The Atiyah–Singer Theorem

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1 Interpretation and Definitions

Definition 1.1. *The index of an elliptic differential operator $\hat{D}$ is the difference between the number of independent solutions of the equation $\hat{D}f = 0$ and the number of independent solutions of the equation $\hat{D}^\dagger g = 0$, where $f \in \Gamma(E)$ and $g \in \Gamma(F)$ — the spaces of smooth sections of vector bundles. In mathematical notation,*

$$I = \dim\ker \hat{D} - \dim\ker \hat{D}^\dagger.$$

Theorem 1.2. *Let X be a compact closed manifold. Let E and F be two vector bundles over X (triples: total space, base and projection), let $\Gamma(E)$ and $\Gamma(F)$ be the spaces of smooth sections of these bundles, and let $\hat{D}$ be an elliptic differential operator (Laplace type, Poisson type, highest part positive definite) acting on sections in $\Gamma(E)$ in such a way that the result lies in $\Gamma(F)$, while acting on any section in $\Gamma(F)$ gives zero. Accordingly, the adjoint operator $\hat{D}^\dagger$ maps sections from $\Gamma(F)$ to $\Gamma(E)$ and annihilates the latter. (Clearly, both $\hat{D}$ and $\hat{D}^\dagger$ are nilpotent.) Then the index of such an operator is determined by the topology of the manifold and its associated structures. Which particular topology and which structures precisely — depends on $\hat{D}$.*

Example 1.3.

Definition 1.4. *The operator*

$$(\not{D}\Phi)_\alpha = (\gamma^M)_{\alpha\beta}\partial_M\Phi_\beta$$

is called the Dirac operator in flat space in the absence of gauge fields. Since γ^M transforms under rotations as a vector — just like the gradient operator ∂_M — it is easy to see that $(\not{D}\Phi)_\alpha$ transforms in the same way as Φ_α, i.e. represents a bispinor.

Definition 1.5. *The Dirac operator on a curved manifold with a gauge field is*

$$\not{D} = e^M_A\gamma^A\left(\partial_M + \frac{1}{4}\omega_{BC,M}\gamma^B\gamma^C + iA_M\right),$$

where $\omega_{BC,M}$ is the spin connection.

Consider an even-dimensional Dirac operator $\hat{D} = \not{D}(1+\gamma^{D+1})$ acting on right-handed spinors Ψ_R satisfying $\gamma^{D+1}\Psi_R = \Psi_R$, converting them into left-handed spinors. The operator $\not{D}(1+\gamma^{D+1})$ annihilates left-handed spinors. The adjoint operator $\hat{D}^\dagger = \not{D}(1-\gamma^{D+1})$ acts on left-handed spinors, converting them into right-handed ones, and annihilates right-handed spinors. Identifying the spaces of right- and left-handed spinors with fibers of bundles E and F, we see that $\hat{D}$ and $\hat{D}^\dagger$ act on sections exactly as required in the theorem.
In this case, the Atiyah–Singer index becomes

$$I_{\not{D}} = n_+^{(0)} - n_-^{(0)},$$

where $n_+^{(0)}$ is the number of zero modes (eigenstates with zero eigenvalue) of $\not{D}$ with positive chirality, and $n_-^{(0)}$ the number of zero modes with negative chirality (i.e. right-handed $\Psi_R : \gamma^5\Psi_R = +\Psi_R$ and left-handed $\Psi_L : \gamma^5\Psi_L = -\Psi_L$ spinors in 4D).

Note that the operators

$$\hat{D} = \not{D}(1+\gamma^{D+1}), \qquad \hat{D}^\dagger = \not{D}(1-\gamma^{D+1})$$

are nothing but supercharges: $\hat{D}$ corresponds to $\hat{Q}^\dagger$ and $\hat{D}^\dagger$ to $\hat{Q}$.

