Queen's University
CISC/CMPE 457
Test 2

October 30, 2017
Duration: 50 minutes

Closed book

Initial of Family Name: ___

Student Number: SOLUTIONS
(Write this at the top of every page.)

There are 4 questions and 26 marks total.

Answer all questions.

This exam paper should have 7 pages,
including this cover page.

1 – Vector and Function Spaces	/ 4
2 – Fourier Bases	/ 9
3 – Fourier Applications	/ 4
4 – Sampling and Filtering	/ 9
Total	/ 26

The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretations of any question that requires a written answer.

1 Vector and Function Spaces — 4 points

Part A — 2 points What are the coordinates of vector $(3, 4)$ in the basis $\langle\ (1, -1), (2, 2)\ \rangle$? Show your work.

USE $\dfrac{v \cdot e}{e \cdot e}\, e$

$$\frac{(3,4)\cdot(1,-1)}{(1,-1)\cdot(1,-1)} = \frac{-1}{2}$$

$$\frac{(3,4)\cdot(2,2)}{(2,2)\cdot(2,2)} = \frac{14}{8} \quad \left(= 7/4\right)$$

Part B — 2 points Show that the functions $f(x) = x$ and $g(x) = x^2 - 1$ orthogonal. Recall that $\int_a^b x^c\ dx = \frac{1}{c+1}\left(b^{c+1} - a^{c+1}\right)$.

$$\int f(x)\, g(x)\, dx = \int x\,(x^2 - 1)\, dx$$

$$= \int (x^3 - x^{\cancel{2}})\, dx$$

$$= \int x^3\, dx - \int x\, dx$$

$$= 0 - 0$$

SINCE BOTH ARE <u>ODD</u> FUNCTIONS

E.G. $\displaystyle\int_{-\infty}^{+\infty} x^3\, dx = \int_{-\infty}^{0} x^3\, dx + \int_{0}^{\infty} x^3\, dx$

$\displaystyle = -\int_0^\infty x^3\, dx + \int_0^\infty x^3\, dx = 0$

... continued

2 Fourier Bases — 9 points

Part A — 1 point Explain why the first two Fourier basis functions are 1 (for a_0) and 0 (for b_0) in the sin/cos representation of the Fourier basis functions.

FIRST TWO BASIS FUNCTIONS ARE

$\cos 0t$ & $\sin 0t$

$\cos 0 = 1$

$\sin 0 = 0$

Part B — 3 points What are the coordinates of $f(t) = e^{i\,2\,(t+7)}$ in the function space spanned by the orthogonal basis functions $\langle e^{i\,k\,t}, k = 0, 1, 2, \ldots \rangle$? Show your work. Explain why the coordinates that you claim are zero are, in fact, zero.

$$e^{i2(t+7)} = e^{i2t + i2 \times 7}$$

$$= e^{i2t} \times e^{i14}$$

SO THE COEFFICIENT OF e^{i2t} IS e^{i14}

ALL OTHER COEFFICIENTS ARE ZERO BECAUSE THE OTHER BASIS FUNCTIONS ARE ORTHOGONAL TO e^{i2t}

... continued

Part C — 3 points What are the coordinates of $2\cos 3t + 4\sin 5t$ in the *complex* Fourier series? That is, what are the corresponding c_k for $k \in \mathbb{Z}$? Show your work. Recall that $\cos kt = \frac{1}{2}(e^{-ikt} + e^{ikt})$ and $\sin kt = \frac{i}{2}(e^{-ikt} - e^{ikt})$.

$2\cos 3t + 4\sin 5t$

i

$= 2 \times \frac{1}{2}(e^{-i3t} + e^{i3t}) + 4 \times \frac{i}{2}(e^{-i5t} - e^{i5t})$

$c_{-3} = 1 \quad c_3 = 1 \quad c_{-5} = 2i \quad c_5 = -2i$

NEGATIVE

Part D — 2 points One representation of the Fourier basis functions has

$$a_k \cos kt + b_k \sin kt$$

as its k^{th} term. For the two other representations discussed in class, show the k^{th} term in the Fourier series of that representation. Put the subscript k on each coefficient of the k^{th} term.

$A_k \cos(t + \theta_k)$

e^{ikt} AND e^{-ikt}

3 Fourier Applications — 4 points

Part A — 2 points Explain how to compute the convolution $f(t) * g(t)$ using the Fourier transform. Explain why this is can be more efficient than computing the convolution directly.

