

Schramm–Loewner Evolution

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1 Introduction

Let there be a domain \mathbb{D}_t (a simply connected domain in the upper half-plane \mathbb{H}) that grows over time along some curve. By the Riemann mapping theorem, for each moment in time one can find a conformal map $f_t : \mathbb{H} \rightarrow \mathbb{D}_t$. Thus, all geometric information about the domain \mathbb{D}_t is encoded in the map f_t .

In 1923 Loewner proposed describing the evolution of domains \mathbb{D}_t by means of a differential equation that governs the evolution of the conformal isomorphism between \mathbb{D}_t and \mathbb{H} . This equation depends on a *driving function* $\zeta(t)$ which takes values on the boundary of \mathbb{H} . Schramm suggested choosing the driving function in the form $\zeta(t) = \sqrt{\kappa}B(t)$, where $B(t)$ is a Brownian motion on the boundary of \mathbb{H} and $\kappa > 0$ is a parameter.

Then the Schramm–Loewner evolution is a probability measure on families of planar curves, obtained as the image of the Wiener measure under such a mapping:

$$\zeta(t) \xrightarrow{\text{Loewner equation}} \gamma(t),$$

where $\gamma(t)$ is a growing curve in the upper half-plane. In this way, at the heart of SLE lies the encoding of boundary growth into conformal correspondence.

2 Complex Analysis

Definition 2.1. A set $E \subset \overline{\mathbb{C}}$ is called *connected* if there do not exist two open sets G_1 and G_2 such that:

- (i) $E \subset G_1 \cup G_2$;
- (ii) $E \cap G_1 \cap G_2 = \emptyset$;
- (iii) $G_1 \cap E \neq \emptyset$, $G_2 \cap E \neq \emptyset$.

Definition 2.2. A domain is a nonempty, connected, open subset strictly contained in \mathbb{C} .

We additionally assume the domain to be simply connected. Such a domain contains no holes and is contractible; it is homotopy equivalent and, moreover, homeomorphic (in \mathbb{C} or \mathbb{R}^2) to an open disk, i.e., it has the same topology. The key result of this section is the **Riemann Mapping Theorem**.

Theorem 2.3. Let D be a simply connected domain in the complex plane \mathbb{C} , with $D \neq \mathbb{C}$, and let $z_0 \in D$. Then there exists a unique holomorphic and injective function f on D which maps D onto the unit disk \mathbb{D} and satisfies $f(z_0) = 0$, $f'(z_0) > 0$.

Proof: Uniqueness is straightforward, while existence requires additional argumentation. A proof may be provided if timing allows in the presentation. \square

Definition 2.4. Such mappings f will be called *uniformizing maps*.

Example 2.5. Consider two domains: the upper half-plane \mathbb{H} and the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and the conformal map

$$f(z) = i \frac{1 - z}{1 + z}. \tag{1}$$

This map sends the unit disk to the upper half-plane, with $f(0) = i$ and $f(1) = 0$.

Next, consider the group of conformal automorphisms of the upper half-plane, $PSL_2(\mathbb{R})$. This group acts on the real projective line (equivalently, on the boundary of \mathbb{H} in the complex sense) by Möbius transformations

$$f(z) = \frac{az + b}{cz + d},$$

with real coefficients a, b, c, d satisfying $ad - bc = 1$. This is a three-dimensional Lie group (a matrix group defined by three real parameters). In particular, there exists a unique automorphism sending any three boundary points to any other three boundary points. Likewise, any pair consisting of an interior point and a boundary point may be transported to any other such pair while preserving the upper half-plane. By the Riemann theorem the same holds for any domain whose conformal map is determined completely by three normalizing conditions.

As stated previously, the Riemann theorem allows one to *encode* the geometry of growing domains by unique conformal mappings. Again, according to this theorem, one may choose any reference domain; for convenience we take the upper half-plane \mathbb{H} .

Definition 2.6. *We say that \mathbb{D} differs from the upper half-plane \mathbb{H} only locally if the set $\mathbb{K} = \mathbb{H} \setminus \mathbb{D}$ is bounded. Such a set \mathbb{K} is called a hull.*

The real points of the closure of \mathbb{K} in \mathbb{C} form a compact set denoted $\mathbb{K}_{\mathbb{R}}$. Let $f : \mathbb{H} \rightarrow \mathbb{D}$ be a conformal bijection and $g : \mathbb{D} \rightarrow \mathbb{H}$ its inverse. For further construction we want f to be holomorphic at infinity and satisfy $f(w) - w = O(1/w)$ as $w \rightarrow \infty$. This can be achieved as follows:

Consider the automorphism group $PSL_2(\mathbb{R})$ of \mathbb{H} . Choose (and fix) an element ϕ of this group so that the composition $f \circ \phi : \mathbb{H} \rightarrow \mathbb{D}$ satisfies the above conditions. This imposes three constraints on f and therefore on ϕ (since f is unique by the Riemann theorem). In particular, the normalization implies $f(\infty) = \infty$, making infinity a simple pole of f . Hence no freedom remains, and we identify $f \circ \phi \equiv f$. This required property is known as *hydrodynamic normalization*. The normalized map is denoted $f_{\mathbb{K}}$, and is determined uniquely by the hull \mathbb{K} : all properties of $f_{\mathbb{K}}$ are intrinsic to \mathbb{K} .

