Kac–Moody Algebras

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1 Finite–Dimensional Semisimple Lie Algebras and Their Generators

Theorem 1.1 (Serre Theorem). *A finite–dimensional semisimple Lie algebra $\mathfrak{g}$ over the field $\mathbb{C}$ is uniquely determined by a set of generators $\{e_i, f_i, h_i \mid i = 1, \ldots, l\}$ with the relations:*

$$[h_i, h_j] = 0, \tag{1}$$
$$[h_i, e_j] = a_{ij} e_j, \tag{2}$$
$$[h_i, f_j] = -a_{ij} f_j, \tag{3}$$
$$[e_i, f_j] = \delta_{ij} h_i, \tag{4}$$
$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad i \neq j, \tag{5}$$
$$(\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0, \quad i \neq j, \tag{6}$$

where $A = (a_{ij})$ is the Cartan matrix. Relations 1–4 are called the Chevalley relations, while 5, 6 are the Serre relations.

The Chevalley relations describe the commutation rules between elements of the Cartan subalgebra $\mathfrak{h}$ and the root vectors e_i, f_i. The Serre relations restrict the repeated action of the operators $\operatorname{ad} e_i$ and $\operatorname{ad} f_i$ on root vectors. They ensure that the root system Φ is generated by simple roots Π with finitely many elements, and that the algebra $\mathfrak{g}$ has finite dimension.

Example 1.2. *Consider the semisimple Lie algebra $\mathfrak{sl}_2$. Its Cartan matrix consists of a single entry: $A = (2)$. The algebra $\mathfrak{sl}_2$ has a basis $\{e, f, h\}$, where:*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here the Serre relations 5, 6 vanish since $l = 1$ (there are no pairs $i \neq j$). We obtain:

$$[h, e] = 2e,$$
$$[h, f] = -2f,$$
$$[e, f] = h.$$

Remark 1.3. *If the Cartan matrix A is degenerate, i.e. $\det A = 0$, then the Lie algebra defined by relations 1–6 becomes infinite–dimensional. Degeneracy of A implies a linear dependence between simple roots, which leads to infinitely many roots in the root system. In this case, Serre relations 5, 6 do not restrict the growth of root vectors, and the algebra becomes infinite–dimensional.*

2 Generalized Cartan Matrix and Kac–Moody Algebras

2.1 Realizations of a Square Matrix

Definition 2.1. *Let A be an $n \times n$ matrix over $\mathbb{C}$. A **realization** of A is a triple $(H, \Pi, \Pi^\vee)$ where:*

- H is a finite–dimensional vector space over $\mathbb{C}$;

- $\Pi^\vee = \{h_1, \ldots, h_n\} \subset H$ is a set of linearly independent vectors;

- $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset H^*$ is a set of linearly independent functionals such that

$$\alpha_j(h_i) = A_{ij} \quad \forall\, i, j.$$

Proposition 2.2. *If $(H, \Pi, \Pi^\vee)$ is a realization of A, then*

$$\dim H \geq 2n - \operatorname{rank} A.$$

Definition 2.3. *A realization of A is called **minimal** if*

$$\dim H = 2n - \operatorname{rank} A.$$

Proposition 2.4 (Example of constructing a minimal realization)**.** *Any $n \times n$ complex matrix admits a minimal realization.*

Proof: Since $\operatorname{rank} A = l$, the matrix A contains a non–singular $l \times l$ submatrix. After permuting rows and columns, we obtain a block matrix:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Consider the matrix

$$C = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-l} \\ 0 & I_{n-l} & 0 \end{pmatrix}.$$

This is a $(2n - l) \times (2n - l)$ matrix, and

$$\det C = \pm \det A_{11} \neq 0,$$

hence C is nondegenerate.

Let H be the space of $(2n - l)$-tuples over $\mathbb{C}$. Define $\alpha_1, \ldots, \alpha_n \in H^*$ by

$$(\lambda_1, \ldots, \lambda_{2n-l}) \mapsto \lambda_i, \quad i = 1, \ldots, n.$$

Let $h_1, \ldots, h_n \in H$ be the first n rows of C. Then the sets $\{\alpha_i\}$ and $\{h_i\}$ are linearly independent, giving a realization of

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with $\dim H = 2n - l$. Undoing the permutation yields a minimal realization of A. $\square$

Definition 2.5. *Let $(H, \Pi, \Pi^\vee)$ and $(H', \Pi', (\Pi')^\vee)$ be two realizations of A. They are called **isomorphic** if there exists an isomorphism*

$$\varphi : H \to H'$$

such that $\varphi(h_i) = h'_i$ and $\varphi^*(\alpha'_i) = \alpha_i$, where $\varphi^* : (H')^* \to H^*$ is the induced dual map.

