Propiedades de la transformada de Fourier.

Sea $f(t)$ y $g(t)$ donde $\mathcal{F}\{f(t)\} = F(jw)$ y $\mathcal{F}\{g(t)\} = G(jw)$

Para α y β constantes

$$\mathcal{F}\{\alpha f(t) + \beta g(t)\} = \alpha F(jw) + \beta G(jw)$$

Derivación respecto al tiempo

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(jw) e^{jwt}\, dw$$

$$\frac{df(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[F(jw) e^{jwt}\right] dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} jw\, F(jw) e^{jwt}\, dw$$

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = (jw)F(jw)$$

Así $\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (jw)^n F(jw)$

Ejemplo: Dada la ecuación diferencial

$$y''(t) + 3y'(t) + 7y(t) = 3u'(t) + 2u(t)$$

obtenga la función $Y(jw)$

$$\mathcal{F}\{y''(t)\} + 3\mathcal{F}\{y'(t)\} + 7\mathcal{F}\{y(t)\} = 3\mathcal{F}\{u'(t)\} + 2\mathcal{F}\{u(t)\}$$

$$(jw)^2 Y(jw) + 3(jw)Y(jw) + 7Y(jw) = 3(jw)U(jw) + 2U(jw)$$

$$(-w^2 + 3jw + 7)Y(jw) = (3jw + 2)U(jw)$$

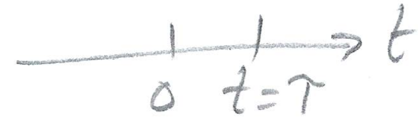
$$Y(jw) = \frac{2 + 3jw}{-w^2 + 3jw + 7} U(jw)$$

Corrimiento con respecto al tiempo

Si $f(t)$ y $\mathcal{F}\{f(t)\} = F(jw)$ y considerando $g(t) = f(t-\tau)$ entonces

$$\mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} g(t) e^{-jwt} dt = \int_{-\infty}^{\infty} f(t-\tau) e^{-jwt} dt$$

si $x = t - \tau$ $\quad t = x + \tau \quad \frac{dt}{dx} = 1$



$$\mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} f(x) e^{-jw(x+\tau)} dx = \int_{-\infty}^{\infty} f(x) e^{-jwx} e^{-jw\tau} dx$$

$$\mathcal{F}\{g(t)\} = e^{-jw\tau} \int_{-\infty}^{\infty} f(x) e^{-jwx} dx = e^{-jw\tau} F(jw)$$

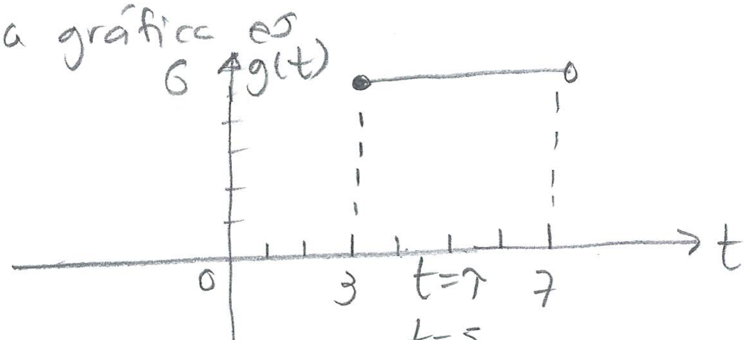
Así

$$\mathcal{F}\{f(t-\tau)\} = e^{-jw\tau} F(jw)$$

Ejemplo: obtenga la transformada de Fourier para el pulso dado por

$$g(t) = \begin{cases} 0 & t < 3 \quad t \geq 7 \\ 6 & 3 \leq t < 7 \end{cases}$$