Definition 2.7. *Consider the domain $\mathbb{D} = \mathbb{H} \setminus \mathbb{K}$, where \mathbb{K} is any hull, and let $f_{\mathbb{K}}$ be its hydrodynamically normalized uniformizing map. Since the boundary of \mathbb{H} is smooth, $f_{\mathbb{K}}$ extends continuously to $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and the set $f_{\mathbb{K}}^{-1}(\mathbb{R} \setminus \mathbb{K}_{\mathbb{R}})$ is a nonempty open subset of $\bar{\mathbb{R}}$ with compact complement. This complement is called the cut of the map $f_{\mathbb{K}}$.*

(!insert examples and images later!) By the Schwarz reflection principle, defining for $\text{Im } z \leq 0$

$$f_{\mathbb{K}}(z) = \overline{f_{\mathbb{K}}(\bar{z})},$$

yields an analytic continuation of f to the whole complex plane \mathbb{C} except along the cut. The jump along the cut is purely imaginary:

$$f(x + i0^+) - f(x - i0^-) = 2i\pi\rho(x),$$

where $\rho(x)$ is the Radon–Nikodym density of a measure $d\mu_{f_{\mathbb{K}}}(x)$ with respect to Lebesgue measure. We call μ the *discontinuity measure*. Then, by the Cauchy integral theorem,

$$f_{\mathbb{K}}(w) = w + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{d\mu_{f_{\mathbb{K}}}(x)}{x - w}, \quad (2)$$

where w lies off the cut.

Thus, instead of studying the evolution of f , one may study the evolution of the positive finite measure $d\mu_{f_{\mathbb{K}}}(x)$.

Definition 2.8. *The capacity of the hull \mathbb{K} is defined by*

$$C_{\mathbb{K}} \equiv \frac{1}{2\pi} \int_{\mathbb{R}} d\mu_{f_{\mathbb{K}}}(x), \quad (3)$$

which is positive whenever $\mathbb{K} \neq \emptyset$.

As $w \rightarrow \infty$, the map $f_{\mathbb{K}}(w)$ has the expansion

$$f_{\mathbb{K}}(w) = w - \frac{C_{\mathbb{K}}}{w} + O\left(\frac{1}{w^2}\right). \quad (4)$$

The usefulness of capacity lies in its well-behaved composition property: if \mathbb{K} and \mathbb{K}' are two hulls, then the union $\mathbb{K} \cup f_{\mathbb{K}}(\mathbb{K}')$ is also a hull, and moreover

$$C_{\mathbb{K} \cup f_{\mathbb{K}}(\mathbb{K}')} = C_{\mathbb{K}} + C_{\mathbb{K}'}$$

This follows by applying expansion (5) to $f_{\mathbb{K}} \circ f_{\mathbb{K}'}$, the map corresponding to the union hull. In particular, capacity is a continuous increasing function on the space of hulls.

3 Loewner Chains

Let $f_t \equiv f_{\mathbb{K}_t}$ be the hydrodynamically normalized conformal map from the upper half-plane \mathbb{H} onto $\mathbb{H} \setminus \mathbb{K}_t$:

$$f_t(w) = w + O\left(\frac{1}{w}\right),$$

and denote by $g_t : \mathbb{H} \setminus \mathbb{K}_t \rightarrow \mathbb{H}$ its inverse. Then

$$g_t(z) = z + O\left(\frac{1}{z}\right), \quad z \rightarrow \infty.$$

To study the evolution of the family of hulls \mathbb{K}_t , fix $\varepsilon \geq 0$ and consider

$$\mathbb{K}_{\varepsilon,t} \equiv g_t(\mathbb{K}_{t+\varepsilon} \setminus \mathbb{K}_t).$$

Define $f_{\varepsilon,t} \equiv f_{\mathbb{K}_{\varepsilon,t}}$. Then on $\mathbb{H} \setminus \mathbb{K}_{t+\varepsilon}$:

$$g_t = f_{\varepsilon,t} \circ g_{t+\varepsilon}.$$

Using the Cauchy-type representation (3) for $f_{\mathbb{K}_{\varepsilon,t}}$, evaluating at $w = g_{t+\varepsilon}(z)$, we obtain

$$g_{t+\varepsilon}(z) - f_{\varepsilon,t} \circ g_{t+\varepsilon}(z) = g_{t+\varepsilon}(z) - g_t(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{d\mu_{f_{\varepsilon,t}}(x)}{g_{t+\varepsilon}(z) - x}. \quad (5)$$

