Solving the linear Boltzmann equation using Monte Carlo methods

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Contents

1 Introduction

1.1 Terms and definitions

The transport equation plays a crucial role in describing the behavior of particles in various physical systems. One of its important derivatives, the linear Boltzmann equation, is governed by the evolution of particle distribution in phase space. In this project, a numerical solution for the linear Boltzmann equation is developed utilizing the Monte Carlo method. The report is structured as follows:

1. **Transport Equation:** The fundamental transport equation is introduced, providing a basis for understanding the derivation of the linear Boltzmann equation.

2. **Linear Boltzmann Equation:** Building upon the transport equation, the linear Boltzmann equation is explored, emphasizing its significance in characterizing particle transport phenomena.

3. **Monte Carlo Method for Linear Boltzmann Equation:** The approach to solving the linear Boltzmann equation using the Monte Carlo method is presented, along with a discussion of the advantages and challenges associated with this numerical technique.

4. **Semi-Analog Monte Carlo Scheme:** A semi-analog Monte Carlo scheme is introduced as an innovative strategy to enhance the efficiency and accuracy of simulations.

1.2 Context and Applications

The linear Boltzmann equation finds applications in a wide range of industrial and research contexts. Notable examples include its use in nuclear reactor physics, radiation shielding design, medical imaging, and astrophysics. Applications extend to industries such as nuclear energy, healthcare, and aerospace, where accurate simulations of particle transport are crucial for optimizing designs and ensuring safety protocols.

In the realm of academic research, the linear Boltzmann equation serves as a cornerstone in the study of rarefied gas dynamics, plasma physics, and neutron transport. Collaborations between universities, research institutions, and companies contribute to advancements in understanding and solving complex problems associated with particle transport phenomena.

This project aims to contribute to the broader understanding and application of the linear Boltzmann equation, with a focus on developing efficient numerical methods for solving this intricate problem.

Note Here is a brief summary of the various existences and uniqueness of solutions based on different cases of equations to be solved. The primary source used is the transport and diffusion document.

2 mathematical analysis

2.1 Transport Equation

In this section, we will examine the solutions of the transport equation, a particular case of the Boltzmann equation, defined below:

Let's $v \in \mathbb{R}^N$

$$\frac{\partial f}{\partial t}(t,x) + v \cdot \nabla_x f(t,x) + a(t,x)f(t,x) = S(t,x), \quad x \in \mathbb{R}^N, \quad t > 0. \tag{1}$$

f is the unknown function, a is the given damping or amplification rate, and S is the given source term.

2.1.1 Cauchy Problem

Special Case without Absorption or Source Term   Let's consider the following Cauchy problem:

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) + v \cdot \nabla_x f(t,x) = 0, & x \in \mathbb{R}^N, \quad t > 0. \\ f(0,x)) = f^{in}(x) \end{cases}$$

If $f^{in} \in \mathcal{C}^1(\mathbb{R}^n)$, then using the method of characteristics, we can easily demonstrate that the Cauchy problem has a unique solution $f \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^n)$. This solution is given as follows:

$$\boxed{f(t,x) = f^{in}(x - tv) \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n}$$

With Absorption and Source Term   This time, we consider damping and the source term, assuming they belong to $\mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^n)$. The considered Cauchy problem is as follows:

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) + v \cdot \nabla_x f(t,x) + a(t,x)f(t,x) = S(t,x), & x \in \mathbb{R}^N, \quad t > 0. \\ f(0,x)) = f^{in}(x) \end{cases}$$

In this case, if $f^{in} \in \mathcal{C}^1(\mathbb{R}^n)$, then using the method of characteristics, we can still demonstrate that the Cauchy problem has a unique solution $f \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^n)$. This solution is given as follows:

$$\boxed{f(t,x) = f^{in}(x - tv)e^{-\int_0^t a(\tau, x+(t-\tau)v)d\tau} + \int_0^t e^{-\int_0^t a(\tau, x+(t-\tau)v)d\tau} S(s, x+(s-t)v)ds \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n}$$

We will demonstrate the previous formula using the characteristics method. We will not demonstrate all folowing expression of the exact solution but the idea would be the same.

