INFDEV026A - Algoritmiek Unit 5

G. Costantini, F. Di Giacomo, G. Maggiore

costg@hr.nl, giacf@hr.nl, maggg@hr.nl – Office H4.204

Today

- ~~Why is my code slow?~~
  - ~~Empirical and complexity analysis~~
- ~~How do I order my data?~~
  - ~~Sorting algorithms~~
- ~~How do I structure my data?~~
  - ~~Linear, tabular, recursive data structures~~
- How do I represent relationship networks?
  - Graphs

More detailed agenda

- What are (di)graphs?
- How do we represent a (di)graph?
  - Adjacency list, adjacency matrix, incidence matrix
- How can we traverse/visit a graph?
  - BFS, DFS
- How can we find the shortest path between two nodes of a graph?
  - Dijkstra’s algorithm

Graphs


MAPQUEST
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Sault Ste. Marie
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Rockford
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Cleveland
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Dayton
Cincinnati
Jefferson City
Frankfort
Evansville
Lexington
Springfield
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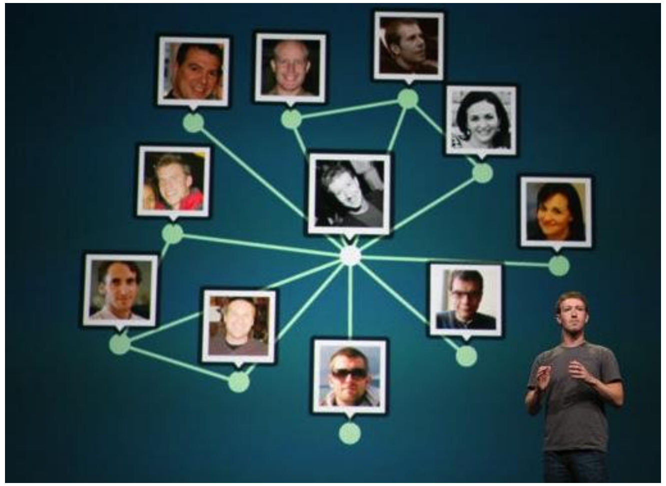
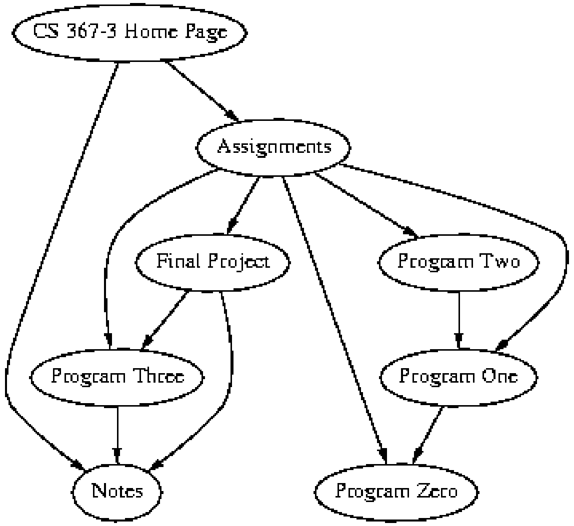
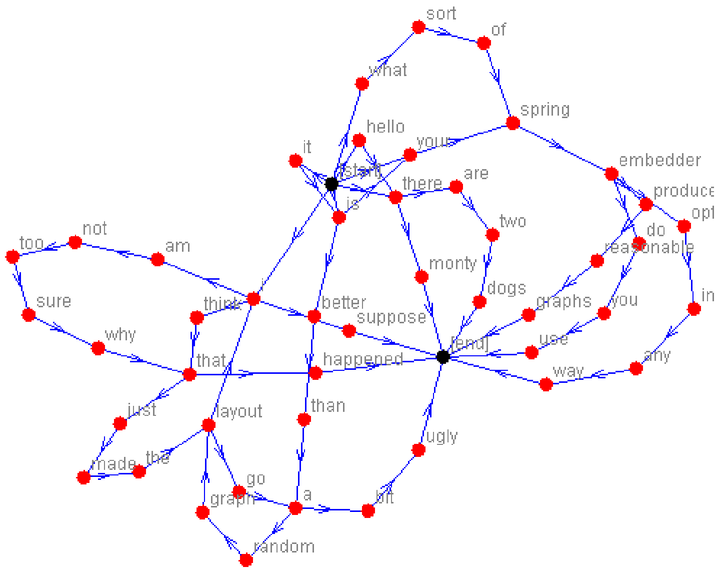



CS 367-3 Home Page
Assignments
Final Project
Program Two
Program Three
Program One
Notes
Program Zero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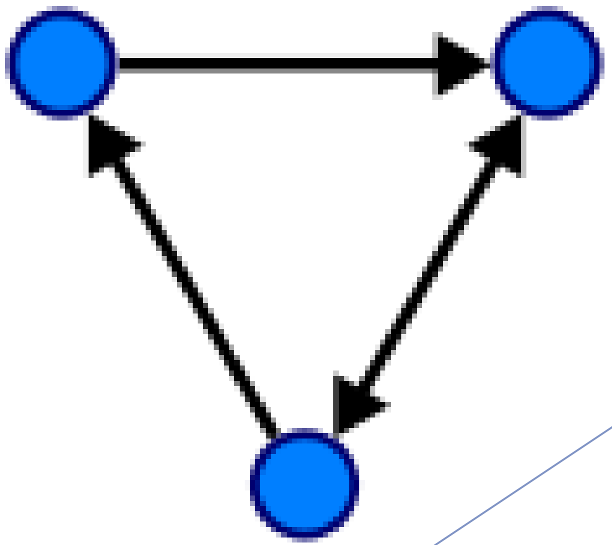
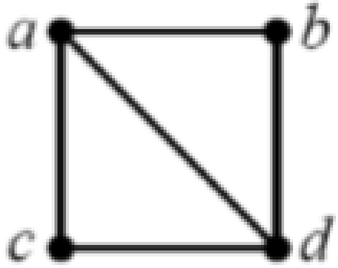
Graphs – Definition

- Nonlinear structure made by
  - finite (and possibly mutable) set of ***nodes*** or **vertices**
  - set of ordered/unordered pairs of these nodes, known as ***edges*** or **arcs**
    - edge (x, y) is said to **point** or **go from** x **to** y
    - may also associate to each edge some edge *value*, such as a symbolic label or a numeric attribute (cost, capacity, length, etc.)



Graphs - Definition

- **Simple** graph → pair $G = (V, E)$ where
  - V and E are finite sets
  - $V =$ vertices (nodes); $E =$ edges (arcs)
  - every element of E is a two-element subset of V



$V = \{a, b, c, d\}$

$E = \{ab, ac, ad, bd, cd\}$

- Graph size → # elements of V → $|V|$
- Given an edge $e = \{a, b\} = ab = ba$
  - e connects/is incident with the two vertices a and b
  - a and b are adjacent/incident upon e/the terminal points of e

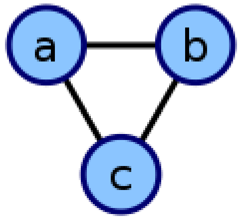
- **Path** from a to b → sequence of edges which form a chain of connected vertices from a to b, with all distinct vertices
  - length of a path = number of edges forming the path

Graphs – Representations

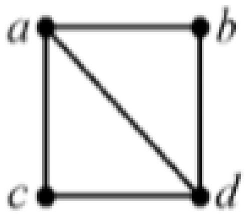
- Possible data structures for the representation of graphs
  - **Adjacency list**
    - Vertices are stored as records or objects, and every vertex stores a list of adjacent vertices
  - **Adjacency matrix**
    - A two-dimensional matrix, in which the rows represent source vertices and columns represent destination vertices
  - **Incidence matrix**
    - A two-dimensional matrix, in which the rows represent the vertices and columns represent the edges

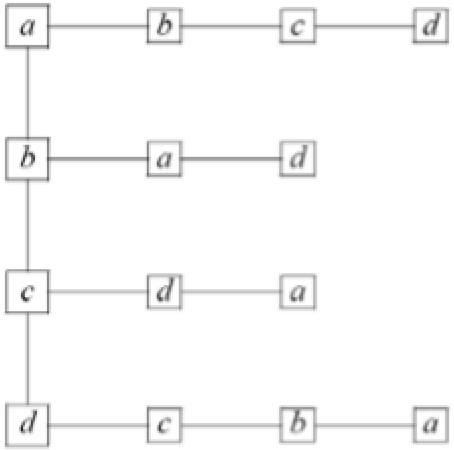
Graphs – Representations

- **Adjacency list**
  - collection of unordered lists, one for each vertex in the graph
  - each list describes the set of neighbors of its vertex



