Queen's University
CISC/CMPE 457
Test 2

October 22, 2020
noon
Duration: 24 hours

Open book

Initial of Family Name: ____

Student Number: ______________________________
(Write this at the top of every page.)

There are 6 questions and 28 marks total.

Answer all questions.

This exam paper should have 8 pages,
including this cover page.

1 – Short Answer	/ 9
2 – Frequency-Domain Filtering	/ 5
3 – Discrete Sampling	/ 2
4 – 2D Fourier Transform	/ 4
5 – Gaussian Blurring	/ 4
6 – De-Blurring	/ 4
Total	/ 28

The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretations of any question that requires a written answer.

1 Short Answer — 9 points

Part A — 1 point In the real Fourier series, why is there no b_0 term?

$b_0 = \sin 0t = 0$

Part B — 1 point Pick any two of the real Fourier basis functions (except the basis function corresponding to b_0) and show that they are orthogonal.

Pick $\cos 0t$ and $\cos 1t$.
Then $\int_0^{2\pi} \cos(0t)\cos(1t)\ dt = \int_0^{2\pi} \cos(1t)\ dt = 0$.
Since the integral of the product is zero, they are orthogonal.

Part C — 1 point What is the fundamental frequency of $2\cos(24t) + 4\sin(9t)$? Explain.

Wavenumbers 24 and 9 correspond to frequencies $\frac{24}{2\pi}$ and $\frac{9}{2\pi}$. The greatest common factor of those is $\frac{3}{2\pi}$, which is the fundamental frequency.

Part D — 1 point In the discrete Fourier transform of an $N \times N$ image, what is the wavenumber of the basis function at position (u, v) in the frequency domain?

$\sqrt{u^2 + v^2}$

Part E — 1 point Why is the *magnitude* plot of the 1D complex Fourier Transform symmetric around the origin (i.e. why is $|F(k)| = |F(-k)|$)?

... continued

The coefficients of the complex series are $c_k = \frac{1}{2}(a_k - ib_k)$ and $c_{-k} = \frac{1}{2}(a_k + ib_k)$, both for $k > 0$. Both c_k and c_{-k} have the same magnitude of $\sqrt{a^2 + b^2}$.

Part F — 1 point What linear function maps x in the range $[0, k)$ onto the range $[0, 2\pi)$? Where is this mapping used when applying the Fourier transform?

$x \rightarrow \frac{2\pi}{k} x$

This is used to map signals of period k to the range $[0, 2\pi)$ before applying the Fourier transform.

Part G — 3 points In the discrete Fourier transform of an $N \times N$ image, what is the maximum frequency among all the 2D basis functions represented in the discrete transform? Explain.

The frequency of the basis function corresponding to position (u, v) is $\frac{\sqrt{u^2+v^2}}{N}$ (this is measured as "cycles per pixel" since the image width is N).

This is maximized at the *corners* of the transform, not at the edges!

The coordinates at the upper-right corner at $(\frac{N}{2}, \frac{N}{2})$ and the corresponding frequency is $\frac{1}{N}\sqrt{(\frac{N}{2})^2 + (\frac{N}{2})^2} = \frac{\sqrt{2}}{N}\frac{N}{2} = \frac{1}{\sqrt{2}}$ cycles per pixel.

2 Frequency-Domain Filtering — 5 points

Part A — 3 points

If $G(u,v) = e^{\frac{-(u^2+v^2)}{2\sigma^2}}$ is the 2D unnormalized Gaussian (i.e. with maximum value 1), what is the effect in the frequency domain of multiplying a frequency-domain image, $F(u,v)$, by $1 - G(u,v)$. What is the corresponding effect in the spatial domain? Explain. Do not consider $F(0,0)$ in your answer.

This will attenuate low-frequency components, with the most attenuation near the origin and less attenuation farther from the origin.

In the spatial domain, this will cause edges (which have high-frequency) to appear more prominently since the low-frequency components have been attenuated.

Part B — 2 points With the multiplication above, $F(0,0) \times (1 - G(0,0)) = 0$. What effect does this have on the image? How should you modify $1 - G(u,v)$ to prevent this effect?

Since $F(0,0)$ is the average value (or intensity) of the image, the multiplication causes the average value to become zero. This causes the image to darken considerably.

To prevent this, instead use

$$G'(u,v) = \begin{cases} 1 & \text{if } (u,v) = (0,0) \\ 1 - G(u,v) & \text{otherwise} \end{cases}$$

