Transformada de Fourier

Considerando la Transformada de Laplace

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$t \geq 0$ $\qquad s = \sigma + j\omega$

$|f(t)| \leq Me^{ct}$, $M>0$, $c>0$ $T>0$, $t>T$

si $f(t)$ continua por partes $t \in (0,\infty)$ y de orden exponencial, entonces $F(s) = \mathcal{L}\{f(t)\}$, así

$$\lim_{s \to \infty} F(s) = 0$$

La transformada de Fourier

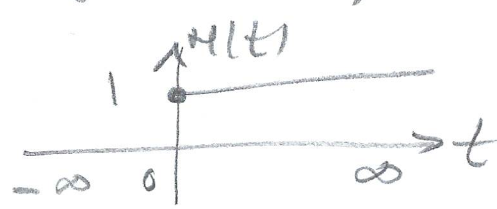
$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = F(j\omega)$$

$$\mathcal{F}^{-1}\{F(j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

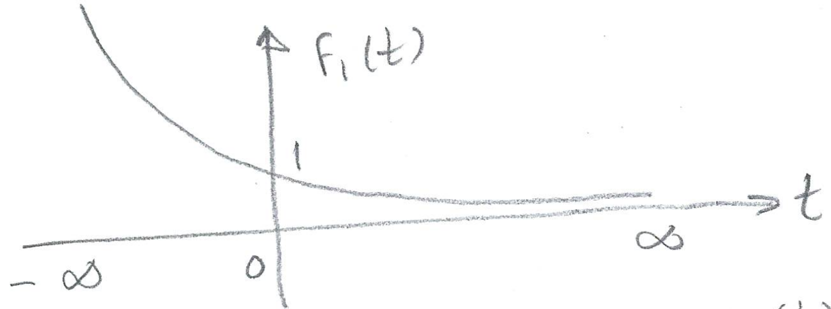
Ejemplo: Obtenga la transformada de Fourier de la función $f(t) = H(t)e^{-at}$ $a>0$ donde $H(t)$ es la función escalón unitario de Heaviside

se pide $\mathcal{F}\{f(t)\} = \mathcal{F}\{H(t)e^{-at}\}$

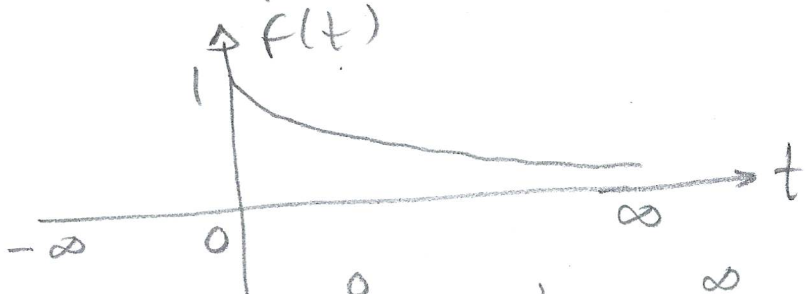
$$H(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$$



Considerando $f_1(t)=e^{-at}$ $\quad a>0$



Haciendo el producto $f(t)=H(t)\,e^{-at}$, la gráfica es



$$\mathcal{F}\{f(t)\}=\int_{-\infty}^{0} 0\, e^{-j\omega t}\,dt+\int_{0}^{\infty} H(t)\,e^{-at}\,e^{-j\omega t}\,dt$$

$$\mathcal{F}\{f(t)\}=\lim_{b\to\infty}\int_{0}^{b} e^{-(a+j\omega)t}\,dt=\lim_{b\to\infty} -\frac{1}{a+j\omega}\,e^{-(a+j\omega)t}\Big|_{0}^{b}$$