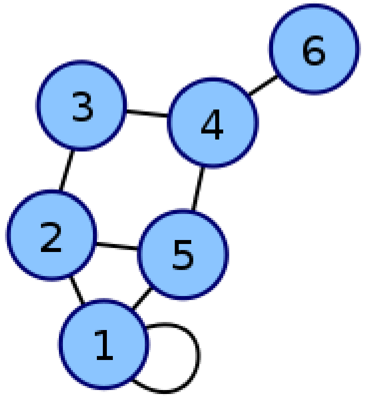
a	adjacent to	b,c
b	adjacent to	a,c
c	adjacent to	a,b





Graphs – Representations

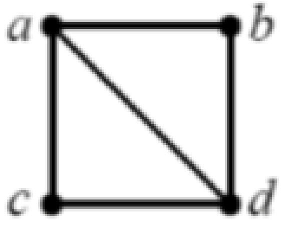
- **Adjacency matrix**
  - represents which vertices of a graph are adjacent to which other vertices
  - rows and columns represent both the vertices
  - given a cell at row i → column j is *True(1)* if there is an edge connecting i to j



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Graphs – Representations

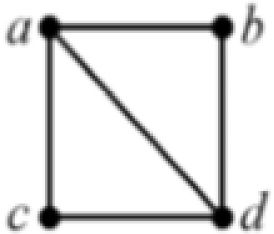
- **Adjacency matrix properties**
  - the matrix is symmetric ($a[i][j] \;==\; a[j][i]$ will be true $\forall i, j$)
  - the number of *True*(1) entries is twice the number of edges
  - different orderings of the vertex set V will result in different adjacency matrices for the same graph
  - preferred representation when the graph is *dense* (= many edges)
    - When the graph is sparse (= few edges), adjacency lists are more efficient



	a	b	c	d
a	F	T	T	T
b	T	F	F	T
c	T	F	F	T
d	T	T	T	F

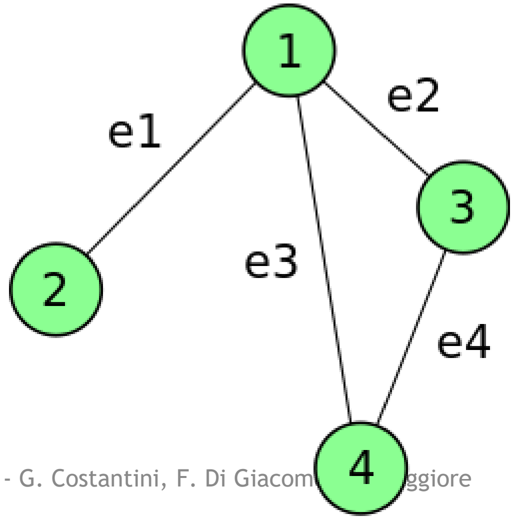
	a	b	c	d
a	0	1	1	1
b	1	0	0	1
c	1	0	0	1
d	1	1	1	0

Graphs – Representations



a	1	1	1	0	0
b	1	0	0	1	0
c	0	1	0	0	1
d	0	0	1	1	1

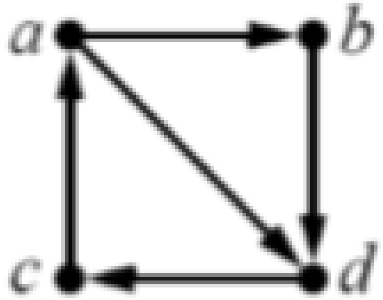
- **Incidence matrix**
  - shows the relationship between two classes of objects: vertices (rows) and edges (columns)
  - given a cell at row i → column j is *True(1)* if vertex i is incident upon edge j



$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

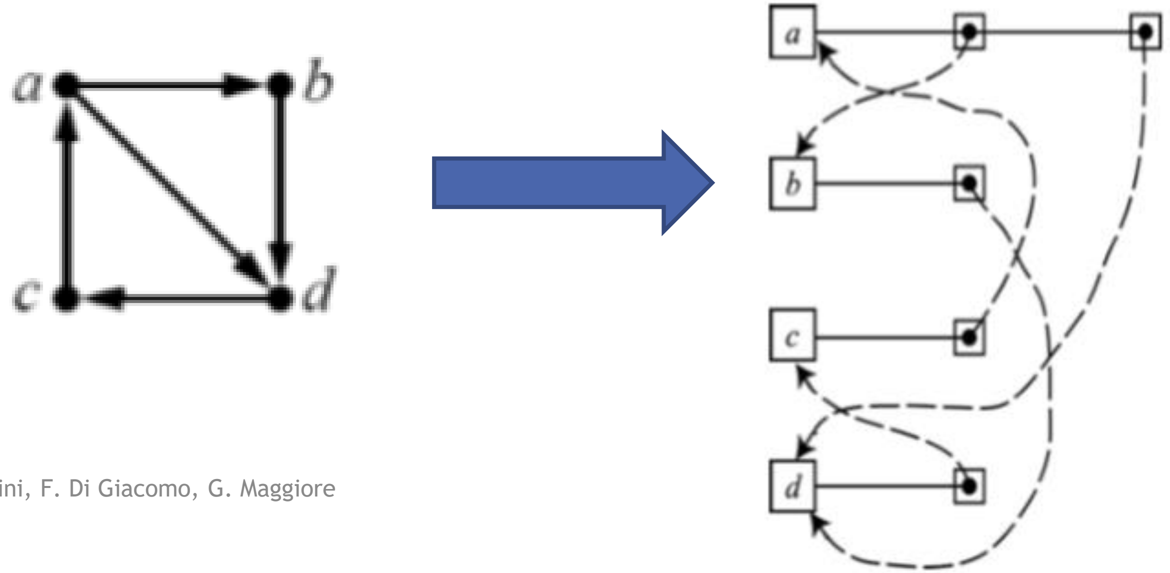
Graphs – Definition of digraph

- A *digraph* (or **directed graph**) is a pair $G = (V, E)$ where V is a finite set and E is a set of <u>ordered</u> pairs of elements of V
  - Difference with simple graphs: edges have a DIRECTION
  - If $e = (a, b)$ …
    - the edge e *emanates/is incident from* vertex a
    - the edge e *terminates/is incident to* vertex b



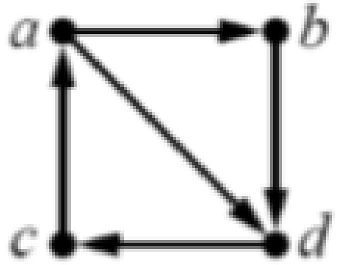
Graphs – Representation of digraphs

- **Adjacency list of a digraph**
  - for each vertex in the graph, store a list containing the edges that <u>emanate</u> from that vertex
  - same as the adjacency list for a graph, except that the links are *not duplicated* unless there are edges going both ways between a pair of vertices



Graphs – Representation of digraphs

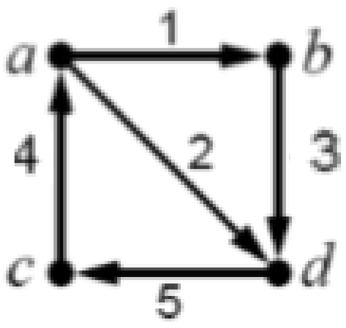
- **Adjacency matrix of a digraph**
  - cell at row i, column j is *True* if there is an edge emanating from vertex i and terminating at vertex j



	a	b	c	d
a	F	T	F	T
b	F	F	F	T
c	T	F	F	F
d	F	F	T	F

Graphs – Representation of digraphs

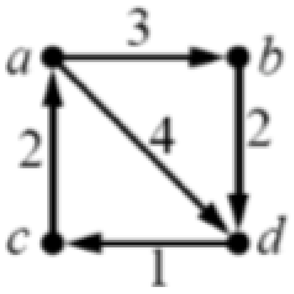
- **Incidence matrix of a digraph**
  - cell at row i, column j has value
    - 1 if edge j emanates from vertex i
    - -1 if edge j terminates at vertex i



a	1	1	0	-1	0
b	-1	0	1	0	0
c	0	0	0	1	-1
d	0	-1	-1	0	1

Graphs - Some more terminology

- Path from a to b in a digraph
  - Same concept as in undirected graphs

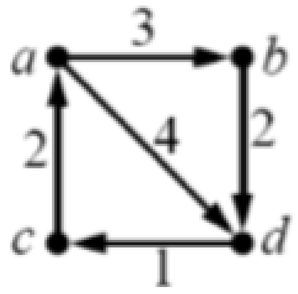


A weighted digraph

- **Weighted digraph** → pair (V, w) where
  - V is a finite set of vertices
  - w is a function that assigns to each pair (x, y) of vertices either a positive integer or ∞ (infinity)
    - called *weight function*
    - cost/time/distance for moving directly from x to y; ∞ means no edge from x to y
  - Weighted graph → w is symmetric ($w(x, y) = w(y, x)$)