When ε is small, the hull $\mathbb{K}_{\varepsilon,t}$ is small and $f_{\varepsilon,t}$ is close to the identity. Therefore, we expect the discontinuity measure to expand to first order as

$$d\mu_{f_{\varepsilon,t}}(x) \simeq \varepsilon \cdot 2\pi d\nu_t(x). \quad (6)$$

Definition 3.1. Taking the limit $\varepsilon \rightarrow 0$ in (5), using (6), we obtain the evolution equation

$$\frac{d}{dt} g_t(z) = \int_{\mathbb{R}} \frac{d\nu_t(x)}{g_t(z) - x}.$$

Such a system is called a **Loewner chain**.

The Loewner measures $d\nu_t$ may depend nonlinearly on g_t and have a simple physical interpretation.

Indeed, $f_{\varepsilon,t}$ uniformizes the hull $\mathbb{K}_{\varepsilon,t}$, which is exactly the image under g_t of the increment $\mathbb{K}_{t+\varepsilon} \setminus \mathbb{K}_t$. Thus $\mathbb{K}_{\varepsilon,t}$ represents the newly grown layer of the hull during time ε , with height of order ε . Since $f_{\varepsilon,t}$ is given (to first order) by formula (2), its Radon–Nikodym derivative $d\nu_t(x)/dx$ is proportional to the height of that boundary portion.

Formally, let f_t (analytic in \mathbb{H}) be the inverse of g_t . Differentiating $\frac{d}{dt} g_t(f_t(w)) = \frac{d}{dt} w = 0$, we obtain:

$$\frac{1}{f_t'(w)} \frac{d}{dt} f_t(w) + \int_{\mathbb{R}} \frac{d\nu_t(x)}{w - x} = 0,$$

hence

$$\frac{d}{dt} f_t(w) = -f_t'(w) \int_{\mathbb{R}} \frac{d\nu_t(x)}{w - x}.$$

This may be viewed as a Riemann–Hilbert problem for

$$\frac{\partial_t f_t(w)}{f'_t(w)},$$

and is equivalent to the boundary condition

$$\lim_{\varepsilon \rightarrow 0^+} \operatorname{Im} \left[\frac{\partial_t f_t(w)}{f'_t(w)} \right]_{w=y+i\varepsilon} = \pi \rho_t(y), \quad d\nu_t(x) = \rho_t(x) dx.$$

By construction, the boundary of the hull is the image of the real line:

$$\partial \mathbb{K}_t = \{f_t(\zeta) \mid \zeta \in \mathbb{R}\}.$$

Its evolution is determined by the normal velocity $v_n(\zeta)$ (tangential velocity has no geometric meaning):

$$v_n(\zeta) = |f'_t(\zeta)| \operatorname{Im} \left[\frac{\partial_t f_t}{f'_t} \right] (\zeta).$$

Rewriting this via $d\nu_t$,

$$v_n(\zeta) d\zeta = \pi |f'_t(\zeta)| d\nu_t(\zeta).$$

Thus $\varepsilon d\nu_t(\zeta)$ encodes the material added to $\mathbb{K}_{\varepsilon,t}$ during time ε , while $|f'_t(\zeta)|$ rescales it into the increment of $\mathbb{K}_{t+\varepsilon}$.

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From now on we restrict to the special case:

Suppose the Loewner measure is concentrated at a single point, so there exists a real function $\xi_t \in \mathbb{R}$ with

$$d\nu_t(x) = 2 \delta(x - \xi_t) dx.$$

Then the Loewner chain equation becomes

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \xi_t}. \quad (7)$$

Time parameterization

The time t has not yet been fixed. In SLE it is standard—indeed essential—to choose time so that it coincides with capacity. Define

$$2t \equiv C_{\mathbb{K}_t},$$

where $C_{\mathbb{K}_t}$ is the capacity of the hull. Then automatically,

$$\int_{\mathbb{R}} d\nu_t(x) = 2.$$

(Addition of hulls ensures consistency):

$$C_{\mathbb{K}_t \cup f_{\mathbb{K}_t}(\mathbb{K}_s)} = t + s.$$

With this time scale, the asymptotics are:

– for $g_t(z)$:

$$g_t(z) = z + \frac{C_{\mathbb{K}_t}}{z} + O\left(\frac{1}{z^2}\right),$$

– for $f_t(w) = g_t^{-1}(w)$:

$$f_t(w) = w - \frac{C_{\mathbb{K}_t}}{w} + O\left(\frac{1}{w^2}\right).$$

Example 3.2. 1. *Hull formed by a vertical slit*

$\mathbb{K} = \{z = iy, y \in [0, a]\}$. The uniformizing map $g_t : \mathbb{H} \setminus [0, ia] \rightarrow \mathbb{H}$ is

$$g_t(z) = \sqrt{z^2 + a^2}, \quad a = a(t).$$

Its hydrodynamic expansion is

$$g_t(z) = z + \frac{a^2}{2z} + O\left(\frac{1}{z^2}\right),$$

so $C_{\mathbb{K}_t} = \frac{a^2}{2}$. From (7), the driving function is

$$\xi_t = g_t(z) - \frac{2}{\dot{g}_t(z)} = 0.$$

2. Hull shaped as a semicircle

First take $\mathbb{K} = \{z \in \mathbb{H} : |z| = 1, 0 \leq \arg z \leq \alpha\}$. A conformal map to \mathbb{H} is a composition of the Joukowski map, a Möbius transform and a square root:

$$g_t(z) = \sqrt{\frac{\frac{1}{2}(z + \frac{1}{z}) - \cos \alpha}{\frac{1}{2}(z + \frac{1}{z}) + 1}}.$$

Its expansion at ∞ does not satisfy hydrodynamic normalization, so apply a Möbius correction

$$\tilde{g}_t(z) = \frac{a g_t(z) + b}{c g_t(z) + d},$$

with

$$c = -d = \frac{2b}{1 - 3 \cos \alpha}, \quad a = \frac{3b - b \cos \alpha}{3 \cos \alpha - 1}, \quad b = 1.$$

Then

$$\tilde{g}_t(z) \approx z + \frac{\sin^2(\alpha/2)(\cos \alpha + 3)}{2z} + O(z^{-2}),$$

so

$$t = \frac{1}{4} \sin^2\left(\frac{\alpha}{2}\right)(\cos \alpha + 3).$$

Hence the Loewner driving term is

$$\xi_t = -2 - 3\sqrt{1 - 2t}.$$

The branch point at $t = \frac{1}{2}$ corresponds to $\alpha = \pi$, where the arc becomes a half-disk. In this case the limit mapping is the Joukowski map:

$$\lim_{t \rightarrow 1/2} \tilde{g}_t(z) = z + \frac{1}{z}.$$

Remark 3.3. It is worth noting that in the arc limit, the conformal map collapses every interior point of the disk to a single point. More generally, if some domain D becomes swallowed by the hull at time $t = t_c$, then on the bounded component of D , the Loewner map g_t converges to a constant sending D to a single point ξ_{t_c} on the real axis.

A small generalization is the infinitesimal hull

$$\mathbb{K}_{\varepsilon; \rho} = \{z = x + i\varepsilon y : 0 < y \leq \pi \rho(x)\},$$

i.e. points between the real axis and the curve $x \mapsto i\varepsilon \pi \rho(x)$, where $0 < \varepsilon \ll 1$ and $x \in \mathbb{R}$. To first order in ε , the hydrodynamically normalized map $\mathbb{H} \setminus \mathbb{K}_{\varepsilon; \rho} \rightarrow \mathbb{H}$ is

$$g(z) = z + \varepsilon \int \frac{\rho(x) dx}{z - x} + \dots, \quad \varepsilon \ll 1. \quad (8)$$

This may be derived by covering the hull with semicircles or by observing directly that the map remains real on $z = x + i\varepsilon \pi \rho(x)$.

4 Schramm's Argument

4.1 Domain Markov Property

Definition 4.1. *An interface is the boundary between two different phases or states in a statistical mechanical model. It is a random curve separating regions with distinct values (for example, in the Ising model it is the boundary between $+1$ and -1 spin clusters).*

Fluctuations of the interface carry essential information about the critical behaviour of the model.

Assume we have fixed an initial segment of the interface $\gamma_{[ac]}$ inside a domain \mathbb{D} , i.e. the portion from a to some point c . Then:

- One may consider the conditional distribution of the remaining part of the interface (from c to b) given the already drawn initial part.
- Alternatively, one may remove the initial part $\gamma_{[ac]}$ from the domain \mathbb{D} , obtaining a new domain $\mathbb{D} \setminus \gamma_{[ac]}$, and study the distribution of the interface inside this new domain starting at c .

We require the equality of these distributions:

$$\gamma_{[ab]} \text{ in } \mathbb{D} \mid \gamma_{[ac]} \stackrel{\text{law}}{=} \gamma_{[cb]} \text{ in } \mathbb{D} \setminus \gamma_{[ac]}. \quad (9)$$

In probabilistic terms:

$$\mathbf{P}_{(\mathbb{D}, a, b)}[\cdot \mid \gamma_{[ac]}] = \mathbf{P}_{(\mathbb{D} \setminus \gamma_{[ac]}, c, b)}[\cdot]. \quad (10)$$

Both probabilities are defined on the same space: simple lattice curves from c to b , inside $\mathbb{D} \setminus \gamma_{[ac]}$.