Proposition 2.6. *Any two minimal realizations of an $n \times n$ matrix A over $\mathbb{C}$ are isomorphic.*

2.2 Lie Algebras $\tilde{L}(A)$ Associated with Complex Matrices

Definition 2.7. *Let A be an $n \times n$ complex matrix of rank l. Let $(H, \Pi, \Pi^\vee)$ be its minimal realization. Then*

$$\dim H = 2n - l, \qquad \Pi^\vee = \{h_1, \ldots, h_n\} \subset H, \qquad \Pi = \{\alpha_1, \ldots, \alpha_n\} \subset H^*, \qquad \alpha_j(h_i) = A_{ij}.$$

Define the Lie algebra $\tilde{L}(A)$ by generators and relations. Let

$$X = \{e_1, \ldots, e_n, \ f_1, \ldots, f_n, \ \tilde{x} \mid x \in H\},$$

and let R be the set of Lie relations:

$$\tilde{x} = \lambda \tilde{y} + \mu \tilde{z}, \quad \text{for all } x, y, z \in H, \ \lambda, \mu \in \mathbb{C}, \ \text{such that } x = \lambda y + \mu z, \tag{7}$$

$$[\tilde{x}, \tilde{y}] = 0, \quad \forall\, x, y \in H, \tag{8}$$

$$[e_i, f_i] = \tilde{h}_i, \quad i = 1, \ldots, n, \tag{9}$$

$$[e_i, f_j] = 0, \quad i \neq j, \tag{10}$$

$$[\tilde{x}, e_i] = \alpha_i(x) e_i, \quad \forall\, x \in H, \ i = 1, \ldots, n, \tag{11}$$

$$[\tilde{x}, f_i] = -\alpha_i(x) f_i, \quad \forall\, x \in H, \ i = 1, \ldots, n. \tag{12}$$

We define $\tilde{L}(A) = L(X; R)$ as the Lie algebra generated by X with relations R.

Lemma 2.8. *Choosing another minimal realization of A does not change $\tilde{L}(A)$ up to isomorphism.*

Proof: Follows from Proposition 2.6. □

Proposition 2.9. *There exists an automorphism $\widetilde{\omega}$ of $\tilde{L}(A)$ uniquely defined by*

$$\widetilde{\omega}(e_i) = -f_i, \quad \widetilde{\omega}(f_i) = -e_i, \quad \widetilde{\omega}(\tilde{x}) = -\tilde{x}$$

for all $x \in H$. Moreover, $\widetilde{\omega}^2 = 1$.

Proof: Let $[F(X)]$ be the Lie algebra constructed from the associative algebra $F(X)$ in the standard way. The set X is viewed as a subset of $[F(X)]$. Let $FL(X)$ be the intersection of all Lie subalgebras of $[F(X)]$ containing X; equivalently, the Lie algebra freely generated by X. Clearly $X \subset FL(X)$, so there exists an embedding

$$i : X \to FL(X).$$

Proposition 2.10. *Let $\theta : X \to L$ be an arbitrary map into a Lie algebra L. Then there exists a unique homomorphism of Lie algebras*

$$\varphi : FL(X) \to L$$

such that $\varphi \circ i = \theta$.

□ There exists a map $\widetilde{\omega} : X \to FL(X)$ defined by the formulas given above. By Proposition 2.10, there exists a unique Lie algebra homomorphism

$$FL(X) \to FL(X)$$

extending this map. We denote it by the same symbol $\widetilde{\omega}$. Moreover,

$$\widetilde{\omega}^2 = 1.$$

Let $\langle R \rangle$ be the ideal of the free Lie algebra $FL(X)$ generated by the set of relations R. Applying $\widetilde{\omega}$ to elements of R, we see that

$$\widetilde{\omega}(\langle R \rangle) \subset \langle R \rangle,$$

and therefore $\widetilde{\omega}$ induces a homomorphism

$$\widetilde{\omega} : FL(X)/\langle R \rangle \to FL(X)/\langle R \rangle.$$

Since $\widetilde{\omega}^2 = 1$, this is an automorphism of the algebra $\tilde{L}(A)$. $\square$

