Delta t}) \tag{4.20}$$

### 4.1.2 Delayed control

When we apply upwind discretization to approximate $\partial_a u$, we choose $\Delta_a^+ U$ (4.8) or (4.9) $\Delta_a^- U$ based on the value of $P_A$. But $P_A$ depends on $\partial_a u$, as seen in (3.46). To work around the circular definition of $\Delta_a^{\pm} U$ we propose the order of computation, as shown in Figure 4.1.2:

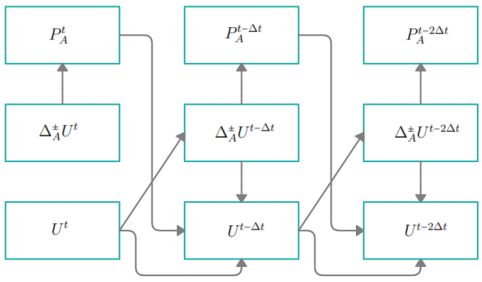


Figure 4.1: The scheme showing the order of computation of $P_A$, $\Delta_A U$ and $U$ at each time layer, starting from the final time $T$ and progressing backwards with step $\Delta t$.

This scheme makes sense from practical considerations. Indeed, in the discrete case, we observe the state of the system at the previous time point $U(x, t - \Delta t)$ to make a decision about $P_A(x, t)$ that drives the system to the desired state $U(x, t)$.

### 4.1.3 Convergence

Consider an arbitrary finite difference scheme on a discretization $\Sigma_{\Delta x}^{\Delta t}$, where $U^{\Delta t}(x, t)$ denotes the approximation of $u(x, t)$:

$$U^{\Delta t}(x, t - \Delta t) = F[U^{\Delta t}(t, \cdot)](x), \tag{4.21}$$

$$U^{\Delta t}(x, T) = \psi(x) \tag{4.22}$$

We will start by recounting the conditions required for a finite difference scheme to be convergent.

Assume the following properties hold for the scheme (4.21):

*Assumptions 1.*

- monotonicity: $F[U_1] \leq F[U_2]$ if $U_1 \leq U_2$
- stability: there exists a solution $U^{\Delta t}$ of (4.21) and a constant $C$, such that $||U^{\Delta t}|| \leq C$

- consistency: for every test function $\phi \in C^{1,2}(\mathbb{R}^{n+1})$

$$\lim_{\Delta t \to 0} \frac{F[\phi(\cdot, t+\Delta t)](x) - \phi(x,t)}{\Delta t} = \partial_t \phi(x,t) - H(x, D_x\phi(x,t), D_x^2\phi(x,t)) \tag{4.23}$$

- $U^{\Delta t}$ assumes the terminal data (4.22) uniformly

$$\lim_{(y,s)\to(x,T),\Delta t\to 0} U^{\Delta t}(y,s) = u_T(x) \tag{4.24}$$

**Theorem 1** ([10], p.334). Let $U^{\Delta t}$ be a solution to (4.21)-(4.22) and $u(x,t)$ be the value function (2.56). Let also assumptions 1 hold. Then

$$\lim_{(y,s)\to(x,t),\Delta t\to 0} U^{\Delta t}(y,s) = u(x,t) \tag{4.25}$$

The rest of this subsection provides details on the conditions when consistency, monotonicity and stability are satisfied.

## Consistency

If we replace the $U^{\Delta t}(x,t)$ in the finite difference scheme by the true solution, evaluated at the same points $u(x,t)$, the result will not equal zero exactly. The discrepancy is called the *local truncation error* $\tau$:

$$\begin{aligned}\tau = &\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \\ &+\tilde{H}\left(x, \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}, \frac{u(x,t) - u(x-\Delta x,t)}{\Delta x}, \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2}\right) \\ &-\partial_t u(x,t) - H(x, \partial_x u(x,t), \partial_x^2 u(x,t))\end{aligned} \tag{4.26}$$

From Taylor expansion we get:

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} = \partial_t u + \mathcal{O}(\Delta t), \tag{4.27}$$

$$\frac{u(x+\Delta x,t)-u(x,t)}{\Delta t} = \partial_x u^+ \mathcal{O}(\Delta x), \tag{4.28}$$

$$\frac{u(x,t)-u(x-\Delta x,t)}{\Delta t} = \partial_x u^+ \mathcal{O}(\Delta x), \tag{4.29}$$

$$\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{\Delta t} = \partial_x^2 u + \mathcal{O}(\Delta x^2), \tag{4.30}$$

Therefore, the local truncation error reads:

$$\begin{aligned}\tau = \partial_t u + \mathcal{O}(\Delta t) + \tilde{H}(x, \partial_x u + \mathcal{O}(\Delta x), \partial_x u + \mathcal{O}(\Delta x), \partial_x^2 u + \mathcal{O}(\Delta x^2)) \\ -\partial_t u - H(x, \partial_x u(x,t), \partial_x^2 u(x,t))\end{aligned} \tag{4.31}$$

Approximating $\tilde{H}(x, \partial_x u + \mathcal{O}(\Delta x), \partial_x u + \mathcal{O}(\Delta x), \partial_x^2 u + \mathcal{O}(\Delta x^2))$ around the point $(x, \partial_x u, \partial_x u, \partial_x^2 u)$ by a tangent plane we have:

$$\begin{aligned}\tilde{H}(x, \partial_x u + \mathcal{O}(\Delta x), \partial_x u + \mathcal{O}(\Delta x), \partial_x^2 u + \mathcal{O}(\Delta x^2)) \approx \\ \tilde{H}(x, \partial_x u, \partial_x u, \partial_x^2 u) + \tilde{H}_{\partial_x u}\mathcal{O}(\Delta x) + \tilde{H}_{\partial_x^2 u}\mathcal{O}(\Delta x^2)\end{aligned} \tag{4.32}$$

Given that $\tilde{H}(x, \partial_x u, \partial_x u, \partial_x^2 u) = H(x, \partial_x u, \partial_x^2 u)$ and substituting the Hamiltonian (4.32) into the expression for the local truncation error (4.31), we get:

$$\tau \approx \mathcal{O}(\Delta t) + \tilde{H}_{\partial_x u}\mathcal{O}(\Delta x) + \tilde{H}_{\partial_x^2 u}\mathcal{O}(\Delta x^2) \tag{4.33}$$

The partial derivatives of the Hamiltonian are $\tilde{H}_{\partial_x u} = g^+ + g^-$ and $\tilde{H}_{\partial_x^2 u} = \frac{\sigma^2}{2}$ and the function $g^+$, $g^-$ and $\sigma^2$ are bounded, therefore the derivatives are bounded. Thus, consistency of the scheme (4.14) immediately follows from (4.33), since $\tau \to 0$

as $\Delta t,\ \Delta x \to 0$.

## Monotonicity

Denote $U_x^t = U^{\Delta t}(x,t)$. Monotonicity of the scheme implies that a monotone sequence at a time layer $t$

$$U_{x-\Delta x}^t \le U_x^t \le U_{x+\Delta x}^t \le ... \tag{4.34}$$

preserves its property on the next time layer

$$U_{x-\Delta x}^{t+\Delta t} \le U_x^{t+\Delta t} \le U_{x+\Delta x}^{t+\Delta t} \le ... \tag{4.35}$$

We will give the definition and the conditions of monotonicity for the forward in time scheme. The result will still apply to the backward in time scheme, since it has the same properties, but it is more difficult to state the monotonicity conditions for such scheme.