(1) CONVERT TO FOURIER DOMAIN

(2) MULTIPLY

(3) CONVERT BACK

$$\begin{tikzcd} f, g \arrow[r, "F.T."] & F, G \arrow[d, "\times"] \\ f * g & F \times G \arrow[l, "F.T.^{-1}"'] \end{tikzcd}$$

TAKES $O(N \log N)$ TIME VS $O(N^2)$ TIME FOR f AND g OF SIZE N.

Part B — 2 points Explain how periodic noise can be removed from an image.

(1) APPLY F.T. TO IMAGE

(2) MAKE ZERO ANY HIGH-VALUED (BRIGHT) SPOTS IN THE F.T.

(3) APPLY F.T.$^{-1}$

4 Sampling and Filtering — 9 points

Part A — 1 point Show a cosine function that has a frequency of 15 cycles per second.

$\cos(2\pi \times 15t)$

AS t GOES $0 \to 1$

$2\pi 15t$ GOES $0 \to 30\pi$

Part B — 2 points What is the fundamental period of the set of functions e^{ikt} for $k = 1, 2, 3, \ldots$? Explain.

FUNDAMENTAL PERIOD HAS FUNDAMENTAL FREQUENCY THAT IS THE GREATEST COMMON FACTOR OF THE FREQUENCIES IN THE SET.

THOSE FREQUENCIES ARE $\frac{1}{2\pi}, \frac{2}{2\pi}, \frac{3}{2\pi}, \ldots$

SO $\frac{1}{2\pi}$ IS THE FUND. FREQ AND 2π THE FUND. PERIOD.

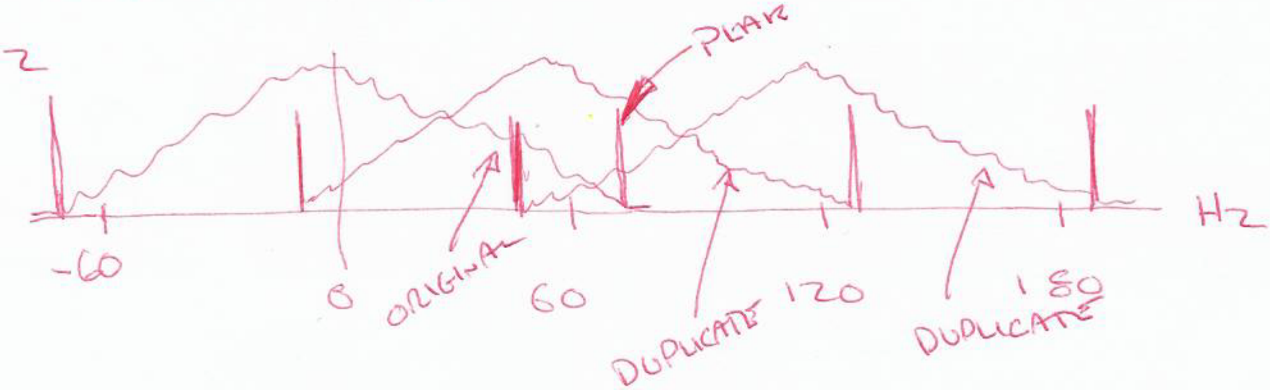
Part C — 2 points What artifact is caused by an ideal low-pass filter? Explain why this occurs.

RINGING OCCURS BECAUSE THE LOBES OF THE SPATIAL DOMAIN SINC (WHICH CORRESPONDS TO THE FREQUENCY DOMAIN ILPF) CAUSE SIGNALS TO BE ECHOED IN THE SIGNAL.

... continued

Part D — 2 points Suppose that a signal has a peak at 80 Hz (i.e. 80 cycles per second), but can be sampled at only 60 Hz. State where the alias of the 80 Hz peak appears in the reconstructed signal. Explain why it appears there.

IN FREQUENCY DOMAIN, SIGNAL IS DUPLICATED EVERY 60 HZ



SO THE SIGNAL ABOVE 60 HZ IS REFLECTED AROUND 60HZ, AND THE ALIAS APPEARS AT 40HZ (= 60 − (80−60))

Part E — 2 points Assuming that a signal can be sampled at only 60 Hz, describe **two** methods by which the alias from an 80 Hz peak can be avoided in the reconstructed signal.

(I.E LOW-PASS FILTER)

① SMOOTH THE SIGNAL BEFORE SAMPLING, IN ORDER TO REMOVE HIGH FREQUENCIES.

② AFTER RECONSTRUCTION, APPLY A NOTCH FILTER AT THE ALIAS (I.E. AT 40 HZ).