Let $\widetilde{H}$ denote the subalgebra of $\tilde{L}(A)$ generated by the elements $\tilde{x}$ for all $x \in H$. Let $\widetilde{N}$ be the subalgebra generated by $e_1, \ldots, e_n$, and $\widetilde{N}^-$ the subalgebra generated by $f_1, \ldots, f_n$. Then

$$\widetilde{\omega}(\widetilde{H}) = \widetilde{H}, \qquad \widetilde{\omega}(\widetilde{N}) = \widetilde{N}^-, \qquad \widetilde{\omega}(\widetilde{N}^-) = \widetilde{N}.$$

Let now V be an n–dimensional vector space over $\mathbb{C}$ with basis $v_1, \ldots, v_n$, and let

$$T(V) = \bigoplus_{s \geq 0} T^s(V)$$

be the tensor algebra of V. Then $T^s(V)$ has a basis of vectors

$$v_{i_1} \otimes \cdots \otimes v_{i_s} = v_{i_1} \ldots v_{i_s},$$

for all $i_1, \ldots, i_s \in \{1, \ldots, n\}$. For each linear functional $\lambda \in H^*$, define a map

$$\theta_\lambda : X \to \operatorname{End} T(V).$$

It is enough to specify its action on basis elements of $T(V)$.

For $x \in H$:

$$\theta_\lambda(\tilde{x}) \cdot 1 = \lambda(x) \cdot 1,$$

$$\theta_\lambda(\tilde{x}) \cdot (v_{i_1} \ldots v_{i_s}) = (\lambda - \alpha_{i_1} - \cdots - \alpha_{i_s})(x) \cdot v_{i_1} \ldots v_{i_s}. \tag{13}$$

For f_i:

$$\theta_\lambda(f_j) \cdot 1 = v_j,$$

$$\theta_\lambda(f_j) \cdot (v_{i_1} \ldots v_{i_s}) = v_j v_{i_1} \ldots v_{i_s}. \tag{14}$$

For e_i, defined inductively in s:

$$\theta_\lambda(e_j) \cdot 1 = 0,$$

$$\theta_\lambda(e_j) \cdot v_i = \delta_{ij} \lambda(h_j) \cdot 1,$$

$$\theta_\lambda(e_j) \cdot (v_{i_1} \ldots v_{i_s}) = v_{i_1} \cdot \big(\theta_\lambda(e_j)(v_{i_2} \ldots v_{i_s})\big) + \delta_{ij}\big(\lambda - \alpha_{i_2} - \cdots - \alpha_{i_s}\big)(h_j) \cdot v_{i_2} \ldots v_{i_s}, \quad s > 1. \tag{15}$$

Proposition 2.11. *The map $\theta_\lambda : X \to \operatorname{End}(T(V))$ extends to a Lie algebra homomorphism*

$$\tilde{L}(A) \to \operatorname{End}(T(V)).$$

Proof: The map $\theta_\lambda : X \to \operatorname{End} T(V)$ extends to a homomorphism $\theta_\lambda : FL(X) \to [\operatorname{End} T(V)]$. From Proposition 2.10 we have $\tilde{L}(A) \cong FL(X)/\langle R \rangle$. Thus, to show that θ_λ induces a map

$$\tilde{L}(A) \to \operatorname{End}(T(V)),$$

it suffices to verify that $\theta_\lambda(r) = 0$ for all $r \in R$. $\square$

Corollary 2.12. *The map $x \mapsto \tilde{x}$ is an isomorphism of vector spaces $H \to \widetilde{H}$.*

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Proof: The subalgebra $\widetilde{H}$ of $\tilde{L}(A)$ is generated by elements $\tilde{x}$ for all $x \in H$. These form a Lie algebra themselves because

$$\tilde{x}_1 + \tilde{x}_2 = \widetilde{x_1 + x_2}, \qquad \lambda\tilde{x} = \widetilde{\lambda x}, \qquad [\tilde{x}_1, \tilde{x}_2] = 0.$$

Thus,

$$\widetilde{H} = \{\tilde{x} \mid x \in H\}.$$

Consider the map

$$H \to \widetilde{H}, \qquad x \mapsto \tilde{x}.$$

It is surjective by construction. To show injectivity, assume $\tilde{x} = 0$ for some $x \in H$. Then

$$\theta_\lambda(\tilde{x}) = 0 \Rightarrow \lambda(x) = 0 \quad \forall \lambda \in H^*.$$

Since this holds for every linear functional, we conclude $x = 0$. $\square$

Proposition 2.13.

$$\tilde{L}(A) = \widetilde{N}^- \oplus \widetilde{H} \oplus \widetilde{N}$$

as a direct sum of subspaces.

