Steady state PASEP From the prism of 1D Magnetic Random Walks

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Abstract

*The Totally Asymmetric Simple Exclusion Process (TASEP) is a stochastic process on a regular one-dimensional lattice with the following dynamics: each site i $(1 < i < N)$ of a one-dimensional lattice of N sites is either occupied by particle $(\tau = 1)$ or empty $(\tau = 0)$. During every infinitesimal time interval dt, each particle in the system has a probability dt of jumping to the next site on its right (for all particles on sites $1 < i < N-1$) if this neighboring site is empty. Furthermore, a particle is added at site $i = 1$ with probability αdt if site 1 is empty and a particle is removed from site N with probability βdt if this site is occupied. The matrix ansatz for such a model is Well-studied: $P_N(\tau_1, \dots \tau_N) = \langle W| \prod_{i=1}^N (\tau_i D + (1-\tau_i)E) |V\rangle$, where matrices E, D satisfy the following algebra: $DE = E + D$. The q-deformation of this algebra is **DE-qED=E+D**. We are interested in a model called **PASEP**, that satisfies this algebra.*

1. PASEP ansatz

Consider the Partially Asymmetric Exclusion Process (PASEP) by determining it's transition rates matrices:

$$h_1 = \begin{pmatrix} -\alpha & \gamma \\ \alpha & \gamma \end{pmatrix}, \quad h_N = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}.$$

where the basis is $\{0, 1\}$ (that means, for example, $\{(0, \tau_2, \dots, \tau_N); (1, \tau_2, \dots, \tau_N)\}$ for h_1).

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Where the basis is $\{(0,0); (0,1); (1,0); (1,1)\}$.

We are interested in **the steady-state**. In this model, the state is clearly determined by $\tau_i \in \{0, 1\}$, so all probabilities satisfy:

$$\begin{aligned} \frac{d}{dt} P_N(\tau_1, \dots, \tau_N) = \sum_{\sigma_1} (h_1)_{\tau_1 \sigma_1} P_N(\sigma_1, \tau_2, \dots, \tau_N) + \\ \sum_{i=1}^{N-1} \sum_{\sigma_i \sigma_{i+1}} (h_1)_{(\tau_i \tau_{i+1})(\sigma_i \sigma_{i+1})} P_N(\dots, \sigma_i, \sigma_{i+1}, \dots) + \\ + \sum_{\sigma_N} (h_N)_{\tau_N \sigma_N} P_N(\tau_1, \dots, \tau_{N-1}, \sigma_N) = 0. \end{aligned} \tag{1}$$

Rewrite (1) in recursive form

- The left boundary:

$$\sum_{\sigma_1} \dots = (2\tau_1 - 1) x_L P_{N-1}(\tau_2, \dots). \tag{2}$$

  Where x_L is some combination of γ and α

- The middle:

$$\sum_{\sigma_i \sigma_{i+1}} \dots = -a(\tau_i, \tau_{i+1}) P_{N-1}(\dots, \tau_{i-1}, \tau_{i+1}, \dots) + a(\tau_{i+1, \tau_i}). \tag{3}$$

  Where $a(\tau, \tau') = \begin{cases} p, & (\tau, \tau') = (1, 0) \\ q, & (\tau, \tau') = (0, 1) \\ 0, & (\tau, \tau') = (0,0)/(1,1) \end{cases}$.

- The right boundary:

$$\sum_{\sigma_N} \dots = -(2\tau_N - 1) x_R P_{N-1}(\tau_1, \dots, \tau_{N-1}). \tag{4}$$

Consider the following ansatz

$$f_N(\tau) = \langle W| \prod_{k=1}^N (\tau_K D + (1 - \tau_k) E) |V\rangle. \tag{5}$$

Substituting (2), (3), (4) into (5) we obtain the following

$$\begin{aligned} pDE - qED = pE - qD, \\ -\alpha \langle W| E + \gamma \langle W| D = -x \langle W|, \\ -\delta E |V\rangle + \beta D |V\rangle = x |V\rangle. \end{aligned} \tag{6}$$

Note: the first in (6) is definitely demanded algebra **DE-qED=E+D** (up to a simple transformation)

