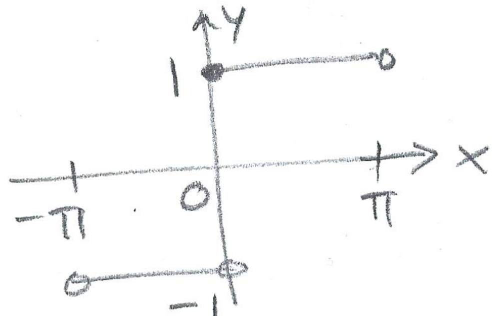
Obtenga la serie de Fourier de la función

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 \leq x < \pi \end{cases}$$

La gráfica de $f(x)$ con $x \in (-\pi, \pi)$ es



Se observa que $f(x)$ es una función impar, por lo que $f(-x) = -f(x)$, es decir, la gráfica de una función impar posee simetría con respecto al origen.

La serie de Fourier es

$$f(x) = \sum_{n=1}^{\infty} b_n \operatorname{sen}\left(\frac{n\pi}{p}x\right) \qquad \begin{array}{l} x \in (-p, p) \\ x \in (-\pi, \pi) \end{array}$$

El coeficiente de Fourier b_n es: $\quad \therefore p = \pi$

$$b_n = \frac{2}{p}\int_0^p f(x) \operatorname{sen}\left(\frac{n\pi}{p}x\right) dx \text{, entonces}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \operatorname{sen} nx \, dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]\Big|_0^{\pi}$$

$$b_n = -\frac{2}{n\pi}\left[\cos(n\pi) - \cos 0\right]; \quad \cos(n\pi) = (-1)^n$$

$$b_n = -\frac{2}{n\pi}\left((-1)^n - 1\right), \text{ así la serie es:}$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \operatorname{sen}(nx)$$

Por otra parte, consideremos un conjunto de puntos de prueba para $n = 1, 2, 3, 4, \dots$

$$f(x) = \frac{2}{\pi}\left[\frac{(-1)^{1+1}+1}{1}\operatorname{sen} x + \frac{(-1)^{2+1}+1}{2}\operatorname{sen}(2x) + \frac{(-1)^{3+1}+1}{3}\operatorname{sen}(3x) + \frac{(-1)^{4+1}+1}{4}\operatorname{sen}(4x) + \dots\right]$$

$$f(x) = \frac{2}{\pi}\left[2\operatorname{sen} x + \frac{2}{3}\operatorname{sen}(3x) + \dots\right]$$

Ahora, tomemos un conjunto de puntos de $x \in (-\pi, \pi)$

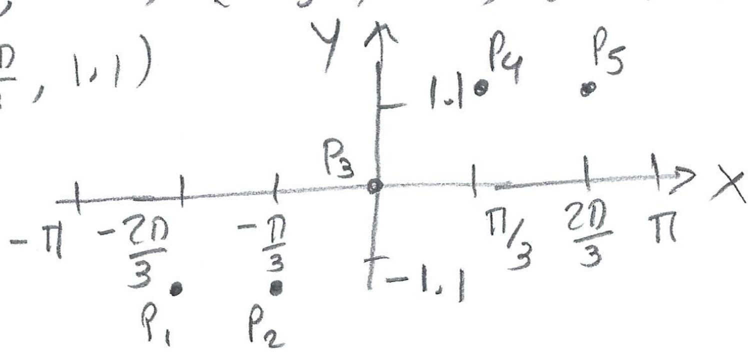
$x_1 = -\frac{2}{3}\pi$, $x_2 = -\frac{1}{3}\pi$, $x_3 = 0$, $x_4 = \frac{1}{3}\pi$, $x_5 = \frac{2}{3}\pi$