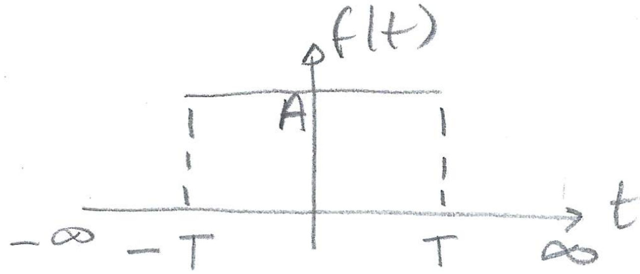
$$\mathcal{F}\{f(t)\}=\lim_{b\to\infty}\left[-\frac{1}{a+j\omega}\,e^{-(a+j\omega)b}+\frac{1}{a+j\omega}\right]$$

$$\mathcal{F}\{f(t)\}=\frac{1}{a+j\omega}$$, esto es

$$\mathcal{F}\{H(t)\,e^{-at}\}=\frac{1}{a+j\omega}\quad a>0$$

Ejemplo: Obtenga la transformada de Fourier del pulso rectangular

$$f(t) = \begin{cases} A & |t| \leq T \\ 0 & |t| > T \end{cases}$$



$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$\mathcal{F}\{f(t)\} = \underbrace{\cancel{\int_{-\infty}^{0} 0\, e^{-j\omega t}\, dt}}_{\text{cero}} + \int_{-T}^{T} A e^{-j\omega t}\, dt + \underbrace{\cancel{\int_{T}^{\infty} 0\, e^{-j\omega t}\, dt}}_{\text{cero}}$$

$$\mathcal{F}\{f(t)\} = -\frac{A}{j\omega} e^{-j\omega t}\Big|_{-T}^{T} = -\frac{A}{j\omega}\left[e^{-j\omega T} - e^{j\omega T}\right]$$

$$e^{-j\omega T} = \cos(-\omega T) + j\,\text{sen}(-\omega T) = \cos(\omega T) - j\,\text{sen}(\omega T)$$
$$e^{j\omega T} = \cos(\omega T) + j\,\text{sen}(\omega T)$$

$$\mathcal{F}\{f(t)\} = -\frac{A}{j\omega}\left[\cos(\omega T) - j\,\text{sen}(\omega T) - \cos(\omega T) - j\,\text{sen}(\omega T)\right]$$

$$\mathcal{F}\{f(t)\} = -\frac{A}{j\omega}\left[-2j\,\text{sen}(\omega T)\right] = \frac{2A}{\omega}\,\text{sen}(\omega T)$$

Ejemplo: Determine los espectros de amplitud y de fase de $f(t) = e^{-at} H(t) \quad a>0$

Dado que el espectro de amplitud y de fase dependen de ω se aplica

$$\mathcal{F}\{f(t)\} = \mathcal{F}\{e^{-at} H(t)\} = \frac{1}{a+j\omega}, \quad a>0$$

Entonces $F(jw) = \frac{1}{a+jw}\left[\frac{a-jw}{a-jw}\right] = \frac{a-jw}{a^2 - ajw + ajw + w^2}$

$$F(jw) = \frac{a-jw}{a^2+w^2} = \frac{a}{a^2+w^2} - \frac{w}{a^2+w^2}j$$

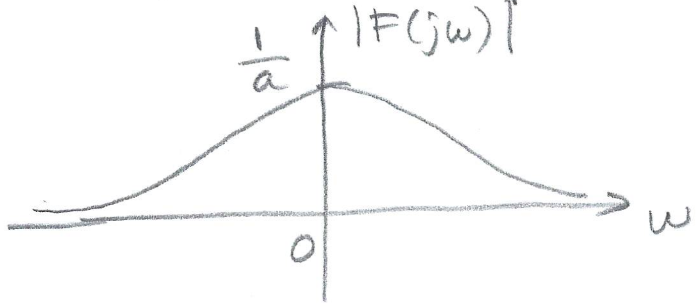
$$|F(jw)| = \sqrt{\left(\frac{a}{a^2+w^2}\right)^2 + \left(-\frac{w}{a^2+w^2}\right)^2} = \sqrt{\frac{a^2}{(a^2+w^2)^2} + \frac{w^2}{(a^2+w^2)^2}}$$

$$|F(jw)| = \sqrt{\frac{a^2+w^2}{(a^2+w^2)^2}} = \sqrt{\frac{1}{a^2+w^2}} = \frac{1}{\sqrt{a^2+w^2}}$$