Let's write the scheme as a function of $U_{x+\Delta x}^t,\ U_x^t,\ U_{x+\Delta x}^t$:

$$U_x^{t+\Delta t} = F(U_{x+\Delta x}^t, U_x^t, U_{x+\Delta x}^t) = U_x^t + \Delta t \tilde{H}(U_{x+\Delta x}^t, U_x^t, U_{x+\Delta x}^t) \tag{4.36}$$

The scheme is monotone, if for all $U_1 \le U_2$ we have $F[U_1] \le F[U_2]$, or alternatively

$$\frac{\partial F}{\partial U_i} \ge 0, \quad \forall i \in \{x - \Delta x, x, x + \Delta x\} \tag{4.37}$$

To determine monotonicity conditions we expand (4.36):

$$U_x^{t+\Delta t} = U_x^t + \Delta t \cdot \min_{a\in A}\left[\left(\frac{\sigma^2}{2\Delta x^2} + \frac{g^-}{\Delta x}\right)U_{x-\Delta x}^t + \left(-\frac{g^+}{\Delta x} - \frac{g^-}{\Delta x} - \frac{\sigma^2}{\Delta x^2}\right)U_x^t + \left(\frac{\sigma^2}{2\Delta x^2} + \frac{g^+}{\Delta x}\right)U_{x+\Delta x}^t + f(x,a)\right] \tag{4.38}$$

Since $g^+$ and $g^-$ are both positive function, we immediately notice, that

$$\frac{\partial F}{\partial U_{x-\Delta x}^t} \geq 0, \quad \frac{\partial F}{\partial U_{x+\Delta x}^t} \geq 0 \tag{4.39}$$

Hence, the scheme (4.36) and therefore the scheme (4.20) is monotone, provided that

$$\frac{\partial F}{\partial U_x^t} = 1 - \Delta t \cdot \max_{a\in A}\left[\frac{g^+}{\Delta x} + \frac{g^-}{\Delta x} + \frac{\sigma^2}{\Delta x^2}\right] \geq 0,$$
$$\Delta t \cdot \max_{a\in A}\left[\frac{g^+}{\Delta x} + \frac{g^-}{\Delta x} + \frac{\sigma^2}{\Delta x^2}\right] \leq 1 \tag{4.40}$$

The expression in (4.40) is called the *CFL (Courant-Friedrichs-Lewy)* condition. Its generalization to higher-dimensional problem $n > 1$ is straightforward:

$$\frac{\Delta t}{\Delta x^2}\max_{a\in A}\sum_{i=1}^{n}[\sigma_i^2(x,a) + \Delta x|g_i(x,a)|] \leq 1 \tag{4.41}$$

In particular, the CFL condition for equation (3.66) reads:

$$\max_{t,A,D,K}\Delta T\left(\frac{\dot{p_k} + \theta_k(p_k - K)}{\Delta K} + \frac{\dot{p_d} + \theta_d(p_d - D)}{\Delta D} + \frac{|P_A^*|}{\overline{A}\Delta\hat{A}} + \right. \tag{4.42}$$
$$\left. + \frac{2\theta_K\alpha_K K(1-K)}{(\Delta K)^2} + \frac{2\theta_D\alpha_D D(1-D)}{(\Delta D)^2}\right) \leq 1 \tag{4.43}$$

In deterministic optimal control with no $\sigma$, (4.41) takes the form:

$$\frac{\Delta t}{\Delta x^2}\max_{a\in A}\sum_{i=1}^{n}\Delta x|g_i(x,a)| \leq 1 \tag{4.44}$$

and particularly for equation (3.47):

$$\Delta T\max_{a\in A}\frac{|P_A^*|}{\overline{A}\Delta A} \leq 1 \tag{4.45}$$

## Stability

Recall the dynamic programming relation (2.57) and the correspondence between the value function and the $u(x,t)$ in the HJB equation (2.78). It is more convenient to show stability of $u(x,t)$ in the DP relation. If we can find $\Delta t$ and $\Delta x$ so that $u(x,t)$ in DP relation is stable, the stability of HJB scheme will follow. To discretize the DP, we will first introduce the notion of a discrete time controlled Markov chain. We can think of the discrete time Markov chain as an approximation of the diffusion process we defined in (2.54). The work of Kushner [12] contains great detail on such approximations. Most important to us is the theorem in Chapter 4.1 of [12] which gives conditions for the local consistency of a discrete time Markov Chain with the diffusion process it approximates. Consider a countable set $\Sigma$ and times $l = k, k+1, ..., M$. We will denote by $x^l \in \Sigma$ the state of the system at time $l$, $a^l \in A$ - the control at time $l$. The dynamics of the system are defined by the one step transition probabilities $p^v(x,y)$. Then our problem is to minimize the cost functional $J_k$:

$$J_k(x,a) = \mathbb{E}_{kx}\left[\sum_{l=k} f_l(x^l,a^l) + u_T(x^M)\right] \tag{4.46}$$

Then for the dynamic programming relation we have (as shown in [12], Chapter 4.2):

$$\hat{U}_k(x) = \min_{a \in A} \left[ \sum_{y \in \Sigma} p^a(x,y) \hat{U}_{k+1}(y) + f_k(x,a) \right], \tag{4.47}$$

$$\hat{U}_M(x) = u_T(x^M) \tag{4.48}$$

We now need to determine the transition probabilities. We know that if the diffusion process is as in (2.54), then the functional defined as (2.56) satisfies the HJB equation (2.78). Just like we did when analyzing monotonicity, we expressed the $U_k(x)$ in HJB as a function of $U_{k+1}$ at the nearest spatial neighbouring points:

$$\begin{aligned} \hat{U}_k(x) = \min_{a \in A} \Bigg[ & \left( 1 - \frac{g^+}{\Delta x} - \frac{g^-}{\Delta x} - \frac{\sigma^2}{\Delta x^2} \right) \hat{U}_{k+1}(x) + \left( \frac{g^+}{\Delta x} + \frac{\sigma^2}{2\Delta x^2} \right) \hat{U}_{k+1}(x + \Delta x) \\ & + \left( \frac{g^-}{\Delta x} + \frac{\sigma^2}{2\Delta x^2} \hat{U}_{k+1}(x - \Delta x) \right) + \Delta t f(x,a) \Bigg] \end{aligned} \tag{4.49}$$

It is natural to interpret the coefficients of $\hat{U}_{k+1}(x \pm \Delta x)$, $\hat{U}_{k+1}(x)$ as transition probabilities $p^v(x,y)$ of jumping from state $x$ to state $y$:

$$p^a(x, x + \Delta x) = \frac{\Delta t}{\Delta x^2} \left[ \frac{\sigma^2(x,a)}{2} + \Delta x g^+(x,a) \right] \tag{4.50}$$

$$p^a(x, x - \Delta x) = \frac{\Delta t}{\Delta x^2} \left[ \frac{\sigma^2(x,a)}{2} + \Delta x g^-(x,a) \right] \tag{4.51}$$

$$p^a(x,x) = 1 - p^a(x, x + \Delta x) - p^a(x, x - \Delta x) \tag{4.52}$$

The discrete time Markov chain approximation is locally consistent with the diffusion process, if we impose certain conditions on the transition probabilities $p^a(x,y)$ and the interpolation interval $\Delta t$. For instance, the transition probabilities should

satisfy:

$$0 \leq p^a(x,y) \leq 1, \quad \forall y = \{x \pm \Delta x, x\}, \tag{4.53}$$

$$p^a(x,y) = 0, \quad \forall y \neq \{x \pm \Delta x, x\}, \tag{4.54}$$

To satisfy (4.53), it is sufficient to have:

$$\frac{\Delta t}{\Delta x^2}\left[\sigma^2(x,a) + \Delta x|g(x,a)|\right] \leq 1 \tag{4.55}$$

Note that this condition is the same as the one we derived for monotonicity. We can generalize it to multiple dimensions in a straightforward way and obtain (4.41).