Tenemos la tabla:

x	$\operatorname{sen} x$	$\operatorname{sen} 3x$	$f(x) = \frac{4}{\pi}\operatorname{sen} x + \frac{4}{3\pi}\operatorname{sen}(3x)$
$-\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{2}$	0	$f\left(-\frac{2\pi}{3}\right) = \frac{4}{\pi}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{2\sqrt{3}}{\pi} = -1.1$
$-\frac{1}{3}\pi$	$-\frac{\sqrt{3}}{2}$	0	$f\left(-\frac{\pi}{3}\right) = \frac{4}{\pi}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{2\sqrt{3}}{\pi} = -1.1$
0	0	0	$f(0) = 0$
$\frac{1}{3}\pi$	$\frac{\sqrt{3}}{2}$	0	$f\left(\frac{\pi}{3}\right) = \frac{4}{\pi}\left(\frac{\sqrt{3}}{2}\right) = \frac{2\sqrt{3}}{\pi} = 1.1$
$\frac{2}{3}\pi$	$\frac{\sqrt{3}}{2}$	0	$f\left(\frac{2\pi}{3}\right) = \frac{4}{\pi}\left(\frac{\sqrt{3}}{2}\right) = \frac{2\sqrt{3}}{\pi} = 1.1$

Construyendo la gráfica con x_1, x_2, x_3, x_4, x_5 se tienen los puntos $P_1\left(-\frac{2\pi}{3}, -1.1\right)$, $P_2\left(-\frac{1}{3}\pi, -1.1\right)$, $P_3(0,0)$, $P_4\left(\frac{\pi}{3}, 1.1\right)$ y $P_5\left(\frac{2\pi}{3}, 1.1\right)$

Aproximación

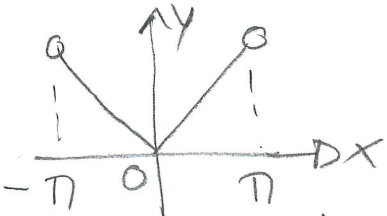


Obtenga la serie de Fourier de la función

$$f(x) = |x| \qquad -\pi < x < \pi$$

La gráfica de $f(x) = |x|$ es el valor absoluto de x, esto es

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



Se observa que $f(x)$ es una función par, es decir en el intervalo simétrico $(-\pi, \pi)$ la gráfica posee simetría con respecto al eje Y. Esto es, $f(-x) = f(x)$

La serie de Fourier es:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p} x\right)$$

donde los coeficientes de Fourier a_0 y a_n son

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \quad \text{y} \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)dx$$

el coeficiente $b_n = 0$

Dado el intervalo $-\pi < x < \pi$, es decir $(-p, p)$ se identifica que $p = \pi$, así

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{1}{2}x^2\Big|_0^{\pi}\right] = \frac{2}{\pi}\left[\frac{\pi^2}{2}\right] = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$, usando integración por partes

$$I = \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$, hacemos

$$u = x \qquad \int dv = \int \cos(nx)\,dx$$

$$\frac{du}{dx} = 1 \qquad v = \frac{1}{n}\operatorname{sen}(nx)$$

$$du = dx$$

Así $I = \frac{x}{n}\operatorname{sen}(nx)\Big|_0^{\pi} - \int_0^{\pi} \frac{1}{n}\operatorname{sen}(nx)\,dx$

$I = \frac{\pi}{n}\cancel{\operatorname{sen}(n\pi)} - 0 - \frac{1}{n}\left[-\frac{1}{n}\cos(nx)\Big|_0^{\pi}\right]$

cero si $n = 1, 2, \ldots$

$I = \frac{1}{n^2}\left[\cos(n\pi) - \cos(0)\right]$; $\cos(n\pi) = (-1)^n$

$I = \frac{1}{n^2}\left((-1)^n - 1\right)$

Entonces

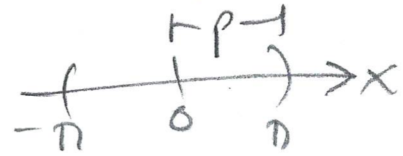
$a_n = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$ de tal forma que la serie es:

$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\left(\frac{n\pi}{\pi}x\right)$, o bien:

$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos(nx) \qquad x \in (-\pi, \pi)$

Obtenga la serie de Fourier de la función

$$f(x)=\begin{cases} 0 & -\pi< x<0 \\ \pi-x & 0\leq x<\pi \end{cases}$$



$p=\pi$

La gráfica de $f(x)$ no es una función impar ni par, por lo que la serie de Fourier es

