0.1 The use of Monte Carlo methods to solve the linear Boltzmann equation.

The transport equation in an infinite medium with its corresponding deterministic collisional component can be expressed as:

$$\partial_t u(x,t,\mathbf{v})+\mathbf{v}\cdot\nabla u(x,t,\mathbf{v})+v\sigma_t(x,t,\mathbf{v})u(x,t,\mathbf{v}) = v\sigma_s(x,t,\mathbf{v})\int P(x,t,\mathbf{v},\mathbf{v}')u(x,t,\mathbf{v}')d\mathbf{v}' \quad (1)$$

Where

$$\sigma_s(x,t,\mathbf{v}) = \int \sigma_s(x,t,\mathbf{v},\mathbf{v}')d\mathbf{v}', \quad P(x,t,\mathbf{v},\mathbf{v}') = \frac{\sigma_s(x,t,\mathbf{v},\mathbf{v}')}{\sigma_s(x,t,\mathbf{v})}$$

The approach involves a series of variable changes. The initial step involves re-expressing the transport equation (1) with respect to a characteristic $x+vt$. As a result, it transforms into:

$$\partial_s u(x+\mathbf{v}s,s,\mathbf{v}) = -v\sigma_t(x+\mathbf{v}s,s,\mathbf{v})u(x+\mathbf{v}s,s,\mathbf{v})+v\sigma_s(x+\mathbf{v}s,s,\mathbf{v})\int P(x+\mathbf{v}s,s,\mathbf{v},\mathbf{v}')u(x+\mathbf{v}s,s,\mathbf{v}')d\mathbf{v}' \quad (2)$$

Let's multiply both sides of the equation by:

$$e^{\int_0^s v\sigma_t(x+\mathbf{v}\alpha,\alpha,v)d\alpha}$$

Following that, we obtain

$$\partial_s[u(x+\mathbf{v}s,s,\mathbf{v})e^{\int_0^s v\sigma_t(x+\mathbf{v}\alpha,\alpha,v)d\alpha}] = e^{\int_0^s v\sigma_t(x+\mathbf{v}\alpha,\alpha,v)d\alpha}v\sigma_s(x+\mathbf{v}s,s,\mathbf{v})\int P(x+\mathbf{v}s,s,\mathbf{v},\mathbf{v}')u(x+\mathbf{v}s,s,\mathbf{v}')d\mathbf{v}'$$

We get after integrating the equation (0.1) between [0, t]:

$$\begin{aligned} u(x+\mathbf{v}t,t,\mathbf{v}) &= u_0(x,\mathbf{v})\exp\left(-\int_0^t v\sigma_t\left(x+\mathbf{v}\alpha,\alpha,\mathbf{v}\right)d\alpha\right) \\ &+\int_0^t\int v\sigma_s\left(x+\mathbf{v}s,s,\mathbf{v}\right)u\left(x+\mathbf{v}s,s,\mathbf{v}'\right)e^{-\int_s^t v\sigma_t(x+\mathbf{v}\alpha,\mathbf{v})d\alpha}P\left(x+\mathbf{v}s,s,\mathbf{v},\mathbf{v}'\right)d\mathbf{v}'ds \end{aligned} \quad (3)$$

We obtain:

$$\begin{aligned} u(x,t,\mathbf{v}) &= u_0(x-\mathbf{v}t,\mathbf{v})\exp\left(-\int_0^t v\sigma_t\left(x-\mathbf{v}(t-\alpha),\alpha,\mathbf{v}\right)d\alpha\right) \\ &+\int_0^t\int v\sigma_s\left(x-\mathbf{v}(t-s),s,\mathbf{v}\right)u\left(x-\mathbf{v}(t-s),s,\mathbf{v}'\right)e^{-\int_s^t v\sigma_t(x-\mathbf{v}(t-\alpha),\mathbf{v})d\alpha}P\left(x-\mathbf{v}(t-s),s,\mathbf{v},\mathbf{v}'\right)d\mathbf{v}'ds \end{aligned} \quad (4)$$

We also have:

$$\begin{aligned} \exp\left(-\int_0^t v\sigma_t\left(x-\mathbf{v}(t-\alpha),\alpha,\mathbf{v}\right)d\alpha\right) &= \exp\left(-\int_0^t v\sigma_t\left(x-\mathbf{v}\alpha,t-\alpha,\mathbf{v}\right)d\alpha\right) \\ &= \int_t^\infty v\sigma_t\left(x-\mathbf{v}s,t-s,\mathbf{v}\right)\exp\left(-\int_0^s v\sigma_t\left(x-\mathbf{v}\alpha,t-\alpha,\mathbf{v}\right)d\alpha\right)ds \end{aligned} \quad (5)$$

Then the integral representation of 1 is provided by:

$$u(x,t,\mathbf{v}) = \int_t^\infty u_0(x-\mathbf{v}t,\mathbf{v}) v\sigma_t\left(x-\mathbf{v}s, t-s, \mathbf{v}\right) \exp\left(-\int_0^s v\sigma_t\left(x-\mathbf{v}\alpha, t-\alpha, \mathbf{v}\right) d\alpha\right) ds$$
$$+\int_0^t \int v\sigma_s\left(x-\mathbf{v}(t-s), s, \mathbf{v}\right) u\left(x-\mathbf{v}(t-s), s, \mathbf{v}'\right) e^{-\int_s^t v\sigma_t(x-\mathbf{v}(t-\alpha),\mathbf{v})d\alpha} P\left(x-\mathbf{v}(t-s), s, \mathbf{v}, \mathbf{v}'\right) d\mathbf{v}' ds \tag{6}$$

Developing a Monte Carlo scheme involves introducing a group of random variables along with their associated probability measure to express 6 as an expectation. The selection of this set of random variables is not unique, resulting in various Monte Carlo schemes with distinct properties.