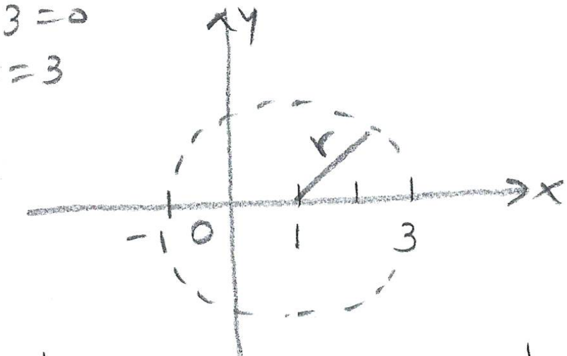
Desarrolle $f(z)=\frac{1}{z-3}$ en una serie de Laurent en potencias de $(z-1)$. Determine el dominio a que la serie converge a $f(z)$

$$f(z)=\sum_{n=-\infty}^{\infty} C_n (z-z_0)^n \qquad C_n=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Si $n=0$ $\quad z_0=1$ $\qquad z-3=0$
$\qquad z=3$



$$C_0=\frac{1}{2\pi i}\oint_C \frac{1}{(z-1)}\left[\frac{1}{z-3}\right]dz \qquad f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$$

$$C_0=\frac{1}{2\pi i}\left[2\pi i\, f(z_0)\Big|_{z_0=1}\right]=\frac{1}{1-3}=-\frac{1}{2}$$

Si $n=1$ $\quad z_0=1$ $\qquad f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$

$$C_1=\frac{1}{2\pi i}\oint \frac{1}{(z-1)^2}\left[\frac{1}{z-3}\right]dz \qquad \begin{matrix} n+1=2 \\ n=1\end{matrix}$$

$$C_1=\frac{1}{2\pi i}\left[\frac{2\pi i}{n!} f'(z_0)\Big|_{z_0=1}\right] \qquad f(z)=\frac{1}{z-3}=(z-3)^{-1}$$

$$f'(z)=-(z-3)^{-2}$$

$$C_1=\frac{1}{2\pi i}\left[\frac{2\pi i}{1!}\left(-\frac{1}{(1-3)^2}\right)\right]=-\frac{1}{4}$$

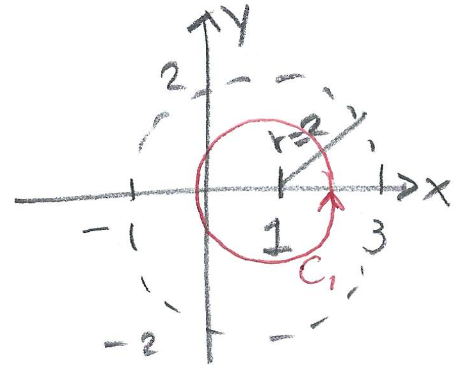
Si $n=2$ $\quad z_0=1$

$$C_2=\frac{1}{2\pi i}\oint_c \frac{1}{(z-1)^3}\left[\frac{1}{z-3}\right]dz \qquad \begin{matrix} n+1=3 \\ n=2\end{matrix} \qquad f''(z)=2(z-3)^{-3}$$

$$f''(1)=\frac{2}{(-2)^3}=-\frac{2}{8}=-\frac{1}{4}$$

$$C_2=\frac{1}{2\pi i}\left[\frac{2\pi i}{2!} f''(z_0)\Big|_{z_0=1}\right]=-\frac{1}{8}$$

Si $n=-1$



$$c_{-1} = \frac{1}{2\pi i}\oint_c \frac{1}{(z-1)^{-1+1}}\left[\frac{1}{z-3}\right]dz$$

$$c_{-1} = \frac{1}{2\pi i}\oint_c \frac{dz}{z-3} = \frac{1}{2\pi i}\oint_c \frac{dz}{z-1-3}$$

$C_1: |z-1| = 1$

$z = x+iy \qquad z(t) = 1+\cos t + i\operatorname{sen} t = 1+e^{it} \qquad 0 \le t \le 2\pi$

$$c_{-1} = \frac{1}{2\pi i}\oint_0^{2\pi} \frac{ie^{it}\,dt}{e^{it}-3} \qquad \begin{matrix} z = 1+e^{it} \\ dz = ie^{it}\,dt \end{matrix} \qquad z-1 = e^{it}$$

$u = e^{it}-3 \qquad$ si $t=0 \quad u=-2$

$\frac{du}{dt} = ie^{it} \qquad t = 2\pi \quad u = e^{2\pi i} - 3 = \cos 2\pi + i\,\overset{\text{cero}}{\cancel{\operatorname{sen} 2\pi}} - 3 = -2$

$$c_{-1} = \frac{1}{2\pi i}\oint_{-2}^{-2}\frac{du}{u} = \frac{1}{2\pi i}\ln|u|\Big|_{-2}^{-2} = \frac{1}{2\pi i}\left[\ln|-2| - \ln|-2|\right] = 0$$

Si $n=-2$

$$c_{-2} = \frac{1}{2\pi i}\oint_c \frac{1}{(z-1)^{-2+1}}\left[\frac{1}{z-3}\right]dz = \frac{1}{2\pi i}\oint_c \frac{z-1}{z-3}\,dz$$

$$c_{-2} = \frac{1}{2\pi i}\left[\oint_c 1 + \frac{2}{z-3}\,dz\right] = \frac{1}{2\pi i}\left[\int_0^{2\pi} ie^{it}\,dt\right] = 0$$

Así

$$\frac{1}{z-3} = \cdots + c_{-2}(z-1)^{-2} + c_{-1}(z-1)^{-1} + c_0 + c_1(z-1) + c_2(z-1)^2 + \cdots$$

$$\frac{1}{z-3} = -\frac{1}{2} - \frac{1}{4}(z-1) - \frac{1}{8}(z-1)^2 - \cdots \qquad 0 < |z-1| < 2$$

Otra opción:

$$f(z) = \frac{1}{z-3} = \frac{1}{z-1-2}$$

i)

$$f(z) = \frac{\frac{1}{z-1}}{\frac{z-1}{z-1} - \frac{2}{z-1}} = \frac{1}{z-1}\left[\frac{1}{1-\frac{2}{z-1}}\right] \qquad \left|\frac{2}{z-1}\right| < 1$$

$$|2| < |z-1|$$

ii)

$$f(z) = \frac{\frac{1}{-2}}{\frac{z-1}{-2} - \frac{2}{-2}} = -\frac{1}{2}\left[\frac{1}{1-\frac{z-1}{2}}\right] \qquad \left|\frac{z-1}{2}\right| < 1$$

$$|z-1| < 2$$

Ayuda: $\frac{1}{1-u} = 1 + u + u^2 + \cdots \quad |u| < 1$

$$f(z) = -\frac{1}{2}\left[1 + \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \cdots\right]$$

$$f(z) = -\frac{1}{2} - \frac{1}{4}(z-1) - \frac{1}{8}(z-1)^2 - \cdots \qquad 0 < |z-1| < 2$$