Proof: Let $I := \widetilde{N}^- + \widetilde{H} + \widetilde{N}$. It is enough to check that I is an ideal of $\tilde{L}(A)$, i.e.

$$\mathrm{ad}e_i(I) \subset I, \quad \mathrm{ad}f_i(I) \subset I, \quad \mathrm{ad}\tilde{x}(I) \subset I.$$

We omit the computations.

To prove directness, suppose

$$w_- + \tilde{x} + w = 0,$$

where $w_- \in \widetilde{N}^-$, $\tilde{x} \in \widetilde{H}$, $w \in \widetilde{N}$. Applying θ_λ gives

$$\theta_\lambda(w_- + \tilde{x} + w) \cdot 1 = 0,$$

so

$$\theta_\lambda(w_-) \cdot 1 + \lambda(x) \cdot 1 = 0.$$

But $\theta_\lambda(w_-) \cdot 1 \in \bigoplus_{s \geq 1} T^s(V)$, while $\lambda(x) \cdot 1 \in T^0(V)$, hence they vanish separately:

$$\theta_\lambda(w_-) = 0, \qquad \lambda(x) = 0 \ \forall \lambda \Rightarrow x = 0.$$

Thus $\tilde{x} = 0$. Since $\varphi : \widetilde{N}^- \to FL(v_1, \ldots, v_n)$ is an isomorphism, $\theta_\lambda(w_-) = 0 \Rightarrow w_- = 0$, and hence $w = 0$.

Therefore

$$\tilde{L}(A) = \widetilde{N}^- \oplus \widetilde{H} \oplus \widetilde{N}.$$

$\square$

Let Q be the subgroup of H^* defined by

$$Q = \{\, \alpha = k_1\alpha_1 + \cdots + k_n\alpha_n \mid k_i \in \mathbb{Z} \,\}.$$

Let $Q^+ = \{\alpha \neq 0 \mid k_i \geq 0\}$ and $Q^- = \{\alpha \neq 0 \mid k_i \leq 0\}$. For each $\alpha \in Q$, define

$$\tilde{L}_\alpha = \{y \in \tilde{L}(A) \mid [\tilde{x}, y] = \alpha(x)y \ \forall x \in H\}.$$

Proposition 2.14. *(i)* $\tilde{L}(A) = \bigoplus_{\alpha \in Q} \tilde{L}_\alpha$.

(ii) $\dim \tilde{L}_\alpha$ *is finite for every* $\alpha \in Q$.

(iii) $\tilde{L}_0 = \widetilde{H}$.

(iv) *If* $\alpha \neq 0$ *and* $\alpha \notin Q^+$, $\alpha \notin Q^-$, *then* $\tilde{L}_\alpha = 0$.

(v) $[\tilde{L}_\alpha, \tilde{L}_\beta] \subset \tilde{L}_{\alpha+\beta}$ *for all* $\alpha, \beta \in Q$.

Proof: To prove $\tilde{L}(A) = \sum_{\alpha \in Q} \tilde{L}_\alpha$, it suffices to show

$$\widetilde{H} \subset \tilde{L}_0, \qquad \widetilde{N} \subset \sum_{\alpha \in Q^+} \tilde{L}_\alpha, \qquad \widetilde{N}^- \subset \sum_{\alpha \in Q^-} \tilde{L}_\alpha.$$

Clearly $\widetilde{H} \subset \tilde{L}_0$. Any monomial w in $e_1, \ldots, e_n$ satisfies

$$[\tilde{x}, w] = \alpha(x)w$$

for some $\alpha \in Q^+$. Closing under brackets shows $\widetilde{N} \subset \sum_{\alpha \in Q^+} \tilde{L}_\alpha$, and similarly $\widetilde{N}^- \subset \sum_{\alpha \in Q^-} \tilde{L}_\alpha$. Thus $\tilde{L}(A) = \sum_{\alpha \in Q} \tilde{L}_\alpha$.

To prove directness, suppose

$$v_1 + \cdots + v_k = 0, \qquad v_i \in \tilde{L}_{\beta_i}, \quad \beta_i \text{ distinct}.$$

Minimality of k yields a contradiction by comparing eigenvalues under $\operatorname{ad}\tilde{x}$, hence the sum is direct:

$$\tilde{L}(A) = \bigoplus_{\alpha \in Q} \tilde{L}_\alpha.$$

From Proposition 2.13 and inclusions

$$\widetilde{N}^- \subset \sum_{\alpha \in Q^-} \tilde{L}_\alpha, \qquad \widetilde{H} \subset \tilde{L}_0, \qquad \widetilde{N} \subset \sum_{\alpha \in Q^+} \tilde{L}_\alpha$$

we obtain (iii),(iv),(v).

