Master Quantitative Finance and Risk & Asset Management

Constant Proportion Portfolio Insurance

Insurance & Finance

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Introduction

Constant Proportion Portfolio Insurance (CPPI) was introduced by ndre F. Perold in 1986 for equity instruments, and has been further analyzed by many scholars. An investor invests in a portfolio and wants to protect the portfolio value from falling below a pre-assigned value. The investor shifts his asset allocation over the investment period among a risk-free asset plus a collection of risky assets.

The CPPI strategy is based on the dynamic portfolio allocation of two basic assets: a riskless asset (usually a treasury bill) and a risky asset (a stock index for example). This strategy relies crucially on the concept of a cushion C, which is defined as the difference between the portfolio value V and the floor F. This later one corresponds to a guaranteed amount at any time t of the management period [0, T]. The key assumption is that the amount e invested on the risky asset, called the exposure, is equal to the cushion multiplied by a fixed coefficient m, called the multiple. The floor and the multiple can be chosen according to the investor's risk tolerance.

Most portfolios contain risky assets. Fluctuations in the values of such assets will generally cause the value of the portfolio in which they are held to change. The asset allocation of the portfolio will also change. If the risky assets increase in value, for example, the proportion of the portfolio they comprise is also likely to increase.

One must decide how to re-balance the portfolio in response to such changes. Dynamic strategies (Buy-and-hold, constant mix, CPPI, OBPI. . .) are explicit rules for doing so. Different rules have different consequences in both long term and short term. A rule preferred by one type of investor may not be preferred by another.In this project, we focus on one of the most important dynamic strategies known by the CPPI strategy.

Chapter 1

Theoretical Framework of CPPI

1.1 Description of CPPI strategy

Constant Proportion Portfolio Insurance (CPPI), introduced by Andre F. Perold in 1986 and improved by Fischer Black and Robert W. Jones in 1987, is a dynamic strategy of portfolio insurance that ensures a minimum guarantee at maturity to the investor and benefits from the market leverage effect. In this first chapter, we present the principle of this strategy its objectives with some numerical examples.

To implement a CPPI strategy, the investor selects the multiplier and a floor below which he does not want the portfolio value to fall. This floor grows at the rate of return on bills and must initially be less than total assets. If we think of the difference between assets and the floor as a "cushion" then the CPPI decision rule is simply to keep the exposure to equities a constant multiple of the cushion.

The product manager will take larger risks when the market is performing well. But if the market is going down he will reduce the risk rapidly.
The following factors play a key role in the risk strategies an investor will take:

- **Price:** the current value of the CPPI. The value at time $t \in [0, T]$ will be denoted as V_t.

- **Floor:** the reference level to which the CPPI is compared. This level will guarantee the possibility of repaying the fixed amount N at maturity T, hence it could be seen as the present value of N at maturity. Typically this is a zero-coupon bond and its price at time t will be denoted as P_t.

- **Cushion:** the cushion is defined as the difference between the price and the floor, Cushion = Price − Floor noted by C_t.

- **Multiplier:** the multiplier is a fixed value which represents the amount of leverage an investor is willing to take.

- **Investment level:** is the percentage invested in the risky asset portfolio; this also known as the exposure and is for each step fixed at: e = Multiplier × Cushion.

- **”GAP” risk:** is the probability that the CPPI value will fall under the Floor. The level of risk an investor will take is equal to the investment level as long as the value of the

CPPI exceeds the floor. For any time t the future investment decision will be made according to the following rule:

- if $\mathbf{V}_t \leq$ **Floor**, we will invest the complete portfolio in a into the zero-coupon bond. The CPPI strategy fails.
- if $\mathbf{V}_t \geq$ **Floor**, we will invest an amount equal to e in the risky asset portfolio.

From the liability view, we have :

- **The floor** corresponds to the present value of the guarantee at maturity (discounting at the rate of the non-risky asset).
- **The cushion** is defined as the difference between the net asset value of the fund and the floor. It is therefore the maximum theoretical amount that can be lost over a period, without compromising the capital guarantee.

From the asset view, we have :

- The portion invested in **risky assets** is initially calculated by multiplying the amount of the cushion by the multiplying coefficient m, representing the manager's risk appetite. This share then changes according to the performance of these risky assets.
- The portion invested in **non-risky assets** is initially calculated by the difference between the fund's NAV and risky asset. This share then changes by revaluation with the rate of the non-risky asset.
- The coefficient m_t at date t is calculated by the ratio between risky asset and cushion.