In discrete statistical mechanics this property follows naturally, e.g. from local interactions. For loop-erased random walks it requires Dirichlet boundary conditions; Neumann boundary conditions are unsuitable because they alter the set of admissible trajectories. In percolation and the Ising model, even more is true: $\mathbf{P}_{(\mathbb{D}, a, b)}$ may be interpreted not only as the law of the interface but as the full probability measure on spin/colour configurations.

Definition 4.2. *Relations (8)–(9) are called the Domain Markov Property.*

Informally: if the interface has already followed the path $\gamma_{[ac]}$, then its continuation from c to b is distributed exactly as if we had restarted the process at c inside the slit domain $\mathbb{D} \setminus \gamma_{[ac]}$. The future depends only on the present state, not on how we arrived there. This property does not rely on conformal symmetry — it arises from locality — but combined with conformal invariance it uniquely leads to SLE.

4.2 Conformal Transport

To study conformally invariant probability measures on simple curves from a to b inside \mathbb{D} , assume that h is a conformal map sending \mathbb{D} to $\hat{\mathbb{D}} = h(\mathbb{D})$. Then the law of interfaces in $(\hat{\mathbb{D}}, h(a), h(b))$ must be the pushforward of the law in (\mathbb{D}, a, b) under h :

$$h(\gamma_{[ab]} \text{ in } \mathbb{D}) \stackrel{\text{law}}{=} \gamma_{[h(a)h(b)]} \text{ in } h(\mathbb{D}).$$

Equivalently,

$$\mathbf{P}_{(\mathbb{D}, a, b)}[\gamma_{[ab]} \subset U] = \mathbf{P}_{(h(\mathbb{D}), h(a), h(b))}[\gamma_{[h(a)h(b)]} \subset h(U)],$$

for any measurable set $U \subset \mathbb{D}$.

Thus, applying a conformal map to a random curve in \mathbb{D} produces a new curve whose law is the same as if it had been generated directly in $h(\mathbb{D})$ from $h(a)$ to $h(b)$.

This condition only requires invariance under the automorphism group preserving (\mathbb{D}, a, b) , but becomes meaningful when comparing scaling limits of discrete models:

Definition 4.3. *If the scaling limit of a discrete interface satisfies conformal transport, we say the critical interfaces are conformally invariant.*

Following Schramm, we restrict attention to random curves — SLE — satisfying **both** conformal invariance and the Domain Markov Property.

4.3 Conformally Invariant Interfaces

We summarise the two structural properties:

Let

- $\gamma_{[ac]}$ — initial portion of the interface in \mathbb{D} ,
- $c \in \mathbb{D}$ — the current tip of the curve,
- hence $\mathbb{D} \setminus \gamma_{[ac]}$ — the updated domain.

Question: How is the remaining part $\gamma'_{[cb]}$ distributed given $\gamma_{[ac]}$?

Answer (Domain Markov): as if we restarted the interface in the slit domain.

Let $h_{\gamma_{[ac]}}$ map $\mathbb{D} \setminus \gamma_{[ac]}$ back onto \mathbb{D} , sending

$$h_{\gamma_{[ac]}}(c) = a, \quad h_{\gamma_{[ac]}}(b) = b.$$

Then the image of $\gamma'_{[cb]}$ under $h_{\gamma_{[ac]}}$ is a full interface from a to b , independent of the past. Probabilistically,

$$\mathbf{P}_{(\mathbb{D}, a, b)} \left[\gamma_{[cb]} \subset U \mid \gamma_{[ac]} \right] = \mathbf{P}_{(\mathbb{D}, a, b)} \left[\gamma_{[ab]} \subset h_{\gamma_{[ac]}}(U) \right].$$

Thus the two defining properties of conformally invariant interfaces are:

1. $h_{\gamma_{[ac]}}(\gamma'_{[cb]})$ does not depend on $\gamma_{[ac]}$ — **Markov property**.
2. This transformed curve has the same distribution as $\gamma_{[ab]}$ — **stationarity of increments**.

Schramm's key observation: A random interface which is simultaneously conformally invariant and domain Markov is **forced** to be an SLE — a memoryless stochastic process.

Remark 4.4. *SLE is strongly conformally invariant — like planar Brownian motion (i.e. $B_t \mapsto f(B_t)$ is again Brownian motion under any local conformal map) — only for $\kappa = 6$. For all other values of κ , SLE is only globally conformally invariant: the law of interfaces is conformally invariant but the Loewner equation itself changes under the mapping.*

5 Chordal SLE

5.1 Definition

The Markov property and stationarity of increments for conformally invariant interfaces imply that, in order to understand the entire curve, it suffices to understand how it grows on an infinitesimal scale. One may therefore describe the growth step-by-step, or more precisely, at infinitesimal increments — via a differential equation. This is exactly what the Loewner evolution does.

