Propositional Calculus (Part 3)

CS 4710 Course Notes

January 23, 2019

Soundness of PC

Recall our statement of the soundness of PC:

If $S \vdash P$, then $S \vDash P$.

Proof by induction on the construction of derivations.

If $S \vdash P$, there exists a sequence (by the definition of $\vdash$ and deduction) $R_1, \ldots, R_n$ where $R_n = P$. We need to demonstrate that $S \vDash R_n$.

Base case (k=1). Prove R_1 is a deduction of R_1 from S. There are three possibilities for R_1's inclusion in such a deduction (by definition):

1. R_1 is a tautology. Note that $Q \vDash R_1$ for any statement Q, by the definition of entailment and the definition of tautology. Thus $S \vDash R_1$. $\blacksquare$
2. $R_1 \in S$. Here, $S \vDash R_1$ as any V satisfying S must satisfy all of its members. $\blacksquare$
3. R_1 is derivable from earlier elements of the series by MP.

Note that case (3) doesn't apply here in the base case (MP requires at least two statements), so it is trivially proven.

Step. Inductive hypothesis: For all $m \leq n$, if $R_1, \ldots, R_m$ is a deduction of R_m from S, then $S \vDash R_m$.

We have the same three cases to deal with as before. The first two cases are trivial to prove (they are practically identical to the base case). The third case, however, *does apply* and must be proven. This proof is left as an exercise on Homework 1 (to be assigned January 28).