In the practice, considering a portfolio composed by a single risky asset and a non-risky one, it is possible to implement the CPPI strategy in order to ensure the investor's capital until maturity.
Let S_t the price at t of the risky and P_t the one for the zero-coupon bond ($P_T = N$ at maturity), with Π_t and $(1-\Pi_t)$ the amount invested in them, respectively. As a consequence we have :

$$P_t = \frac{N}{(1+r_f)^{T-t}}$$

N : Notional
r_f **:** risk-free rate
T **:** maturity

and

$$V_t = \Pi_t S_t + (1-\Pi_t)P_t$$

with V_t portfolio value.
Until maturity, the insurer can re-balance the portfolio at re-balancing date t, when it is possible :

- $V_t > P_t$ **(better situation):** the insurer invests $E_t = m(V_t - P_t)$ in the risky asset, with m multiplier chosen at the beginning by the client and the insurer with respect to the leverage and the risk profile of the client.Then, the rest $M_t = V_t - P_t$ is invested in non-risky asset.

- $V_t < P_t$ **(worst situation):** the insurer invests V_t in non-risky asset and waits until maturity T. In this case the capital N is not guaranteed and as a result the CPPI strategy fails.
$$V_T = V_t(1 + r_f)^{T-t} \leq P_t(1 + r_f)^{T-t} = P_T = N$$

1.2 Advantages of the CPPI strategy

- **Very transparent :** in contrast to options-based portfolio insurance strategies the investor always has a very transparent overview of the investments.

- **Straightforward : CPPI** explicitly rules out market timing and is quite simple to apply. There is no need for extensive research on companies and markets.

- **Flexible :** Floor and multiplier can be arbitrarily chosen and changed at any time. However, it should be avoided to use this flexibility too extensively in order not to dilute the CPPI strategy.

- **Profits can be locked in :** With the use of a ratchet strategy, the floor can be increased after market rises.

- **Strategy can be terminated at any time :** As derivatives are not used, the investments can be sold at any time. Also, the protection runs during the whole time and not only to a certain redemption date.

1.3 Drawbacks of the CPPI strategy

- **Protection can be breached in the event of a market crash :** If there is a sudden drop in the market such that the investor is not able to re-balance his portfolio adequately, the floor can be breached. The multiplier determines the amount of market loss permissible, being its reciprocal value. With a multiplier of 5, the market would be allowed a 20 % overnight drop, which is unlikely but not impossible.

- **Cashing out effect :** The exposure to the risky asset reaches zero as the portfolio approaches the floor. If this happens shortly after the initial investment, re-entering the stock market can be difficult if not impossible.

Chapter 2

Practical Applications

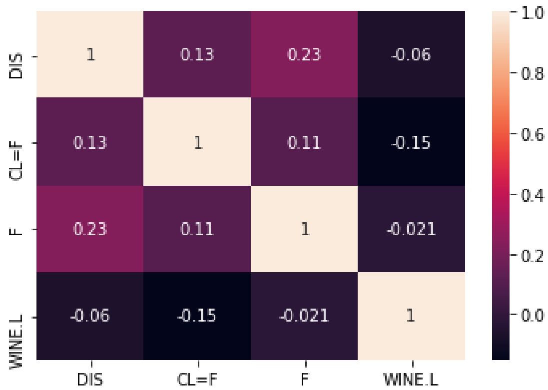
2.1 Data presentation

In this section, we will present our portfolio composed by **4** stocks:

- **DIS :** Disney Land
- **CL=F:** Crude Oil
- **F :** Ford
- **WINE.L :** Naked Wines PLC

We have chosen yahoo finance **daily** data starting from 01/02/2019 to 01/02/2020.

To be sure that our portfolio is well diversified, we present the correlation matrix:



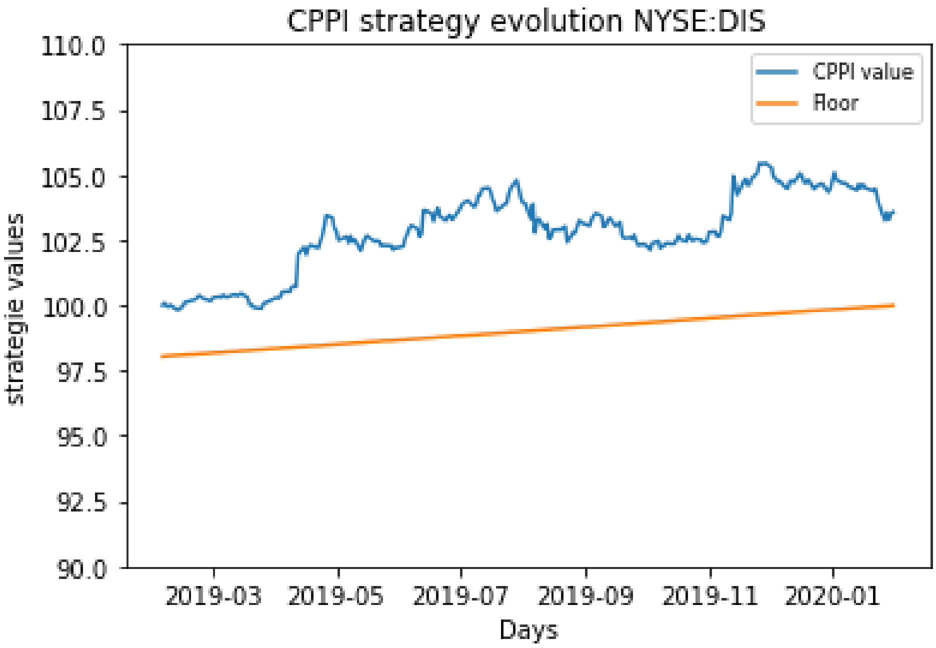
As we can see, all coefficients are less than 0.5, so we can consider the four stocks almost uncorrelated.