Graphs - Some more terminology

- **Weighted path length**
  - sum of the weights of the edges along the path



A weighted digraph

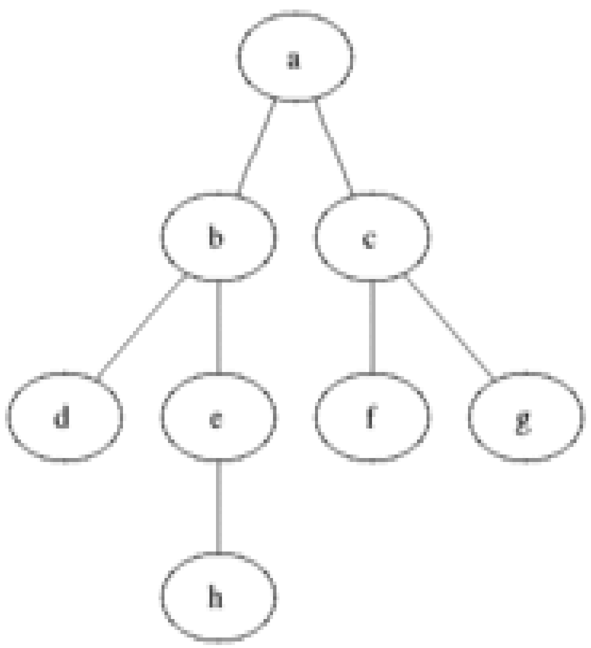
- **Shortest distance** from x to y
  - minimum weighted path length among all the paths from x to y
  - *Dijkstra's Shortest Path Algorithm* → finding the shortest path from one vertex to each other vertex in a (di)graph

Graphs – Traversal algorithms

- *Graph traversal* → visiting all the nodes in a graph in a particular manner, updating and/or checking their values along the way

- Possible algorithms
  - **BFS** (Breadth First Search)
    - Inspect all neighbors of a node; then for each neighbor inspect all its unvisited neighbors, etc…
  - **DFS** (Depth First Search)
    - Start from one neighbor and go as far as possible in that direction before continuing with exploring the other neighbors

Graph – BFS traversal algorithm



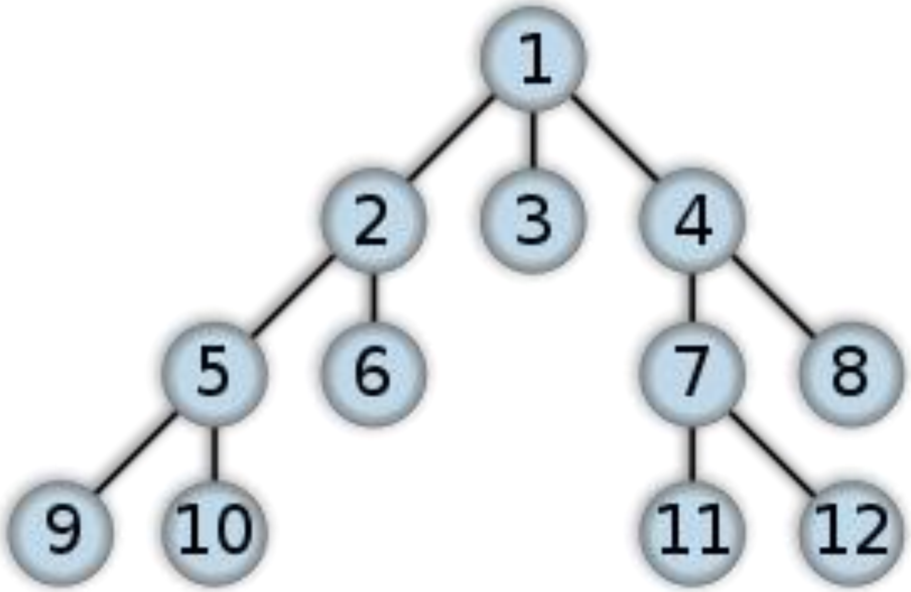
- Search is limited to essentially two operations
  - visit and inspect a node of a graph
  - gain access to visit the nodes that neighbor the currently visited node
- Algorithm
  - begins at a root node and inspects all the neighboring nodes
  - for each of those neighbor nodes in turn, it inspects their neighbor nodes which were unvisited, and so on
- Complexity → $O(|V| + |E|)$

Graph – BFS traversal algorithm

- **Queue** data structure used to store intermediate results as it traverses the graph
  1. Enqueue the root node
  2. Dequeue a node and examine it
     - [If the element sought is found in this node, quit the search and return a result]
     - Otherwise enqueue any successors (the direct child nodes) that have not yet been discovered
  3. If the queue is empty, every node on the graph has been examined [quit the search and return "not found“]
  4. If the queue is not empty, repeat from Step 2

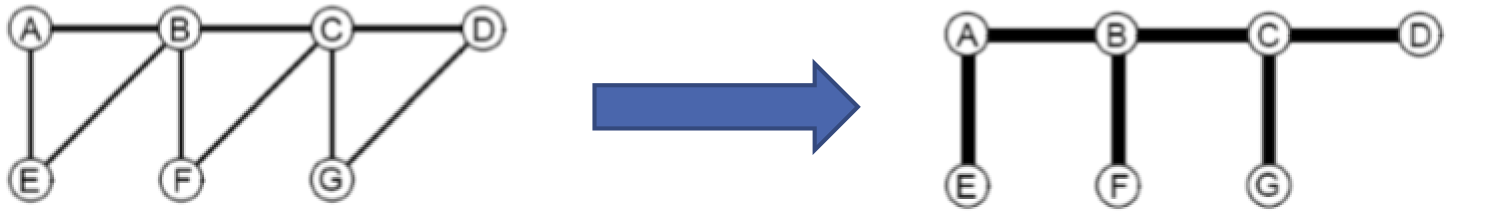
Graph – BFS traversal algorithm

- Result of a BFS traversal



Graph – BFS traversal algorithm

- BFS traversal
  - Returned list of visited vertices: A, B, E, C, F, D, G



Graph – BFS traversal algorithm

Algorithm pseudocode:

```
Breadth-First-Search(Graph, root)
    for each node n in Graph:
        n.distance = INFINITY
        n.parent = NIL
    create empty queue Q
    root.distance = 0
    Q.enqueue(root)
    while Q is not empty:
        current = Q.dequeue()
        for each node n that is adjacent to current:
            if n.distance == INFINITY:
                n.distance = current.distance + 1
                n.parent = current
                Q.enqueue(n)
```

Graph – DFS traversal algorithm

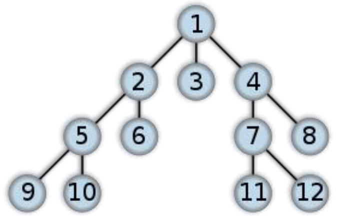
- Algorithm
  - Starts at a root node
  - Explores as far as possible along each branch before backtracking

- Complexity → $O(|V| + |E|)$

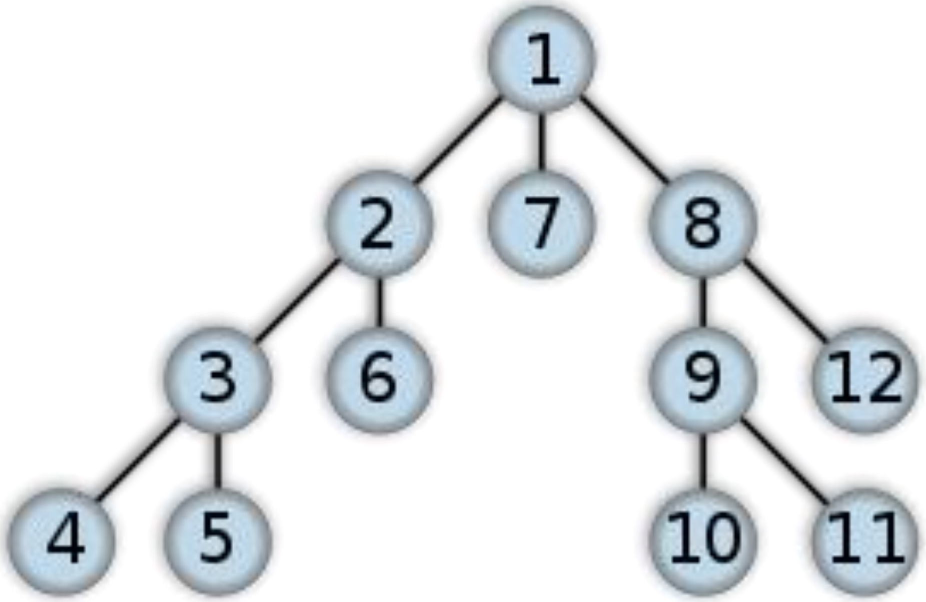
- Difference with BFS
  - DSF uses a **stack** instead of a queue
    - *Push* only the first unvisited neigbour of the top element of the stack
    - *Pop* from the stack if there are no other unvisited neighbours
  - A recursive implementation is possible