To complete the stability analysis, we bound the cost functional (4.46)

$$|J_k(x,a)| \leq (T - t_0)||f||_\infty + ||u_T||_\infty \tag{4.56}$$

Since $|J_k(x,a)|$ is bounded, so is $|\hat{U}_k(x,t)|$ in (4.47), and therefore, $|U^{\Delta t}(x,t)|$ in (4.13). By choosing $K = (T-t_0)||f||_\infty + ||u_T||_\infty$, stability is achieved under conditions (4.55).

## Convergence speed

We've shown that the scheme (4.14) is monotone, consistent and stable, thus it converges to the exact solution by Theorem 1. We conclude this subsection by a brief discussion of the speed of convergence. It is known [10], that monotone schemes are first order accurate in time and first or second order accurate in space. From LTE in (4.33) we conclude that with the scheme (4.20) we achieve first-order convergence both in time and space.

### 4.1.4 Operator splitting method

We now introduce the operator splitting method. Recall the Hamilton-Jacobi-Bellman equation we derived for the system described by the jump-diffusion process (2.82):

$$\partial_t u(x,t) + H(x, u, \partial_x u, \partial_{xx} u) = 0, \tag{4.57}$$

$$u(x,T) = u_T(x) \tag{4.58}$$

where the Hamiltonian reads:

$$\begin{aligned} H(x, u, \partial_x u, \partial_{xx} u) = \min_{a \in A} [f(x,a) + \partial_x u(x,t) \cdot g(x,a) + \frac{1}{2} \cdot \partial_{xx} u(x,t) \sigma^2(x,a) \\ + \lambda \sum_{z_i \in \mathcal{S}} (u(x+z_i, t) - u(x,t)) p_Z(z_i)] \end{aligned} \tag{4.59}$$

We will split the Hamiltonian in (4.59) into two parts $H(x, u, \partial_x u, \partial_{xx} u)$ and rewrite (4.57):

$$\partial_t u + \mathcal{A}(u) + \mathcal{B}(u) = 0, \tag{4.60}$$

$$u(x,T) = u_T(x) \tag{4.61}$$

where the operators $A$ and $B$ are defined as follows:

$$\mathcal{A}(u) = \min_{a \in A} [f(x,a) + \partial_x u(x,t) \cdot g(x,a) + \frac{1}{2} \cdot \partial_{xx} u(x,t) \sigma^2(x,a)], \tag{4.62}$$

$$\mathcal{B}(u) = \lambda \sum_{z_i \in \mathcal{S}} (u(x+z_i, t) - u(x,t)) p_Z(z_i) \tag{4.63}$$

Let's discretize the domain with step $\Delta t$ in time into $N = T/\Delta t$ intervals and $\Delta x$

in space into $M = x_M/\Delta x$ intervals:

$$t_0 < t_0 + \Delta t < ... < t_N - \Delta t < t_N = T, \tag{4.64}$$

$$x_0 < x_0 + \Delta x < ... < x_M - \Delta x < x_M \tag{4.65}$$

Suppose we use backward in time Forward Euler to approximate the solution of (4.60):

$$\frac{U(x, t + \Delta t) - U(x, t)}{\Delta t} + (A + B)U(x, t + \Delta t) = 0, \tag{4.66}$$

$$U(x, t) = (I + \Delta t(A + B))U(x, t + \Delta t) \tag{4.67}$$

where $U(x, t)$ denotes the numerical approximation of $u(x, t)$ and $A$ and $B$ are the numerical versions of operators $\mathcal{A}$ and $\mathcal{B}$.

The operator splitting method implies that at each time step we include the operators $A$ and $B$ separately and successively. We will replace (4.60) by the following system:

$$\begin{cases} \partial_t \overline{u}(x, t) + \mathcal{A}(\overline{u}) = 0, \quad t \in [t_n, t_{n+1}], \quad \overline{u}(t_{n+1}) = \overline{u}_{sp}^{n+1}, \quad \forall n = 0, ..., N \\ \partial_t \overline{v}(x, t) + \mathcal{B}(\overline{v}) = 0, \quad t \in [t_n, t_{n+1}], \quad \overline{v}(t_{n+1}) = \overline{u}(t_n), \quad \forall n = 0, ..., N \end{cases} \tag{4.68}$$

For $n = N$, $\overline{u}_{sp}^{n+1}$ is equal to the final condition $u_T$, for $n < N$, $\overline{u}_{sp}^{n+1} = \overline{v}(t_n)$, which is computed on the previous iteration. Thus, the equations in the system (4.68) are connected via the final conditions.

We compute the Forward Euler approximation $\overline{U}$ and $\overline{V}$ of the system (4.68):

$$\begin{cases} \frac{\overline{U}(x,t+\Delta t)-\overline{U}(x,t)}{\Delta t} + A\overline{U}(x,t+\Delta t) = 0, \\ \frac{\overline{V}(x,t+\Delta t)-\overline{V}(x,t)}{\Delta t} + B\overline{V}(x,t+\Delta t) = 0, \\ \overline{V}(x,t+\Delta t) = \overline{U}(x,t) \end{cases} \tag{4.69}$$

or, rewriting explicitly for $\overline{U}(x,t)$ and $\overline{V}(x,t)$:

$$\begin{cases} \overline{U}(x,t) = (I+\Delta tA)\overline{U}(x,t+\Delta t), \\ \overline{V}(x,t) = (I+\Delta tB)\overline{V}(x,t+\Delta t) = (I+\Delta tA)(I+\Delta tB)\overline{U}(x,t+\Delta t) \end{cases} \tag{4.70}$$

The approximation $\overline{V}$ of (4.68):

$$\overline{V}(t,x) = (I+\Delta tA)(I+\Delta tB)\overline{U}(t+\Delta t,x) = (I+\Delta t(A+B))\overline{U}(t+\Delta t,x) + O(\Delta t^2) \tag{4.71}$$

Comparing (4.66) and (4.71), we notice that the new approximation has an additional error of order $\Delta t^2$ at each time step. We've seen that the numerical solution of HJB is only first order accurate, therefore operator splitting method gives a way to compute (4.60) without destroying the accuracy achieved with scheme (4.14).