Matrix realization

To obtain an exact matrix realization of algebra (6) let's recall a *q-boson algebra*

$$aa^\dagger - qa^\dagger a = 1.$$

With basis $|n\rangle, \quad n = 0, 1, 2, \dots$, on which a and $a^\dagger$ act as usual

$$a|n\rangle = \sqrt{1 - q^n} |n-1\rangle, \quad (n \geq 1), \quad a|0\rangle = 0,$$
$$a^\dagger = \sqrt{1 - q^{n+1}} |n+1\rangle.$$

Therefore, required representation in operator and matrix form respectively

$$D = a^\dagger + \frac{1}{1-q}\mathbb{I}, \quad E = \frac{1}{1-q} a + \frac{1}{1-q}\mathbb{I} \tag{7}$$

$$D = \begin{pmatrix} \frac{1}{1-q} & \sqrt{1-q^1} & 0 & 0 & \cdots \\ 0 & \frac{1}{1-q} & \sqrt{1-q^2} & 0 & \cdots \\ 0 & 0 & \frac{1}{1-q} & \sqrt{1-q^3} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

$$E = \begin{pmatrix} \frac{1}{1-q} & 0 & 0 & 0 & \cdots \\ \frac{\sqrt{1-q^1}}{1-q} & \frac{1}{1-q} & 0 & 0 & \cdots \\ 0 & \frac{\sqrt{1-q^2}}{1-q} & \frac{1}{1-q} & 0 & \cdots \\ 0 & 0 & \frac{\sqrt{1-q^3}}{1-q} & \frac{1}{1-q} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

2. Mockin path view

Define the transfer-matrix of 1-D Mockin random-walk as $T = D + E$. Explicitly:

$$T = \begin{pmatrix} \frac{1}{1-q} & \sqrt{1-q^1} & 0 & 0 & \cdots \\ \frac{\sqrt{1-q^1}}{1-q} & \frac{1}{1-q} & \sqrt{1-q^2} & 0 & \cdots \\ 0 & \frac{\sqrt{1-q^2}}{1-q} & \frac{1}{1-q} & \sqrt{1-q^3} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \tag{8}$$

In terms of Mockin path it means that the rates of moving up, down and horizontally (u_h, d_h and f_h respectively) are:

$$u_h = T_{h,h+1} = \sqrt{1 - q^{h+1}},$$
$$d_h = T_{h,h-1} = \frac{\sqrt{1-q^h}}{1-q}, \quad (h \geq 1), \quad T_{0,-1} = 0,$$
$$f_h = T_{h,h} = \frac{2}{1-q}.$$

This matrix may be symmetrized by transform $\widetilde{T} = GTG^{-1}$, where $G = diag((1-q)^{\frac{h}{2}})$.

Using Farvard's and spectral theorems, we may diagonalize $\widetilde{T}$ with the help of **continuous q-Hermite polynomials**

$$\boxed{\widetilde{T} \overset{U}{\cong} \left(\frac{2}{1-q}\mathbb{I} + \frac{2}{\sqrt{1-q}} M_x \right) \text{ on } L^2([-1, 1], d\mu(x))} \tag{9}$$

Where $U : l^2 \to L^2(\mu), \quad (Uf)(x) = \sum_{n \geq 0} f_n p_n(x)$ with $p_n(x) = \kappa_n H_n(x \mid q)$,

$$\frac{\kappa_{n+1}}{\kappa_n} = \frac{1}{1 - q^{n+1}}, \quad \frac{\kappa_{n-1}}{\kappa_n} = \sqrt{1 - q^n}.$$

The orthogonality measure:

$$d\mu(x) = \frac{1}{2\pi} \frac{w(x \mid q)}{\sqrt{1 - x^2}} dx.$$

Where

$$w(x \mid q) = \left| \left(e^{2i\theta}; q \right)_\infty \right|^2 = h(x, 1) h(x, -1) h\left(x, q^{\frac{1}{2}}\right) h\left(x, -q^{\frac{1}{2}}\right),$$

with

$$h(x, \alpha) := \prod_{k=0}^{\infty} \left[1 - 2\alpha x q^k + \alpha^2 q^{2k} \right] = \left(\alpha e^{i\theta}, \alpha e^{-i\theta}; q \right)_\infty,$$
$$x = \cos\theta.$$

3. The phase diagram

How boundary conditions – 2 last eq. in (6) – affect the matrix representation of algebra?

Note that the representation that we are going to assume is just one of many possible ones.