We will demonstrate the previous formula using the method of characteristics. The proof of the following expression for the exact solution will not be elaborated upon, but the underlying idea would be analogous. The interested reader is invited to see Mettre la source allaire!!!!!.

Let $y \in \mathbb{R}$, and consider the characteristic curve emanating from y at $t = 0$, given by :

$$\{(t, \gamma(t)), |t \geq 0\}, \text{ where } \gamma(t)y + tv$$

If $f \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^n$ is the solution of the previous Cauchy problem, then $t \mapsto f(t, \gamma(t))$ is of $\mathcal{C}^1$ class on $\mathbb{R}_+$, and

$$\begin{aligned} \frac{df}{dt}(t, \gamma(t)) &= (\frac{\partial f}{\partial t} v \cdot \nabla_x f(t,x))(t, \gamma(t)) \\ &= S(t, \gamma(t)) - a(t, \gamma(t))f(t, \gamma(t)), \quad t > 0. \end{aligned}$$

The previous partial differential equation reduces to an ordinary differential equation (ODE) of the form:

$$\begin{cases} u'(t) + \alpha(t)u(t) = \Sigma(t), & t > 0, \\ u(0) = u^{in} \end{cases}$$

The well-known solution is calculated using the constant variation method:

$$u(t) = u^{in}e^{-A(t)} + \int_0^t e^{-A(t)-A(s)}\Sigma(s)ds, \text{ with } A(t) = \int_0^t \alpha(\tau)d\tau.$$

Then by applying this to the Cauchy problem along the characteristic curve from y at $t = 0$, we obtain:

$$f(t, y+tv) = f^{in}(y)e^{\int_0^t a(\tau, y+\tau v)d\tau} + \int_0^t e^{\int_0^t a(\tau, y+\tau v)d\tau} S(s, y+sv)ds, \quad y \in \mathbb{R}^n, t \geq 0.$$

By replacing the dummy variable y with $x - tv$ we arrive the expected formula.

2.1.2 Boundary Value Problems

In the previous section, we presented results applicable when considering complete freedom for the spatial variable in $\mathbb{R}^n$. However, for a significant portion of physical problems, this may not be the case. That is why we are more interested in a problem of the boundary value type. In this section, we will consider the set Ω, which is an open set with a $\mathcal{C}^1$ boundary in $\mathbb{R}^n$. Additionally, we take $v \in \mathbb{R}^n \backslash \{0\}$.

Special Case without Absorption or Source Term

If, moreover, Ω is convex, and let $f_b^- \in \mathcal{C}^1(\mathbb{R}_+ \times \partial\Omega^-)$ and $f^{in} \in \mathcal{C}^1(\overline{\Omega})$, where $\partial\Omega^- = \{x \in \partial\Omega \mid v \cdot n_x < 0\}$, then the problem:

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) + v \cdot \nabla_x f(t,x) = 0, & x \in \mathbb{R}^N, \quad t > 0. \\ f|_{\partial\Omega} = f_b^- \\ f|_{t=0} = f^{in}(x) \end{cases}$$

has a unique solution $f \in \mathcal{C}^1(\mathbb{R}_+ \times \Omega)$ and $f \in \mathcal{C}^0(\mathbb{R}_+ \times \overline{\Omega})$ if and only if $\forall y \in \partial\Omega^-$,

$$f_b^-(0,y) = f^{in}(y), \quad \frac{\partial f_b^-}{\partial t}(t,y) + v \cdot \nabla f^{in}(y) = 0.$$

The expression of the solution is then given by:

$$\boxed{f(t,x) = \begin{cases} f^{in}(x - tv) & si \ t \leq \tau_x, \\ f_b^-(t - \tau_x, x - \tau_x v) & si \ t > \tau_x, \end{cases}}$$