Graph – DFS traversal algorithm

BFS was…

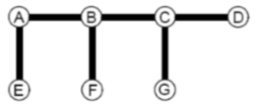


- Result of a DFS traversal

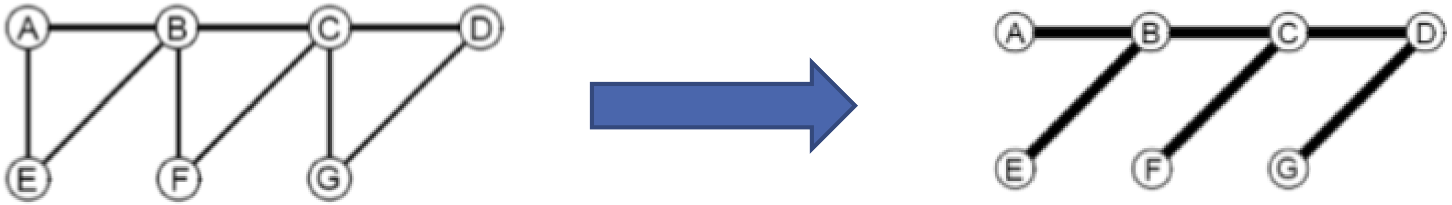


Graph – DFS traversal algorithm

BFS was…



- DFS traversal
  - Returned list of visited vertices: A, B, C, D, G, F, E



Graph – DFS traversal algorithm

Algorithm pseudocode (recursive)

```
procedure DFS(G,v):
     label v as discovered
     for all edges from v to w in G.adjacentEdges(v) do
        if vertex w is not labeled as discovered then
        recursively call DFS(G,w)
```

Graph – DFS traversal algorithm

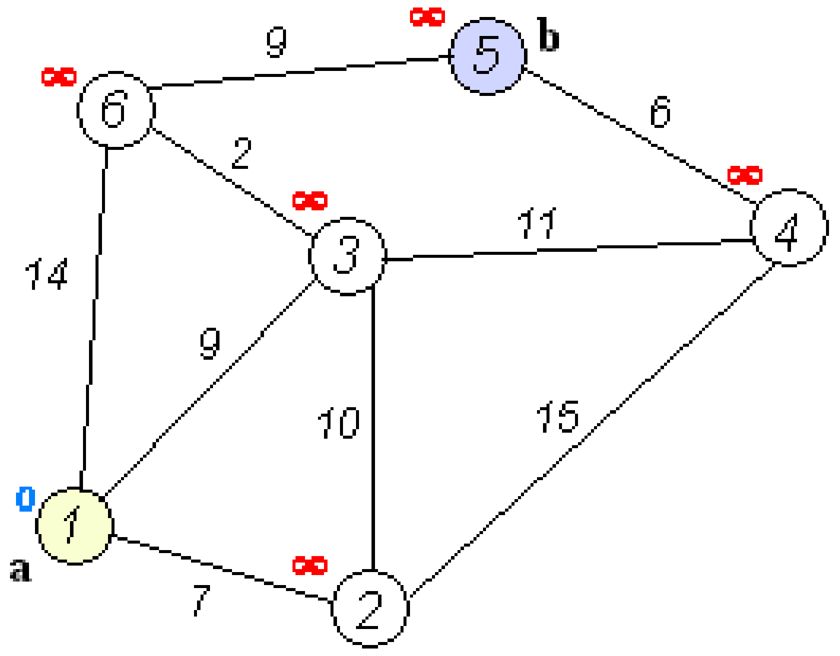
Algorithm pseudocode (iterative)

```
procedure DFS-iterative(G,v):
  let S be a stack
  S.push(v)
  while S is not empty
      v = S.pop()
      if v is not labeled as discovered:
          label v as discovered
          for all edges from v to w in G.adjacentEdges(v) do
             S.push(w)
```

Graphs – Dijkstra’s algorithm

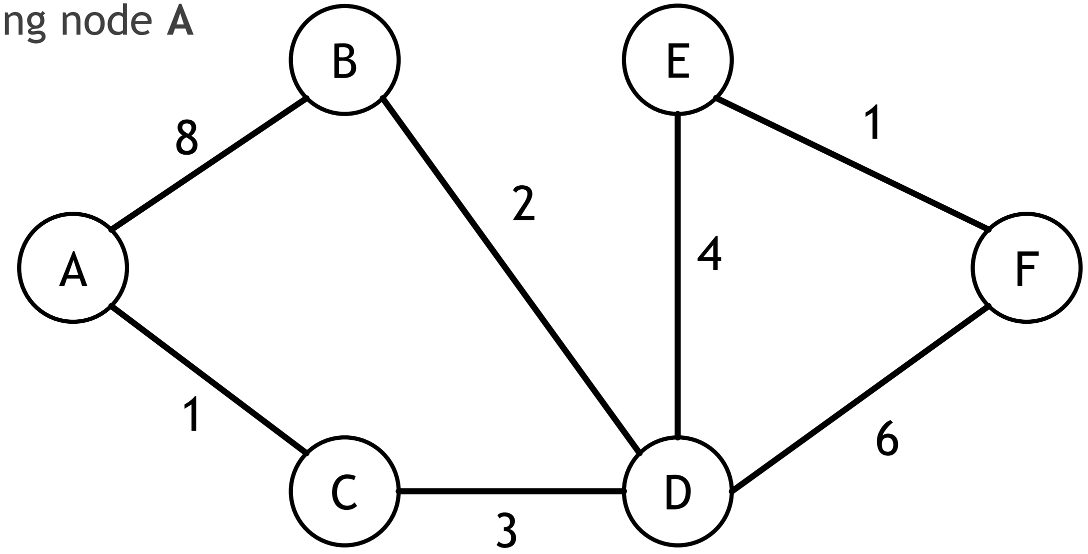
- Single-source shortest path problem
  - for a given source vertex (node) in the graph, the algorithm finds the path with lowest cost (i.e., the shortest path) between that vertex and every other vertex

- Informal steps of the algorithm
  - Pick the unvisited vertex with the lowest-distance
  - Calculate the distance through it to each unvisited neighbor
  - Update the neighbor's distance if smaller
  - Mark as visited when done with neighbors



Graphs – Dijkstra's algorithm

- Example
  - Starting node **A**



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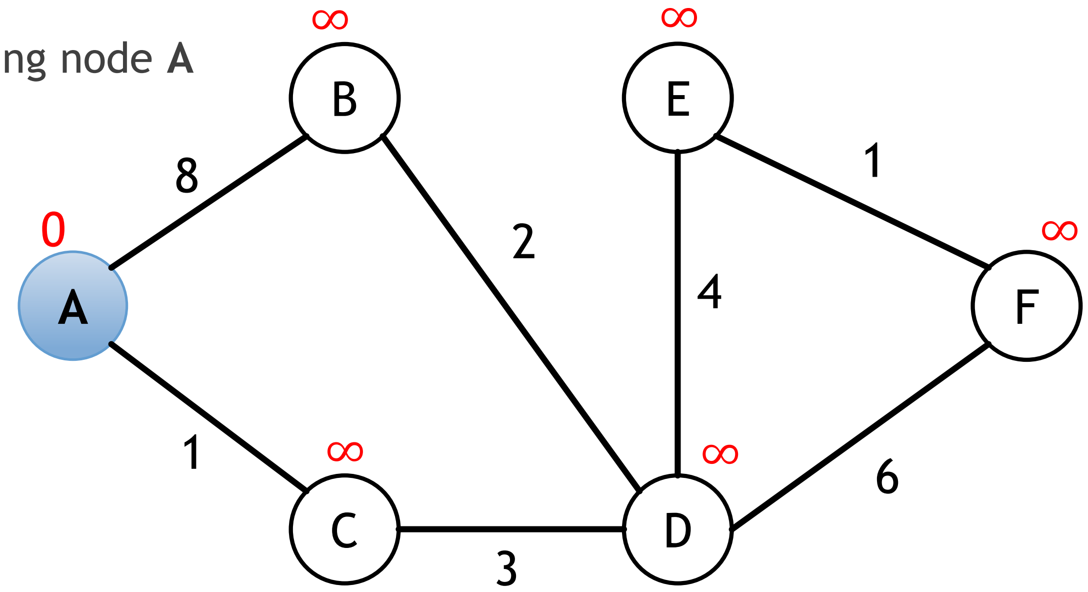