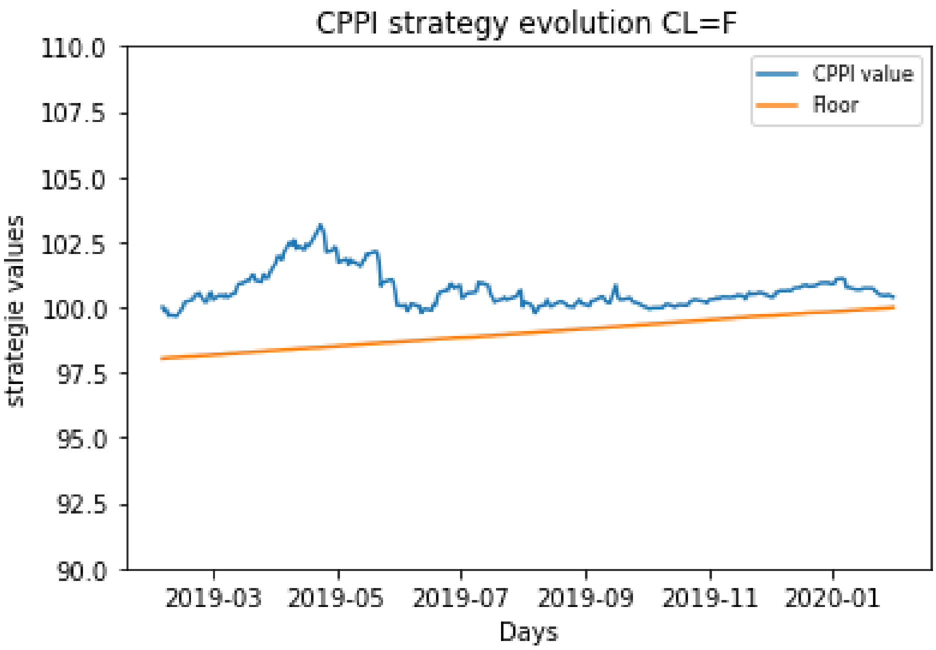
We suppose that:

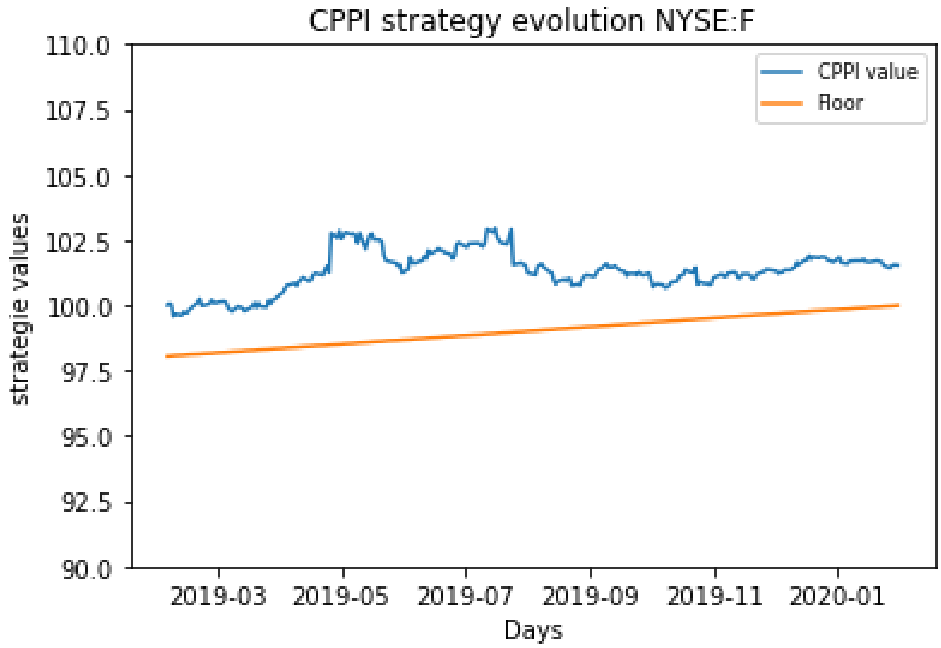
- **risk-free rate** : $r_f = 2\%$
- **strike** : $K = 100$
- **the tolerance for re-balancing**: $tol = 10\%$
- **multiplier**: $m = 5$