Finally, $\dim \tilde{L}_\alpha < \infty$: if $\alpha = k_1\alpha_1 + \cdots + k_n\alpha_n$ with $k_i \geq 0$, then monomials containing e_i exactly k_i times span $\tilde{L}_\alpha$, and there are finitely many of them; similarly for Q^-.

In particular:
$$\dim \tilde{L}_{\alpha_i} = 1, \qquad \dim \tilde{L}_{-\alpha_i} = 1,$$
$$\dim \tilde{L}_{k\alpha_i} = 0, \quad \dim \tilde{L}_{-k\alpha_i} = 0, \quad k > 1.$$

$\square$ And now we finally come close to the definition of the Kac–Moody algebra. But first we need to prove one more important statement — and for that, naturally, we need a lemma.

Lemma 2.15. *Let H be a finite–dimensional abelian Lie algebra, and let V be an H–module such that*

$$V = \bigoplus_{\lambda \in H^*} V_\lambda,$$

where

$$V_\lambda = \{v \in V \mid xv = \lambda(x)v \text{ for all } x \in H\}.$$

Let U be a submodule of V. Then

$$U = \bigoplus_{\lambda \in H^*} (U \cap V_\lambda).$$

Proposition 2.16. *The algebra $\tilde{L}(A)$ contains a unique ideal I, maximal with respect to the condition*

$$I \cap \widetilde{H} = 0.$$

Proof: Let J be an arbitrary ideal in $\tilde{L}(A)$ satisfying $J \cap \widetilde{H} = 0$. By Proposition 2.14,

$$\tilde{L}(A) = \bigoplus_{\alpha \in H^*} \tilde{L}_\alpha.$$

View $\tilde{L}(A)$ as an $\widetilde{H}$–module. By Lemma 2.15,

$$J = \bigoplus_{\alpha \in H^*} (\tilde{L}_\alpha \cap J).$$

Each $\tilde{L}_\alpha$ with $\alpha \neq 0$ lies in $\widetilde{N}$ or $\widetilde{N}^-$, hence

$$J = (\widetilde{N}^- \cap J) \oplus (\widetilde{N} \cap J).$$

In particular, $J \subset \widetilde{N}^- \oplus \widetilde{N}$.

Now let I be the ideal generated by all ideals J such that $J \cap \widetilde{H} = 0$. All such J lie in $\widetilde{N}^- \oplus \widetilde{N}$, hence $I \subset \widetilde{N}^- \oplus \widetilde{N}$, and therefore $I \cap \widetilde{H} = 0$.

Thus, I is the unique ideal of $\tilde{L}(A)$ maximal under the condition $I \cap \widetilde{H} = 0$. $\square$

2.3   The Kac–Moody algebra $L(A)$

Definition 2.17. *An $n \times n$ matrix $A = (A_{ij})$ is called a **generalized Cartan matrix (GCM)** if:*

$$A_{ii} = 2, \qquad i = 1, \ldots, n,$$

$$A_{ij} \in \mathbb{Z}, \qquad A_{ij} \leq 0 \ for \ i \neq j,$$

$$A_{ij} = 0 \Rightarrow A_{ji} = 0.$$

The Cartan matrix of any finite–dimensional simple Lie algebra is a GCM.

Definition 2.18. *From now on we assume that A is a GCM. Let $\tilde{L}(A)$ be the Lie algebra associated to A, and let I be the unique maximal ideal with $I \cap \widetilde{H} = 0$. Define*

$$L(A) = \tilde{L}(A)/I. \tag{16}$$

*The Lie algebra $L(A)$ is called the **Kac–Moody algebra** associated with the GCM A.*

There is a natural homomorphism

$$\theta : \tilde{L}(A) \to L(A).$$

Set $N = \theta(\widetilde{N})$ and $N^- = \theta(\widetilde{N}^-)$.

Example 2.19. *Consider the GCM*

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

This matrix corresponds to the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Let $(H, \Pi, \Pi^\vee)$ be its minimal realization with $\Pi = \{\alpha_0, \alpha_1\}$ — the set of simple roots, $\Pi^\vee = \{h_0, h_1\}$. The relations $\alpha_j(h_i) = A_{ij}$ give:

$$\alpha_0(h_0) = 2, \quad \alpha_0(h_1) = -2, \quad \alpha_1(h_0) = -2, \quad \alpha_1(h_1) = 2.$$

The dimension of the Cartan subalgebra is

$$\dim H = 2n - \operatorname{rank}(A) = 4 - 1 = 3.$$

Let $\{h_0, h_1, d\}$ be a basis of H, where d is called the degree derivation, chosen so that $\alpha_0(d) = \alpha_1(d) = 1$.

