Resolution of the linear Boltzmann equation by Monte Carlo method

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Boltzmann equation

The transport equation in an infinite medium with its corresponding deterministic collisional component can be expressed as:

$$\partial_t u(x,t,\mathbf{v})+\mathbf{v}\cdot\nabla u(x,t,\mathbf{v})+v\sigma_t(x,t,\mathbf{v})u(x,t,\mathbf{v}) = v\sigma_s(x,t,\mathbf{v})\int P(x,t,\mathbf{v},\mathbf{v}')$$

Where

$$\sigma_s(x,t,\mathbf{v}) = \int \sigma_s(x,t,\mathbf{v},\mathbf{v}')d\mathbf{v}', \quad P(x,t,\mathbf{v},\mathbf{v}') = \frac{\sigma_s(x,t,\mathbf{v},\mathbf{v}')}{\sigma_s(x,t,\mathbf{v})}$$

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Example of the calculation of π

We draw a circle with a radius of $r = 1$ inscribed in a square of side length 2. We throw stones in the air and count the number of stones within the circle. An estimate of π is given by

$$\pi_N = \frac{\text{Number of stones within the circle}}{4N}$$

pi.png

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Variable change

The approach involves a series of variable changes. The initial step is re-expressing the transport equation with respect to a characteristic $x + vt$. As a result, it transforms into:

$$\partial_s u(x + \mathbf{v}s, s, \mathbf{v}) = -v\sigma_t(x + \mathbf{v}s, s, \mathbf{v})u(x + \mathbf{v}s, s, \mathbf{v}) \\ + v\sigma_s(x + \mathbf{v}s, s, \mathbf{v}) \int P(x + \mathbf{v}s, s, \mathbf{v}, \mathbf{v}')u(x + \mathbf{v}s, s, \mathbf{v}')d\mathbf{v}'$$

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Equation Transformation with Multiplication and Integration

After multiplying both sides of the equation by:

$$e^{\int_0^s v\sigma_t(x+\mathbf{v}\alpha,\alpha,v)d\alpha}$$

Following that, we obtain

$$\partial_s[u(x+\mathbf{v}s,s,\mathbf{v})e^{\int_0^s v\sigma_t(x+\mathbf{v}\alpha,\alpha,v)d\alpha}]$$
$$= e^{\int_0^s v\sigma_t(x+\mathbf{v}\alpha,\alpha,v)d\alpha} v\sigma_s(x+\mathbf{v}s,s,\mathbf{v})\int P(x+\mathbf{v}s,s,\mathbf{v},\mathbf{v}')u(x+\mathbf{v}s,s,\mathbf{v}')d\mathbf{v}'$$

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Integration of the equation

We get after integrating the equation between [0, t]:

$$u(x+\mathbf{v}t,t,\mathbf{v}) = u_0(x,\mathbf{v})\exp\left(-\int_0^t v\sigma_t\left(x+\mathbf{v}\alpha,\alpha,\mathbf{v}\right)d\alpha\right)$$
$$+\int_0^t\int v\sigma_s\left(x+\mathbf{v}s,s,\mathbf{v}\right)u\left(x+\mathbf{v}s,s,\mathbf{v}'\right)e^{-\int_s^t v\sigma_t(x+\mathbf{v}\alpha,\mathbf{v})d\alpha}P\left(x+\mathbf{v}s,s,\mathbf{v},\mathbf{v}'\right)d\mathbf{v}'ds$$

After a variable change, we obtain:

$$u(x,t,\mathbf{v}) = u_0(x-\mathbf{v}t,\mathbf{v})\exp\left(-\int_0^t v\sigma_t\left(x-\mathbf{v}(t-\alpha),\alpha,\mathbf{v}\right)d\alpha\right)$$
$$+\int_0^t\int v\sigma_s\left(x-\mathbf{v}(t-s),s,\mathbf{v}\right)u\left(x-\mathbf{v}(t-s),s,\mathbf{v}'\right)e^{-\int_s^t v\sigma_t(x-\mathbf{v}(t-\alpha),\mathbf{v})d\alpha}P\left(x-\mathbf{v}(t-s),s,\mathbf{v},\mathbf{v}'\right)d\mathbf{v}'ds$$