We will use the following notation:

$$\langle W|n\rangle = w_n, \quad |n\rangle = 0, 1, 2 \dots,$$
$$\langle n|V\rangle = v_n.$$

We will demand (7) stay the same, therefore:

$$\begin{aligned} w_{n+1} = \frac{\gamma}{\alpha}(1-q) \sqrt{\frac{1-q^n}{1-q^{n+1}}} w_{n-1} + \frac{1 - (\alpha - \gamma)}{\alpha \sqrt{1 - q^{n+1}}} w_n, \\ v_{n+1} = \frac{\delta}{\beta} \frac{1}{(1-q)} \sqrt{\frac{1-q^n}{1-q^{n+1}}} v_{n-1} + \frac{1 - (\beta - \delta)}{\beta(1-q)\sqrt{1-q^{n+1}}} v_n. \end{aligned} \tag{10}$$

The phase transition occurs in the poles of $\langle W| T^N |V\rangle$. Using (9) we obtain:

$$\langle W| T^N |V\rangle = \int_{-1}^{1} \left(\frac{2}{1-q} + \frac{2}{\sqrt{1-q}} x \right)^N \widetilde{W}(x) \widetilde{V}(x) d\mu(x).$$

Where $\widetilde{W}(x) = \sum_{n \geq 0} (G^{-1}W)_n p_n(x)$,
$\widetilde{V}(x) = \sum_{n \geq 0} (GV)_n p_n(x)$.

If $q^n \to 0$ (n is large) our linear recursion on w_n, v_n (10) becomes constant, so $w_n \sim r^n, \quad v_n \sim r^n$ for some r, therefore:

$$r^2 - Br - A(1-q) = 0,$$
$$r^2 - Dr - C = 0.$$

Where $A = \frac{\gamma}{\alpha}, \quad B = \frac{1-(\alpha-\gamma)}{\alpha}, \quad C = \frac{\delta}{\beta(1-q)}, \quad D = \frac{1-(\beta-\delta)}{\beta(1-q)}$.

Let us denote $x_{eff} = \frac{r + r^{-1}}{2}$, **then the phase transition occurs when $x_{eff} = \pm 1$, or equally $r = \pm 1$.**

Then we analyze asymptotics of an integral, using the Laplace method:

- $\alpha > \frac{1+\gamma(2-q)}{2} \backsim \beta > \frac{1+2\delta}{2-q} \to \nu = \frac{3}{2}$,
- $\alpha < \frac{1+\gamma(2-q)}{2}$ and $\lambda_L > \max(\lambda(1), \lambda_R) \to \nu = \frac{1}{2}$,
- $\beta < \frac{1+2\delta}{2-q}$ and $\lambda_R > \max(\lambda(1), \lambda_L) \to \nu = \frac{1}{2}$,
- $\alpha = \frac{1+\gamma(2-q)}{2}$ or $\beta = \frac{1+2\delta}{2-q} \to \nu = \frac{1}{2}$,
- if both $\alpha = \frac{1+\gamma(2-q)}{2}, \quad \beta = \frac{1+2\delta}{2-q} \to \nu = 0$.

4. Hydrodynamic limit

In [1] Timo Seppalainen proves the formula for the flux function of a process with ergodic initial distribution and the expectation $\mathbb{E}(\eta(i, 0)) = \rho$

$$\begin{aligned} f(\rho) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \Big[p\, \mathbb{P}\Big(\eta(i, s) = 1, \eta(i+1, s) = 0\Big) - \\ -q\, \mathbb{P}\Big(\eta(i, s) = 0, \eta(i+1, s) = 0\Big) \Big]. \end{aligned} \tag{11}$$

One may show that $\nu_\rho(\eta_i = 1) = \rho$ is the invariant measure for such a process, i.e satisfies the condition: $\int Lf(\eta) \nu_\rho(d\eta) = 0 \quad \forall f \in C_0$. Therefore, we should substitute ν_ρ in (11) ($\nu_\rho = P_n$). Thus we get

$$f(\rho) = (p - q)\rho(1 - \rho).$$

Therefore, PDE:

$$\partial_t u + \partial_x((p-q)u(1-u)) = 0.$$

Well-known results for other models:

1. TASEP

$$f(\rho) = \rho(1 - \rho).$$

2. q-TASEP

$$f_\alpha(\rho) = \alpha \cdot \rho, \quad \rho = \frac{\log q}{\log q + \log(1-q) + \Psi_q(\log_q \alpha)}.$$

   Where α is the parameter of the q-geometric distribution, $\Psi_q(z) = \frac{\partial}{\partial z} \log \Gamma_q(z)$

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