The algebra $\tilde{L}(A)$ is generated as a free Lie algebra by

$$\{e_0, e_1, f_0, f_1\} \cup \{\tilde{x} \mid x \in H\}$$

with relations

$$[e_0, f_0] = h_0, \qquad [e_1, f_1] = h_1, \qquad [e_0, f_1] = [e_1, f_0] = 0,$$

$$[h_0, e_0] = 2e_0, \qquad [h_0, e_1] = -2e_1, \qquad [h_0, f_0] = -2f_0, \qquad [h_0, f_1] = 2f_1,$$

$$[h_1, e_0] = -2e_0, \qquad [h_1, e_1] = 2e_1, \qquad [h_1, f_0] = 2f_0, \qquad [h_1, f_1] = -2f_1,$$

$$[d, e_i] = e_i, \qquad [d, f_i] = -f_i, \qquad [d, h_i] = 0.$$

To compute the maximal ideal I with $I \cap \widetilde{H} = 0$, one identifies all elements of I with 0 under the projection

$$\widetilde{L}(A) \to L(A) = \widetilde{L}(A)/I.$$

The commutation relations then become

$$[e_0, f_0] = h_0 + K, \qquad [e_1, f_1] = h_1 - K,$$

$$[e_i, f_j] = 0 \ (i \neq j),$$

$$[h_0, e_0] = 2e_0, \quad [h_0, e_1] = -2e_1, \qquad [h_1, e_0] = -2e_0, \quad [h_1, e_1] = 2e_1,$$

$$[h_0, f_0] = -2f_0, \quad [h_0, f_1] = 2f_1, \qquad [h_1, f_0] = 2f_0, \quad [h_1, f_1] = -2f_1,$$

$$[d, e_i] = e_i, \qquad [d, f_i] = -f_i, \qquad [d, h_i] = 0,$$

$$[K, \cdot] = 0 \ \textit{(central element)},$$

and the higher commutators vanish:

$$[e_0, e_1] = 0, \qquad [f_0, f_1] = 0.$$

Proposition 2.20.

$$L(A) = N^- \oplus \theta(\widetilde{H}) \oplus N.$$

Moreover, $\theta : \widetilde{H} \to \theta(\widetilde{H})$ is an isomorphism.

Proof: From Proposition 2.16,

$$I = (\widetilde{N}^- \cap I) \oplus (\widetilde{N} \cap I).$$

Since

$$\tilde{L}(A) = \widetilde{N}^- \oplus \widetilde{H} \oplus \widetilde{N},$$

we obtain

$$L(A) = N^- \oplus \theta(\widetilde{H}) \oplus N,$$

and θ is an isomorphism on $\widetilde{H}$. $\square$

By Corollary 2.12 there is a natural isomorphism $H \to \widetilde{H}$. Composing with θ gives an isomorphism $H \to \theta(\widetilde{H})$. We henceforth identify $\theta(\widetilde{H})$ with H, writing

$$L(A) = N^- \oplus H \oplus N.$$

A key structural theorem is the following.

Proposition 2.21. *Let $A = (A_{ij})$ be a GCM of size $n \times n$. Let L be a Lie algebra over $\mathbb{C}$, and let H be a finite–dimensional abelian subalgebra with $\dim H = 2n - \mathrm{rank}(A)$. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset H^*$ and $\Pi^\vee = \{h_1, \ldots, h_n\} \subset H$ be linearly independent sets with $\alpha_j(h_i) = A_{ij}$.*

Let $e_1, \ldots, e_n, f_1, \ldots, f_n \in L$ satisfy

$$[e_i, f_i] = h_i, \qquad [e_i, f_j] = 0 \ (i \neq j),$$

$$[x, e_i] = \alpha_i(x) e_i, \qquad [x, f_i] = -\alpha_i(x) f_i \quad (x \in H),$$

and suppose that e_i, f_i, H generate L, and L contains no nontrivial ideal intersecting H trivially. Then L is isomorphic to the Kac–Moody algebra $L(A)$.

