Derivation of the Distribution of the Compound Poisson Process ($S(t)$).

Consider the compound process:

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

where,

$N(t)$ is a poisson process with rate λ.

X_i are iid exponential random variables with rate μ.

$N(t)$ and (X_i) are independent.

Given $N(t) = n$, we have

$$S(t) \mid N(t) = n = X_1 + X_2 + \dots + X_n$$

Since each $X_i \sim Exp(\mu)$, their sum follows a Gamma distribution.

$$S(t) \mid N(t) = n \sim Gamma(n, \mu)$$

with density,

$$f_{S(t) \mid N(t)=n}(s) = \frac{\mu^n s^{n-1} e^{-\mu s}}{(n-1)!}, \quad s > 0$$

Also the poisson pmf is,

$$P(N(t) = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

Using the law of total probability;

$$f_{S(t)}(s) = \sum_{n=1}^{\infty} f_{S(t) \mid N(t)=n}(s) \, P(N(t) = n).$$

Substituting:

$$f_{S(t)}(s) = \sum_{n=1}^{\infty} \frac{\mu^n s^{n-1} e^{\mu s}}{(n-1)!} \cdot \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

Let $k = n-1$, Then the series becomes;

$$f_{S(t)}(s) = e^{-(\lambda t + \mu s)} \sum_{k=0}^{\infty} \frac{(\lambda t \mu)^k}{k!\,(k+1)!} s^k (\lambda t)$$

Using the known Bessel function identity;

$$\sum_{k=0}^{\infty} \frac{(xy)^k}{k!\,(k+1)!} = \frac{I_1(2\sqrt{xy})}{\sqrt{xy}}$$

We obtain the closed form;

$$f_{S(t)}(s) = e^{-(\lambda t + \mu s)} \frac{\lambda t}{\sqrt{\lambda t \mu s}} I_1\left(2\sqrt{\lambda t \mu s}\right), \; s > 0$$

The only way $S(t) = 0$ is if $N(t) = 0$, thus:

$$P(S(t) = 0) = e^{-\lambda t}$$

The laplace transform of $S(t)$ is;

$$L_{S(t)}(v) = E\left[e^{-v S(t)}\right]$$

$$= e^{-\lambda t \left(1 - \frac{\mu}{\mu + v}\right)}$$

$$= e^{-\lambda t \left(\frac{v}{\mu + v}\right)}$$