Due to conformal invariance, we may reduce to the standard setting:

$$(\mathbb{D}, a, b) = (\mathbb{H}, 0, \infty),$$

that is, we consider a curve γ in the upper half-plane beginning at 0 and tending to ∞ .

Let $\gamma_{[0,t]}$ denote the initial segment of the interface from 0 to some point γ_t . Then the domain $\mathbb{H} \setminus \gamma_{[0,t]}$ is what remains after the curve has grown up to time t . We wish to describe how the domain changes as the curve grows — equivalently, how the conformal map sending

$$\mathbb{H} \setminus \gamma_{[0,t]} \longrightarrow \mathbb{H}$$

changes in time. As in the first section, we use capacity as our time parameter: $2t := C_{\gamma_{0,t}}$. Let $f_t : \mathbb{H} \rightarrow \mathbb{H} \setminus \gamma_{[0,t]}$ be the conformal map, normalised at infinity by

$$f_t(w) = w - \frac{2t}{w} + O\left(\frac{1}{w^2}\right).$$

Then its inverse

$$g_t = f_t^{-1} : \mathbb{H} \setminus \gamma_{[0,t]} \rightarrow \mathbb{H} \quad \text{satisfies} \quad g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right).$$

Recall Loewner's equation

$$\frac{d}{dt}g_t(z) = \frac{2}{g_t(z) - \xi_t}.$$

This Loewner ODE describes the growth of the hull $\gamma_{[0,t]}$, driven by a real-valued function ξ_t . The tip of the curve at time t is

$$\gamma_t = \lim_{\varepsilon \rightarrow 0^+} g_t^{-1}(\xi_t + i\varepsilon),$$

so that ξ_t is the image of the curve tip on the real axis.

If ξ_t is smooth, the curve γ_t is smooth; if ξ_t is irregular (e.g. continuous but nowhere differentiable like Brownian motion), the curve becomes fractal.

To shift the tip back to 0 (working in \mathbb{H} with marked points $0, \infty$), define

$$h_t(z) = g_t(z) - \xi_t.$$

Its expansion at infinity is $h_t(z) = z - \xi_t + \frac{2t}{z} + O(z^{-2})$, i.e. hydrodynamic normalisation shifted by $-\xi_t$. For $s > t$ the additional hull segment pushed forward by h_t ,

$$h_t(\mathbb{K}_s \setminus \mathbb{K}_t),$$

is independent of $\mathbb{K}_{t'}$ for $t' \leq t$ (Markov property) and has the same law as a fresh hull of capacity $s - t$ (stationarity of increments). Since the hull determines the map, the composition

$$h_s \circ h_t^{-1}$$

which sends $h_t(\mathbb{K}_s \setminus \mathbb{K}_t)$ back to \mathbb{H} , is independent of the past and distributed like h_{s-t} . Its asymptotics is

$$h_s \circ h_t^{-1}(z) = z - (\xi_s - \xi_t) + \frac{2(s-t)}{z} + \dots,$$

so the increment $\xi_s - \xi_t$ is the only driving parameter.

Two physical assumptions now determine the law of ξ_t :

- The curve should not branch — growth over short times should occur near previous growth. Hence ξ_t has continuous paths.
- Symmetry under reflection across $\text{Im } z = 0$ implies

$$g_t(z) \stackrel{d}{=} -\overline{g_t(-\bar{z})} \quad \Rightarrow \quad \xi_t \stackrel{d}{=} -\xi_t,$$

so ξ_t is symmetrically distributed about 0.

Theorem 5.1. Any one-dimensional Markov process with continuous paths, stationary increments and symmetry $\xi_t \stackrel{d}{=} -\xi_t$ is a rescaled Brownian motion.

Proof: See Lecture 2. \square

Corollary 5.2. There exists $\kappa > 0$ such that

$$\xi_t = \sqrt{\kappa} B_t,$$

where B_t is standard Brownian motion with covariance $\mathbb{E}[B_s B_t] = \min(s, t)$.

Remark 5.3. Even without using capacity as the time-parameter, one still finds that ξ_t is a continuous martingale, hence a time-changed Brownian motion.

Definition 5.4. The equation

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z,$$

defines **chordal SLE** with parameter κ in the upper half-plane \mathbb{H} from 0 to ∞ .

Definition 5.5. For any $z \in \mathbb{H}$ one solves the equation up to the time τ_z such that

$$g_{\tau_z}(z) = \xi_{\tau_z}.$$

Then τ_z is called the swallowing time of z .

Two key properties of chordal SLE:

1. **Scale invariance:**

$$g_t(z) \stackrel{d}{=} \lambda^{-1} g_{\lambda^2 t}(\lambda z),$$

a consequence of Brownian scaling.