With $\tau_x = \inf\{t \geq 0 \mid x - tv \notin \overline{\Omega}\}$

With Absorption and Source Term

We still consider Ω to be convex, with $f_b^- \in \mathcal{C}^1(\mathbb{R}_+ \times \partial\Omega^-)$, $f^{in} \in \mathcal{C}^1(\overline{\Omega})$, a, and $S \in \mathcal{C}^1(\mathbb{R}_+ \times \overline{\Omega})$. If we assume that $\forall y \in \partial\Omega^-$,

$$f_b^-(0,y) = f^{in}(y),$$

$$\frac{\partial f_b^-}{\partial t}(0,y) + v \cdot \nabla_x f^{in}(y) + a(0,y)f^{in}(y) = S(0,y)$$

In this case, the problem becomes:

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) + v \cdot \nabla_x f(t,x) + a(t,x)f(t,x) = S(t,x), & x \in \mathbb{R}^N, \quad t > 0. \\ f|_{\partial\Omega} = f_b^- \\ f|_{t=0} = f^{in}(x) \end{cases}$$

has a unique solution $f \in \mathcal{C}^1(\mathbb{R}_+ \times \Omega)$ and $f \in \mathcal{C}^0(\mathbb{R}_+ \times \overline{\Omega})$. The expression for f is then:

$$\boxed{\begin{aligned} f(t,x) = \ &\mathbf{1}_{t \leq \tau_x} f^{in}(x - tv) \exp\left(-\int_0^t a(s, x + (s-t)v)ds\right) \\ &+ \mathbf{1}_{t > \tau_x} f_b^-(t - \tau_x, x - \tau_x v) \exp\left(-\int_{t-\tau_x}^t a(s, x + (s-t)v)ds\right) \\ &+ \int_{t-\tau_x}^t \exp\left(-\int_s^t a(\tau, x + (\tau-t)v)d\tau\right) S(s, x + (s-t)v)ds \end{aligned}}$$

2.1.3 Generalized Solutions

In this section, Ω is not necessarily convex. This implies that the continuity of potential solutions is not guaranteed. However, in practice, these cases must be addressed. Below, we define the concept of generalized solutions, which allows us to handle this issue without resorting to the theory of distributions.

Definition Let Ω be an open set with a $\mathcal{C}^1$ boundary in $\mathbb{R}^n$, and let a and S be continuous on $\mathbb{R}_+ \times \Omega$. We say that f is a generalized solution of the problem

$$\frac{\partial f}{\partial t}(t,x) + v \cdot \nabla_x f(t,x) + a(t,x)f(t,x) = S(t,x), \quad x \in \mathbb{R}^N, \quad t > 0.$$

if and only if the function
$s \mapsto f(t+s, x+sv)$ is of class $\mathcal{C}^1$ almost everywhere in $(t,x) \in \mathbb{R}_+ \times \Omega$ and satisfies
$\frac{d}{ds}f(t+s, x+sv) + a(t+s, x+sv)f(t+s, x+sv) = S(t+s, x+sv) \quad (t+s, x+sv) \in \mathbb{R}_+^* \times \Omega.$

Let Ω be an open set with a $\mathcal{C}^1$ boundary in $\mathbb{R}^n$, and let a and S be continuous on $\mathbb{R}_+ \times \Omega$. Let $f_b^- \in \mathbf{L}_{loc}^\infty(\mathbb{R}_+ \times \partial\Omega^-)$ and $f^{in} \in \mathbf{L}_{loc}^\infty(\overline{\Omega})$.