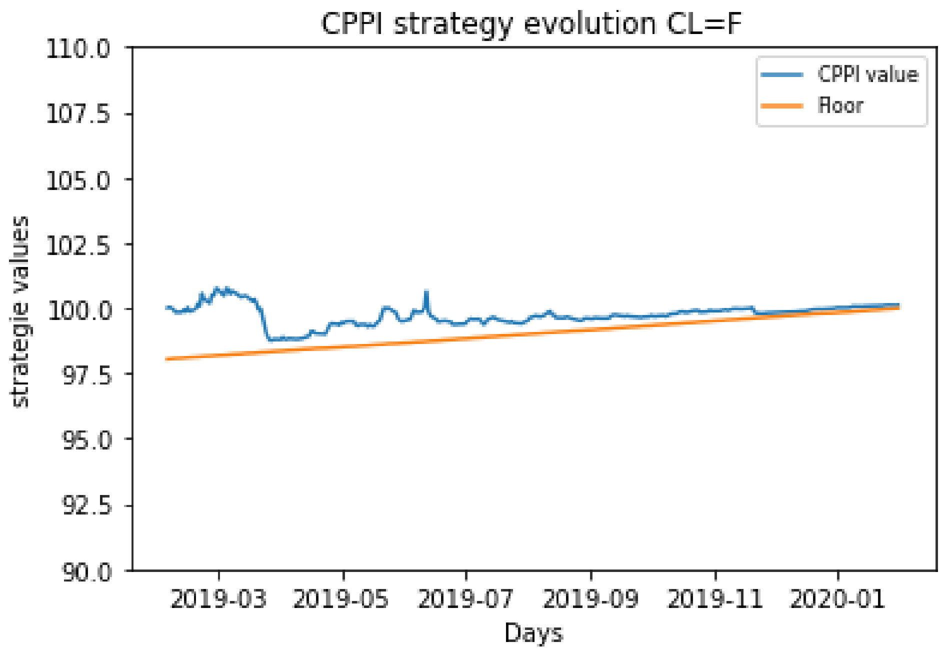
2.2 Numerical application using one single asset

We have implemented the CPPI strategy on Python and tested it using the real data of the stocks mentioned above. We constructed four different portfolios composed by one of those stocks and a risk-less asset in order to value their individual behaviour while used to ensure a capital using the CPPI strategy. We have obtained that:







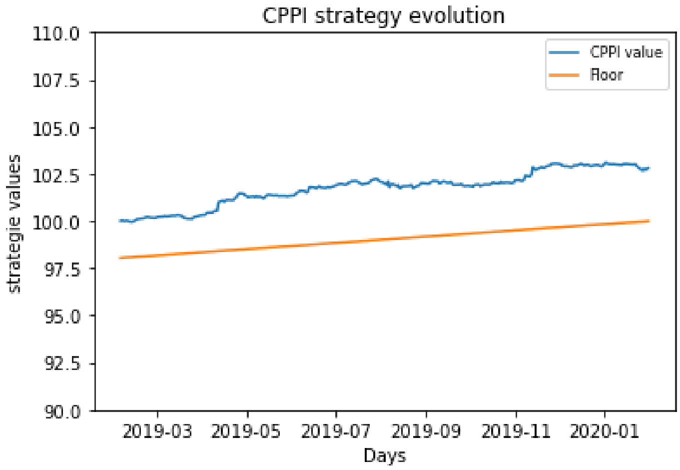


As we can see from these graphs, the strategy succeeds using all the portfolios. This can be easily noticed since the portfolios value is always bigger than the floor until maturity, so the strategy has not failed.

Choice of m

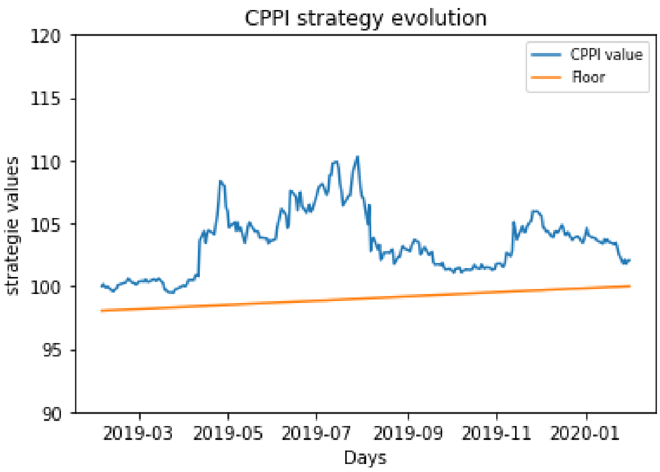
Another key aspect of the CPPI strategy is the choice of m. This decision is mainly based on how the investor or insurer is ready to risk to have a larger profit at maturity.

- **Averse to risk**: In this first case we will choose $m = 2$ (averse to risk situation) supposing that the VaR or the stress index represents 50%. Using a very low multiplier means that the investor had a had a negative view about the economic situation.