2. **Markov property:**

$$h_t(z) = g_t(z) - \xi_t \quad \Rightarrow \quad h_s \circ h_t^{-1} \stackrel{d}{=} h_{s-t}.$$

5.2 Properties

We list key properties of SLE, all understood almost surely.

For any κ , the preimage of the driving term,

$$\gamma_t = \lim_{\varepsilon \rightarrow 0^+} g_t^{-1}(\sqrt{\kappa} B_t + i\varepsilon),$$

is a continuous curve — the *SLE trace*. It begins at 0 and tends to ∞ as $t \rightarrow \infty$:

$$\lim_{t \rightarrow \infty} \gamma_t = \infty.$$

The trace has no self-intersections for small κ .

The geometry of the trace depends on κ :

For $x \in \mathbb{R}$ define

$$Y_t = \frac{h_t(x)}{\sqrt{\kappa}} = \frac{g_t(x) - \xi_t}{\sqrt{\kappa}}.$$

Applying Itô,

$$dY_t = \frac{2/\kappa}{Y_t} dt + dB_t,$$

a Bessel process with $\frac{n-1}{2} = 2/\kappa$. Using known recurrence criteria:

If $n < 2$ the process hits 0 infinitely often; if $n > 2$ it avoids 0.

Since

$$\kappa = \frac{4}{n-1},$$

we conclude:

- $\kappa > 4 \Rightarrow n < 2$: $Y_t \rightarrow 0$ a.s., the trace hits the real line infinitely often.
- $\kappa < 4 \Rightarrow n > 2$: the trace does not hit the boundary.

Thus SLE is simple for $\kappa < 4$, self-touching for $4 < \kappa < 8$, and space-filling for $\kappa \geq 8$ (phase transition at $\kappa = 4$):

- $0 \leq \kappa \leq 4$: simple non-self-intersecting trace,
- $4 < \kappa < 8$: double points occur,
- $\kappa \geq 8$: the curve is dense in \mathbb{H} .

Proposition 5.6. *Fractal dimension of the trace:*

$$d_\kappa = \begin{cases} 1 + \frac{\kappa}{8}, & \kappa \leq 8, \\ 2, & \kappa \geq 8. \end{cases}$$

Examples:

$$\kappa = 2 \Rightarrow d = \frac{5}{4}, \quad \kappa = 6 \Rightarrow d = \frac{7}{4}, \quad \kappa = 8 \Rightarrow d = 2.$$

Simulation links to lattice models:

- LERW $\leftrightarrow \kappa = 2 \Rightarrow d = 5/4$,
- Percolation $\leftrightarrow \kappa = 6 \Rightarrow d = 7/4$.

Definition 5.7. *The hull \mathbb{K}_t is*

$$\mathbb{K}_t = \mathbb{H} \setminus g_t^{-1}(\mathbb{H}) = \{z \in \mathbb{H} : \tau_z < t\}.$$

Properties of hulls:

- \mathbb{K}_t is the complement of the unbounded component of $\mathbb{H} \setminus \gamma_{(0,t]}$,
- For $\kappa \leq 4$, $\mathbb{K}_t = \gamma_{[0,t]}$ (a simple curve),
- For $\kappa > 4$, \mathbb{K}_t has interior.

Theorem 5.8 (Duality Conjecture). *For $\kappa > 4$, the boundary of \mathbb{K}_t locally looks like an SLE with dual parameter*

$$\kappa^* = \frac{16}{\kappa} < 4.$$

(Not proven in general.)

Theorem 5.9. *Let Ω be a bounded simply connected domain with marked boundary points a, b , and γ^δ the interface in the critical Ising model on the lattice $\delta\mathbb{Z}^2$ with Dobrushin boundary conditions. Then as $\delta \rightarrow 0$, γ^δ converges in law to chordal SLE(3) from a to b in Ω , with respect to the curve metric*

$$d(\gamma_1, \gamma_2) = \inf_{\text{reparam.}} \sup_{t \in [0,1]} |\gamma_1(\phi_1(t)) - \gamma_2(\phi_2(t))|.$$

Theorem 5.10. *For the critical FK-Ising model with Dobrushin boundary conditions, γ^δ converges in law to SLE(16/3).*

Proof: [Sketch] Requires three ingredients:

1. **Condition G (Geometric crossing bound).**

There exists $C > 1$ such that for any annulus $A(z_0, r, R)$ with $R/r > C$ and $\partial B(z_0, r) \cap \partial\Omega^\delta \neq \emptyset$,

$$\mathbb{P}[\gamma^\delta \text{ makes an unforced crossing of } A(z_0, r, R)] < \frac{1}{2}.$$

2. Compactness & convergence of driving terms.

If $\{\gamma^\delta\}$ satisfies Condition G, then:

- $\{\gamma^\delta\}$ is pre-compact,
- Any subsequential limit is a Loewner evolution with driving process ξ_t ,
- $\xi_t^\delta \rightarrow \xi_t$ uniformly, with ξ_t a continuous martingale and $\mathbb{E}[e^{\epsilon|\xi_t|/\sqrt{t}}] < \infty$.