In this case, the problem:

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) + v \cdot \nabla_x f(t,x) + a(t,x)f(t,x) = S(t,x), \quad x \in \mathbb{R}^N, \quad t > 0. \\ f|_{\partial\Omega} = f_b^- \\ f|_{t=0} = f^{in}(x) \end{cases}$$

has a unique generalized solution. The formula for this solution is the same as in the previous case, i.e., almost everywhere in $(t,x) \in \mathbb{R}_+ \times \Omega$, we have:

$$\begin{aligned} f(t,x) = \mathbf{1}_{t \leq \tau_x} f^{in}(x-tv) &\exp\left(-\int_0^t a(s, x+(s-t)v)ds\right) \\ + \mathbf{1}_{t > \tau_x} f_b^-(t-\tau_x, x-\tau_x v) &\exp\left(-\int_{t-\tau_x}^t a(s, x+(s-t)v)ds\right) \\ + \int_{t-\tau_x}^t \exp\left(-\int_s^t a(\tau, x+(\tau-t)v)d\tau\right) &S(s, x+(s-t)v)ds \end{aligned}$$

2.2 Linear Boltzmann Equation

In this section, we will focus on the actual Boltzmann equation. This integro-differential equation is described in **??**.

$$\frac{\partial f}{\partial t}(t,x,v) + v \cdot \nabla_x f(t,x,v) + a(t,x)f(t,x,v) = \int_{\mathbb{R}^n} k(t,x,v,w)f(t,x,w)d\mu(w) + Q(t,x,v). \quad (2)$$

To simplify notations, this equation will be written as:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + af = \mathcal{K}f + Q$$

Here, v and k are given continuous functions. The range of possibilities for the velocity v is very diverse due to the variety of physical problems that can be considered. Therefore, the notation $d\mu(w)$ encompasses all these possibilities by incorporating a weight function. The major difference from the transport equation is that, this time, we consider exchange phenomena, and thus, v cannot be seen as a simple parameter.

In this section, we cannot highlight the exact expression of the solutions because the method of characteristics yields another integral equation involving the solution.

Definition Let Q be continuous on $]0, T[\times\Omega \times \mathbb{R}^n$. We say that f is a generalized solution of the problem, continuous on $]0, T[\times\Omega \times \mathbb{R}^n$,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + af = \mathcal{K}f + Q$$

if and only if, for all $(t, x, v) \in]0, T[\times\Omega \times \mathbb{R}^n$, the function

$$s \mapsto f(t+s, x+sv, v)$$

is of class $\mathcal{C}^1$ for all $x + sv \in \Omega$ and satisfies, for all $s \in \mathbb{R}$ such that $x + sv \in \Omega$,
$\frac{d}{ds}f(t+s, x+sv, v) + a(t+s, x+sv, v)f(t+s, x+sv, v) = (\mathcal{K}f + Q)(t+s, x+sv, v)$.

2.2.1 Cauchy Problem

Let $f^{in} \in \mathbf{C}_b(\mathbb{R}^n \times \mathbb{R}^n)$ and $Q \in \mathbf{C}_b(]0, T[\times\mathbb{R}^n \times \mathbb{R}^n)$. Also, let $0 \le a \in \mathbf{C}_b(]0, T[\times\mathbb{R}_x^n \times \mathbb{R}_v^n)$ and $0 \le k \in \mathbf{C}_b(]0, T[\times\mathbb{R}_x^n \times \mathbb{R}_v^n \times \mathbb{R}_w^n)$. The notation $\mathbf{C}_b(X)$ denotes the set of bounded continuous functions on X with values in $\mathbb{R}$.

We assume that:

$$\sup_{(t,x,v)\in]0,T[\times\mathbb{R}^n\times\mathbb{R}^n} \int_{\mathbb{R}^n} k(t, x, v, w) \, d\mu(w) < +\infty$$

Then the problem is:

$$\begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f + af = \mathcal{K}f + Q & (t, x, v) \in]0, T[\times\mathbb{R}^n \times \mathbb{R}^n \\ f|_{t=0} = f^{in} \end{cases}$$

has a unique generalized solution $f \in \mathbf{C}_b(]0, T[\times\mathbb{R}^n \times \mathbb{R}^n)$.