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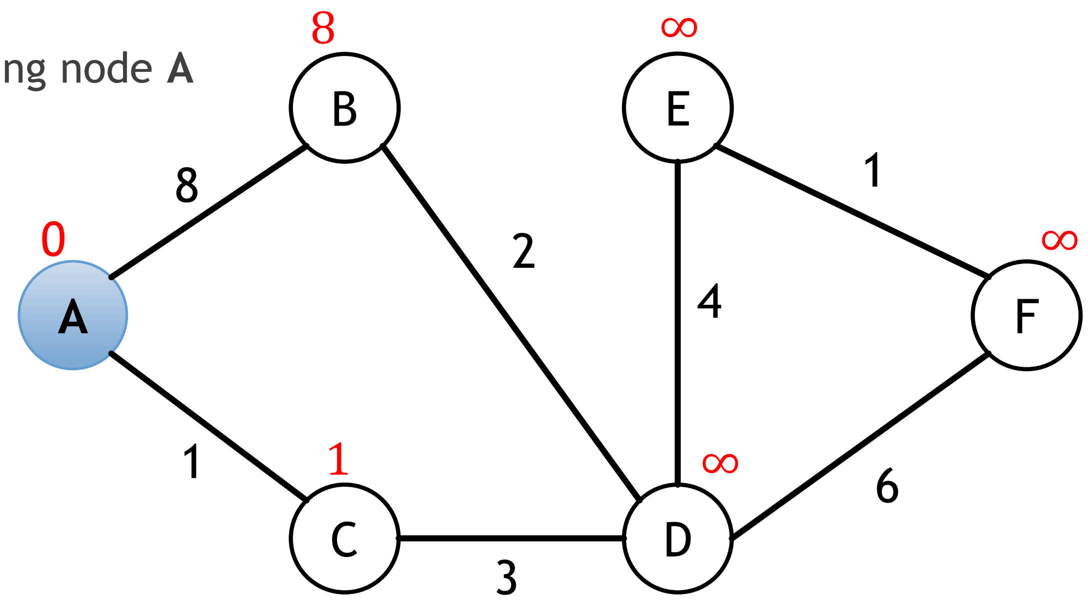

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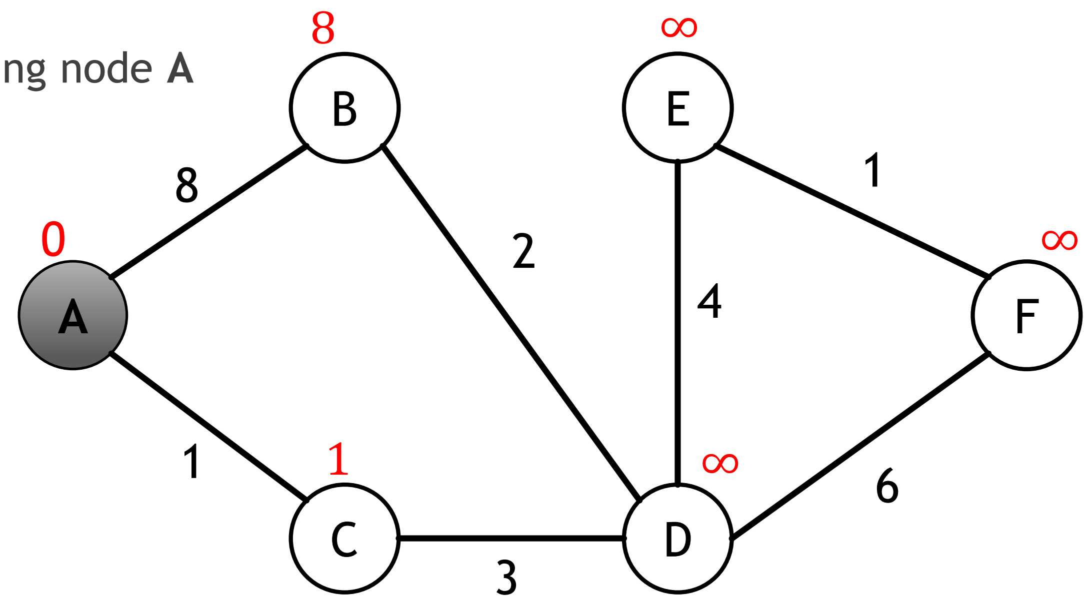

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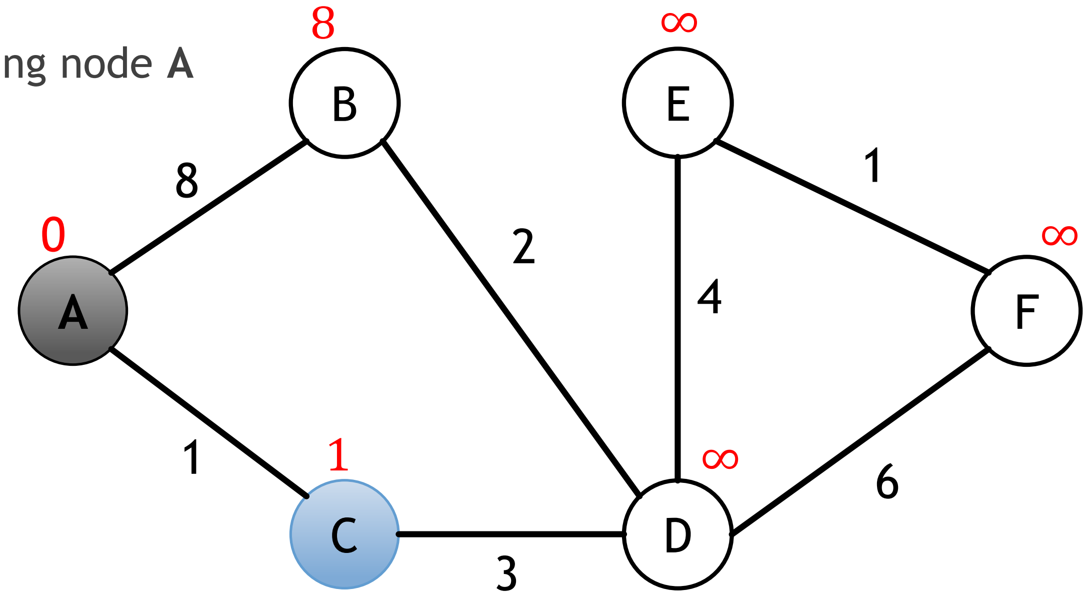

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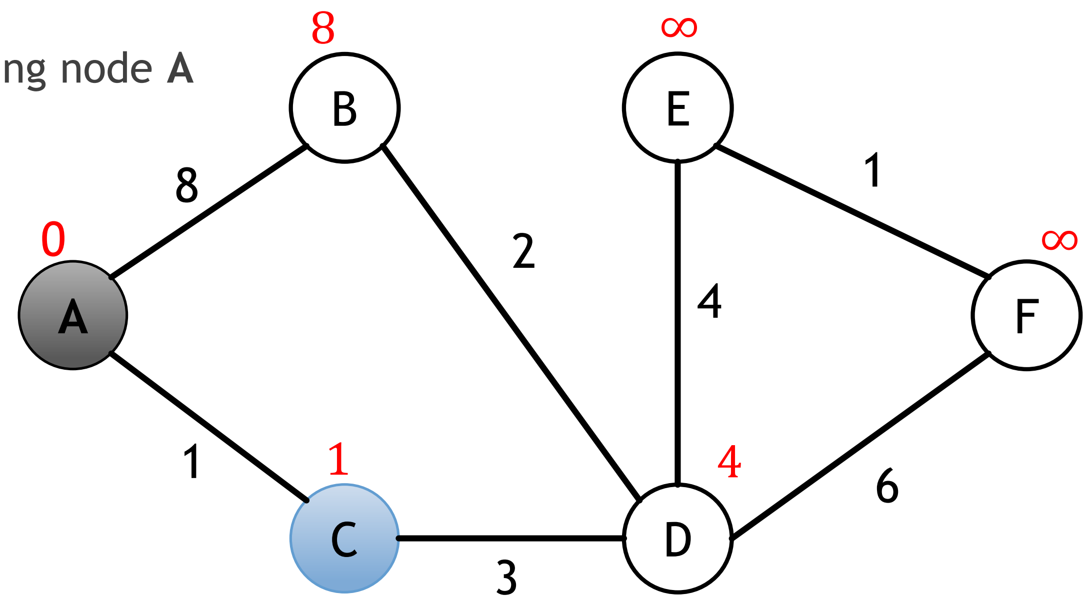

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Graphs – Dijkstra's algorithm

