$$p(w \mid x, t) \propto \underbrace{p(t \mid X, w, \beta)}_{\text{likelihood}} \underbrace{p(w)}_{\text{prior}}$$

$$p(w) = \mathcal{N}(w \mid m_0, S_0)$$

$$p(t \mid X, w, \beta) = \prod_{n=1}^{N} \mathcal{N}(t_n \mid w^T \phi(x_n), \beta^{-1})$$

so

$$p(w \mid X, t) \propto \exp\left[-\frac{1}{2}(w - m_0)^T S_0^{-1} (w - m_0)\right] \cdot \exp\left[-\frac{\beta}{2}(t - \phi w)^T (t - \phi w)\right]$$

so

$$\log p(w \mid X, t) = -\frac{\beta}{2}(t - \phi w)^T (t - \phi w) - \frac{1}{2}(w - m_0)^T S_0^{-1} (w - m_0)$$

quadratic term:

$$-\frac{\beta}{2}(w^T \phi^T \phi w) - \frac{1}{2} w^T S_0^{-1} w$$

linear term:

$$-\frac{\beta}{2}(-2 t^T \phi w) - \frac{1}{2}(-2 m_0^T S_0^{-1} w)$$

because $\log p(w \mid X, t) = \mathcal{N}(w \mid m_N, S_N)$

$$= -\frac{1}{2}(w - m_N)^T S_N^{-1} (w - m_N)$$

quadratic term:

$$-\frac{1}{2} w^T S_N^{-1} w$$

linear term:

$$-\frac{1}{2}(-2 m_N^T S_N^{-1} w)$$

because their corresponding terms are the same

so

$$-\frac{1}{2}(\beta w^T \phi^T \phi w + w^T S_0^{-1} w)$$

$$= -\frac{1}{2} w^T (\beta \phi^T \phi + S_0^{-1}) w \equiv -\frac{1}{2} w^T S_N^{-1} w$$

thus: $S_N^{-1} = \beta \phi^T \phi + S_0^{-1}$

besides:

$$\beta t^T \phi w + m_0^T S_0^{-1} w = m_N^T S_N^{-1} w$$

$$\beta t^T \phi + m_0^T S_0^{-1} = m_N^T S_N^{-1}$$

$$m_N^T = (\beta t^T \phi + m_0^T S_0^{-1}) S_N$$

because S_0, S_N are symmetric

so

$$(S_0^{-1})^T = S_0^{-1}$$

$$m_N = S_N(\beta \phi^T t + S_0^{-1} m_0)$$

$$S_N^T = S_N$$