CS 4710, Homework 1
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Throughout this assignment the ~ symbol may be used to indicate negation (¬). The values T and F are also used to represent true (⊤) and false (⊥), respectively. In the final question, longer expressions are sometimes given variable names (e.g. M = (P ↓ P) ↓ (Q ↓ Q)) to enhance legibility.

1. Prove that {¬, ∧} is an expressively complete set of logical operators for PC.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
P	Q	~P	~Q	P ^ Q	~P ^ Q	P ^ ~Q	~P ^ ~Q	P ^ ~ P	~(P ^ ~ P)	~(~P^Q)	~(P^~Q)	(P^Q) ^ (~P ^ ~Q)	(P^~Q) ^ (~P ^ Q)	~(~P^~Q)	~(P^Q)
T	T	F	F	T	F	F	F	F	T	T	T	T	F	T	F
T	F	F	T	F	F	T	F	F	T	T	F	F	T	T	T
F	T	T	F	F	T	F	F	F	T	F	T	F	T	T	T
F	F	T	T	F	F	F	T	F	T	T	T	T	F	F	T

We have 16 possible valuations for any statement, all of which can be constructed with {~,^}:

1. TTTT - through combination J
2. FFFF - through combination I
3. TTTF - through combination O
4. FTTT - through combination P
5. TTFT - through combination K
6. TFTT - through combination L
7. TFFT - through combination M
8. FTTF - through combination N
9. TTFF - through combination A
10. TFTF - through combination B
11. FTFT - through combination D
12. FFTT - through combination C
13. TFFF - through combination E
14. FTFF - through combination G
15. FFTF - through combination F
16. FFFT - through combination H

2. Prove that modus tollens is a valid inference rule in propositional calculus.

P	Q	~P	~Q	P→ Q	(P→Q) ^ ~Q	((P→Q) ^ ~Q) → ~P
T	T	F	F	T	F	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	F	**T**
F	F	T	T	T	T	**T**

Given ¬Q and P→ Q are true, we find (P→Q) ^ ¬Q → ¬P is also always true, meaning it is a tautology. This proves modus tollens - that P → Q, ¬Q ⊢ ¬P.

3. Prove soundness for PC (if S ⊢ P, then S ⊨ P) using mathematical induction.

Proof by induction on the construction of derivations:
Fact: If S ⊢ P, there exists a sequence (by the definition of ⊢ and deduction) $R_1, \ldots, R_n$ where R_n = P. We need to demonstrate that S ⊨ R_n.

Base case (k=1). Prove R_1 is a deduction of R_1 from S. There are three possibilities for R_1's inclusion in such a deduction (by definition):

1. R_1 is a tautology. Note that Q ⊨ R_1 for any statement Q, by the definition of entailment and the definition of tautology. Thus S ⊨ R_1.
2. R_1 ∈ S. Here, S ⊨ R_1 as any V satisfying S must satisfy all of its members.
3. R_1 is derivable from earlier elements of the series by MP. Note that R_1 is the only element in the series, therefore this case is trivially proven (MP requires two statements).

Step. Inductive hypothesis: For all m ≤ n, if $R_1, \ldots, R_m$ is a deduction of R_m from S, then S ⊨ R_m. We have the same three cases to deal with as before.

1. R_m is a tautology. Note that Q ⊨ R_m for any statement Q, by the definition of entailment and the definition of tautology. Thus S ⊨ R_m.
2. R_m ∈ S. Here, S ⊨ R_m as any V satisfying S must satisfy all of its members.
3. R_m is derivable from earlier elements of the series by MP.
   Given a number m ≤ n, R_m is in the sequence R_1, ... R_n. We have already derived R_1 meaning that S ⊢ R_1, and because R_m is in the sequence R_1, ... R_n, R_m is a deduction from R_1. We can show by modus ponens (MP), the definition of ⊢, and induction that S ⊢ R_m and R_m is thus derivable from earlier elements of the series.

Now given $S \vdash R_m \Rightarrow S \vDash R_m$ in our inductive hypothesis, we can show $S \vdash R_{m+1} \Rightarrow S \vDash R_{m+1}$.

Any number $m+1 \leq n$ is still within the sequence $R_1, \ldots R_n$ and is therefore a deduction from R_1, meaning it too will be derivable from earlier elements of the series via modus ponens such that $S \vdash R_{m+1}$. Because we have found $S \vdash R_{m+1}$ and $S \vdash R_m$ via modus ponens, by the definition of MP there must be two statements in the sequence, which we will R_i and R_j. By the definition of modus ponens R_i represents a preceding statement of the sequence P, and R_j represents that $P \rightarrow R_{m+1.}$ R_i and R_j are elements of the sequence that are derivable from previous statements ($S \vdash R_i$, $S \vdash R_j$), and so given our inductive hypothesis we know $S \vDash R_i$, $S \vDash R_j$. Thus the only relevant valuations of R_i and R_j are "true" when we come to construct our truth table. Wherever R_i and R_j are true, R_{m+1} will also be true via MP, so $S \vDash R_{m+1}$ and thus soundness in the inductive step ($S \vdash R_{m+1} \Rightarrow S \vDash R_{m+1}$) holds.

R_i (P)	R_j ($P \rightarrow R_{m+1}$)	R_{m+1}
T	T	T

4. Let A be a set of atomic statements, and |A| denote the number of elements in A (its cardinality). Let V_{all} denote the set of all possible valuations for A. Prove that $|V_{all}| = 2^{|A|}$.

Proof:
We can write A as a set of atomic statements $A = \{a_1, a_2, a_3, \ldots, a_n\}$ where the cardinality of A is n ($|A| = n$). Due to the principle of bivalence, the valuation of any one atomic statement is "one (and only one) member of $\{\top, \bot\}$." The cardinality of $\{\top, \bot\}$ is 2, meaning every atomic statement can take on two possible values. We can then use the fundamental counting principle, which says that to find the total outcomes of a scenario one must multiply the total number of outcomes for each individual event. Therefore, each additional statement added to A doubles the number of possible valuations in V_{all}.

Knowing V_{all} denotes the set of all possible valuations for A, $|V_{all}| = 2 * 2 * 2 \ldots * 2$ where there are n distinct elements, meaning 2 is multiplied by itself n times. Therefore, $|V_{all}| = 2^n = 2^{|A|}$.

5. Prove that the Pierce arrow is expressively complete.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
P	Q	P↓Q	(P↓Q) ↓Q	(P↓Q) ↓P	P↓P	Q↓Q	(P↓Q) ↓ (P↓Q)	(P↓P) ↓ P	I ↓ I	E ↓ E	D ↓ D	(P↓P)↓ (Q↓Q)	M↓M	M↓C	O↓O
T	T	F	F	F	F	F	T	F	T	T	T	T	F	F	T
T	F	F	T	F	F	T	T	F	T	T	F	F	T	T	F
F	T	F	F	T	T	F	T	F	T	F	T	F	T	T	F
F	F	T	F	F	T	T	F	F	T	T	T	F	T	F	T

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10. TFTF - through combination B
11. FTFT - through combination G
12. FFTT - through combination F
13. TFFF - through combination M
14. FTFF - through combination D
15. FFTF - through combination E
16. FFFT - through combination C