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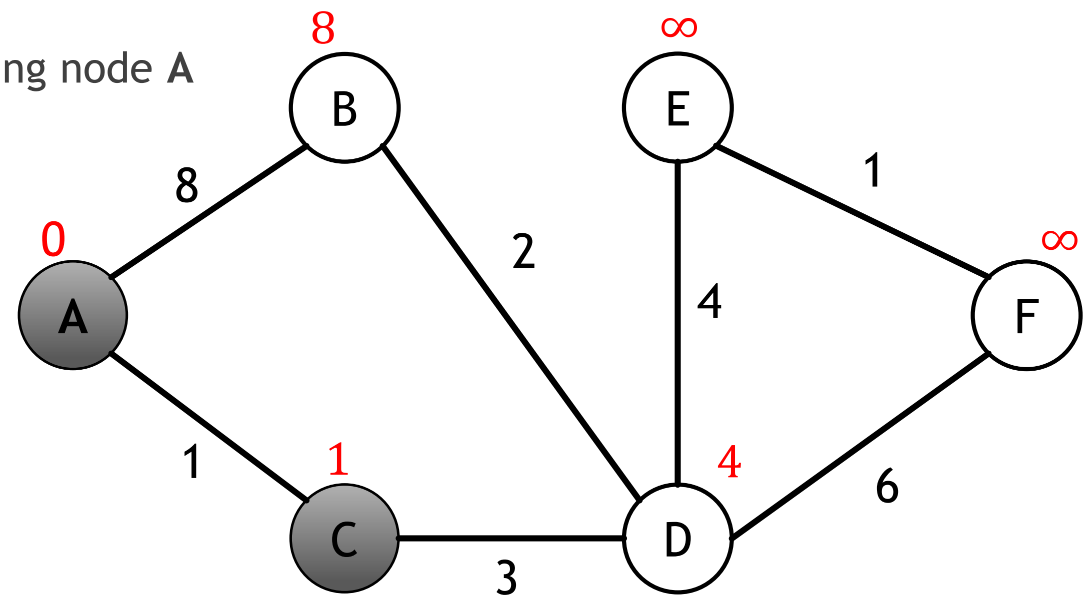

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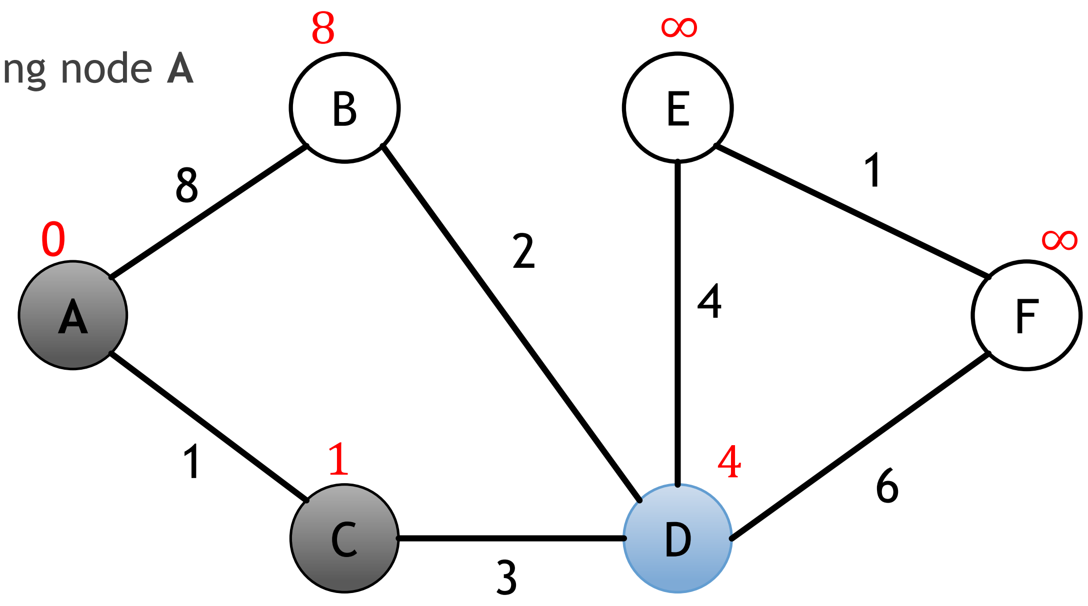

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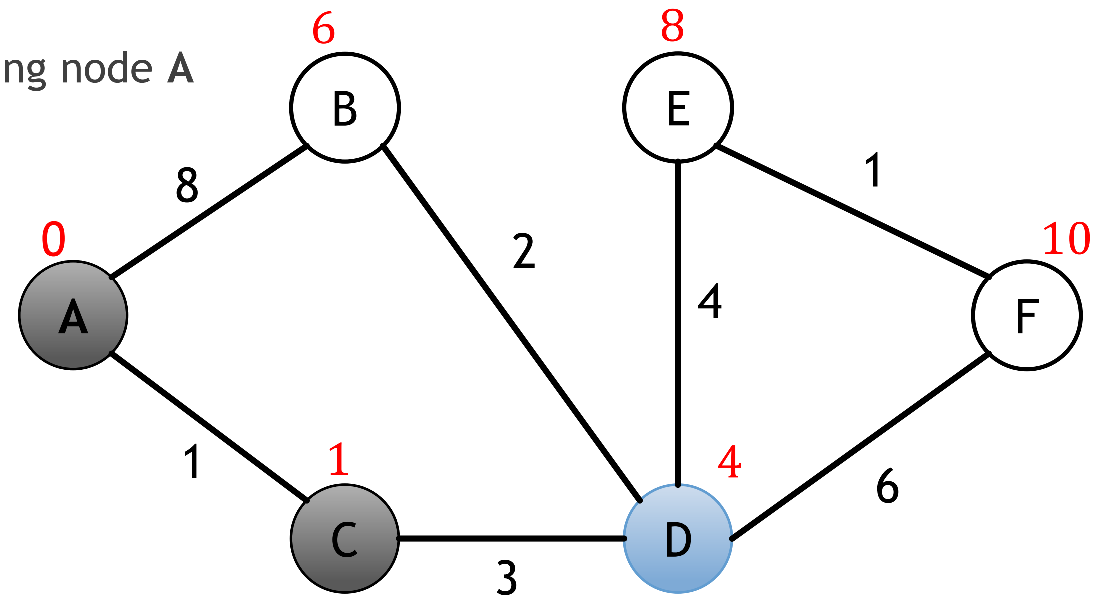

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Graphs – Dijkstra’s algorithm

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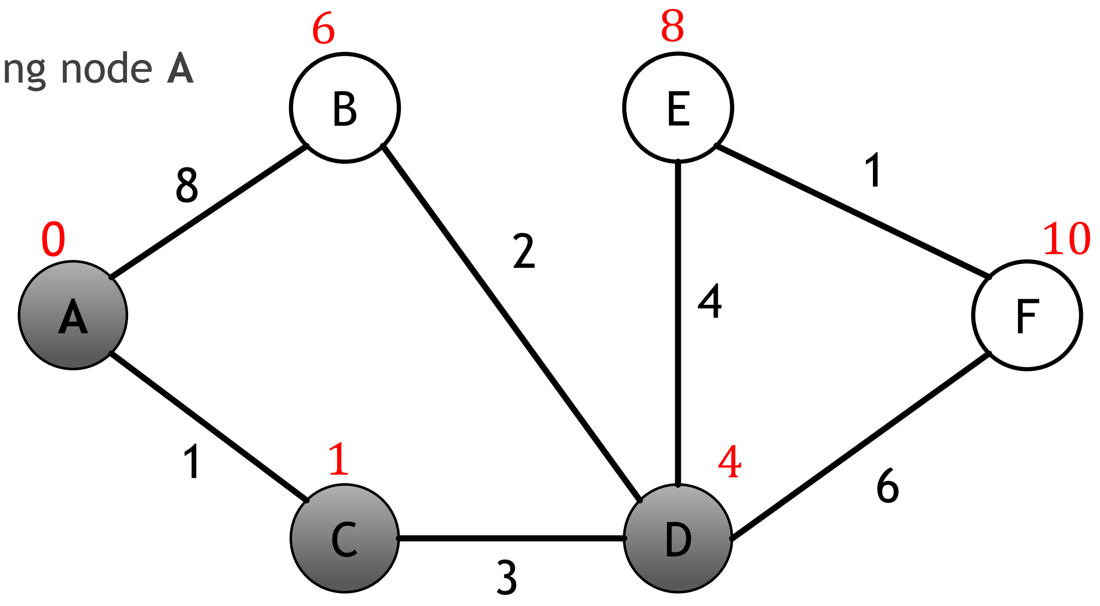

