# Derivation of the Distribution of the Compound Poisson Process S(t)

We consider the compound process:

$$S(t) = X_1 + X_2 + \dots + X_{N(t)},$$

where: $N(t)$ is a Poisson process with rate $\lambda$, due to exponential interarrival times. $\{X_i\}$ are i.i.d. exponential random variables with rate $\mu$. $N(t)$ and $X_i$ are independent.

## 1. Conditional Distribution

Given **N(t) = n**, the sum

$$S(t) \mid N(t)=n = X_1 + \dots + X_n$$

follows a **Gamma(n, $\mu$)** distribution with density:

$$f(s \mid n) = \mu\blacksquare\ s\blacksquare\blacksquare^{1}\ e^{-\mu s} / (n-1)!\ ,\ s > 0.$$

The Poisson probability mass function is:

$$P(N(t)=n) = e^{-\lambda t} (\lambda t)\blacksquare / n!.$$

## 2. Unconditional Density of S(t)

Using the law of total probability:

$$f_{S(t)}(s) = \Sigma\ f(s|n)\ P(N(t)=n).$$

Evaluating this series yields the closed-form Bessel density:

$$\mathbf{f_{S(t)}(s) = \exp(-(\lambda t + \mu s)) \cdot (\lambda t / \sqrt{(\lambda t\ \mu\ s)}) \cdot I\blacksquare(\ 2\sqrt{(\lambda t\ \mu\ s)}\ ),}$$

where $I\blacksquare$ is the modified Bessel function of the first kind.

## 3. Atom at Zero

Since $S(t)=0$ only when the Poisson count is zero:

**$P(S(t)=0) = e^{-\lambda t}$.**

## 4. Laplace Transform (Moment Generating Tool)

The Laplace transform of S(t) is:

$$L(v) = E[e^{-vS(t)}] = \exp( -\lambda t \cdot v/(\mu + v) ).$$

This expression confirms that S(t) is a Poisson–Gamma mixture, also known as the Bessel distribution.