2.2.2 Boundary Value Problem

We define the set Γ_- as $\Gamma_- = \{(x, v) \in \partial\Omega \times \mathbb{R}^n \mid v \cdot n_x < 0\}$
Let $f^{in} \in \mathbf{C}_b(\overline{\Omega} \times \mathbb{R}^n)$, $f_b^- \in \mathbf{C}_b([0, T] \times \Gamma_-)$ such that $f_b^-(0, y, v) = f^{in}(y, v) \quad \forall(y, v) \in \Gamma_-$
Let $Q \in \mathbf{C}_b([0, T] \times \overline{\Omega} \times \mathbb{R}^n)$.
Let $0 \le a \in \mathbf{C}_b(\mathbb{R}_+ \times \Omega_x \times \mathbb{R}_v^n)$ and $0 \le k \in \mathbf{C}_b(\mathbb{R}_+ \times \Omega_x^n \times \mathbb{R}_v^n \times \mathbb{R}_w^n)$.
Then the problem

$$\begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f + af = \mathcal{K}f + Q & (t, x, v) \in]0, T[\times\Omega \times \mathbb{R}^n \\ f|_{t=0} = f^{in} \\ f|_{\Gamma^-} = f_b^- \end{cases}$$

has a unique generalized solution $f \in \mathbf{C}_b([0, T] \times \Omega \times \mathbb{R}^n)$.

3 The use of Monte Carlo methods to solve the linear Boltzmann equation

To facilitate the introduction of a numerical Scheme let introduce new notation. The transport equation in an infinite medium with its corresponding deterministic collisional component can be expressed as:

$$\partial_t u(x, t, \mathbf{v}) + \mathbf{v} \cdot \nabla u(x, t, \mathbf{v}) + v\sigma_t(x, t, \mathbf{v})u(x, t, \mathbf{v}) = v\sigma_s(x, t, \mathbf{v}) \int P(x, t, \mathbf{v}, \mathbf{v}')u(x, t, \mathbf{v}')d\mathbf{v}' \quad (3)$$

Where
$$\sigma_s(x, t, \mathbf{v}) = \int \sigma_s(x, t, \mathbf{v}, \mathbf{v}')d\mathbf{v}', \quad P(x, t, \mathbf{v}, \mathbf{v}') = \frac{\sigma_s(x, t, \mathbf{v}, \mathbf{v}')}{\sigma_s(x, t, \mathbf{v})}$$

The approach involves a series of variable changes. The initial step involves re-expressing the transport equation **??** with respect to a characteristic $x + vt$. As a result, it transforms into:

$$\partial_s u(x+\mathbf{v}s, s, \mathbf{v}) = -v\sigma_t(x+\mathbf{v}s, s, \mathbf{v})u(x+\mathbf{v}s, s, \mathbf{v})+v\sigma_s(x+\mathbf{v}s, s, \mathbf{v}) \int P(x+\mathbf{v}s, s, \mathbf{v}, \mathbf{v}')u(x+\mathbf{v}s, s, \mathbf{v}')d\mathbf{v}'$$
(4)

Let's multiply both sides of the equation by:

$$e^{\int_0^s v\sigma_t(x+\mathbf{v}\alpha, \alpha, v)d\alpha}$$

Following that, we obtain

$$\partial_s[u(x+\mathbf{v}s, s, \mathbf{v})e^{\int_0^s v\sigma_t(x+\mathbf{v}\alpha, \alpha, v)d\alpha}] = e^{\int_0^s v\sigma_t(x+\mathbf{v}\alpha, \alpha, v)d\alpha}v\sigma_s(x+\mathbf{v}s, s, \mathbf{v}) \int P(x+\mathbf{v}s, s, \mathbf{v}, \mathbf{v}')u(x+\mathbf{v}s, s, \mathbf{v}')d\mathbf{v}'$$

We get after integrating the equation **??** between [0, t]:

$$u(x + \mathbf{v}t, t, \mathbf{v}) = u_0(x, \mathbf{v}) \exp\left(-\int_0^t v\sigma_t (x + \mathbf{v}\alpha, \alpha, \mathbf{v}) \, d\alpha\right)$$
$$+ \int_0^t \int v\sigma_s (x + \mathbf{v}s, s, \mathbf{v}) u (x + \mathbf{v}s, s, \mathbf{v}') e^{-\int_s^t v\sigma_t(x+\mathbf{v}\alpha, \mathbf{v})d\alpha} P (x + \mathbf{v}s, s, \mathbf{v}, \mathbf{v}') \, d\mathbf{v}'ds \quad (5)$$