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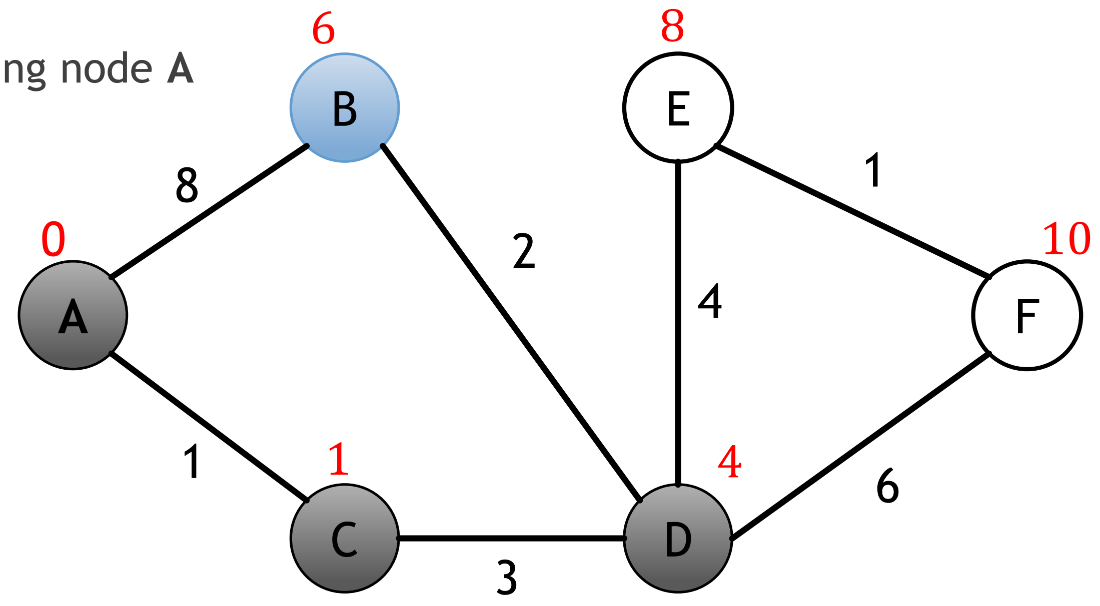

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- Example
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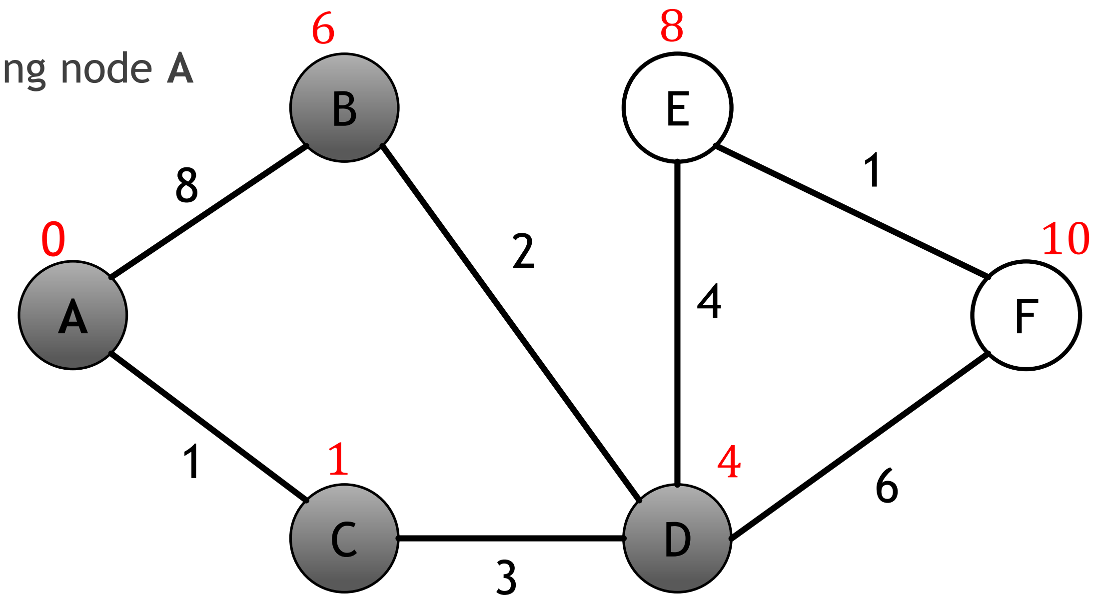

- Pick the unvisited vertex with the lowest-distance
- Calculate the distance through it to each unvisited neighbor
- Update the neighbor's distance if smaller
- Mark as visited when done with neighbors

Graphs – Dijkstra's algorithm

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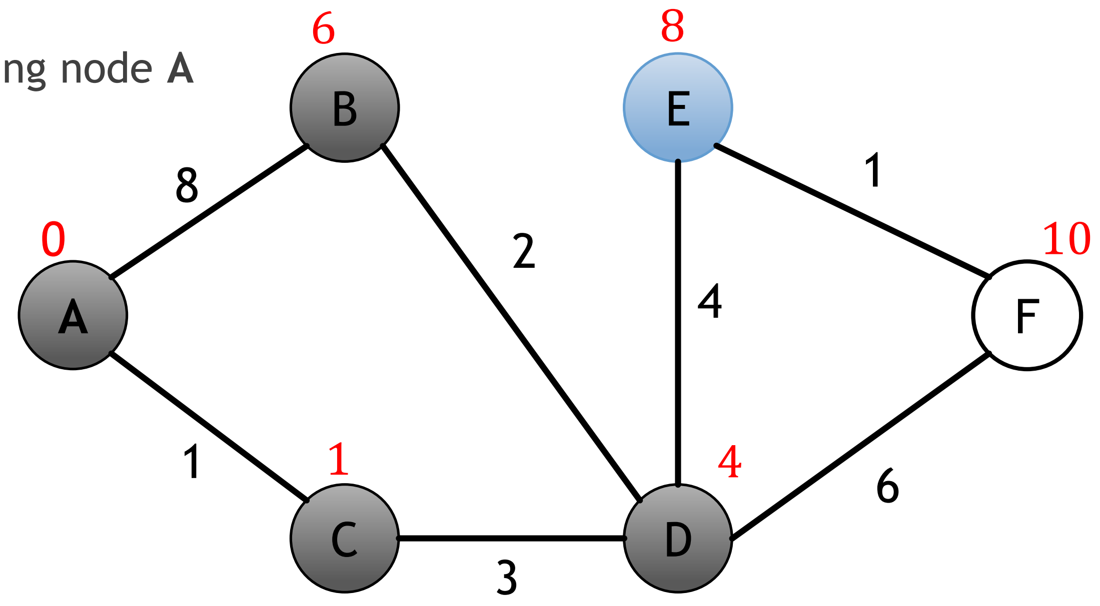

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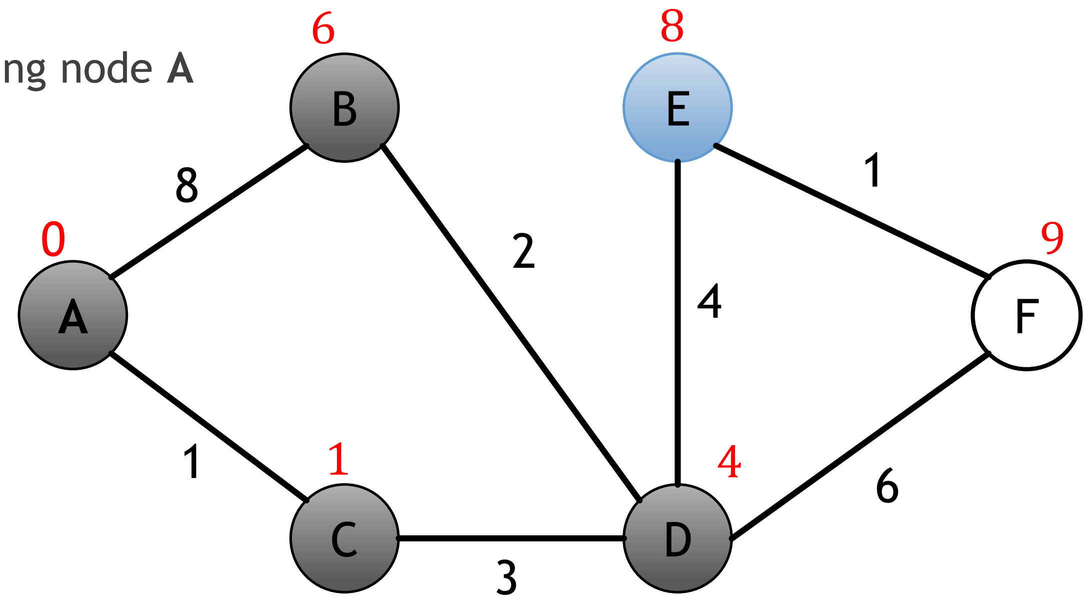