Theorem 1.6. *Consider the Dirac operator (13.6) defined on an even-dimensional manifold with Abelian or non-Abelian gauge field $A_M(x)$. Then the operators*

$$\hat{Q} = -\frac{i}{2\sqrt{2}}\,\not{D}(1-\gamma^{D+1}), \qquad \hat{Q}^\dagger = -\frac{i}{2\sqrt{2}}\,\not{D}(1+\gamma^{D+1}),$$

where

$$\gamma^{D+1} = \frac{(-i)^{D/2}}{D!}\varepsilon_{A_1\ldots A_D}\gamma^{A_1}\ldots\gamma^{A_D}$$

(or simply $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in 4D) satisfy the supersymmetry algebra (1.7) with Hamiltonian

$$\hat{H} = -\frac{\hat{\phi}^2}{2}.$$

Proof: This follows immediately from the facts that γ^{D+1} anticommutes with all γ^{A_j} and that $(\gamma^{D+1})^2 = \not{\mathbb{1}}$. □

Operators $\hat{D}$ and $\hat{D}^\dagger$ satisfying the theorem's assumptions can be interpreted as supercharges in full generality. Being nilpotent, they obey (1.7), where the operator $\{\hat{D}, \hat{D}^\dagger\}$ plays the role of the Hamiltonian. Therefore, the previously defined index of $\hat{D}$ is, up to a sign, precisely the Witten index (1.8)!

Essentially, the Atiyah–Singer theorem is a statement about the structure of the spectrum of a supersymmetric Hamiltonian.

Notes

Definition 1.7. *A quantum system is called supersymmetric if its Hamiltonian can be represented as the anticommutator of two nilpotent mutually adjoint operators called supercharges:*

$$\hat{Q}^2 = (\hat{Q}^\dagger)^2 = 0, \qquad \{\hat{Q}, \hat{Q}^\dagger\} = \hat{H}.$$

Definition 1.8. *The Witten index is defined as the difference between the number of bosonic and fermionic non-degenerate zero-energy states in a supersymmetric quantum system. It is written as:*

$$\tilde{Z}(\beta) \equiv I_W = n_B^{E=0} - n_F^{E=0}.$$

Atiyah–Singer theorem	Supersymmetry
$\Gamma(E)$	bosonic (or fermionic) states
$\Gamma(F)$	fermionic (or bosonic) states
$\hat{D}$, $\hat{D}^\dagger$	supercharges
$\{\hat{D}, \hat{D}^\dagger\}$	Hamiltonian
Atiyah–Singer index	Witten index

2 Hirzebruch Signature

Definition 2.1. *If*

$$\alpha = \alpha_{M_1\ldots M_p}\, dx^{M_1} \wedge \ldots \wedge dx^{M_p},$$

then

$$\star\alpha = \frac{1}{(D-p)!}\, E^{M_1\ldots M_p M_{p+1}\ldots M_D}\, \alpha_{M_1\ldots M_p}\, dx^{M_{p+1}} \wedge \ldots \wedge dx^{M_D},$$

where $E_{M_1\ldots M_D}$ is the covariant totally antisymmetric tensor $E_{M_1\ldots M_D} = \sqrt{g}\,\varepsilon_{M_1\ldots M_D}$.

With this definition one has the property

$$\star\star\,\alpha_p = (-1)^{p(D-p)}\,\alpha_p.$$

Consider the de Rham complex on a manifold of dimension $D = 4k$. Let us look at the Hodge duality operator. Note that its square is sometimes $+1$ and sometimes -1 — which is inconvenient, but motivates the following definition.

Definition 2.2. *The signature operator is the composition*

$$\hat{\tau} = (-1)^{p(p+1)/2} \circ \star \tag{1}$$

(which is equivalent to $\hat{\tau}\alpha_p = (-1)^{(4k-p)(4k-p+1)/2} \star \alpha_p = (-1)^{p(p-1)/2} \star \alpha_p$), whose square is identity and which anticommutes with the operator $\hat{R} = d + d^\dagger$ (and therefore commutes with the de Rham Laplacian $\Delta = -dd^\dagger - d^\dagger d = -\hat{R}^2$).

Thus the eigenvalues of $\hat{\tau}$ are $+1$ and -1. Hence all p-forms can be split into two subsets:
• S_+ — forms α for which $\hat{\tau}\alpha = \alpha$ (positive signature), • S_- — forms with negative signature.
The condition $\{\hat{R}, \hat{\tau}\} = 0$ means that $\hat{R}$ maps positive-signature states into negative-signature states and vice versa.