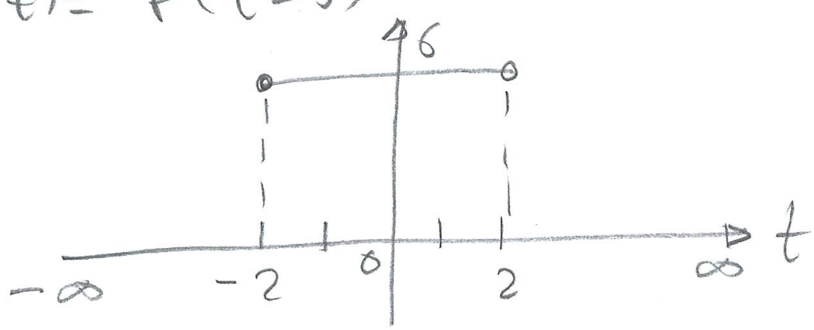
La gráfica es



Recorriendo $g(t)$ 5 unidades a la izquierda, obtenemos

$$g(t) = f(t-5)$$

Identificamos que $t = \tau = 5$



usando $\mathcal{F}\{f(t-\tau)\} = e^{-j\omega\tau} F(j\omega)$

Entonces $\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} 6\, e^{-j\omega t}\, dt = \cancel{\int_{-\infty}^{-2} 0\, e^{-j\omega t}\, dt} + \int_{-2}^{2} 6\, e^{-j\omega t}\, dt + \cancel{\int_{2}^{\infty} 0\, e^{-j\omega t}\, dt}$

cero $\qquad$ cero

$$\mathcal{F}\{f(t)\} = 6\left(-\frac{1}{j\omega} e^{-j\omega t}\right)\Bigg|_{-2}^{2}$$

$$\mathcal{F}\{f(t)\} = -\frac{6}{j\omega}\left[e^{-2j\omega} - e^{2j\omega}\right] = \frac{6}{j\omega}\left[e^{2j\omega} - e^{-2j\omega}\right]$$

$$\mathcal{F}\{f(t)\} = \frac{6\times 2}{\omega}\left[\frac{e^{2j\omega} - e^{-2j\omega}}{2j}\right] = \frac{12}{\omega}\operatorname{sen}(2\omega) = F(j\omega)$$

Así como $\tau = 5$

$$\mathcal{F}\{f(t-\tau)\} = e^{-5j\omega}\left[\frac{12}{\omega}\operatorname{sen}(2\omega)\right]$$

Por otra parte $\mathcal{F}^{-1}\{e^{-j\omega\tau} F(j\omega)\} = f(t-\tau)$

Ejemplo: obtenga la transformada de Fourier inversa

$$\mathcal{F}^{-1}\left\{\frac{e^{2j\omega}}{5+j\omega}\right\}$$

Se escribe

$$\mathcal{F}^{-1}\left\{\frac{e^{-(-2j\omega)}}{5+j\omega}\right\} = f(t-(-2))$$

$e^{2j\omega} = e^{-j\omega\tau}$ $\qquad$ $2j\omega = -j\omega\tau$

$\tau = -2$

$$f(t) = \mathcal{F}^{-1}\left\{\frac{1}{5+j\omega}\right\} = H(t)\, e^{-5t}$$

$$\mathcal{F}^{-1}\left\{\frac{e^{2j\omega}}{5+j\omega}\right\} = H(t+2)\, e^{-5(t+2)}$$

Corrimiento con respecto a la frecuencia

$$\mathcal{F}\{g(t)\} = \int_{\infty}^{\infty} e^{jw_0 t} f(t) e^{-jwt} dt = \int_{\infty}^{\infty} f(t) e^{-j(w-w_0)t} dt$$

$$\mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j\tilde{w}t} dt \quad , \quad \tilde{w} = w - w_0$$

$$\mathcal{F}\{g(t)\} = F(j\tilde{w}) \quad , \text{ entonces}$$

$$\mathcal{F}\{e^{jw_0 t} f(t)\} = F(j(w-w_0))$$

Ejemplo: obtenga la transformada de Fourier

$$\mathcal{F}\left\{ \frac{\operatorname{sen} 3t}{4+t^2} \right\}$$

Se usa $\operatorname{sen} 3t = \dfrac{e^{3ti} - e^{-3ti}}{2i} \left[\dfrac{-i}{-i}\right] = \dfrac{1}{2} i \left(e^{-3ti} - e^{3ti}\right)$

$$\mathcal{F}\left\{ \frac{1}{2} i \left(e^{-3ti} - e^{3ti}\right)\left(\frac{1}{2^2+t^2}\right)\right\}$$

$$\frac{1}{2} i \left[\mathcal{F}\left\{\frac{e^{-3ti}}{2^2+t^2}\right\} - \mathcal{F}\left\{\frac{e^{3ti}}{2^2+t^2}\right\}\right] \quad , \text{ Ayuda } \mathcal{F}\left\{\frac{1}{a^2+t^2}\right\} = \frac{\pi}{a} e^{-a|w|}$$

$$\frac{1}{2} i \left[\frac{\pi}{2} e^{-2|w+3|} - \frac{\pi}{2} e^{-2|w-3|}\right]$$

$$F(w) = \frac{\pi}{4} i \left[e^{-2|w+3|} - e^{-2|w-3|}\right]$$

Ejemplo:
Obtenga $\mathcal{F}^{-1}\left\{ \frac{10 \operatorname{sen} 3w}{w+\pi} \right\}$