### Stability

Applying the operator splitting method, we should also ensure that the stability requirements are satisfied. Consider the equation

$$\partial_t u(x,t) + \lambda \sum_{z_i \in \mathcal{S}} (u(x+z_i) - u(x)) p_Z(z_i) = 0, \tag{4.72}$$

$$u(x,T) = u_T(x) \tag{4.73}$$

and its Forward Euler approximation on a partition (4.64)-(4.65):

$$\frac{U(t+\Delta t,x) - U(t,x)}{\Delta t} + \lambda \Delta t \sum_i (U(t+\Delta t, x+z_i) - U(t+\Delta t,x)) p(x+z_i | x \in \Omega_j) = 0 \tag{4.74}$$

We can get the condition from (4.74) after rearranging

$$U(t,x) = U(t+\Delta t,x)(1-\lambda\Delta t) + \lambda \Delta t \sum_i U(t+\Delta t, x+z_i) p(x+z_i|x \in \Omega_j) \tag{4.75}$$

$$\lambda \Delta t \leq 1 \tag{4.76}$$

## 4.2 Numerical solution for optimum battery capacity

Each point of the one-dimensional function that we aim to optimize in subsection 3.4 is a actually a value function $u(t_0, A_0, K_0, D_0)$, computed for the corresponding battery capacity $\overline{A}$. The derivative and Hessian of this function are unknown. There is a number of derivative-free optimization methods, for example, direct search method, coordinate descent, constrained optimization by linear approximation (COBYLA).

The computational complexity though turned out to be too demanding. Indeed, the goal is to compute the HJB equation for several grids $\Delta A \cdot \overline{A} = 2^{-3}, 2^{-4}, ..., 2^{-8}$ to see the convergence rate and determine the limiting value as $\Delta A \to 0$. Keeping $\Delta K = \Delta D = 2^{-3}$ fixed, from the CFL condition for $\Delta A \cdot \overline{A} = 2^{-3}$ we get $\Delta T = 2^{-8}$, for $\Delta A \cdot \overline{A} = 2^{-8}$ we get $\Delta T = 2^{-14}$. Thus, to solve the problem on the finest grid means computing the HJB equation on $N = 1.2 \cdot 10^{11}$ points for a number of different values of $\overline{A}$. However, having computed a number of points, we noticed that the curve resembles a polynomial. Instead of using the optimization algorithm, we will compute a number of points of $u(t_0, A_0, K_0, D_0)$ and approximate it with a polynomial regression model:

$$y(\overline{A}) = \beta_0 + \beta_1 \overline{A} + \beta_2 \overline{A}^2 + ... + \beta_n \overline{A}^n + \epsilon \tag{4.77}$$

The coefficients $\beta_i$ are computed using the method of least squares (which is realized by python's function *polyfit*).

The optimum battery size $\overline{A}$ will then be determined by optimizing the polynomial approximation:

$$\overline{A}^* \approx \arg\min_{\overline{A}>0} y(u_{\overline{A}}), \tag{4.78}$$

$$u_{\overline{A}^*}(t_0, A_0, K_0, D_0) \approx y(\overline{A}^*) \tag{4.79}$$

# Chapter 5

# Results

## 5.1 Reference point model

This section provides the computation of the energy cost for the case where there is no battery in the system. This result will serve as a reference point for us to compare the cost of energy with and without the battery in the system and evaluate the cost savings.

### Deterministic model

In the deterministic case the cost of energy is obtained by a simple straightforward computation:

$$\hat{u} = \int_0^T p_K(s) p_D(s) ds = 52.72 \text{ c.u.} \tag{5.1}$$

### Stochastic model

To get a reference cost value in stochastic case we solve equation (3.30) (subsection 2.1.4) using scheme (4.20). With the following discretization

$$\Delta t = 2^{-10}, \Delta K = \Delta D = 2^{-3} \tag{5.2}$$

we get a solution shown in Figure 5.1. Each point on the surface $\hat{u}(0, K_0, D_0)$ represents the cost of energy given the initial conditions $K_0$ and $D_0$. Since the initial values of price $p_K(0) = 0.5$ and demand $p_D(0) = 0.5$ we determine the cost of energy $\hat{u}(0, 0.5, 0.5) = 49.81$.

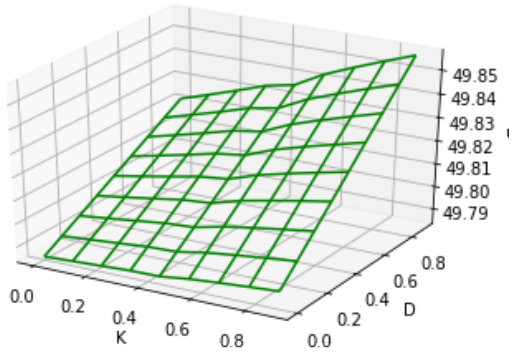


Figure 5.1: Numerical solution of the Kolmogorov Backward equation at time $t = 0$.

## 5.2 Deterministic model

This section presents the approximate solution of the HJB equations (3.47) and (3.52)-(3.54), obtained with the numerical scheme (4.20) programmed in python.

### Model without battery penalization

First we consider the model with no penalization of battery use (problem (3.32)-(3.39)). The parameters of discretization are as follows:

$$\Delta t = 2^{-5}, \Delta A \cdot \overline{A} = 2^{-3}, \overline{A} = 6.75 \tag{5.3}$$

$\overline{A} = 6.75$ is the optimum battery size for given discretization, as computed in section 5.4 of this chapter.

The plot in Figure 5.2 shows the value function $u(t, A)$. Each point of the surface

represents the optimal cost of energy for the time period of $[t, T]$, given that at time $t$ the level of charge in the battery is $A$. In particular, when the battery is new and there is no initial charge, we get the cost of energy $u(0, 0) = 30.4$. This is a good improvement in comparison with no battery case, where the cost is 52.72 (see equation (5.1)).

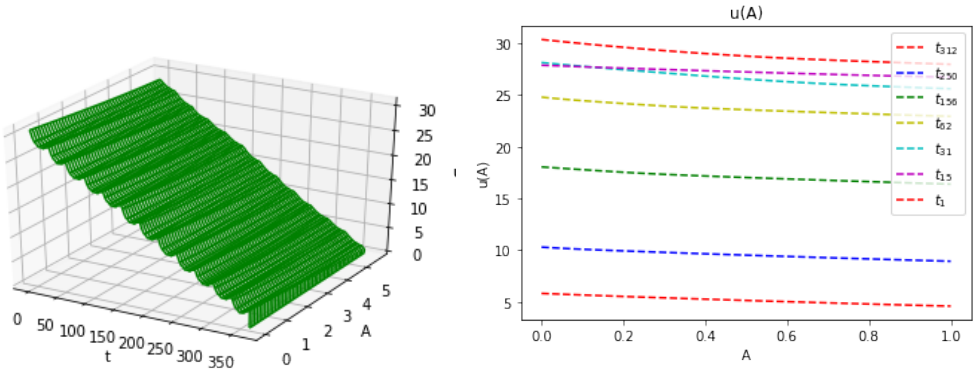


Figure 5.2: The value function $u(t, A)$.

As the plot on the right of Figure 5.2 shows, the value function is monotonically decreasing in $A$. This is indeed correct, since the more charge we have in the battery at any given moment, the less energy we will buy from the grid, hence the smaller production cost.