  This situation helps us to avoid losing our initial amount (100) and to take advantage from the momentum of the risky asset, which means less profit. As we can see, the maximum profit is approximately 102.5, which means that we guarantee our capital but without generating any profit.

- **Taking risk**: In this second case we will choose $m = 10$ supposing that the VaR is equal to 10%. Using a very low multiplier means that the investor had a had a optimistic view about the economic situation.

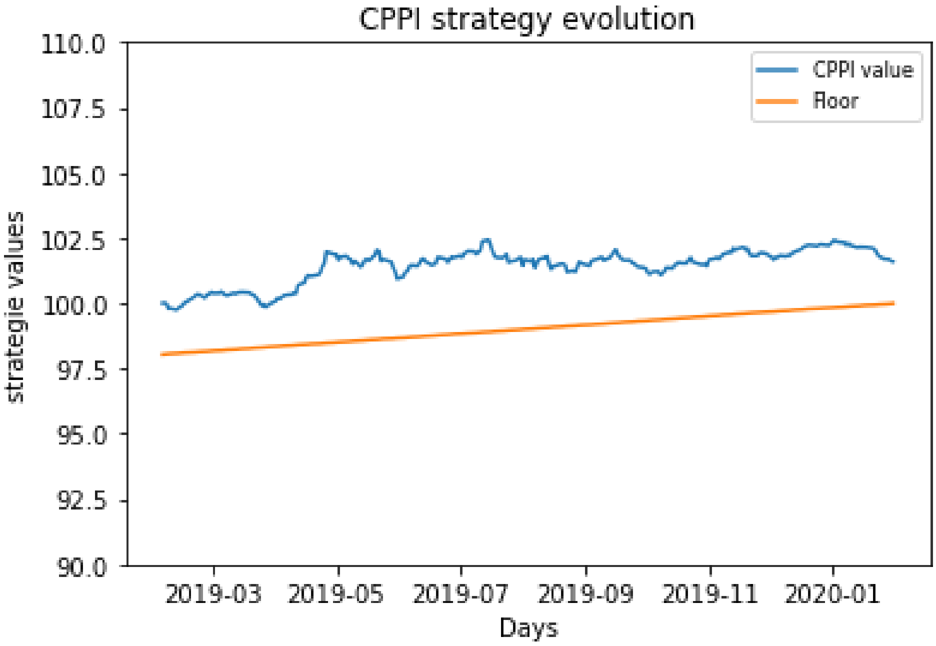


  As we can see, the profit of our strategy becomes more important, having the highest value achieved at 110.

In conclusion, it is very important to price well the multiplier which plays the main role to take or not risks. It refers the situation of the investor. we can conclude that it represents the amplitude of maximum losses/profits that we can accept on risky asset between two rebalancing days.

2.3 Numerical application using more than a single asset

In this section, we present the result of the CPPI strategy on the portfolio, composed by all the four stocks mentioned above, supposing at the beginning equal weights in order to let them have the same initial influence on the portfolio.

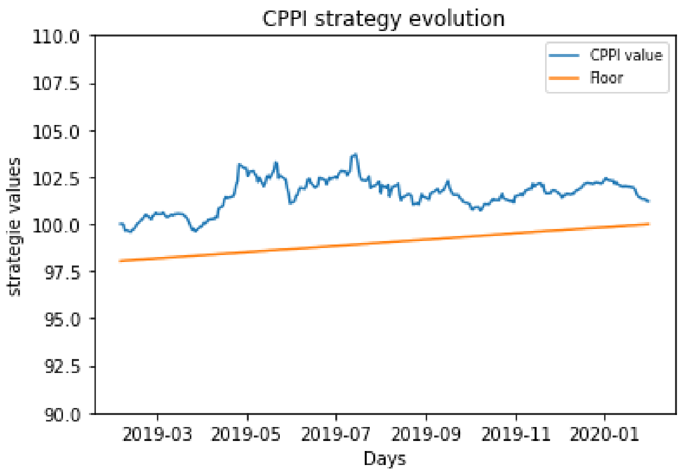


With a multiplier = 5, we can see that our strategy is working well and that the notional is guaranteed at the maturity. Also, the graphic shows that the maximal profit achieved using the CPPI strategy is 107.