Proof: These generators satisfy all defining relations of $\tilde{L}(A)$, giving a surjective homomorphism

$$\theta : \tilde{L}(A) \to L.$$

Since $\theta : \widetilde{H} \to H$ is an isomorphism (Corollary 2.12), $\ker \theta \cap \widetilde{H} = 0$. Hence $\ker \theta \subset I$, where I is the maximal ideal with trivial intersection with $\widetilde{H}$. Since L contains no such nontrivial ideal, we must have $\ker \theta = I$, hence

$$L \cong \tilde{L}(A)/I = L(A).$$

$\square$

Corollary 2.22. *If A is an ordinary Cartan matrix, then $L(A)$ is a finite–dimensional semisimple Lie algebra with Cartan matrix A.*

    Proof: Here $\operatorname{rank}(A) = n$, so $\dim H = n$. A finite–dimensional semisimple Lie algebra satisfies the hypotheses of Proposition 2.21, hence it is isomorphic to $L(A)$. $\square$

    This shows that Kac–Moody algebras generalize the theory of finite–dimensional semisimple Lie algebras.

    To confuse notation pleasantly, we continue to denote by e_i, f_i, h_i the images of the same elements of $\tilde{L}(A)$ in $L(A)$.

Proposition 2.23. *There exists an automorphism ω of $L(A)$ with $\omega^2 = 1$, such that*

$$\omega(e_i) = -f_i, \qquad \omega(f_i) = -e_i, \qquad \omega(x) = -x \ (x \in H).$$

    Proof: By Proposition 2.9, $\tilde{L}(A)$ admits an involution $\widetilde{\omega}$ with $\widetilde{\omega}^2 = 1$. Since $\widetilde{\omega}(\widetilde{H}) = \widetilde{H}$, it preserves the maximal ideal I, so passes to the quotient, inducing ω on $L(A)$. $\square$

    As in Proposition 2.14, for each $\alpha \in Q$ define

$$L_\alpha = \{y \in L(A) \mid [x,y] = \alpha(x)y \ \forall x \in H\}. \tag{17}$$

Proposition 2.24.   *(i) $L(A) = \bigoplus_{\alpha \in Q} L_\alpha$,*

  (ii) $\dim L_\alpha < \infty$ for all $\alpha \in Q$,

  (iii) $L_0 = H$,

  (iv) If $\alpha \neq 0$ and $\alpha \notin Q^+$ or $\alpha \notin Q^-$, then $L_\alpha = 0$,

  (v) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.

    Proof: Let $\theta : \tilde{L}(A) \to L(A)$ be the canonical map. By Proposition 2.14,

$$\tilde{L}(A) = \bigoplus_{\alpha \in Q} \tilde{L}_\alpha.$$

By Lemma 2.15,

$$I = \bigoplus_{\alpha \in Q} (I \cap \tilde{L}_\alpha).$$

Hence

$$L(A) = \bigoplus_{\alpha \in Q} \theta(\tilde{L}_\alpha), \qquad L_\alpha = \theta(\tilde{L}_\alpha).$$

From Proposition 2.20,

$$L(A) = N^- \oplus H \oplus N,$$

and

$$N^- = \bigoplus_{\alpha \in Q^-} L_\alpha, \qquad H = L_0, \qquad N = \bigoplus_{\alpha \in Q^+} L_\alpha.$$

Finite dimensionality and bracket compatibility follow since $\theta(\tilde{L}_\alpha)$ is finite–dimensional and Jacobi gives

$$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}.$$

$\square$

Definition 2.25. *The subalgebra $H \subset L(A)$ is called the **Cartan subalgebra**. A nonzero $\alpha \in H^*$ is called a **root** if $L_\alpha \neq 0$. Roots in Q^+ are **positive**, those in Q^- are **negative**. The space L_α is the **root space** of α, and $\dim L_\alpha$ its **multiplicity**.*

For an ordinary Cartan matrix all root multiplicities equal 1, but for GCM's this is no longer true.

Proposition 2.26. *(i) $\dim L_{\alpha_i} = 1$ and $\dim L_{-\alpha_i} = 1$,*

(ii) If $k > 1$, then $L_{k\alpha_i} = 0$ and $L_{-k\alpha_i} = 0$.

Proof: Since $L_{\alpha_i} = \theta(\widetilde{L}_{\alpha_i})$ and $\dim \widetilde{L}_{\alpha_i} = 1$, $\dim L_{\alpha_i} \leq 1$. If $\dim L_{\alpha_i} = 0$ then $e_i \in I$, so $[e_i, f_i] = h_i \in I$, contradicting $I \cap \widetilde{H} = 0$. Thus $\dim L_{\alpha_i} = 1$. The same argument gives $\dim L_{-\alpha_i} = 1$.