Frame Title

We also have:

$$\exp\left(-\int_0^t v\sigma_t\left(x-\mathbf{v}(t-\alpha),\alpha,\mathbf{v}\right)d\alpha\right)$$
$$=\exp\left(-\int_0^t v\sigma_t\left(x-\mathbf{v}\alpha,t-\alpha,\mathbf{v}\right)d\alpha\right)$$
$$=\int_t^\infty v\sigma_t\left(x-\mathbf{v}s,t-s,\mathbf{v}\right)\exp\left(-\int_0^s v\sigma_t\left(x-\mathbf{v}\alpha,t-\alpha,\mathbf{v}\right)d\alpha\right)ds$$

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The integral form of the Boltzmann equation

Then the integral representation of the transport equation is provided by:

$$
\begin{aligned}
u(x,t,\mathbf{v}) = {} & \int_t^\infty u_0(x-\mathbf{v}t,\mathbf{v}) v\sigma_t\left(x-\mathbf{v}s, t-s, \mathbf{v}\right) \\
& \qquad \exp\left(-\int_0^s v\sigma_t\left(x-\mathbf{v}\alpha, t-\alpha, \mathbf{v}\right) d\alpha\right) ds \\
& + \int_0^t \int v\sigma_s\left(x-\mathbf{v}(t-s), s, \mathbf{v}\right) u\left(x-\mathbf{v}(t-s), s, \mathbf{v}'\right) \\
& \qquad e^{-\int_s^t v\sigma_t(x-\mathbf{v}(t-\alpha),\mathbf{v})d\alpha} P\left(x-\mathbf{v}(t-s), s, \mathbf{v}, \mathbf{v}'\right) d\mathbf{v}' ds
\end{aligned}
$$

Semi-analog scheme

Developing a Monte Carlo scheme involves introducing random variables and their associated probability measure to express the equation as an expectation. The choice of this set of random variables is not unique, leading to different Monte Carlo schemes with distinct properties.
For the semi-analog scheme, we introduce the probability measure of the interaction time:

$$f_\tau(\mathbf{x}, t, \mathbf{v}, s)ds = 1_{[0,\infty[}(s)\, v\sigma_t(\mathbf{x} - \mathbf{v}s, t - s, v)e^{-\int_0^s v\sigma_t(\mathbf{x}-\mathbf{v}\alpha, t-\alpha, v)\, d\alpha} ds$$

$\forall(x, t, v) \in D \times [0, T] \times \mathbb{R}^3$
We introduce the specified random variables corresponding to the previously identified probability measures.

$$\begin{cases} \tau \text{ with probability measure } f_\tau(\mathbf{x}, t, \mathbf{v})\, ds, \\ \mathbf{V}' \text{ with probability measure } P^s_{\mathbf{V}'}(\mathbf{x}, t, s, \mathbf{v}, \mathbf{v}')\, dv' \end{cases}$$

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Expression of the solution

We found the following expectation value:

$$u(x,t,v) = E\Big[1_{[t,\infty[}(\tau)u_0(\mathbf{x}-\mathbf{v}t,\mathbf{v}) \\ +1_{[0,t[}(\tau)\frac{\sigma_s(\mathbf{x}-\mathbf{v}\tau,t-\tau,v)}{\sigma_t(\mathbf{x}-\mathbf{v}\tau,t-\tau,v)}u(\mathbf{x}-\mathbf{v}\tau,t-\tau,\mathbf{V}')\Big]$$

Essentially, the process of constructing a Monte Carlo scheme is based on searching for solutions of expectation that possess this specific structures:

$$u_p(\mathbf{x},t,\mathbf{v}) = w_p(t)\delta_x(\mathbf{x}_p(t))\delta_{\mathbf{v}(\mathbf{v}_p(t))}$$

Replacing u_p in the equation yields:

$$\begin{cases} w_p(t) = 1_{[0,\infty[}(\tau)w_p(0) + 1_{[0,t[}(\tau)\frac{\sigma_s}{\sigma_t}(x_p(t-\tau),t-\tau,\mathbf{v}_p(t-\tau))w_p(t-\tau), \\ x_p(t) = 1_{[0,\infty[}(\tau)(\mathbf{x}_0-\mathbf{v}t) + 1_{[0,t[}(\tau)(x_{t-\tau}-\mathbf{v}\tau), \\ v_p(t) = 1_{[0,\infty[}(\tau)\mathbf{v} + 1_{[0,t[}(\tau)\mathbf{V}'. \end{cases}$$

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Numerical result