Choice of m

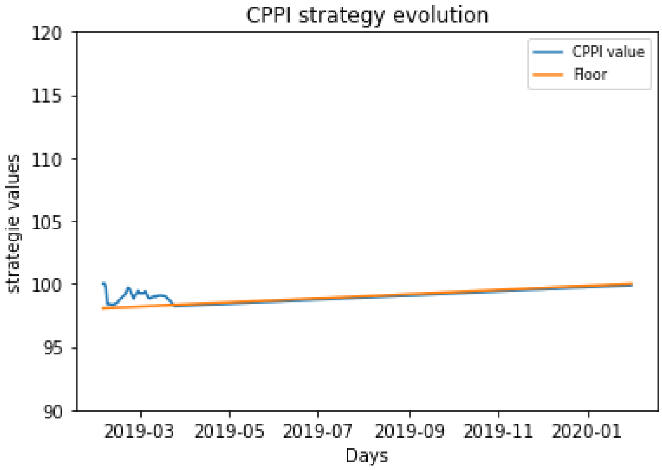
As for the single risky asset case, the value chosen for m can largely influence the success/failure of the CPPI strategy using a portfolio composed by more than a risky asset. Therefore, the choice of it, it is extremely important and depends on how much risk the investor or insurer wants to take:

- **Averse to risk**: In this first case we will choose $m = 2$ (averse to risk situation), as before, supposing that the VaR or the stress index represents 50%. Using a very low multiplier means that the investor had a had a negative view about the economic situation.



As we have seen in the case of one asset,using a very low multiplier allows to avoid losing our initial amount (100). As we can see, the maximum profit is approximately 103, which means that we guarantee our capital but without generating high profit.

- **Taking risk**: In this case we will choose $m = 10$ supposing that the VAR is equal to 10%. Using a very low multiplier means that the investor had a had a optimistic view about the economic situation.



As we can see, our strategy fails after few days. This means that being very optimistic in this presented period is a very risky vision. Actually, we can explain that at a rebalancing date between March 2019 and May 2019, the market crash, so the value of the portfolio becomes less than the floor. The investor have to stop the strategy and invest all his money in non-risky asset.

In conclusion, it is very important to price well the multiplier which plays the main role to take or not risks. In this case, the multiplier choice could seriously influence the success of the strategy leading to a failure using a large value of m.

Chapter 3

Theoretical Framework in continuous time

The CPPI (Constant Proportion Portfolio Insurance) strategy is based on a dynamic portfolio allocation on two basic assets: a risk-less asset (usually a treasury bill) and a risky asset (a stock index for example).

This strategy depends crucially on the cushion C, which is defined as the difference between the portfolio value V and the floor F. This later one corresponds to a guaranteed amount at any time t of the management period [0, T]. The key assumption is that the amount e invested on the risky asset, called the exposure, is equal to the cushion multiplied by a fixed coefficient m, called the multiple. The floor and the multiple can be chosen according to the investors risk tolerance.

The risk-aversion investor will choose a small multiple or/and a high floor and vice versa. The higher the multiple, the more the investor will benefit from increases in stock prices. Nevertheless, the higher the multiple, the higher the risk that the portfolio value becomes smaller than the floor if the risky asset price drops suddenly. As the cushion value is approximately equal to zero, exposure is near zero too.

3.1 Black and Scholes model

In the continuous-time case, we assume that the **risky asset** satisfies the Black and Scholes model, so characterized by this dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

with market trend μ, volatility σ and Brownian Motion W_t.

Solving the SDE above, it is possible to find that:

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

Regarding the risk-less asset, we have the following equation :

$$dBt = rB_t \quad dt \quad ; \quad B_0 = 1$$

with risk-free rate r and solving it, we obtain $B_t = e^{rt}$.

Then from the self-financing portfolio, we have that:

$$dV_t = r_k dt + E_t \frac{dS_t}{S_t}$$

$$dV_t = r\left(V_t - E_t\right)dt + E_t\frac{dS_t}{S_t}$$

The main goal of CPPI strategy in Black Scholes framework is to compute the cushion C_t dynamically and observe a non-negative process at each time t. Moreover, in this setting we are pretty sure that the nominal is guaranteed at maturity. From the CPPI strategy, we know that:

$$V_t = mC_t + \left(V_t - mC_t\right)$$

$$1 = \left(\tfrac{mC_t}{V_t} + \left(1 - \tfrac{mC_t}{V_t}\right)\right)$$

and by the assumption of self-financing portfolio, we have

$$dV_t = V_t\left(\frac{mC_t}{V_t}\frac{dS_t}{S_t} + \left(1 - \frac{mC_t}{V_t}\right)\frac{dB_t}{B_t}\right)$$

thus

$$dC_t = d\left(V_t - F_t\right)$$

$$dC_t = V_t\left(\frac{mC_t}{V_t}\frac{dS_t}{S_t} + \left(1 - \frac{mC_t}{V_t}\right)\frac{dB_t}{B_t}\right) - F_t\frac{dB_t}{B_t}$$

$$\mathrm{d}\ \mathrm{C}_t = C_t\left(\tfrac{mdS_t}{S_t} - (m-1)rdt\right)$$

$$dC_t\left(m\left(\mu dt + \sigma dW_t\right) - (m-1)rdt\right)$$

$$dC_t = C_t\left((r + m(\mu - r))dt + m\sigma dW_t\right)$$

Then

$$C_t = C_0 e^{(r+m(\mu-r))t - \frac{m^2\sigma^2 t}{2} + m\sigma W_t}$$

It's clear that the cushion C_t is a Geometric Brownian Motion.