## Optimal path computation

Computation of the optimal path requires the knowledge of the derivative of the value function $\partial_A u$, since the battery dynamics read:

$$\frac{dA(t)}{dt} = -P_A(t) \tag{5.4}$$

which in the numerical solution translates to:

$$A^{t+\Delta t} = -P_A^t \Delta t + A^t \tag{5.5}$$

To obtain the optimal trajectory we start at $A(0)$ and use numerical battery dynamics (5.5) to calculate the next optimal point. But $A^{t+\Delta t}$ computed this way is not generally a grid point. Suppose $A_i < A^{t+\Delta t} < A_{i+1}$. We will choose the nearest grid point to be the optimal:

$$\widetilde{A}^{t+\Delta t} = \begin{cases} A_i, & \text{if } A^{t+\Delta t} - A_i \le A_{i+1} - A^{t+\Delta t}, \\ A_{i+1}, & \text{if } A^{t+\Delta t} - A_i > A_{i+1} - A^{t+\Delta t} \end{cases} \tag{5.6}$$

$$\text{error: } \; h = \widetilde{A}^{t+\Delta t} - (A^t - P_A^t \, \Delta t) \tag{5.7}$$

To compute the control at this point we use interpolated values of $\widetilde{u}_A$ to counteract the error:

$$\widetilde{P}_A^t = \begin{cases} \min\{D^t, \overline{P}_A\}, & \text{if } \; \widetilde{u}_A^t + K^t \ge 0 \\ \underline{P}_A^t, & \text{if } \; \widetilde{u}_A^t + K^t < 0, \end{cases} \tag{5.8}$$

where

$$\widetilde{u}_A^t = (u_{A_{i+1}}^t + u_{A_i}^t)h + u_{A_i}^t, \qquad \text{if } \; h > 0 \tag{5.9}$$

$$\widetilde{u}_A^t = (u_{A_i}^t + u_{A_{i-1}}^t)(1 - |h|) + u_{A_i}^t, \quad \text{if } \; h < 0 \tag{5.10}$$

The procedure (5.6)-(5.9), along with the path smoothing technique (section 3.3), allows to compute the optimal controls $P_A^*(t)$ starting from $A(0) = 0$.

The optimal control $P_A^*(t)$ that minimizes the cost of energy and the level of charge $A(t)$ that corresponds to it are shown in Figures 5.3 (for 30 days) and 5.4 (for the year).

Let's see how we choose the optimal path. Recall the expression for the optimal

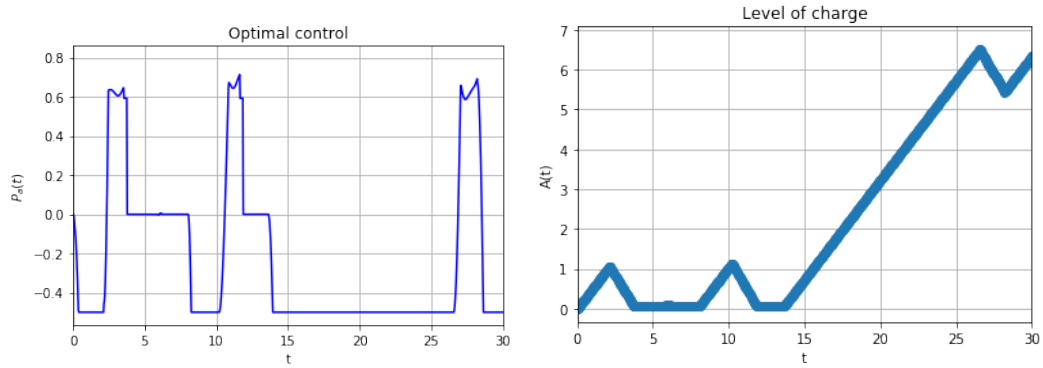


Figure 5.3: The optimal control path (left) and the corresponding level of charge (right) shown for the period of 30 days, which minimize the energy cost over the year.

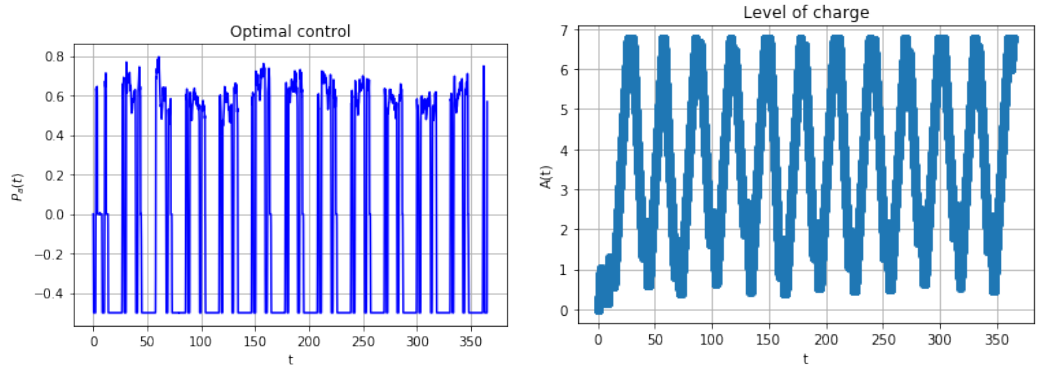


Figure 5.4: The optimal control path (left) and the corresponding level of charge (right) the year.

control:

$$P_A^* = -\mathbb{1}_{\{\partial_A u + K < 0\}} \underline{P}_A(A) + \mathbb{1}_{\{\partial_A u + K \geq 0\}} \min\{p_D(t), \overline{P}_A(A)\} \tag{5.11}$$

The plots in Figure 5.5 show when and why the control switches based on the condition $\partial_A u + p_K$, current demand $p_D(t)$ and level of charge $A(t)$.

## Convergence estimation

We showed in chapter 3, that the numerical solution converges to the exact solution with order one both in time and space. Figure 5.6 shows that our numerical solution

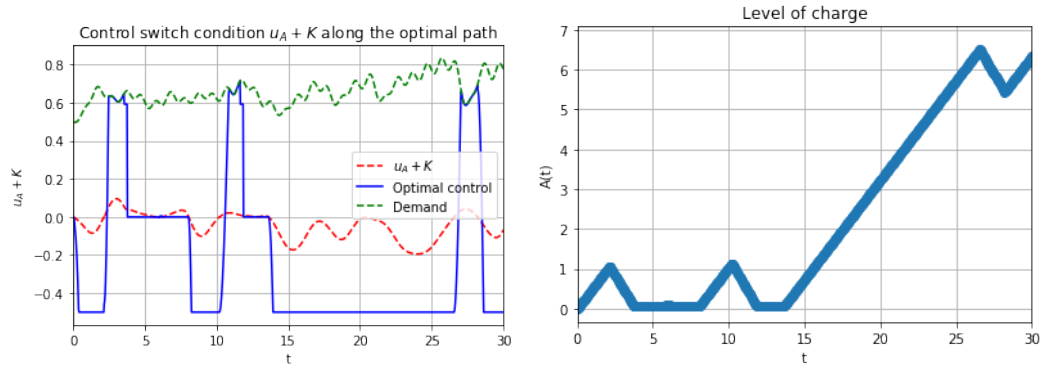


Figure 5.5: The left plot shows the control along with the condition $\partial_A u + p_K$ and the demand. When $\partial_A u + p_K < 0$ (red line), the battery is charged ($P_A = -0.5$); when $\partial_A u + p_K \geq 0$, we discharge at a rate equal to the demand (green line) ($P_A = \min\{p_D, \overline{P}_A\}$); $P_A = 0$ in the intervals [3.3,7.6] and [12.0,13.5] because even though $\partial_A u + p_K \geq 0$, the battery is depleted, as we can see on the right plot.

is in agreement with this result.