We obtain:

$$u(x, t, \mathbf{v}) = u_0(x - \mathbf{v}t, \mathbf{v}) \exp\left(-\int_0^t v\sigma_t (x - \mathbf{v}(t - \alpha), \alpha, \mathbf{v}) \, d\alpha\right)$$
$$+ \int_0^t \int v\sigma_s (x - \mathbf{v}(t - s), s, \mathbf{v}) u (x - \mathbf{v}(t - s), s, \mathbf{v}') e^{-\int_s^t v\sigma_t(x-\mathbf{v}(t-\alpha), \mathbf{v})d\alpha} P (x - \mathbf{v}(t - s), s, \mathbf{v}, \mathbf{v}') \, d\mathbf{v}'ds$$
(6)

We also have:

$$\exp\left(-\int_0^t v\sigma_t (x - \mathbf{v}(t - \alpha), \alpha, \mathbf{v}) \, d\alpha\right) = \exp\left(-\int_0^t v\sigma_t (x - \mathbf{v}\alpha, t - \alpha, \mathbf{v}) \, d\alpha\right)$$
$$= \int_t^\infty v\sigma_t (x - \mathbf{v}s, t - s, \mathbf{v}) \exp\left(-\int_0^s v\sigma_t (x - \mathbf{v}\alpha, t - \alpha, \mathbf{v}) \, d\alpha\right) ds \quad (7)$$

Then the integral representation of **??** is provided by:

$$u(x, t, \mathbf{v}) = \int_t^\infty u_0(x - \mathbf{v}t, \mathbf{v})v\sigma_t (x - \mathbf{v}s, t - s, \mathbf{v}) \exp\left(-\int_0^s v\sigma_t (x - \mathbf{v}\alpha, t - \alpha, \mathbf{v}) \, d\alpha\right) ds$$
$$+ \int_0^t \int v\sigma_s (x - \mathbf{v}(t - s), s, \mathbf{v}) u (x - \mathbf{v}(t - s), s, \mathbf{v}') e^{-\int_s^t v\sigma_t(x-\mathbf{v}(t-\alpha), \mathbf{v})d\alpha} P (x - \mathbf{v}(t - s), s, \mathbf{v}, \mathbf{v}') \, d\mathbf{v}'ds$$
(8)

4 Semi analog MC scheme

Developing a Monte Carlo scheme involves introducing a group of random variables along with their associated probability measure to express **??** as an expectation. The selection of this set of random variables is not unique, resulting in various Monte Carlo schemes with distinct properties. In this project we used one of them, the so-called Semi-analog scheme. The algorithm 1 describe this method.

This method is described as backward. In fact, we first initialize S_p, the remaining lifetime of the particle, to t, and it will decrease to zero. Here, we adopt a general form for the boundary conditions. In the program we developed, we have tried various conditions: Periodic boundary conditions (which are not truly significant in a physical problem but can be interpreted as an infinite environment), and Elastic boundary conditions.

At line 11, we sample τ using the function: $\tau = \frac{-\ln(\mathcal{U}(0,1))}{\sigma_t \sqrt{v^2}}$. If τ is greater than s_p, then the particle is moved, its lifetime is set to zero, and its contribution is updated. Otherwise, the particle is moved, its speed is sampled, its lifetime is updated, and then its weight is changed.