Definition 2.3. *The Hirzebruch signature of a $4k$-dimensional manifold is defined as*

$$\mathrm{sign} = b_+ - b_-, \tag{2}$$

where $b_\pm$ are the numbers of linearly independent closed but non-exact forms belonging to $S_\pm$.

Remark 2.4.
- *In fact, only forms of degree $p = D/2 = 2k$ contribute to (2). The duality operator maps p-forms to $(D-p)$-forms, and the same holds for $\hat{\tau}$. Therefore, for any $\alpha_{p\neq D/2}$ the forms $\alpha_p + \hat{\tau}\alpha_p$ belong to S_+ and $\alpha_p - \hat{\tau}\alpha_p$ to S_-, giving equal contributions to b_+ and b_- that cancel in the difference (2).*
- *The Hirzebruch signature is a special case of the Atiyah–Singer index and is a topological invariant.*

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Now move from geometry to dynamics. Consider the $N = 2$ supersymmetric sigma model on a manifold with dimension divisible by four, with Hamiltonian

$$\hat{H}^{\mathrm{cov}} = -\frac{1}{2\sqrt{g}} D_M\left(\sqrt{g}\, g^{MN} D_N\right) - \frac{1}{2} R_{ABCD}\, \hat{\bar{\psi}}_A \psi_B \hat{\bar{\psi}}_C \psi_D,$$

where $\hat{\bar{\psi}}_A = \partial/\partial\psi_A$.

Theorem 2.5. *In this language, the operator (1) has a beautiful representation:*

$$\hat{\tau} = \prod_{M=1}^{4k} \left(\frac{\partial}{\partial\psi^M} - \psi^M \right). \tag{3}$$

Proof: Write operator (3) as a product of k blocks:

$$\hat{\tau} = \prod_{M=1}^{4} \left(\frac{\partial}{\partial\psi^M} - \psi^M \right) \times \prod_{M=5}^{8} \left(\frac{\partial}{\partial\psi^M} - \psi^M \right) \times \ldots$$

Each block commutes with "foreign" fermionic variables (e.g. the first block commutes with $\psi^{M=5,\ldots,4k}$, since differentiation annihilates variables not present and multiplication by ψ^M leaves them unchanged). Checking its action on $\psi^{M=1,2,3,4}$ one sees:

- Acting on 1 (the empty state) it creates the antisymmetric basis:

$$\Big(\frac{\partial}{\partial\psi^1} - \psi^1\Big)1 = -\psi^1.$$

- Acting successively, it builds all fermionic degrees, for example:

$$\Big(\frac{\partial}{\partial\psi^2} - \psi^2\Big)\psi^1 = \frac{\partial}{\partial\psi^2}\psi^1 - \psi^2\psi^1 = -\psi^2\psi^1.$$

- For polynomials, differentiation lowers degree while multiplication increases it.

Thus the full operator $\hat{\tau}$ antisymmetrizes all combinations of variables. □

Proposition 2.6. *The Hirzebruch signature is*

$$\text{sign} = \text{Tr}^*\Big\{\hat{\tau}\, e^{-\beta\hat{H}}\Big\},$$

where Tr^* *denotes the operator trace in the Hilbert space.*

Such a statistical sum is known to be given by a (continuum) path integral:

$$Z = \int dq\, \mathcal{K}(q, q; -i\beta),$$

where $\mathcal{K}(q, q; -i\beta)$ is the Euclidean evolution operator in imaginary time $-i\beta$ with coincident endpoints.