Since $\widetilde{L}_{k\alpha_i} = 0$ for $k > 1$, also $L_{k\alpha_i} = 0$. $\square$

The elements $\alpha_1, \ldots, \alpha_n$ are called the **fundamental roots** of $L(A)$.

3 Classification of Generalized Cartan Matrices

Definition 3.1. *Two matrices A, A' are **equivalent** if they have the same size n and there exists a permutation σ of $\{1, \ldots, n\}$ such that*

$$A'_{ij} = A_{\sigma(i)\sigma(j)}.$$

Definition 3.2. *A matrix A is **indecomposable** if it is not equivalent to a block diagonal sum*

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

of two smaller GCM's. Equivalently, A is indecomposable iff A^t is.

Let $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. We write $v \geq 0$ if all $v_i \geq 0$, and $v > 0$ if all $v_i > 0$.

Definition 3.3. *A GCM A is of **finite type** if:*

(i) $\det A \neq 0$,

(ii) $\exists u > 0$ such that $Au > 0$,

(iii) if $Au \geq 0$ then $u > 0$ or $u = 0$.

*A is of **affine type** if:*

(i) $\operatorname{corank}(A) = 1$ (rank$(A) = n - 1$),

(ii) $\exists u > 0$ such that $Au = 0$,

(iii) if $Au \geq 0$ then $Au = 0$.

*A is of **indefinite type** if:*

(i) $\exists u > 0$ such that $Au < 0$,

(ii) if $Au \geq 0$ and $u \geq 0$, then $u = 0$.

Example 3.4. *1. Finite type matrices $-$ semisimple Lie algebras, corresponding to positive definite quadratic form.*

2. Affine type matrices $-$ affine Kac–Moody algebras, quadratic form is degenerate.

3. Indefinite type $-$ hyperbolic Kac–Moody algebras, quadratic form is indefinite.

Theorem 3.5. *Let A be an indecomposable GCM. Then exactly one of the following holds:*

(a) A is of finite type;

(b) *A is of affine type;*

(c) *A is of indefinite type.*

Moreover, A^t has the same type as A.

Theorem 3.6. *Let A be an indecomposable GCM. Then A is of finite type if and only if A is a Cartan matrix.*

We now define the Dynkin diagram $\Delta(A)$ associated to a GCM A. The vertices are $1, \ldots, n$. If $i \neq j$, the edge joining them is determined by (A_{ij}, A_{ji}):

1. $A_{ij}A_{ji} = 0$ — no edge,

2. $A_{ij}A_{ji} = 1$ — single edge,

3. $A_{ij}A_{ji} = 2$, $A_{ij} = -1$, $A_{ji} = -2$ — double edge with arrow $j \to i$,

4. $A_{ij}A_{ji} = 3$, $A_{ij} = -1$, $A_{ji} = -3$ — triple edge with arrow $j \to i$,

5. $A_{ij}A_{ji} = 4$, $A_{ij} = -1$, $A_{ji} = -4$ — quadruple edge with arrow $j \to i$,

6. $A_{ij}A_{ji} = 4$, $A_{ij} = -2$, $A_{ji} = -2$ — double edge with two arrows (both directions),

7. $A_{ij}A_{ji} \geq 5$ — edge marked with integers $|A_{ij}|, |A_{ji}|$.

Clearly A is completely determined by its Dynkin diagram $\Delta(A)$. Also, A is indecomposable iff $\Delta(A)$ is connected.

Proposition 3.7. *If A is a GCM whose Dynkin diagram lies in the affine list, then $\det A = 0$.*

Theorem 3.8. *Let A be an indecomposable GCM. Then A is affine if and only if its Dynkin diagram lies in the* **affine list***.*

Proof: The proof proceeds by complete case analysis:

- One vertex: $A = (2)$ — finite type $\mathfrak{sl}_2$, no affine diagrams.

- Two vertices:
$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}, \qquad a, b \in \mathbb{Z}_{>0}.$$

Affine condition: $\det A = 4 - ab = 0 \Rightarrow ab = 4$.
$$(a, b) = (1, 4), (4, 1) \Rightarrow \widetilde{A}_1, \qquad (a, b) = (2, 2) \Rightarrow \widetilde{A}_1'.$$

- For 3 or more vertices:

  - If the diagram contains a cycle, it is affine of type $\widetilde{A}_\ell$ ($\ell \geq 2$).
  - The only cases with triple edges:
  $$A = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \implies \widetilde{G}_2,$$

  or its transpose $\widetilde{G}_2'$. We verify $\det A = 0$ and all proper subdiagrams are finite.
  - Double and single edge families yield the remaining affine types.

$\square$