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{p}x\right)+b_n\operatorname{sen}\left(\frac{n\pi}{p}x\right)\right] \text{ donde}$$

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)\,dx\,;\quad a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\left(\frac{n\pi}{p}x\right)dx$$

$$\text{y}\quad b_n=\frac{1}{p}\int_{-p}^{p}f(x)\operatorname{sen}\left(\frac{n\pi}{p}x\right)dx$$

Entonces los coeficientes de Fourier se calculan:

$$a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\,dx+\int_{0}^{\pi}\pi-x\,dx\right]=\frac{1}{\pi}\left[\pi x-\frac{1}{2}x^2\right]\Big|_0^{\pi}$$

$$a_0=\frac{1}{\pi}\left[\pi^2-\frac{\pi^2}{2}\right]=\frac{1}{2}\pi$$

$$a_n=\frac{1}{\pi}\left[\underbrace{\cancel{\int_{-\pi}^{0}0\cos(nx)\,dx}}_{\text{cero}}+\int_{0}^{\pi}(\pi-x)\cos(nx)\,dx\right]$$

$$a_n=\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\cos(nx)\,dx,\quad \text{integrando por partes}$$

$$u=\pi-x \qquad \int dv=\int\cos(nx)\,dx$$

$$\frac{du}{dx}=-1 \qquad v=\frac{1}{n}\operatorname{sen}(nx)$$

$$du=-dx$$

$$a_n=\frac{1}{\pi}\left[\frac{\pi-x}{n}\operatorname{sen}(nx)\Big|_0^{\pi}-\int_0^{\pi}\frac{1}{n}\operatorname{sen}(nx)(-dx)\right]$$

$$a_n = \frac{1}{\pi}\Big[\underbrace{\cancel{\frac{\pi-\pi}{n}\,sen(n\pi) - 0}}_{\text{cero}} + \frac{1}{n}\int_0^{\pi} sen\,(nx)\,dx\Big]$$

$n = 1, 2, \ldots$

$\cos(n\pi) = (-1)^n$

$$a_n = \frac{1}{\pi}\Big[\frac{1}{n}\Big(-\frac{1}{n}\cos(nx)\Big)\Big|_0^{\pi}\Big] = -\frac{1}{\pi n^2}\big[\cos(n\pi) - \cos 0\big]$$

$$a_n = -\frac{1}{\pi n^2}\big[(-1)^n - 1\big] = \frac{1-(-1)^n}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\Big[\underbrace{\cancel{\int_{-\pi}^{0} 0\; sen(nx)\,dx}}_{\text{cero}} + \int_0^{\pi}(\pi - x)\,sen(nx)\,dx\Big]$$

$$b_n = \frac{1}{\pi}\Big[\int_0^{\pi}(\pi - x)\,sen(nx)\,dx\Big]$$, integrando por partes

$u = \pi - x \qquad \int dv = \int sen(nx)\,dx$

$\frac{du}{dx} = -1 \qquad v = -\frac{1}{n}\cos(nx)$

$du = -dx$

$$b_n = \frac{1}{\pi}\Big[-\frac{(\pi - x)}{n}\cos(nx)\Big|_0^{\pi} - \int_0^{\pi} -\frac{1}{n}\cos(nx)\,(-dx)\Big]$$

$$b_n = \frac{1}{\pi}\Big[\frac{\pi}{n} - \frac{1}{n}\Big(\frac{1}{n}\,sen(nx)\Big)\Big|_0^{\pi}\Big] = \frac{1}{\pi}\Big[\frac{\pi}{n} - 0\Big] = \frac{1}{n}$$

La serie de Fourier es

$$f(x) = \frac{\frac{\pi}{2}}{2} + \sum_{n=1}^{\infty}\Big[\frac{1-(-1)^n}{\pi n^2}\cos(nx) + \frac{1}{n}\,sen(nx)\Big]$$, entonces:

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\Big[\frac{1-(-1)^n}{\pi n^2}\cos(nx) + \frac{1}{n}\,sen(nx)\Big]$$