Theorem 2.7. *The Hirzebruch signature is given by the integral*

$$\text{sign} = \int \det{}^{-1/2}\left[\frac{\tan(\mathcal{R}/2\pi)}{\mathcal{R}/2\pi}\right], \tag{4}$$

where $\mathcal{R}$ is the curvature two-form:

$$\mathcal{R}^{AB} = \frac{1}{2}R^{AB}{}_{MN}\, dx^M \wedge dx^N.$$

Proof: References the original works [59]. ****To be done if time permits. □

The integrand in (4) should be read as a Taylor series:

$$\det{}^{-1/2}\left[\frac{\tan(\mathcal{R}/2\pi)}{\mathcal{R}/2\pi}\right] = 1 - \frac{1}{24\pi^2}\,\text{Tr}\{\mathcal{R}\wedge\mathcal{R}\} + \text{higher terms}.$$

The series terminates at the term $\sim \overbrace{\mathcal{R}\wedge\ldots\wedge\mathcal{R}}^{D/2}$, proportional to the volume form. Only this term contributes to the integral. For a 4-dimensional manifold,

$$\text{sign} = -\frac{1}{24\pi^2}\int \text{Tr}\{\mathcal{R}\wedge\mathcal{R}\}.$$

3 Dirac Index

Why specifically the Dirac operator? As discussed earlier, the Dirac operator generates a pair of supersymmetric charges, whose action (see corresponding theorems) is isomorphic to the action of the exterior derivative and its adjoint in the real case, or the holomorphic derivative and its adjoint in the complex case. These in turn form the de Rham and Dolbeault complexes, respectively. Thus, the index of the Dirac operator can be related to the index of the de Rham or Dolbeault complexes. In essence, we obtain a topological invariant from an analytic quantity. Moreover, as we have also seen, the Dirac index is connected to the Witten index.

3.1 Kähler Manifolds

We now begin the computation of the Dirac index

$$I_D = \mathrm{Tr}^*\{\gamma^{D+1} e^{\beta \not{\mathcal{D}}/2}\}$$

on curved manifolds. We first consider the simpler case where the geometry is Kähler and the gauge field is Abelian.

Theorem 3.1. *For Kähler manifolds, the Dirac operator (1.5) with gauge field $A_M = (i\partial_m W, -i\partial_{\bar{m}} W)$, where W is the superpotential (gauge potential), admits a decomposition into holomorphic and antiholomorphic parts $\not{\mathcal{D}} = \not{\mathcal{D}}^{hol} - (\not{\mathcal{D}}^{hol})^\dagger$, where*

$$\not{\mathcal{D}}^{hol} = \sqrt{2}\, e_a^m \psi^a \left[\partial_m + \frac{1}{2}\omega_{\bar{b}c,m}(\hat{\bar{\psi}}^{\bar{b}}\psi^c - \psi^c\hat{\bar{\psi}}^{\bar{b}}) + iA_m\right]$$

$$(\not{\mathcal{D}}^{hol})^\dagger = -\sqrt{2}\, e_{\bar{a}}^{\bar{m}} \hat{\bar{\psi}}^{\bar{a}} \left[\partial_{\bar{m}} + \frac{1}{2}\omega_{b\bar{c},\bar{m}}(\psi^b\hat{\bar{\psi}}^{\bar{c}} - \hat{\bar{\psi}}^{\bar{c}}\psi^b) + iA_{\bar{m}}\right].$$

The holomorphic part may be mapped to the holomorphic exterior derivative (5) of the twisted Dolbeault complex. The antiholomorphic part maps to the exterior antiholomorphic derivative (6) of the twisted anti-Dolbeault complex (see Theorem 3.2).

Theorem 3.2. *The action of Q_W on wavefunctions $\Psi(z^m, \bar{z}^m; \psi^a)$ is isomorphic to the action of the nilpotent operator*

$$\partial_W = \partial - \partial\left(W - \frac{1}{4}\ln\det h\right)\wedge \tag{5}$$

on Dolbeault complex forms. The action of $\hat{\bar{Q}}_W$ on dual wavefunctions $\Psi(z^m, \bar{z}^m, \bar{\psi}^a)$ is isomorphic to the operator

$$\bar{\partial}_W = \bar{\partial} + \bar{\partial}\left(W + \frac{1}{4}\ln\det h\right)\wedge \tag{6}$$

in the anti-Dolbeault complex.