3. Discrete fermionic observable $F_n^\delta(z)$.

It is a martingale for the interface and converges to

$$M_t(z) = (\partial_z[-h_t(w(z))^{-1}])^{1/2}.$$

This convergence is uniform on compact sets of Ω for all admissible interfaces.

Hence the limit γ has driving term $W_t = \sqrt{3}B_t$, so the scaling limit is SLE(3); uniqueness gives full convergence.

□

Remark 5.11. *Weak convergence of γ^δ to SLE means weak convergence of probability measures on the space of curves: for every bounded continuous functional F ,*

$$\mathbb{E}[F(\gamma^\delta)] \rightarrow \mathbb{E}[F(\gamma)] \quad \text{as } \delta \rightarrow 0.$$

6 Group-Theoretic Formulation of SLE

The goal of this section is to develop an alternative group-theoretic formulation of SLE processes, which can later be used to connect SLE with the representation-theoretic framework of Conformal Field Theory (CFT).

To establish this connection, we introduce vector fields w_{-1} and w_{-2} in the standard complex coordinate z on the upper half-plane \mathbb{H} . If needed, these objects may be transported to any other domain via conformal maps.

If a stochastic differential equation on a Riemann surface Σ has the form

$$dh_t(z) = \sigma(h_t(z)) dt + \rho(h_t(z)) d\xi_t,$$

then under a change of coordinates $w = \varphi(z)$, where φ is conformal, the transformed map

$$h_t^\varphi = \varphi \circ h_t \circ \varphi^{-1}$$

satisfies

$$dh_t^\varphi = \sigma^\varphi(h_t^\varphi) dt + \rho^\varphi(h_t^\varphi) d\xi_t,$$

where

$$\rho^\varphi \circ \varphi = \varphi' \rho, \quad \sigma^\varphi \circ \varphi = \varphi' \sigma + \frac{\kappa}{2} \varphi'' \rho^2.$$

It follows that the corresponding vector fields

$$w_{-1} = \rho(z) \partial_z, \quad w_{-2} = \frac{1}{2} \left(-\sigma(z) + \frac{\kappa}{2} \rho(z) \rho'(z) \right) \partial_z,$$

are holomorphic and form a subalgebra of the Witt algebra, extendable to the Virasoro algebra.

As discussed earlier, in the chordal case it is convenient to introduce

$$h_t(z) \equiv g_t(z) - \xi_t,$$

which satisfies the stochastic Loewner equation

$$dh_t = \frac{2}{h_t} dt - d\xi_t,$$

where g_t is the conformal map from \mathbb{H} with the slit $\gamma_{[0,t]}$ back to \mathbb{H} , and $\xi_t = \sqrt{\kappa} B_t$ is the driving function expressed via Brownian motion B_t .

The germ of h_t near infinity lies in the group N_- — the group of holomorphic germs at ∞ of the form

$$z + \sum_{m \leq -1} h_m z^{m+1},$$

with real coefficients. These maps fix ∞ and have unit derivative there.

This group acts (anti-)compositionally on the space O_- — the space of germs of holomorphic functions fixing ∞ but without derivative normalization. To each h_t we associate an element $\mathfrak{g}_{h_t} \in N_-$ satisfying the stochastic differential equation:

$$\mathfrak{g}_{h_t}^{-1} d\mathfrak{g}_{h_t} = dt \left(\frac{2}{z} \partial_z + \frac{\kappa}{2} \partial_z^2 \right) - d\xi_t \partial_z.$$

Relation to the Witt/Virasoro Algebra

From this we extract:

$$w_{-1} = -\partial_z \equiv \ell_{-1}, \quad w_{-2} = -\frac{1}{z} \partial_z \equiv \ell_{-2},$$

where $\ell_n = -z^{n+1} \partial_z$ are the standard generators of the Witt algebra — a subalgebra of the Virasoro algebra.

- w_{-1} is holomorphic on all of \mathbb{H} , tangent to the boundary, and hence extends to \mathbb{C} via Schwarz reflection.
- w_{-2} is also holomorphic in \mathbb{H} but has a pole at 0, extending to $\mathbb{C} \setminus \{0\}$ with a simple pole of residue 2.

Both vector fields vanish at infinity (w_{-1} to second order, w_{-2} to third order), matching the asymptotic behaviour $h_t(z) = z + O(1/z)$ as $z \rightarrow \infty$. The absence of other common zeros is a geometric reason why the SLE trace escapes to infinity as $t \rightarrow \infty$.

To define Brownian motion along the evolving curve, one needs a parametrization. The vector field ℓ_{-1} generates a one-parameter family of automorphisms of \mathbb{H} extending to the boundary, and thus plays the role of a natural time parametrization.