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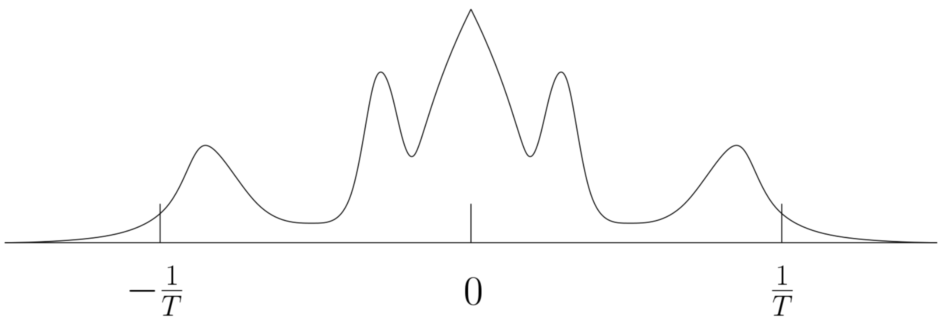
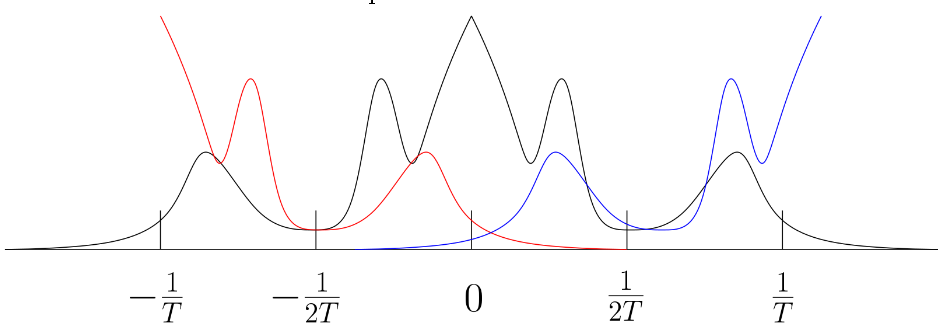
3 Discrete Sampling — 2 points

Below is the true frequency spectrum of a signal. Draw what this spectrum would look like if it were discretely sampled at a rate of $\frac{1}{T}$.



Should be reflected around $\frac{1}{2T}$ *and* around $-\frac{1}{2T}$ and added to original spectrum.

Here are the reflections of the spectrum:



The sum of these is left to the imagination.

4 2D Fourier Transform — 4 points

Suppose you have a function `FFT_1D`(f, N), which takes a discrete signal $f[0], f[1], \ldots f[N-1]$ and replaces the elements of the f array with the Fourier transform $F[0], F[1], \ldots F[N-1]$, where $F[k]$ is the Fourier transform of f for wavenumber k.

Provide pseudocode that uses `FFT_1D()` to implement the 2D Fourier transform. The 2D function is `FFT_2D`(f, N), where f is an $N \times N$ image array.

Assume that all values in f (1D or 2D) are complex. In your code, you can specify row r of the 2D f as $f[-, r]$ and can specify a column c as $f[c, -]$.

```
FFT_2D( f, N )
    # Compute FTs of rows
    for r = 0 ... N-1
        FFT_1D( f[-,r], N )
    # Compute FTs of columns
    for c = 0 ... N-1
        FFT_1D( f[c,-], N )
    # Could do columns first, then rows.
    # Could have 1-indexed arrays.
```

... continued

5 Gaussian Blurring — 4 points

The Fourier Transform of a Gaussian with standard deviation σ in the spatial domain is a Gaussian with standard deviation $\frac{1}{\sigma}$ in the frequency domain.

Explain, with respect to the amount of blurring in the spatial domain and the attenuation of frequencies in the frequency domain, why this recipricol relationship makes sense.

The high frequencies in the frequency domain are attenuated in proportion to a Gaussian of standard deviation $\frac{1}{\sigma}$.

As σ gets larger, *more* blurring occurs in the spatial domain because the wider Gaussian "averages" over a greater area before the Gaussian's value becomes very close to zero.

In the frequency domain, however, the corresponding Gaussian becomes *narrower* as its standard deviation $\frac{1}{\sigma}$ gets smaller.

A narrower Gaussian, when multiplying in the frequency domain, will attenuate more frequencies because its has more values near zero close to the origin than does a wider Gaussian. Attenuation of more frequencies correspond to more blurring in the spatial domain.

6 De-Blurring — 4 points

Suppose you have an image I' that is the result of blurring an image, I, by a Gaussian filter, G, like this:

$$I' = I * G.$$

If you know only I' and G, how would you use the Fourier Transform to remove the blurring from I'? In essence, you are recovering the original I, knowing only I' and G.

Prove algebraically that your method is correct. Hint: Consider the convolution theorem and note that the inverse of multiplication is division in the frequency domain.

Divide $FT(I')$ by $FT(G)$, then apply the inverse Fourier transform to the result to get I.

In the following, FT() is the Fourier transform and $\mathrm{FT}^{-1}()$ is the inverse Fourier transform.

$$\begin{aligned}
\mathrm{FT}(I') &= \mathrm{FT}(I * G) \\
&= \mathrm{FT}(I) \times \mathrm{FT}(G) \\[1ex]
\mathrm{FT}(I) &= \mathrm{FT}(I') \div \mathrm{FT}(G) \\[1ex]
I &= \mathrm{FT}^{-1}(\ \mathrm{FT}(I') \div \mathrm{FT}(G)\ )
\end{aligned}$$