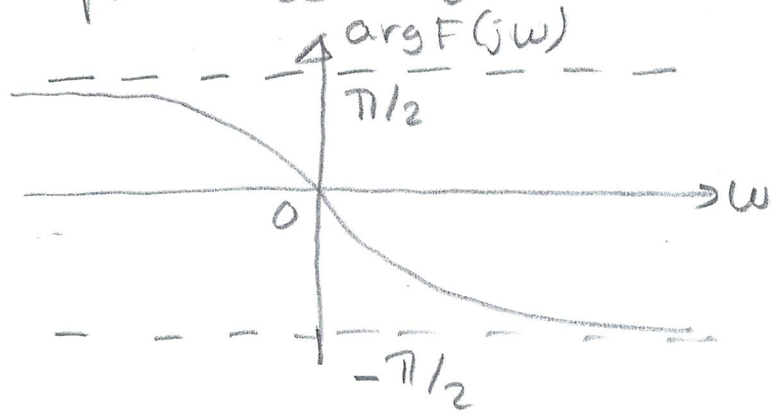
$$\tan\theta = \frac{\frac{-w}{a^2+w^2}}{\frac{a}{a^2+w^2}} = -\frac{w(a^2+w^2)}{a(a^2+w^2)} = -\frac{w}{a}$$

$$\theta = \arg F(jw) = \arctan\left(-\frac{w}{a}\right) = -\arctan\left(\frac{w}{a}\right), \quad a>0$$

Espectro de amplitud



Espectro de fase



Si $|F(jw)| = \frac{1}{\sqrt{a^2+w^2}}$

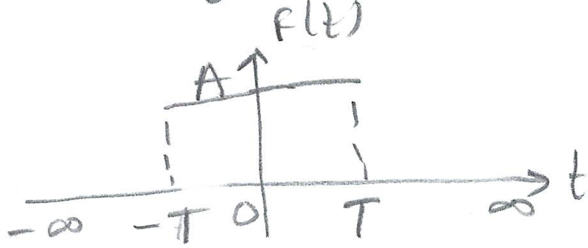
$w = 0 \Rightarrow |F(jw)| = \frac{1}{a}$

$w \to \pm\infty \Rightarrow |F(jw)| \to 0$

$-\frac{\pi}{2} < \arg F(jw) < \frac{\pi}{2}$

Ejemplo: Obtenga los espectros de amplitud y de fase del pulso rectangular dado por

$$f(t) = \begin{cases} A & |t| \leq T \\ 0 & |t| > T \end{cases}$$



El espectro de amplitud y de fase dependen de ω, entonces

$\mathcal{F}\{f(t)\} = \frac{2A}{\omega} \operatorname{sen}(\omega T)$, multiplicando por $1 = \frac{T}{T}$

$$F(j\omega) = \frac{2AT}{\omega T} \operatorname{sen}(\omega T) = 2AT \operatorname{senc}(\omega T)$$

$$\omega = \frac{2\pi}{\text{periodo}}$$

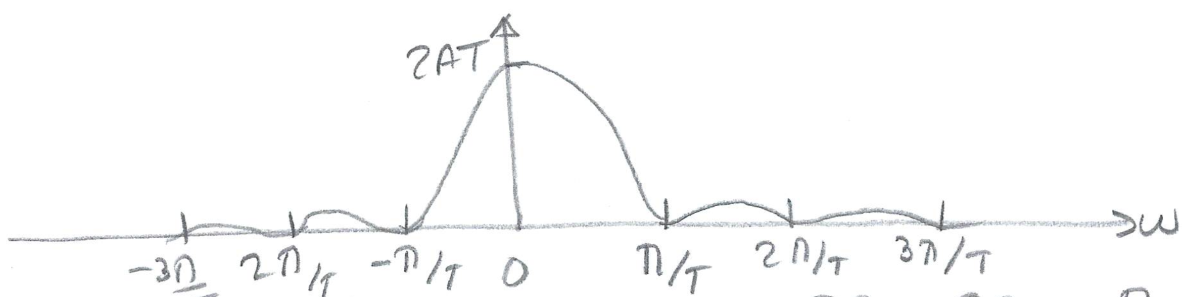
$$\omega = \frac{2\pi}{2T} = \frac{\pi}{T}$$