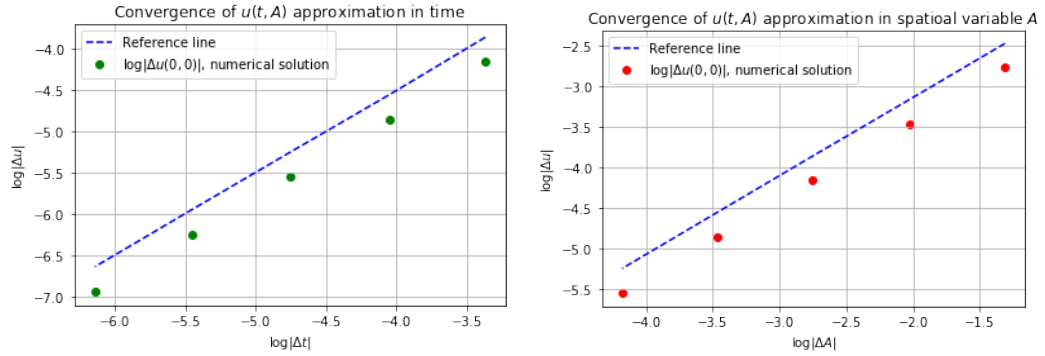


Figure 5.6: Estimation of convergence of the approximate solution to the exact one in time and space. The line with slope one is given for reference.

## Model with battery penalization

Let us see how the solution of the deterministic model changes when we consider the effect of battery wear off (problem (3.48)-(3.49)). The plots in Figure 5.7 show the optimal control, the value of $\partial_A u + p_K$ that determines how the control switches and the current level of charge in the battery. In general, the choice of control follows a similar logic as we described in the model with no battery penalization: charge when

$\partial_A u + p_K < -C_A$ at a rate $P_A = 0.5$, discharge when $\partial_A u + p_K > -C_A$ at a rate equal to the demand, or no discharge, when the battery is empty. But the difference can be seen at the interval $t \in [12, 14]$, for example, where we choose not to use the battery, even though there is a nonzero charge left, as seen on the right plot. We choose $P_A = 0$ here, because $-C_A \leq \partial_A u + p_K < C_A$.

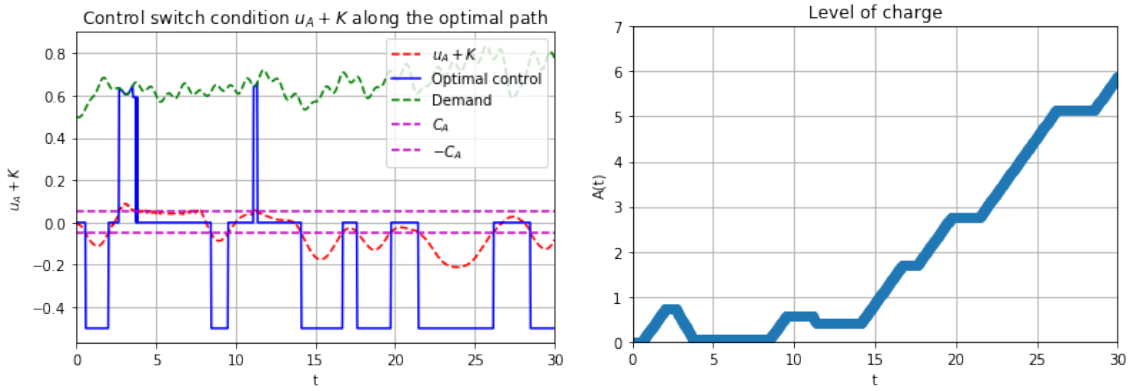


Figure 5.7: The optimal control (left) and the level of charge (right) in the model with battery penalization.

If we compare with previous results, we'll notice that penalizing the use of the battery leads to more cautious use of it, choosing $P_A = 0$ as control even when there is charge in the battery (whenever the condition $-C_A < \partial_A u + K < C_A$ holds). This model gives a naturally higher optimal cost of energy, but it accounts for more real life factors and is therefore more accurate.

## 5.3 Stochastic model

This section presents the approximate solution of problem (3.56)-(3.61), obtained with numerical scheme (4.20) programmed in python. The parameters of discretization are as follows:

$$\Delta t = 2^{-10}, \Delta K = \Delta D = 2^{-3}, \Delta A \cdot \overline{A} = 2^{-3}, \overline{A} = 5.6 \tag{5.12}$$

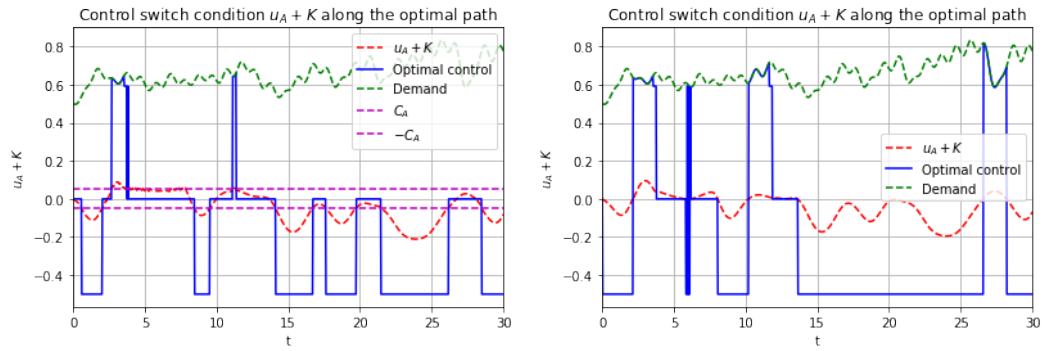


Figure 5.8: Comparison of the optimal control for model with (left) and without (right) penalization. When $\partial_A u + p_k$ (red line) falls in the band $[-C_A, C_A]$ (magenta lines), the battery is switched off and the control is zero ($P_A = 0$). Penalization leads to more careful use of the battery.

The value of the maximum capacity of the battery is optimal and computed in section 4.3 of this chapter. Figure 5.9 shows the plots of the surface $u$ at time $t = 0$ as three two-dimensional projections onto the axis $A - D$, $A - K$, $D - K$. Given the initial conditions of price $K_0 = 0.5$, demand $D_0 = 0.5$ and empty battery $A_0 = 0$, the cost of energy is given by $u(0, 0, 0.5, 0.5) = 22.45$. As we can see, we have significantly reduced the cost of energy by adding the battery, since the cost without the battery is $\hat{u}(0, 0.5, 0.5) = 49.81$.