Remark 3.3. *In the case of a Kähler manifold, under $\gamma^A \to \sqrt{2}\hat{\Psi}^A$, the Dirac operator becomes*

$$\not{\mathcal{D}} = \sqrt{2}\, e_A^M\, \hat{\Psi}^A \left(\partial_M + \frac{1}{2}\omega_{BC,M}\hat{\Psi}^B\hat{\Psi}^C + iA_M\right).$$

The twisted Dolbeault complex is described by the $N=2$ sigma-model

$$S = \int d\bar{\theta}d\theta dt \left[\frac{1}{4}h_{m\bar{n}}(Z,\bar{Z})\,\bar{D}\bar{Z}^{\bar{n}}DZ^m + W(Z,\bar{Z})\right],$$

(a set of d holomorphic chiral $(2,2,0)$ multiplets $Z^m(\bar{D}Z^m = 0)$ and conjugate antiholomorphic multiplets $\bar{Z}^{\bar{m}}$, with Hermitian $h_{m\bar{n}}$ and real W).

We attempt to compute the Witten index in the small–β limit:

$$I_W = \lim_{\beta\to 0}\int \prod_j \frac{dp_j dq_j}{2\pi}\prod_\alpha d\psi_\alpha d\bar{\psi}_\alpha\, e^{-\beta H(p_j,q_j;\psi_a,\bar{\psi}_a)}. \tag{7}$$

The classical Hamiltonian for a general complex manifold is

$$H_W = h^{\bar{n}m}(\mathcal{P}_m + i\partial_m W)(\overline{\mathcal{P}}_{\bar{n}} - i\partial_{\bar{n}}W) - e_a^k e_c^m e_{\bar{b}}^{\bar{l}} e_{\bar{d}}^{\bar{n}}(\partial_k\partial_{\bar{l}}h_{m\bar{n}})\psi^a\psi^c\bar{\psi}^{\bar{b}}\bar{\psi}^{\bar{d}} - 2\partial_m\partial_{\bar{n}}W\,\psi^m\bar{\psi}^{\bar{n}}.$$

For Kähler manifolds the 4-fermion term is absent, and the Bismut connection reduces to Levi-Civita. Using also that $\omega_{ab,M}$ and $\omega_{\bar{a}\bar{b},M}$ vanish in the Kähler case,

$$H_W^{\text{Kähler}} = h^{\bar{k}j}\big(P_j - i\omega_{a\bar{b},j}\psi^a\psi^{\bar{b}} + i\partial_j W\big)\big(\bar{P}_{\bar{k}} - i\omega_{c\bar{d},\bar{k}}\psi^c\psi^{\bar{d}} - i\partial_{\bar{k}}W\big) - 2\partial_j\partial_{\bar{k}}W\,\psi^j\bar{\psi}^{\bar{k}}. \tag{8}$$

The corresponding Lagrangian is

$$L_W^{\text{Kähler}} = h_{j\bar{k}}\dot{z}^j\dot{\bar{z}}^{\bar{k}} + \frac{i}{2}\left(\psi^a\dot{\bar{\psi}}^{\bar{a}} - \dot{\psi}^a\bar{\psi}^{\bar{a}}\right) + i(\dot{\bar{z}}^{\bar{k}}\omega_{a\bar{b},\bar{k}} + \dot{z}^j\omega_{a\bar{b},j})\psi^a\bar{\psi}^{\bar{b}}$$
$$+ i(\dot{\bar{z}}^{\bar{k}}\partial_{\bar{k}}W - \dot{z}^j\partial_j W) + 2\partial_j\partial_{\bar{k}}W\,\psi^j\bar{\psi}^{\bar{k}}.$$

Compute (7) for $\mathbb{C}P^n$ choosing

$$W = \frac{q}{2(n+1)}\ln\det h = -\frac{q}{2}\ln(1+\bar{z}^k z^k),$$

since the natural metric is $h_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j}$ with $K = \ln(1+z\bar{z})$. Substituting into (8) gives

$$I_W = \int \prod_{k=1}^{n} \frac{dP_k d\bar{P}_k dz^k d\bar{z}^k}{(2\pi)^2} \prod_{a=1}^{n} d\psi^a d\bar{\psi}^a\, e^{-\beta H_W^{\text{Kähler}}}.$$

The result is

$$I_W = \left(\frac{q}{2\pi}\right)^n \int \frac{1}{(1+\bar{z}^l z^l)^{n+1}} \prod_k (dz^k d\bar{z}^k) = \frac{q^n}{n!}. \tag{9}$$

For $n = 1$, i.e. $\mathbb{C}P^1 \equiv S^2$, (9) yields the integer $I_W = q$, which is correct. But for $n \geq 2$ the index becomes fractional, contradicting integrality. A theorem states that q must be integer for odd n and half-integer for even n to define a well-posed field. Tree-level evaluation is insufficient — one-loop contributions are required to obtain the correct value.