We Know that S_t is also a Brownian Motion, let's see if there's a relation between S_t and C_t

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$
$$\Rightarrow \ln\left(\tfrac{S_t}{5_0}\right) = \left(\mu - \tfrac{\sigma^2}{3}\right)t + \sigma W_t$$
$$\Rightarrow \quad \tfrac{1}{\sigma}\left[\ln\left(\tfrac{S_t}{S_0}\right) - \left(\mu - \tfrac{\sigma^2}{2}\right)t\right] = W_t$$

Remember that :

$$C_t = C_0 e^{(r+m(\mu-r))t-\frac{m^2\sigma^2 t}{2}+m\sigma W_t}$$

by plugging the equation above in the (3.9) we have :

$$C_t = C_0 \left(\frac{S_t}{S_0}\right)^m e^{((1-m)\left(r+\frac{m\sigma^2}{2}\right)t)}.$$

$$\mathrm{C}_t = \alpha_t S_t^m$$

with:

$$\alpha_t = e^{((1-m)\left(r+\frac{m\sigma^2}{2}\right)t)}.$$

3.2 Stochastic jumps and CPPI

Stochastic volatility and the leverage effect are stylized (empirical) facts found in a number of markets. Another important stylized fact is the existence of jumps in asset prices and, for example, volatility. In 1976, Merton published his jump diffusion model, enhancing the Black-Scholes-Merton setup through a model component generating jumps with log-normal distribution. The risk-neutral SDE is presented with the following equation:

$$dS_t = (r - r_j)\, S_t dt + \sigma S_t dZ_t + J_t S_t dN_t$$

with :

$$r_J \equiv \lambda \cdot \left(e^{\mu_j+\delta^2/2} - 1\right)$$

- λ : Jump intensity
- μ_j: Expected jump size
- δ: Standard deviation of jump size

For completeness, here is an overview of the variables and parameters meaning:

- S_t: stock value at time t
- r : Constant risk-less short rate
- r_j: drift correction for jumps
- Z_t: Standard Brownian motion
- J_t: Jump at date t with the distribution
  -
  $$\log(1 + J_t) \approx \mathbf{N}\left(\log(1+\mu_J) - \frac{\delta^2}{2}, \delta^2\right)$$

  with N as the cumulative distribution function of a standard normal random variable.

Euler discretization

To solve the SDE presented in (3.11) we have chosen the Euler discretization presented as follows:

$$S_t = S_{t-\Delta t}\left(e^{\left(r-r_J-\sigma^2/2\right)\Delta t+\sigma\sqrt{\Delta t}z_t^1} + \left(e^{\mu_j+\delta z_t^2}-1\right)y_t\right)$$

the z_t^n are standard normally distributed and the y_t are Poisson distributed with intensity λ.

Since there's jumps, it's not anymore guaranteed that $C_t > 0$.

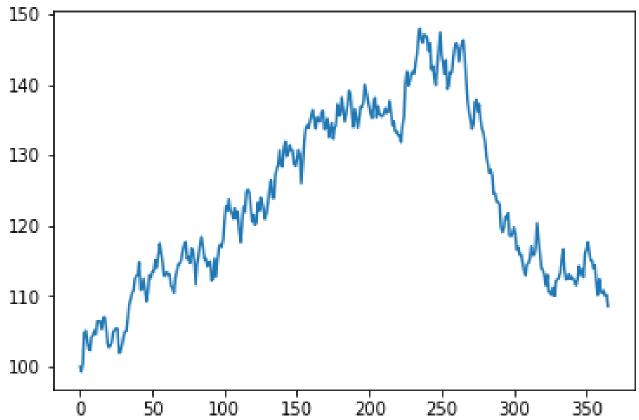
Worse we can have $C_\tau < 0$.

Chapter 4

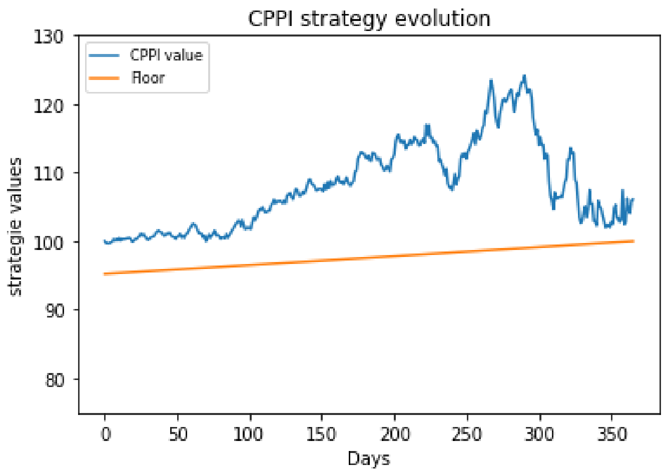
Practical Applications in continuous time