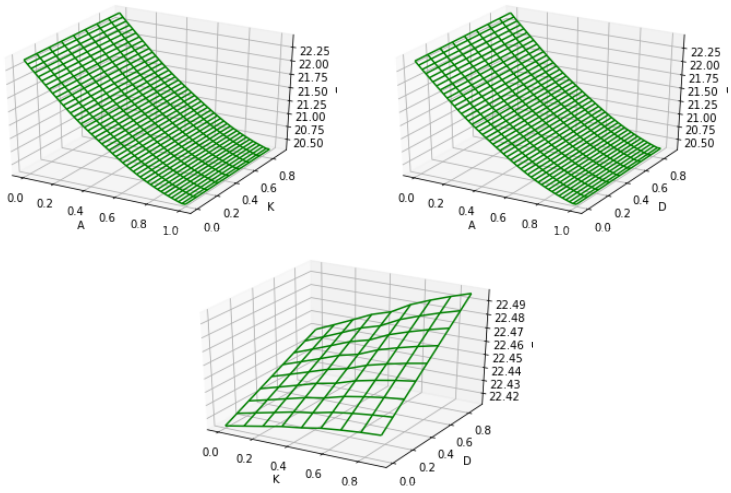


Figure 5.9: Numerical solution of $u(t_0, A_0, K_0, D_0)$

The plot in $K-D$ is shown for $A=0$. The value function $u(\cdot,\cdot,K,D)$ increases as $K$ and $D$ grow, since both the price of electricity and the demand volume translate to higher cost of production. The plots in $A-D$ and $A-K$ are qualitatively similar to each other and show that $\partial_A u \leq 0$, as discussed in deterministic case, and $\partial_D u,\ \partial_K u \geq 0$, as expected.

To compute the optimal control we must first simulate a real world situation. We generate price and demand processes for the year (Figure 5.10). The corresponding control and the level of charge of the battery are shown in Figure 5.11.

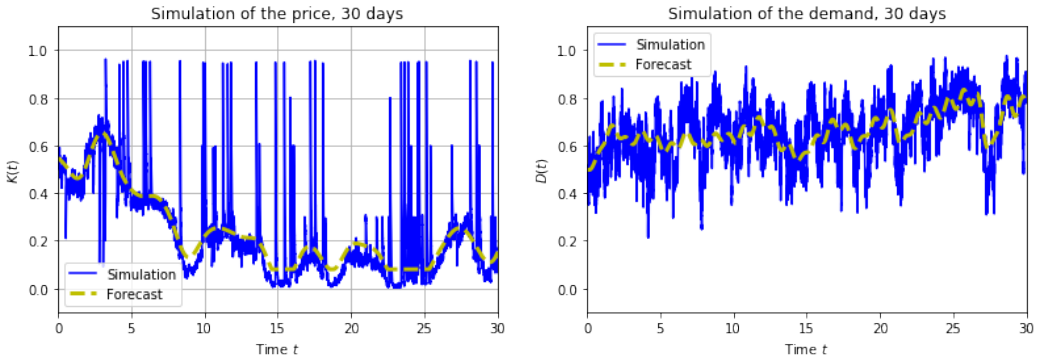


Figure 5.10: Simulations of spot price (left) and demand (right), which are used to compute the optimal battery decision policy and the corresponding cost. 30 days shown for better readability.

The optimal control path calculation is done with the optimal path smoothing technique described in subsection (2.3.3).

The advantage of smoothing is seen if we take a small period and compare the smooth and non-smooth optimal control paths (Figure 5.12).

## Convergence estimation

We expect order one convergence in time and space for the stochastic model as well as for the deterministic one. Figure 5.13 shows that it is indeed the case.

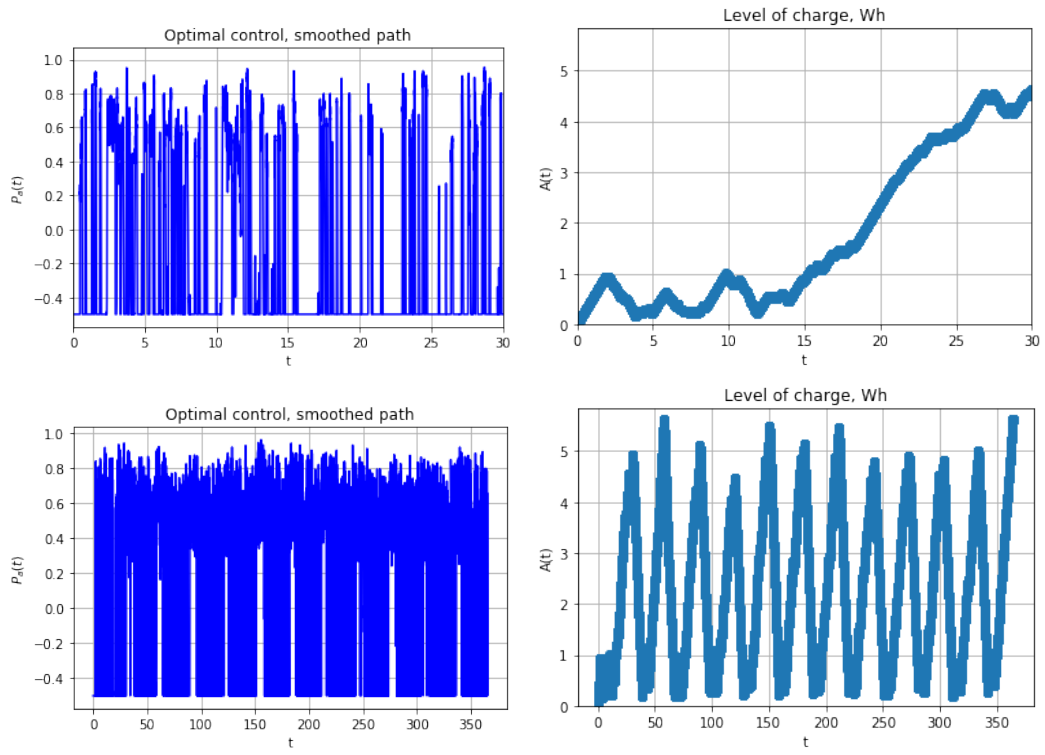


Figure 5.11: Optimal control path (left) and the level of charge in the battery (right), optimize the cost of energy.

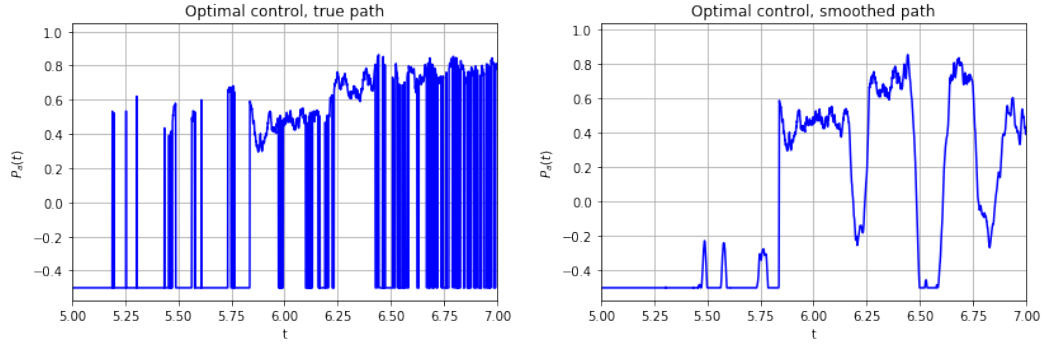


Figure 5.12: True optimal control path (left) and smoothed path (right).

## 5.4 Numerical solution for optimum battery capacity

This section provides results on the one-dimensional optimization problem, which chooses the optimal battery size.

