Integral de Fourier

$f(x) \in (-\infty, \infty)$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}[A(\alpha)\cos\alpha x + B(\alpha)\,\text{sen}\,\alpha x]\,d\alpha$$

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\,\text{sen}\,\alpha x\,dx$$

Ejemplo: obtenga la integral de Fourier de

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 2 \\ 0 & x > 2 \end{cases}$$

$$A(\alpha) = \underbrace{\cancel{\int_{-\infty}^{0} 0\cos\alpha x\,dx}}_{\text{cero}} + \int_0^2 \cos\alpha x\,dx + \underbrace{\cancel{\int_2^{\infty} 0\cos\alpha x\,dx}}_{\text{cero}}$$

$$A(\alpha) = \frac{1}{\alpha}\,\text{sen}(\alpha x)\Big|_0^2 = \frac{1}{\alpha}(\text{sen}(2\alpha) - \underbrace{\cancel{\text{sen}(0)}}_{\text{cero}}) = \frac{1}{\alpha}\,\text{sen}(2\alpha)$$

$$B(\alpha) = \underbrace{\cancel{\int_{-\infty}^{0} 0\,\text{sen}\,\alpha x\,dx}}_{\text{cero}} + \int_0^2 \text{sen}\,\alpha x\,dx + \underbrace{\cancel{\int_2^{\infty} 0\,\text{sen}\,\alpha x\,dx}}_{\text{cero}}$$

$$B(\alpha) = -\frac{1}{\alpha}\cos(\alpha x)\Big|_0^2 = -\frac{1}{\alpha}\left[\cos(2\alpha) - \cos(0)\right] = -\frac{1}{\alpha}[\cos(2\alpha)+1]$$

Entonces

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\frac{1}{\alpha}\,\text{sen}(2\alpha)\cos\alpha x + \left(-\frac{1}{\alpha}\{\cos(2\alpha)+1\}\right)\text{sen}\,\alpha x\right]d\alpha$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\frac{1}{\alpha}\operatorname{sen}(2\alpha)\cos\alpha x - \frac{1}{\alpha}(\cos(2\alpha)+1)\operatorname{sen}\alpha x\right]d\alpha$$

Simplificando

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\operatorname{sen}\alpha\cos\alpha(x-1)}{\alpha}\,d\alpha$$

Si $X=1$

$$f(1) = \frac{2}{\pi}\int_0^{\infty}\frac{\operatorname{sen}\alpha}{\alpha}\,d\alpha = \frac{2}{\pi}\lim_{b\to\infty}\int_0^{b}\frac{\operatorname{sen}\alpha}{\alpha}\,d\alpha$$

Método numérico de los trapecios

$$\int_0^{b}\frac{\operatorname{sen}\alpha}{\alpha}\,d\alpha \approx \frac{b-0}{2n}\left[f(\alpha_0)+2f(\alpha_1)+2f(\alpha_2)+\dots+2f(\alpha_{n-1}) + f(\alpha_n)\right]$$

$\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \rightarrow \alpha$ $\qquad n=4$

$\alpha_0 = 0.1 \quad \frac{\pi}{4} \quad \frac{\pi}{2} \quad \frac{3\pi}{4} \quad \pi$

$$\frac{\operatorname{sen}(\alpha_0)}{\alpha_0} = 0.9983\ ; \quad \frac{\operatorname{sen}(\alpha_1)}{\alpha_1} = 0.9\ ; \quad \frac{\operatorname{sen}(\alpha_2)}{\alpha_2} = 0.63$$

$$\frac{\operatorname{sen}(\alpha_3)}{\alpha_3} = 0.3\ ; \quad \frac{\operatorname{sen}(\alpha_4)}{\alpha_4} = 0$$

$$I = \frac{b}{8}\left[0.99 + 2(0.9) + 2(0.63) + 2(0.3) + 0\right] \qquad \text{Si } b=2$$

$$I = \frac{b}{8}\left[0.99 + 1.8 + 1.26 + 0.6 + 0\right] = \frac{4.65\,b}{8} \approx \frac{93}{160}b =$$

$$f(x) = \frac{2}{\pi}\left[\frac{93}{80}\right] = 1.79$$

Ejercicio: Obtenga la integral de Fourier de $f(x) = \begin{cases} 0 & x<0 \\ x & 0<x<3 \\ 0 & x>3 \end{cases}$

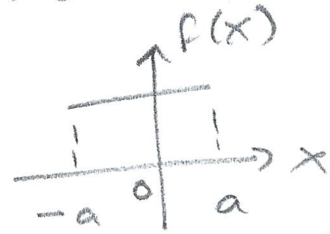
Integral de Fourier de una función par en $(-\infty, \infty)$ es la integral coseno

$$f(x) = \frac{2}{\pi}\int_0^{\infty} A(\alpha)\cos\alpha x \, d\alpha$$

donde $A(\alpha) = \int_0^{\infty} f(x)\cos\alpha x \, dx$

Ejemplo: obtenga la integral de Fourier de

$$f(x) = 1 \qquad |x| \leq a$$
$$f(x) = 0 \qquad |x| > a$$



Entonces

$$A(\alpha) = \int_0^a \cos\alpha x \, dx = \frac{1}{\alpha}\operatorname{sen}(\alpha x)\Big|_0^a$$

$$A(\alpha) = \frac{1}{\alpha}\left[\operatorname{sen}(\alpha a) - \operatorname{sen}(0)\right] = \frac{1}{\alpha}\operatorname{sen}(\alpha a)$$