Algorithm 1 Semi analog scheme algorithm

1: $u(x,t,v) \leftarrow 0$
2: **for** $p = 1$ to N_{MC} **do**
3:     $s_p \leftarrow t$
4:     $x_p \leftarrow x$
5:     $v_p \leftarrow v$
6:     $w_p \leftarrow \frac{1}{N_{MC}}$
7:     **while** $s_p > 0$ AND $w_p > 0$ **do**
8:         **if** $x_p \notin D$ **then**
9:             apply_boundary_conditions(x_p, s_p, v_p)
10:         **end if**
11:         Sample τ.
12:         **if** $\tau > s_p$ **then**
13:             $x_p \leftarrow -v_p s_p$
14:             $s_p \leftarrow 0$
15:             $u(x,t,v) \leftarrow w_p \times u_0(x_p, v_p)$
16:         **else**
17:             $x_p \leftarrow -v_p \tau$
18:             Sample the velocity of particle p from $P(x_p, s_p, v_p, dv')$
19:             $v_p \leftarrow v'$
20:             $s_p \leftarrow -\tau$
21:             $w_p \leftarrow \frac{\sigma(x_p, s_p, v')}{\sigma(x_p, s_p, v_p)} \times w_p$
22:         **end if**
23:     **end while**
24: **end for**

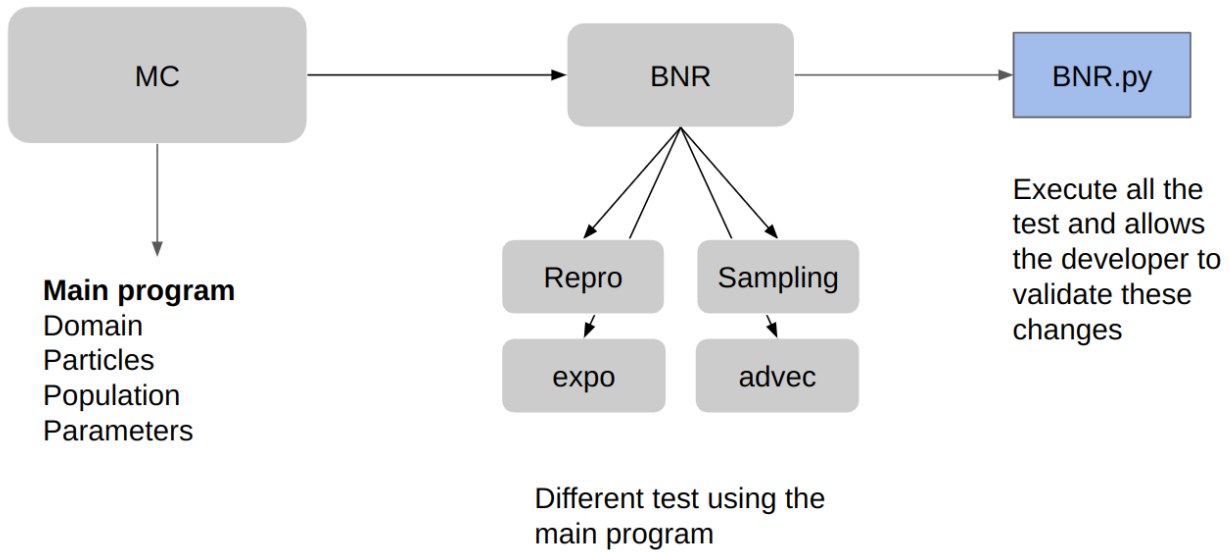# 5 Numerical tests/results

5.1 Code architecture

Figure 1: Code architecture

The figure **??** represent the structure of our implementation. The **Main program** is used for several computation, **Domain** set up the boundary condition, initial condition and the sampling of v and τ. **Particles** set up the semi analog scheme,**Population** Move all the particles and **parameters** initialize the parameters.

6 Conclusion

First and foremost, we would like to express our sincere appreciation to our supervisor, Gael Poette, whose exceptional support was evident through his constant presence and responsiveness throughout our research.

This project provided a unique opportunity to work on Monte Carlo methods. Accustomed to working with partial differential equations, the adoption of this new method for solving equations presented a stimulating challenge over the course of these four months. Collaborating in a group of five within such a limited timeframe posed its own set of challenges, but we successfully brought the project to completion.

The work we accomplished enabled us to gain a deeper understanding of high-dimensional equation solving, a skill set that will undoubtedly prove invaluable in our future professional lifes.