Sumando cero

$$\operatorname{sen} 3(w+\pi-\pi) = \operatorname{sen}[3(w+\pi)-3\pi] = \operatorname{sen}[3(w+\pi)+(-3\pi)]$$

Luego $\operatorname{sen}(A+B) = \operatorname{sen} A \cos B + \cos A \operatorname{sen} B$

$$\operatorname{sen}[3(w+\pi)+(-3\pi)] = \operatorname{sen}(3(w+\pi))\cos(-3\pi) + \underbrace{\cancel{\cos(3(w+\pi))\operatorname{sen}(-3\pi)}}_{\text{cero}}$$

$$\operatorname{sen}[3(w+\pi)+(-3\pi)] = -\operatorname{sen}(3w+3\pi) = -\operatorname{sen}(3(w+\pi))$$

$$\mathcal{F}^{-1}\left\{ \frac{10 \operatorname{sen} 3w}{w+\pi} \right\} = -10\,\mathcal{F}^{-1}\left\{ \frac{\operatorname{sen} 3(w+\pi)}{w+\pi} \right\}; \quad \begin{matrix} w-(-\pi) \\ w-w_0 \end{matrix}$$

Ayuda:

$$\mathcal{F}^{-1}\left\{ \frac{2k}{w} \operatorname{sen} aw \right\} = k\,[u(t+a)-u(t-a)]$$

usando

$$\mathcal{F}^{-1}\{F(j(w-w_0))\} = e^{jw_0 t} f(t)$$

$$f(t) = -5e^{-\pi j t}[u(t+3)-u(t-3)]$$

Ejemplo:
Obtenga $\mathcal{F}^{-1}\left\{ w e^{-\frac{w^2}{16}} \right\}$

Se observa que

$$\frac{d}{dw} e^{-\frac{w^2}{16}} = -\frac{2w}{16} e^{-\frac{w^2}{16}} = -\frac{1}{8} w e^{-\frac{w^2}{16}}$$

Entonces

$$\mathcal{F}^{-1}\left\{ w e^{-\frac{w^2}{16}} \right\} = \mathcal{F}^{-1}\left\{ -8\left(-\frac{1}{8} w e^{-\frac{w^2}{16}}\right) \right\} = \mathcal{F}^{-1}\left\{ -8 \frac{d}{dw} e^{-\frac{w^2}{16}} \right\}$$

Entonces, ayuda:

$$t^n f(t) = \mathcal{F}^{-1}\{i^n F^{(n)}(\omega)\}$$

$$\mathcal{F}^{-1}\{w e^{-\frac{w^2}{16}}\} = \mathcal{F}^{-1}\{-8 \frac{d}{dw} e^{-\frac{w^2}{16}}\} = 8i\, \mathcal{F}^{-1}\{i \frac{d}{dw} e^{-\frac{w^2}{16}}\}$$

Así $F(\omega) = e^{-\frac{w^2}{16}}$ se requiere $f_1(t)$

Ayuda:

$$\mathcal{F}^{-1}\{\sqrt{\frac{\pi}{a}}\, e^{-\frac{w^2}{4a}}\} = e^{-at^2}, \quad a>0$$

Si $16 = 4a \rightarrow a = 4$

$$\frac{\sqrt{4}}{\sqrt{\pi}} \mathcal{F}^{-1}\{\frac{\sqrt{\pi}}{\sqrt{4}} e^{-\frac{w^2}{4a}}\} = \frac{2}{\sqrt{\pi}} e^{-4t^2} = f_1(t)$$

$n=1$

Así

$$\mathcal{F}^{-1}\{w e^{-\frac{w^2}{16}}\} = 8i[t f_1(t)] = 8it\left[\frac{2}{\sqrt{\pi}} e^{-4t^2}\right]$$

$$f(t) = \frac{16it}{\sqrt{\pi}} e^{-4t^2}$$

Tarea: 1) Obtener $\mathcal{F}\{8 e^{-2t^2} \operatorname{sen}(3t)\}$

Tarea: 2) Dada $f(t) = \frac{1}{a^2+t^2}$, $a>0$, el teorema de Parseval es: $\mathcal{F}\{f(t)\} = F(\omega)$ y $\int_{-\infty}^{\infty} [f(t)]^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$

Demostrar que: $\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{\pi}{2a^3}$