The current battery price of 156 USD/kWh is quite high comparing to the electricity prices, which leads to the fact that it is not reasonable to purchase a battery with

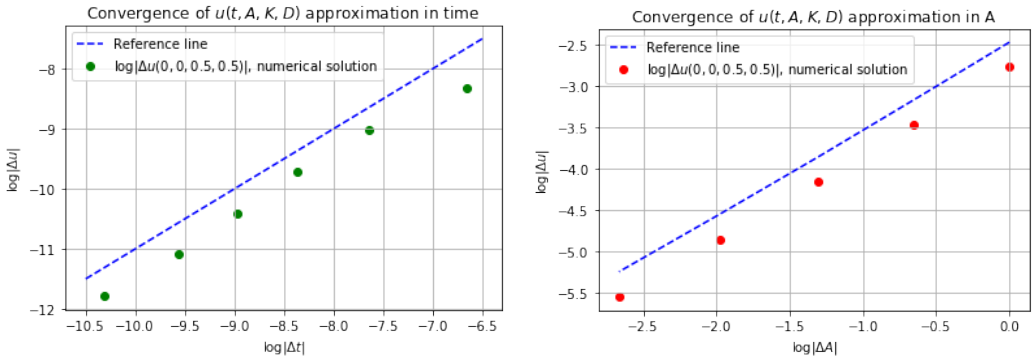


Figure 5.13: Estimation of convergence of the approximate solution to the exact one for the stochastic model.

power $\overline{P}_A = D_{max}$. However, the battery price has decreased almost ten times in the last decade. The technologies are developing rapidly, which leads us to believe, that the greater price decrease can be achieved in the foreseeable future. The coefficient $K_A$ is set to 7.7, which corresponds to the battery price lowering to the level of 0.310 USD/kWh and the payback period of 10 years.

The computations are performed on the following grids:

$$\Delta t = 2^{-14},\ \Delta K = \Delta D = 2^{-3},\ \Delta A \cdot \overline{A} = 2^{-3}, 2^{-4}, ..., 2^{-8} \tag{5.13}$$

To estimate convergence rate we need to compute the solution on several grids. The value of time step $\Delta t = 2^{-14}$ was chosen based on the CFL factor, so that the solution on the finest grid $\Delta A \cdot \overline{A} = 2^{-8}$ is still stable (max CFL = 0.87).

We will determine the optimal point by optimizing a polynomial fitted to the computed data, as described in section 3.2. Figure 5.14 shows fitting a polynomial of 3rd degree to the points computed on the finest grid.

The polynomial approximations of the solutions of $u_{\overline{A}}(t_0, A_0, K_0, D_0)$ computed on grids (5.13) are gathered on Figure 5.15. We can estimate the convergence rate and the limit value of the battery size $\overline{A}^*$ and the optimal energy cost $u_{\overline{A}^*}$.

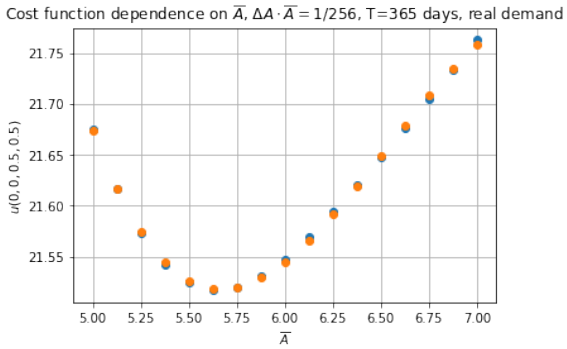


Figure 5.14: Polynomial regression model (3rd degree) which approximates the optimum energy cost as a function of the battery size.

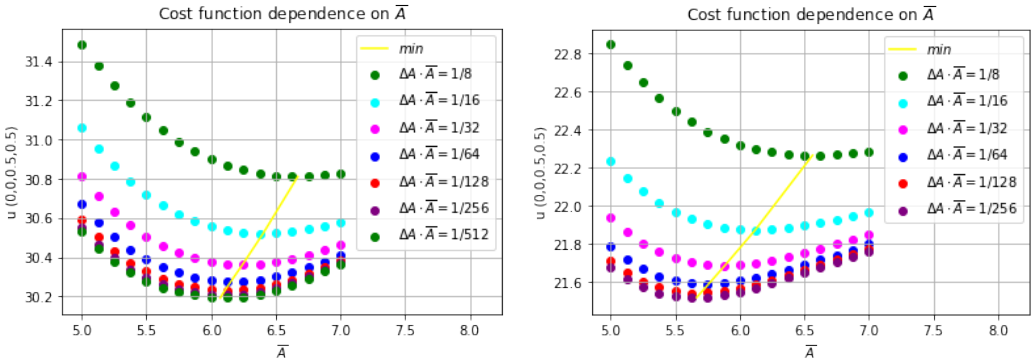


Figure 5.15: The approximations of $u_{\overline{A}}(t_0, A_0)$ for deterministic problem (left) and $u_{\overline{A}}(t_0, A_0, K_0, D_0)$ for stochastic problem (right), computed on several grids. The yellow line joins the minimum points of the polynomials.

The table 5.1 shows the optimum value of $\overline{A}^*$ for different grids and the difference between those values for successive grids. As we reduce $\Delta A \cdot \overline{A}$ by 2, $\Delta \overline{A}$ decreases by a value approaching 2, which suggests convergence of order one. Using this information we can estimate the true minimum value as $\overline{A}^* = 6.06$. The corresponding value function $u_{\overline{A}^*} = 30.18$.

The table 5.2 shows the optimum value of $\overline{A}^*$ for different grids and the difference between those values for two successive grids. As we reduce $\Delta A \cdot \overline{A}$ by 2, $\Delta \overline{A}$ decreases by a value approaching 2, which suggests convergence of order one. Using this information we can estimate the true minimum value as $\overline{A}^* = 5.63$. The corresponding value function $u_{\overline{A}^*} = 21.49$.

| $\Delta A_p \cdot \overline{A}$ | $\overline{A}_p^*$ | $\Delta\overline{A}_p^*$ | $\Delta\overline{A}_{p-1}^*/\Delta\overline{A}_p^*$ |
|---|---|---|---|
| 1/8, p=1 | 6.67 | - | - |
| 1/16, p=2 | 6.40 | 0.268 | - |
| 1/32, p=3 | 6.24 | 0.159 | 1.69 |
| 1/64, p=4 | 6.15 | 0.089 | 1.79 |
| 1/128, p=5 | 6.11 | 0.047 | 1.89 |
| 1/256, p=6 | 6.08 | 0.024 | 2.95 |
| 1/52, p=6 | 6.07 | 0.012 | 2.01 |

Table 5.1: Deterministic model. Estimation of the convergence rate of battery size optimization problem. As the step size decreases by 2, so does the change in the optimum battery size, as seen in the last column, which indicates first-order convergence.

| $\Delta A_p \cdot \overline{A}$ | $\overline{A}_p^*$ | $\Delta\overline{A}_p^*$ | $\Delta\overline{A}_{p-1}^*/\Delta\overline{A}_p^*$ |
|---|---|---|---|
| 1/8, p=1 | 6.56 | - | - |
| 1/16, p=2 | 6.12 | 0.445 | - |
| 1/32, p=3 | 5.88 | 0.236 | 1.89 |
| 1/64, p=4 | 5.76 | 0.123 | 1.91 |
| 1/128, p=5 | 5.70 | 0.063 | 1.96 |
| 1/256, p=6 | 5.67 | 0.031 | 2.00 |

Table 5.2: Stochastic model. Estimation of the convergence rate of battery size optimization problem. As the step size decreases by 2, so does the change in the optimum battery size, as seen in the last column, which indicates first-order convergence.

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