donde

$$\operatorname{senc}(\omega T) = \begin{cases} \frac{\operatorname{sen} \omega T}{\omega T} & \omega \neq 0 \\ 1 & \omega = 0 \end{cases}$$

Entonces, el espectro de amplitud

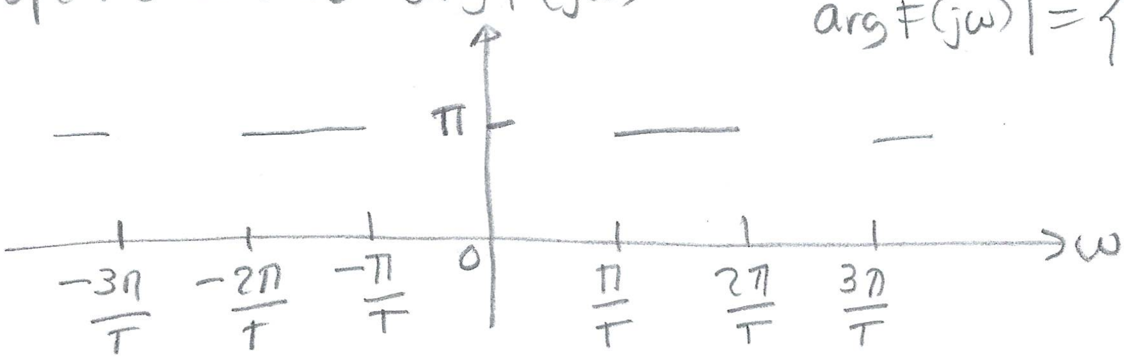
$$|F(j\omega)| = 2AT\,|\operatorname{senc}(\omega T)|$$



nota: periodo del pulso $2T$ $\quad \omega = \frac{2\pi}{T} = \frac{2\pi}{2T} = \frac{\pi}{T}$

Espectro de fase $\arg F(j\omega)$

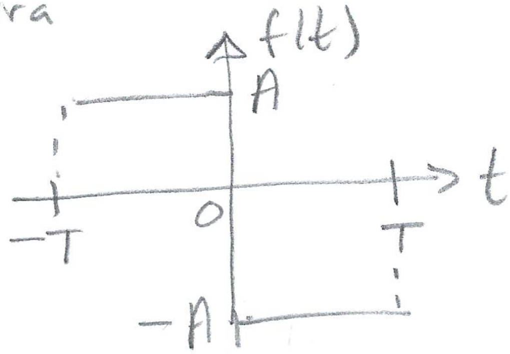
$$\arg F(j\omega)| = \begin{cases} 0 & \operatorname{senc}(\omega T) \geq 0 \\ \pi & \operatorname{senc}(\omega T) < 0 \end{cases}$$



Ejercicio: Obtenga la transformada de Fourier del pulso exponencial bilateral dado por

$$f(t) = \begin{cases} e^{at} & t \leq 0 \\ e^{-at} & t > 0 \end{cases} \qquad a > 0$$

Ejercicio: Obtenga la transformada de Fourier del pulso "encendido-apagado" mostrado en la figura



Ejercicio: Un pulso triangular está definido por

$$f(t) = \begin{cases} \frac{A}{T}t + A & -T \leq t \leq 0 \\ -\frac{A}{T}t + A & 0 < t \leq T \end{cases}$$

Obtenga la transformada de Fourier de f(t)