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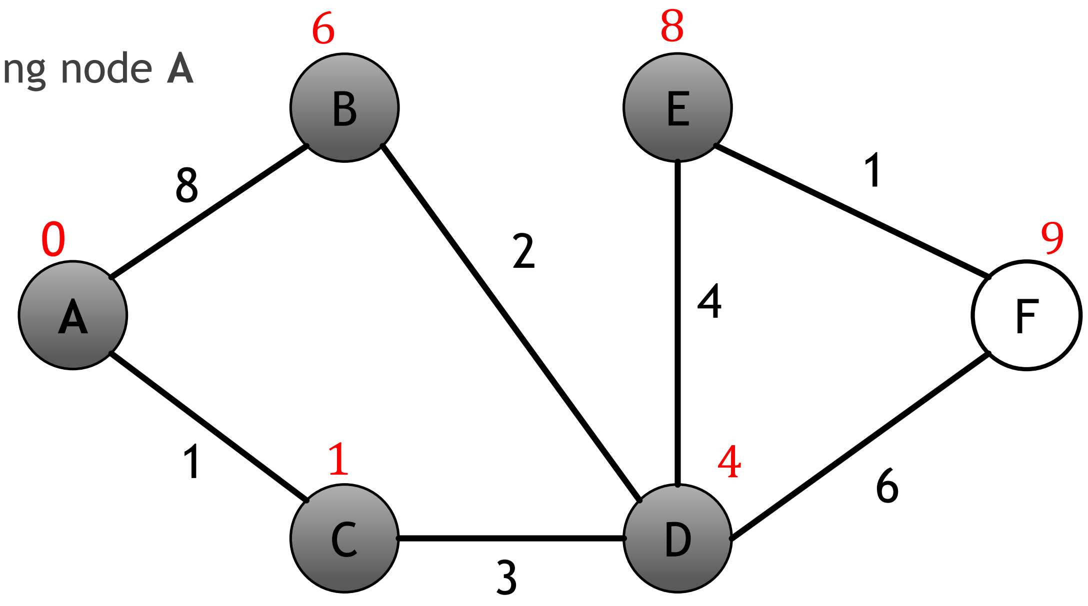

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Graphs – Dijkstra's algorithm

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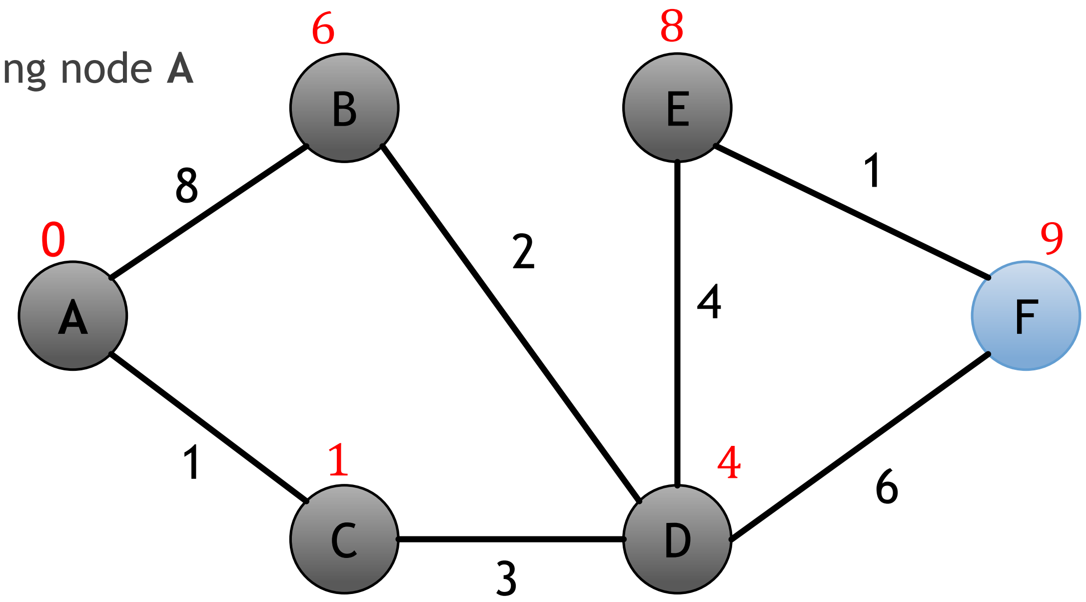

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Graphs – Dijkstra's algorithm

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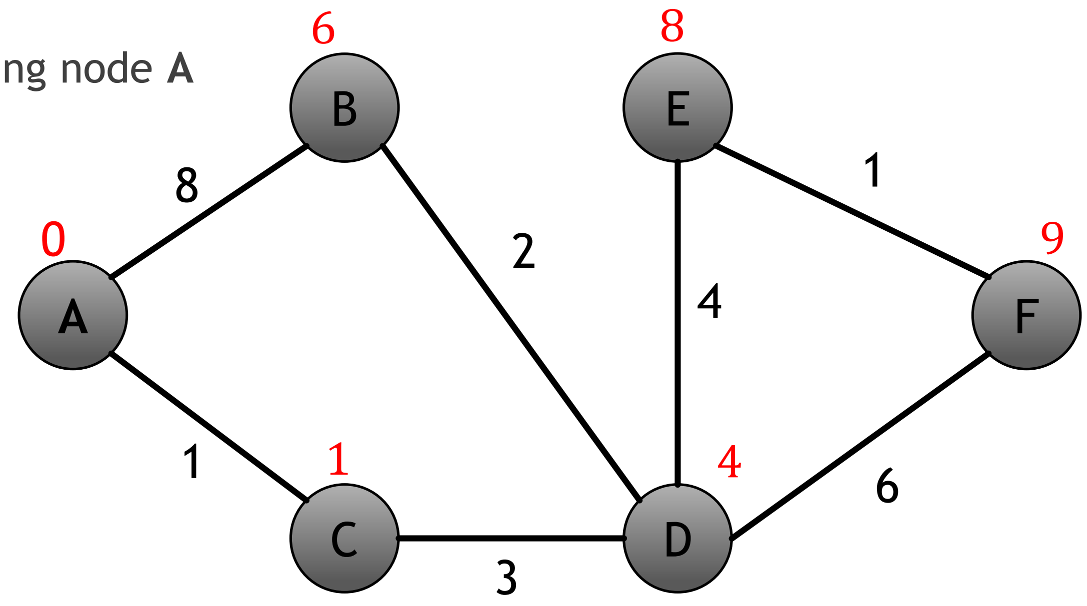

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Graphs – Dijkstra's algorithm (pseudocode)

```
function Dijkstra(Graph, source):
    create vertex set Q
    for each vertex v in Graph:            // Initialization
        dist[v] ← INFINITY                 // Unknown distance from source to v
        prev[v] ← UNDEFINED                // Previous node in optimal path from source
        add v to Q                         // All nodes initially in Q (unvisited nodes)
    dist[source] ← 0                       // Distance from source to source
    while Q is not empty:
       u ← vertex in Q with min dist[u]    // Source node will be selected first
        remove u from Q

       for each neighbor v of u:           // where v is still in Q.
           alt ← dist[u] + length(u, v)
           if alt < dist[v]:               // A shorter path to v has been found
               dist[v] ← alt
               prev[v] ← u
    return dist[], prev[]
```

Graphs – Dijkstra’s algorithm

- Main steps of the algorithm
  - Pick the unvisited vertex with the lowest-distance
  - Calculate the distance through it to each unvisited neighbor
  - Update the neighbor's distance if smaller
  - Mark visited when done with neighbors

Graphs – Dijkstra's algorithm

1. Assign to every node a tentative distance value: set it to zero for the initial node and to infinity (∞) for all other nodes.
2. Set the initial node as current. Mark all other nodes unvisited. Create a set of all the unvisited nodes called the *unvisited set*.
3. For the current node, consider all of its unvisited neighbors and **calculate their *tentative* distances**. Compare the newly calculated *tentative* distance to the current assigned value and **assign the smaller one**.
   - For example, if the current node *A* is marked with a distance of 6, and the edge connecting it with a neighbor *B* has length 2, then the distance to *B* (through *A*) will be 6 + 2 = 8. If B was previously marked with a distance greater than 8 then change it to 8. Otherwise, keep the current value.
4. When we are done considering all of the neighbors of the current node, **mark the current node as visited** and remove it from the *unvisited set*. A visited node will never be checked again.
5. **Select the unvisited node that is marked with the smallest tentative distance**, and set it as the new "current node" then go back to step 3.

Homework

- ***GO ON WITH THE ASSIGNMENT!!!***
  - *Exercise 1 should be* ***completed***
  - *Exercise 2 should be* ***in progress***
    - Tip: look for binary tree code from N@tschool, as inspiration
  - *[for the fastest] Exercise 3 can now be* ***started***
    - creation of the adjacency matrix of the graph