# Wild Automorphisms of the Complex Field of Finite and Infinite Orders

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**Summary**

A wild automorphism is an automorphism of the complex field that does not leave the real numbers fixed, unlike the two ‘classic’ automorphisms of the complex numbers: the identity automorphism and conjugation. In this paper we use Zorn’s Lemma to prove that any automorphism of a subfield of the complex field that has a finite order can be extended to an automorphism of the entire complex field of the same order. It then follows that there exists automorphisms of the complex field of order n for every finite n. Finally, we show that there exists an automorphism of the complex field of infinite order.

In Theorem 1.1, we are going to apply Zorn's Lemma to a set of functions. If we consider an automorphism, $\partial$, and the set of automorphisms which are extensions of $\partial$, we may call this set a family, $\mathcal{F}$.

Each of the automorphisms in this set can be thought of as subsets of ordered pairs. Since $\mathcal{F}$ is a family of automorphisms, the domain of any function in $\mathcal{F}$ is equal to its range. If $f \in \mathcal{F}$, then the set $f \subseteq X \times X$ is a function such that if $(a,b),(a,c) \in f$, then $b = c$. It follows that if $f, g \in \mathcal{F}$ and $f$ is a subset of $g$, then $\text{dom}(f) \subseteq \text{dom}(g)$. This is because if $a \in \text{dom}(f)$, then $f(a) = g(a)$. Thus, $\mathcal{F}$ may be partially ordered by inclusion.

Theorem 1.1 will also rely on Lemma 3.27 and 5.24 from [1]. These two lemmas allow us to extend automorphisms.

**Theorem 1.1** *Any order n automorphism of a subfield of $\mathbb{C}$ can be extended to an order n automorphism of $\mathbb{C}$, where $n \in \mathbb{N}$.*

Theorem 1.1 asserts that any order n automorphism of a subfield of $\mathbb{C}$ can be extended to an order n automorphism of $\mathbb{C}$, but in order to extend such an automorphism to $\mathbb{C}$, we must have that there exists at least one automorphism of order $n$ $\forall n \in \mathbb{N}$, so that $\mathcal{F}$ is non empty and we may apply Zorn's Lemma. We may show this by noting that if $\text{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ is cyclic, then $\mathbb{Q}(\alpha)$ is called a cyclic extension. It is certainly true that there exists cyclic extensions of $\mathbb{Q}$ that have elements of order $n$ $\forall n \in \mathbb{N}$, and indeed Hilbert proved that $S_n$ is always a Galois group. Thus, there exists automorphisms of the subfields of $\mathbb{C}$ that are of order $n$ $\forall n \in \mathbb{N}$, and Corollary 1.1 immediately follows from Theorem 1.1.

**Corollary 1.1** *There exists an automorphism of $\mathbb{C}$ of order $n$ for all $n \in \mathbb{N}$.*

If there exists automorphisms of $\mathbb{C}$ of order $n$ for all of $\mathbb{N}$, then it is natural to ask if there exists automorphisms of order $\mathbb{C}$ for a larger order. Theorem 1.2 uses the fact that if we pick two numbers that are transcendental over $\mathbb{Q}$, $\alpha, \beta$, then $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\beta)$ by an isomorphism $\phi$ where $\phi(\alpha) = \beta$.

**Theorem 1.2** *There exists an automorphism of $\mathbb{C}$ that is of infinite order.*

*Proof.* Consider $\phi : \mathbb{Q}(\pi) \to \mathbb{Q}(\pi + 1)$, where $\phi$ leaves $\mathbb{Q}$ fixed, and $\phi(\pi) = \pi + 1$. Then $\phi$ is a function, and $\phi$ is at least an isomorphism. The function $\phi$ is of infinite order, since $\phi^n(\pi) = \pi + n \neq \pi$, $\forall n \in \mathbb{N}$, $n \neq 0$. Notice that $\mathbb{Q}(\pi) = \mathbb{Q}(\pi + 1)$. Thus, $\phi$ is an automorphism.

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