Queen's University
CISC/CMPE 457
Test 2

October 31, 2019
Duration: 50 minutes

Closed book

Initial of Family Name: ____

Student Number: ____________________
(Write this at the top of every page.)

There are 3 questions and 17 marks total.

Answer all questions.

This exam paper should have 7 pages,
including this cover page.

1 – Projections and Function Spaces	/ 5
2 – Fourier Transforms	/ 7
3 – Sampling and Filtering	/ 5
Total	/ 17

The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretations of any question that requires a written answer.

1 Projections and Function Spaces — 5 points

Part A — 1 point Show an expression for the *function* that is the projection of $f(t)$ onto $g(t)$.

$$\left(\frac{\int f(t)\ g(t)\ dt}{\int g(t)\ g(t)\ dt} \right) g(t)$$

Part B — 2 points Show that the functions $1, x, x^2, x^3, \ldots$ are *not* orthogonal. It is sufficient to pick any two of these functions and show that those two are not orthogonal.

Compute the integral in $[-\infty, +\infty]$ of the product of two of these functions, such that the product is odd.

For example, $\int (1 \times x^2)\ dx \neq 0$.

State that they are not orthogonal because the integral is not zero.

Part C — 2 points We showed in class that $A\cos(t+\theta) = (-A\sin\theta)\cos t + (A\cos\theta)\sin t$. Draw a diagram of the $\langle \cos t, \sin t\rangle$ function space and show where $2\cos(t+\frac{\pi}{2})$ is in that space. Show all calculations.

$A = 2$

$\theta = \frac{\pi}{2}$

Coefficient of $\cos t$ is $-A\sin\theta = -2\sin\frac{\pi}{2} = -2$

Coefficient of $\sin t$ is $A\cos\theta = 2\cos\frac{\pi}{2} = 0$

Plot $(-2, 0)$ on axes with $\cos t$ on the horizontal and $\sin t$ on the vertical.

Note that the provided formula, $A\cos(t+\theta) = (-A\sin\theta)\cos t + (A\cos\theta)\sin t$, is **incorrect**. The correct right-hand side is $(A\cos\theta)\cos t + (-A\sin\theta)\sin t$. Using the correct right-hand side yields a point at $(0, -2)$.

Either answer is acceptable.

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2 Fourier Transforms — 7 points

Part A — 1 point Name one important property of the Fourier basis functions.

One of:

- each pair is orthogonal
- all periodic functions can be written as a linear combination of the basis functions

Part B — 1 point What two equations describe the relationship between convolution and multiplication with respect to the spatial and frequency domains? Use the notations $FT(f)$ and $FT(g)$ to represent the Fourier Transform of spatial-domain functions f and g.

$$\mathrm{FT}(f * g) = \mathrm{FT}(f) \times \mathrm{FT}(g)$$

$$\mathrm{FT}(f \times g) = \mathrm{FT}(f) * \mathrm{FT}(g)$$

Part C — 2 points What are all of a_k, b_k coefficients of the Fourier series of $f(t) = 1 + 2\cos(3\,t)$. Explain your answer.

a_0 is the average, which is $a_0 = 1$ since the cos term has an average of zero.

The a_k term is the coefficient of the cosine term with wavenumber k, so $a_3 = 2$ and all other a_k and b_k are zero.

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Part D — 2 points What is the **direction** and **frequency** of the 2D Fourier basis function with coefficient $F(u, v)$ in an image that is $N \times N$ pixels? Specify the frequency in cycles per pixels.

direction (u, v) (can be normalized or not)

frequency $\sqrt{u^2 + v^2}$ cycles per N pixels

Part E — 1 point Suppose that the Fourier Transform of an image has very large magnitudes at $(40, 0), (80, 0), (120, 0), (160, 0)$, and so on. This is caused by equally-spaced vertical lines in the image. What is the horizontal spacing between those vertical lines? Explain and show your work.

The fundamental frequency of the vertical lines is 40 cycles across the width of the image, since that's the closest-to-the-origin of the peaks in the frequency domain.

So the spacing of the vertical lines is $\frac{1}{40}$ of the width of the image.

3 Sampling and Filtering — 5 points

Part A — 2 points For a signal with maximum frequency $\frac{11}{7}$ Hz and fundamental period 7, what is the maximum sampling period to avoid aliasing? Explain with reference to the Sampling Theorem.

The Sampling Theorem says that the sampling frequency must be at least twice the highest frequency in the signal.

So sampling frequency must be at least $2 \times \frac{11}{7} = \frac{22}{7}$.

So sampling period must be at most $\frac{7}{22}$.

Part B — 2 points Carefully explain what happens in the frequency domain to cause aliasing of a sampled function in the spatial domain. Include in your explanation the impulse train used for sampling and the harmonics of the function that exist in the frequency domain.

In the frequency domain, harmonic copies of the frequency-domain function, F, overlap.

Areas of overlap cause F to be added to itself, thus losing the original F.

When transformed back to the spatial domain, a different signal results.

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Part C — 1 point What filter can be applied to a signal in the spatial domain to reduce aliasing? Explain, with reference to what happens in the frequency domain, why this filter reduces aliasing.

Apply a low-pass filter like Butterworth or Gaussian (but not "ideal")

This removes high frequencies, so the highest frequency in the signal can be reduced to below half of the sampling frequency.