4.1 Numerical application without jumps

First, we have decided to implement the stochastic evolution of a stock without jumps, fixing risk-free rate $r = 0.05$, value of the stock at starting date $S_0 = 100$, volatility $\sigma = 25\%$, maturity $T = 1$ and number of discretization steps $M = 365$, obtaining:

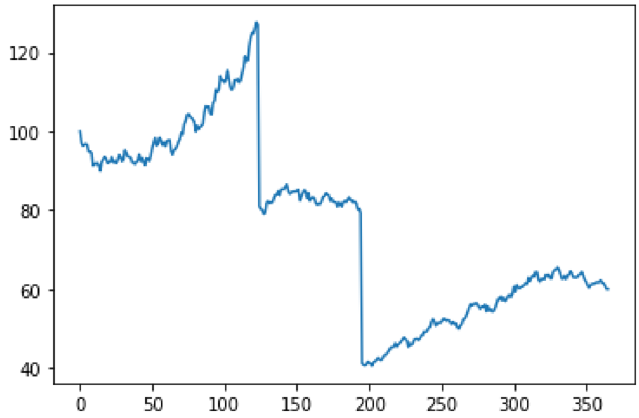


Then, we have applied the CPPI strategy with dynamic cushion.

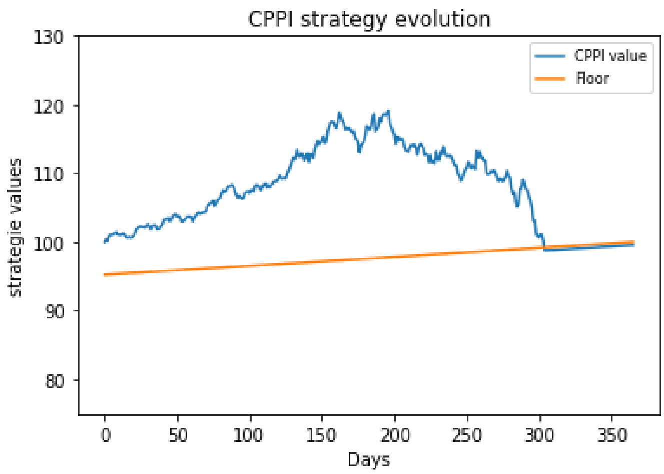


4.2 Numerical application with jumps

Using the same fixed values as in the case above, we have implemented the stochastic evolution of a stock with jumps letting jump intensity $\lambda = 0.75$, expected jump size $\mu = -0.06$ and standard deviation of the jump size $\delta = 0.25$. We have simulated the stock many times ($N = 100$), since not all the stock evolution will present jumps but only some of them, and then chosen one of the ones with a considerable jump.



Consequently, we have applied to the selected stock dynamic the CPPI strategy with dynamic cushion, obtaining:



As we can notice, the strategy has failed with the chosen stock evolution and this is due to the presence of a huge negative jump.

4.3 Monte Carlo simulation

Since with jumps and stochastic evolution we have many sources of randomness, we can easily evaluate from few simulations the goodness of the strategy. For this reason we have decided to implement a Monte Carlo method to approximate the final value of our portfolio using CPPI strategy to check if it is bigger or smaller than the floor at maturity, which implies the success or failure of the strategy.
A Monte Carlo method is a technique based on the estimation of a value as the mean of its simulations.

$$\Theta_m = \frac{1}{m}\sum_{j=1}^{m}\Theta_j$$

It is not needed to check the convergence of the Monte Carlo method is given by the Law of Large Numbers as the average of independent samples converges to the expected value. In addition, we know that the distribution of the error converges to a properly scaled normal distribution for the Central Limit Theorem. Both of them need proper assumptions that in our case are verified.

After applying the Monte Carlo method to the CPPI strategy in the case with more sources of randomness (Stock evolution with jumps), we have found using the number of simulations $M = 1000$ that its value converges to:

$$\mathbb{E}[V_T] = 105.3834$$

Then, we have computed the corresponding variance and confidence interval with $\alpha = 95\%$ since know they can be found as:

$$\sigma^2(\Theta_m) = \frac{1}{m-1}\left[\frac{1}{m}\sum_{j=1}^{m}(\Theta_j^2 - \Theta_m^2)\right]$$

$$CI(\Theta_m) = [\Theta_m - z_{0.95}\sigma(\Theta_m), \Theta_m + z_{0.95}\sigma(\Theta_m)]$$

with $z_{0.95} \approx 1.96$.

Therefore, we have obtained that:

$$\sigma^2(V_T) = 1.3888$$

$$CI(V_T) = [103.0736, 107.6932]$$

As we can see, the approximated value of the portfolio value at maturity is bigger than 100 with a considerably law variance and a good confidence interval (even the lower bound bigger than 100), so we can finally say that the CPPI strategy is profitable and it is a good method to ensure a capital until maturity.

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