Así

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\operatorname{sen}(\alpha a)\cos(\alpha x)}{\alpha} \, d\alpha$$

Integral de Fourier de una función impar en $(-\infty, \infty)$ es la integral seno

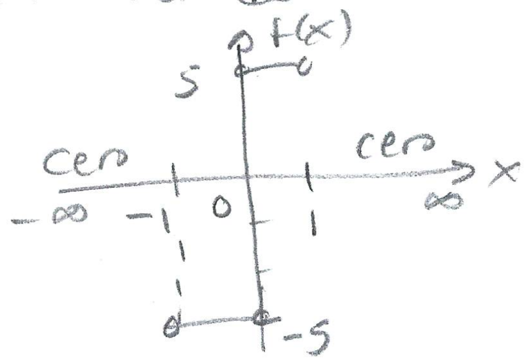
$$f(x) = \frac{2}{\pi}\int_0^{\infty} B(\alpha)\operatorname{sen}\alpha x \, d\alpha$$

donde

$$B(\alpha) = \int_0^{\infty} f(x)\operatorname{sen}\alpha x \, dx$$

Ejemplo: obtenga la integral de Fourier de

$$f(x) = \begin{cases} 0 & x < -1 \\ -5 & -1 < x < 0 \\ 5 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$



Se pide:

$$f(x) = \frac{2}{\pi}\int_0^\infty B(\alpha)\,\text{sen}\,\alpha x\,d\alpha$$

$$B(\alpha) = \int_0^\infty f(x)\,\text{sen}\,\alpha x\,dx = \int_0^1 5\,\text{sen}\,\alpha x\,dx$$

$$B(\alpha) = -\frac{5}{\alpha}\cos(\alpha x)\Big|_0^1 = -\frac{5}{\alpha}\left[\cos\alpha - \cos 0\right]$$

$$B(\alpha) = -\frac{5}{\alpha}\left[\cos\alpha - 1\right]$$

$$f(x) = \frac{2}{\pi}\int_0^\infty -\frac{5}{\alpha}(\cos\alpha - 1)\,\text{sen}\,\alpha x\,d\alpha$$

$$f(x) = \frac{10}{\pi}\int_0^\infty \frac{(1-\cos\alpha)\,\text{sen}\,\alpha x}{\alpha}\,d\alpha$$

Ejercicios: Utilizar la integral seno o coseno de Fourier para representar $f(x)$

a) $f(x) = \begin{cases} |x|, & |x| < \pi \\ 0 & |x| > \pi \end{cases}$

b) $f(x) = \begin{cases} x, & |x| < \pi \\ 0 & |x| > \pi \end{cases}$

Formulación compleja de la integral de Fourier

Si $f(x)$, $x \in (-\infty, \infty)$ y

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\operatorname{sen}\alpha x]\, d\alpha$$

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx \quad y \quad B(\alpha) = \int_{-\infty}^{\infty} f(x)\operatorname{sen}\alpha x\, dx$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\left\{\int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx\right\}\cos\alpha x + \left\{\int_{-\infty}^{\infty} f(x)\operatorname{sen}\alpha x\, dx\right\}\operatorname{sen}\alpha x\right] d\alpha$$

Llamando $x = t$ para la integral impropia $\frac{dx}{dt} = 1$ $dx = dt$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\left\{\int_{-\infty}^{\infty} f(t)\cos\alpha t\, dt\right\}\cos\alpha x + \left\{\int_{-\infty}^{\infty} f(t)\operatorname{sen}\alpha t\, dt\right\}\operatorname{sen}\alpha x\right] d\alpha$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(t)[\cos\alpha t\cos\alpha x + \operatorname{sen}\alpha t\operatorname{sen}\alpha x]\, dt\right] d\alpha$$

Ayuda $\cos(u - v) = \cos u\cos v + \operatorname{sen} u\operatorname{sen} v$

Así $u = \alpha t$ $v = \alpha x$

$$\cos\alpha t\cos\alpha x + \operatorname{sen}\alpha t\operatorname{sen}\alpha x = \cos(\alpha t - \alpha x) = \cos(\alpha(t - x))$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos(\alpha(t-x))\, dt\, d\alpha$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos(\alpha(t-x))\, dt\, d\alpha,$$ el integrando función par de α

Sumando cero al integrando

$$i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\operatorname{sen}(\alpha(t-x)\, dt\, d\alpha = 0$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\left[\cos(\alpha(t-x))+i\,\mathrm{sen}\,(\alpha(t-x))\right] dt\, d\alpha$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\alpha(t-x)}\, dt\, d\alpha$$

Se puede escribir

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\alpha t}\, e^{-i\alpha x}\, dt\, d\alpha$$

Agrupando respecto a las variables de integración

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\, e^{i\alpha t}\, dt\right] e^{-i\alpha x}\, d\alpha$$

La integral impropia del integrando depende de α

$$C(\alpha)=\int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

Así

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha)\, e^{-i\alpha x}\, d\alpha$$

Ejercicio: Resolver la ecuación integral dada para la función $f(x)$

a) $\displaystyle\int_{0}^{\infty} f(x)\cos \alpha x\, dx = e^{-\alpha}$

b) Verificar que $f(x)=\frac{2}{\pi}\left[\frac{1}{1+x^2}\right]$

con $x>0$ es solución de la ecuación integral