A correct computation requires:

- Impose periodic boundary conditions

$$z^k(\beta) = z^k(0), \qquad \psi^a(\beta) = \psi^a(0).$$

- Expand all fields in Fourier modes

$$z^k(\tau) = z^k_{(0)} + \sum_{m\neq 0} z^k_{(m)} e^{2\pi i m\tau/\beta} \qquad \text{and similarly for } \psi^a(\tau).$$

- Substitute into

$$\tilde{Z}(\beta) = \int \prod_{j,\tau} dq_j(\tau) \prod_{\alpha,\tau} d\psi_\alpha(\tau) d\bar{\psi}_\alpha(\tau)\, \exp\Big[-\int_0^\beta L_E(q_j,\psi_\alpha,\bar{\psi}_\alpha)d\tau\Big].$$

- Integrate out all non-zero Fourier modes in Gaussian approximation.

The correct Witten index for a Kähler manifold then becomes

$$I_W = \int e^{\mathcal{F}/2\pi}\,\det{}^{1/2}\left[\frac{\mathcal{R}/4\pi}{\sin(\mathcal{R}/4\pi)}\right], \tag{10}$$

where $\mathcal{F}$ is the field-strength two-form,

$$\mathcal{F} = \frac{1}{2}F_{MN}dx^M \wedge dx^N = -2i\,\partial_j\partial_{\bar{k}}W\,dz^j \wedge d\bar{z}^{\bar{k}},$$

and $\mathcal{R}$ was defined in Theorem (2.7).

The expansion

$$e^{\mathcal{F}/2\pi} = 1 + \frac{\mathcal{F}}{2\pi} + \frac{\mathcal{F}\wedge\mathcal{F}}{8\pi^2} + \dots$$

and

$$\det{}^{1/2}\left[\frac{\mathcal{R}/4\pi}{\sin(\mathcal{R}/4\pi)}\right] = 1 + \frac{1}{192\pi^2}\int \mathrm{Tr}(\mathcal{R}\wedge\mathcal{R}) + \dots$$

contribute only in degree $2n$.

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For $\mathbb{C}P^1$ the determinant term vanishes, leaving only the flux contribution. For $\mathbb{C}P^2$, the index contains two contributions:

$$I_{\mathbb{C}P^2} = \frac{1}{8\pi^2}\int \mathcal{F}\wedge\mathcal{F} + \frac{1}{192\pi^2}\int \mathrm{Tr}(\mathcal{R}\wedge\mathcal{R}). \tag{11}$$

The first term (11) is the Abelian Pontryagin number, the second equals the Hirzebruch signature of $\mathbb{C}P^2$ multiplied by $-1/8$. For higher n, many topological invariants contribute.

Remark 3.4.

$$I_D = \int e^{-\mathcal{F}/2\pi}\,\det{}^{1/2}\left[\frac{\mathcal{R}/4\pi}{\sin(\mathcal{R}/4\pi)}\right].$$

In other words: $I_D = I_W$ for $4k$-dimensional manifolds, $I_D = -I_W$ for $(4k+2)$-dimensional manifolds (due to conventions in sign choices).

3.2 Non-Kähler Manifolds

Theorem 3.5. *Torsion does not affect the Dirac index.*

Proof: This follows immediately from the fact that the Dirac operator on any even-dimensional manifold with spin structure possesses supersymmetry generated by the Hermitian pair $-i\not{\mathcal{D}}$ and $\not{\mathcal{D}}\gamma^{D+1}$, independent of whether torsion is present or not. Supersymmetry is preserved under continuous deformation of the torsion tensor C_{MNL}, hence torsion can be continuously "untwisted" to zero without changing the (discrete) value of the index. □